REGULARITY OF AREA MINIMIZING CURRENTS II: CENTER MANIFOLD

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Abstract. This is the second paper of a series of three on the regularity of higher codimension area minimizing integral currents. Here we perform the second main step in the analysis of the singularities, namely the construction of a *center manifold*, i.e. an approximate average of the sheets of an almost flat area minimizing current. Such center manifold is complemented with a Lipschitz multi-valued map on its normal bundle, which approximates the current with a highe degree of accuracy. In the third and final paper these objects are used to conclude a new proof of Almgren's celebrated dimension bound on the singular set.

0. Introduction

In this second paper on the regularity of area minimizing integer rectifiable currents (we refer to the Foreword of [6] for the precise statement of the final theorem and on overview of its proof) we address one of the main steps in the analysis of the singularities, namely the construction of what Almgren calls *center manifold*. Unlike the case of hypersurfaces, singularities in higher codimension currents can appear as "higher order" perturbation of smooth minimal submanifolds. In order to illustrate this phenomenon, we can consider the examples of area minimizing currents induced by complex varieties of $\mathbb{C}^n$, as explained in the Foreword of [6]. Take, for instance, the complex curve:

$$\mathcal{V} := \left\{(z,w) : (z-w^2)^2 = w^5\right\} \subset \mathbb{C}^2.$$

The point $0 \in \mathcal{V}$ is clearly a singular point. Nevertheless, in every sufficiently small neighborhood of the origin, $\mathcal{V}$ looks like a small perturbation of the smooth minimal surface $\{z = w^2\}$: roughly speaking, $\mathcal{V} = \{z = w^2 \pm w^{5/2}\}$. One of the main issues of the regularity of area minimizing currents is to understand this phenomenon of "higher order singularities". Following the pioneering work of Almgren [2], a way to deal with it is to approximate the minimizing current with the graph of a multiple valued function on the normal bundle of a suitable, curved, manifold. Such manifold must be close to the "average of the sheets" of the current (from this the name *center manifold*): the hope is that such a property will guarantee a singular "first order expansion" of the corresponding approximating map.

A "center manifold" with such an approximation property is clearly very far from being uniquely defined and moreover the relevant estimates are fully justified only by the concluding arguments, which will appear in [7]. In this paper, building upon the works [4, 5, 6], we provide a construction of a center manifold $\mathcal{M}$ and of an associated approximation of the corresponding area minimizing current via a multiple valued function $F : \mathcal{M} \to \mathcal{A}_Q(\mathbb{R}^{m+n})$.

The corresponding construction of Almgren is given in [2, Chapter 4]. Unfortunately, we do not understand this portion of Almgren's monograph deeply enough to make a rigorous comparison between the two constructions. Even a comparison between the statements is prohibitive, since the main ones of Almgren (cf. [2, 4.30 & 4.33]) are rather involved and seem to require a thorough understanding of most of the chapter (which by itself has the size of a rather big monograph). At a first sight, our approach seems to be much simpler and to deliver better estimates.

In the rest of this introduction we will explain some of the main aspects of our construction.

0.1. **Whitney-type decomposition.** The center manifold is the graph of a classical function over an m-dimensional plane with respect to which the excess of the minimizing current is sufficiently small. To achieve a suitable accuracy in the approximation of the average of the sheets of the current, it is necessary to define the function and an appropriate scale, which varies locally. Around any given point such scale is morally the first at which the sheets of the current cease to be close. This leads to a Whitney-type decomposition of the reference m-plane, where the refining algorithm is stopped according to three conditions. In each cube of the decomposition the center manifold is then a smoothing of the average of the Lipschitz multiple valued approximation constructed in [4], performed in a suitable orthonormal system of coordinates, which changes from cube to cube.

0.2. **$C^{3,\kappa}$-regularity of $\mathcal{M}$.** The arguments of [7] require that the center manifold is at least C^3-regular. As it is the case of Almgren's center manifold, we prove actually $C^{3,\kappa}$ estimates, which are a natural outcome of some Schauder estimates. It is interesting to notice that, if the current has multiplicity one everywhere (i.e., roughly speaking, is made of a single sheet), then the center manifold coincides with it and, hence, we can conclude directly a higher regularity than the one given by the usual De Giorgi-type (or Allard-type) argument. This is already remarked in the introduction of [2] and it has been proved in our paper [3] with a relatively simple and short direct argument. The interested reader might find useful to consult that reference as well, since many of the estimates of this note appear there in a much more elementary form.

0.3. **Approximation on $\mathcal{M}$.** Having defined a center manifold, we then give a multivalued map on its normal bundle which approximates the current. The relevant estimates on this map and its approximation properties are then given locally for each cube of the Whitney decomposition used in the construction of the center manifold. We follow a simple principle: at each scale where the refinement of the Whitney decomposition has stopped, the image of such function coincides (on a large set) with the Lipschitz multiple valued approximation constructed in [6], i.e. the same map whose smoothed average has been used to construct the center manifold. As a result, the graph of F is well centered, i.e. the average of F is very close (compared to its Dirichlet energy and its L^2 norm) to be the manifold $\mathcal{M}$ itself. As far as we understand Almgren is not following this principle and it seems very difficult to separate his construction of the center manifold from the one of the approximating map.

0.4. **Splitting before tilting.** The regularity of the center manifold $\mathcal{M}$ and the centering of the approximating map F are not the only properties needed to conclude our proof in [7]. Another ingredient plays a crucial role. Assume that around a certain point, at all scales larger than a given one, say s, the excess decays and the sheets stay very close. If at scale s the excess is not decaying anymore, then the sheets must separate as well. In other words, since the tilting of the current is under control up to scale s, the current must in some sense "split before tilting". We borrow the terminology from a remarkable work of Rivière [10], where first this phenomenon was investigated independently of Almgren's monograph in the case of 2-dimensional area-minimizing currents. Rivière's approach relies on a clever "lower epiperimetric inequality", which unfortunately seems limited to the 2-dimensional context.

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1. Construction algorithm and main existence theorem

The goal of this section is to specify the algorithm leading to the center manifold. This algorithm will involve a few parameters. Their choice plays an important role and for a couple of them it will be fully specified later. In fact the algorithm can be performed only when these parameters satisfy certain inequalities: these will instead be declared rather soon.

In what follows let $\Sigma \subset \mathbb{R}^{m+n}$ be a C^{3,ε_0} embedded submanifold of dimension $m+\bar{n}$, and T an integer rectifiable current of dimension m in Σ. Moreover, set $l := n - \bar{n}$. For balls in $\mathbb{R}^{m+n}$ we will use $\mathbf{B}_r(p)$. If $\pi \subset \mathbb{R}^{m+n}$ is a subspace, $\mathbf{p}_\pi$ will denote the orthogonal projection onto it. $B_r(q,\pi)$ is the m-dimensional ball in the affine plane $q+\pi$ with center q and radius r, i.e. $\mathbf{B}_r(q)\cap(q+\pi)$, whereas $\mathbf{C}_r(p,\pi)$ will be used for the cylinder $\{(x+y) : x \in \mathbf{B}_r(p), y \in \pi^\perp\}$. The points p and q will be omitted if they are the origin and the plane π will be omitted if it is $\pi_0 := \mathbb{R}^m \times \{0\}$. In what follows we also assume that π is always *oriented* by an m-vector $\vec{\pi} := v_1 \wedge \ldots \wedge v_m$ (thereby making a distinction when the same plane is given opposite orientations).

Definition 1.1. Given a current T, we define the *excess* of T in balls and cylinders as

$$\mathbf{E}(T, \mathbf{B}_r(x), \pi) := (2\omega_m\, r^m)^{-1} \int_{\mathbf{B}_r(x)} |\vec{T} - \vec{\pi}|^2\, d\|T\|, \tag{1.1}$$

$$\mathbf{E}(T, \mathbf{C}_r(x,\pi), \pi') := (2\omega_m\, r^m)^{-1} \int_{\mathbf{C}_r(x,\pi)} |\vec{T} - \vec{\pi}'|^2\, d\|T\|\,, \tag{1.2}$$

and the *height functions* in a set A as

$$\mathbf{h}(T,A,\pi) := \sup_{x,y\,\in\,\mathrm{spt}(T)\,\cap\,A} |\mathbf{p}_{\pi^\perp}(x)-\mathbf{p}_{\pi^\perp}(y)|,$$

with $\pi^\perp$ the orthogonal complement to π. The shortened notation $\mathbf{E}(T,\mathbf{C}_r(x,\pi))$ will be used for $\mathbf{E}(T,\mathbf{C}_r(x,\pi),\pi)$, which is the cylindrical excess as defined in [6] provided $(\mathbf{p}_\pi)_\sharp T \llcorner \mathbf{C}_r(x,\pi) = Q\,[\![B_r(\mathbf{p}_\pi(x),\pi)]\!]$. With a slight abuse of notation, we also write $|\pi_2-\pi_1|$ for $|\vec\pi_2-\vec\pi_1|$. We say that an m-dimensional plane π is *optimal* for T in a ball $\mathbf{B}_r(x)$ if

$$\mathbf{E}(T,\mathbf{B}_r(x)) := \min_\tau \mathbf{E}(T,\mathbf{B}_r(x),\tau) = \mathbf{E}(T,\mathbf{B}_r(x),\pi), \tag{1.3}$$

and

$$\mathbf{h}(T,\mathbf{B}_r(x)) := \min\big\{\mathbf{h}(T,\mathbf{B}_r(x),\tau) : \tau \text{ satisfies (1.3)}\big\} = \mathbf{h}(T,\mathbf{B}_r(x),\pi).$$

In other words, π is optimal if it minimizes the excess and, among all the minimizers of the excess, it also minimizes the height: in particular if the minimizer τ in (1.3) were unique, then the second requirement would be redundant. In any case we do not claim any uniqueness (which in general would be false). We are now ready to summarize the main assumptions of our theorems.

Assumption 1.2. We assume that, for each $p\in\Sigma$, Σ is the graph of a C^{3,ε_0} map $\Psi_p : T_p\Sigma \to T_p\Sigma^\perp$. We denote by $\mathbf{c}(\Sigma)$ the number $\sup_{p\in\Sigma}\|D^2\Psi_p\|_{C^{1,\varepsilon_0}}$. T^0 is an m-dimensional integral current supported in Σ and area minimizing in $\Sigma\setminus\mathrm{spt}(\partial T^0)$ such that

$$\Theta(0,T^0)=Q \quad\text{and}\quad \partial T^0\llcorner\mathbf{B}_{6\sqrt{m}}=0, \tag{1.4}$$

$$\|T^0\|(\mathbf{B}_{6\sqrt{m}\rho}) \le \left(\omega_m Q(6\sqrt{m})^m+\varepsilon_2^2\right)\rho^m \quad \forall\rho\le 1, \tag{1.5}$$

$$\mathbf{E}\left(T^0,\mathbf{B}_{6\sqrt{m}}\right)=\mathbf{E}\left(T^0,\mathbf{B}_{6\sqrt{m}},\pi_0\right), \tag{1.6}$$

$$\boldsymbol{m}_0 := \max\left\{\mathbf{c}(\Sigma)^2, \mathbf{E}\left(T^0,\mathbf{B}_{6\sqrt{m}}\right)\right\} \le \varepsilon_2^2, \tag{1.7}$$

where ε_2 is a small positive constant. To simplify the notation later it is convenient to set $T := T^0\llcorner\overline{\mathbf{B}}_{23/4\sqrt{m}}$.

Remark 1.3. Note that (1.7) implies $\mathbf{A} := \|A_\Sigma\|_{C^0(\Sigma)} \le C\boldsymbol{m}_0^{1/2}$, where A_Σ denotes the second fundamental form of Σ and C is a geometric constant. Moreover, since $D\Psi_p(p)=0$, we also infer $\|D\Psi_p\|_{C^1}\le C\boldsymbol{m}_0^{1/2}$ in the ball of radius $6\sqrt{m}$. Similarly the oscillation of Ψ_p is controlled, in $B_{6\sqrt{m}}$, by $C\boldsymbol{m}_0^{1/2}$.

We will sometimes parametrize Σ as a graph of a function Ψ over a plane which is not tangent to Σ but tilted by at most $C\boldsymbol{m}_0^{1/2}$. By Lemma B.1 we then still have $\|D\Psi\|_{C^{2,\varepsilon_0}}\le C\boldsymbol{m}_0^{1/2}$. We show now that, without loss of generality, we can make this assumption for the plane $\mathbb{R}^{m+\bar n}\times\{0\}$. Indeed, by (1.7) and the monotonicity formula there is a point $p\in\Sigma\cap\mathbf{B}_{6\sqrt{m}}$ such that the distance between π_0 and its projection $\mathbf{p}_{T_p\Sigma}(\pi_0)$ is at most $C\boldsymbol{m}_0^{1/2}$. Therefore, by possibly rotating the coordinates orthogonal to π_0, we can assume that $\mathbb{R}^{m+\bar n}\times\{0\}$ is tilted by at most $C\boldsymbol{m}_0^{1/2}$ compared to $T_0\Sigma$.

We specify next some notation which will be recurrent in the paper when dealing with cubes of π_0. For each $j \in \mathbb{N}$, $\mathscr{C}^j$ denotes the closed cubes L of π_0 of the form

$$[a_1, a_1 + 2\ell] \times \ldots \times [a_m, a_m + 2\ell] \times \{0\} \subset \pi_0 \times \pi_0^\perp, \tag{1.8}$$

where $2\,\ell = 2^{1-j} =: 2\,\ell(L)$ is the side-length of the cube, $a_i \in 2^{1-j}\mathbb{Z}\ \forall i$ and we require in addition $L \subset [-4,4]^m$. The *center of* L is $x_L = (a_1 + \ell, \ldots, a_m + \ell)$ and we write $\mathscr{C} = \bigcup_{j\in\mathbb{N}} \mathscr{C}^j$. In what follows, to avoid cumbersome notation, we will usually drop the factor $\{0\}$ in (1.8). If $H \in \mathscr{C}^j$, we call *father of* H the unique cube $L \in \mathscr{C}^{j-1}$ which contains it and likewise we say that H is *a son* of L. In general, if H and L are two cubes in $\mathscr{C}$ with $H \subset L$, then we call L an *ancestor* of H and H a *descendant* of L.

Definition 1.4. A Whitney decomposition of $[-4,4]^m \subset \pi_0$ consists of a closed set $\mathbf{\Gamma} \subset [-4,4]^m$ and a family $\mathscr{W} \subset \mathscr{C}$ of dyadic cubes satisfying the following properties:

(w1) $\mathbf{\Gamma} \cup \bigcup_{L\in\mathscr{W}} L = [-4,4]^m$ and $\mathbf{\Gamma}$ does not intersect any element of $\mathscr{W}$;
(w2) the interiors of any pair of distinct cubes $L_1, L_2 \in \mathscr{W}$ are disjoint;
(w3) if $L_1, L_2 \in \mathscr{W}$ have nonempty intersection, then $\frac{1}{2}\ell(L_1) \le \ell(L_2) \le 2\,\ell(L_1)$.

Observe that (w1) - (w3) imply $\mathrm{dist}(\mathbf{\Gamma}, L) \ge 2\ell(L)$ for every $L \in \mathscr{W}$. However, we do *not* require any inequality of the form $\mathrm{dist}(\mathbf{\Gamma}, L) \le C\ell(L)$, although this would be customary for what is commonly called Whitney decomposition in the literature.

1.1. **The Whitney decomposition.** For every $L \in \mathscr{C}$, set $r_L := M_0\sqrt{m}\,\ell(L)$ and choose a ball $\mathbf{B}_L$ of radius $64r_L$, where the constant M_0 will be specified later, and center $p_L := (x_L, y_L) \in \mathrm{spt}(T)$ (recall that x_L is the center of the cube). The existence of p_L is in fact guaranteed by Assumption 1.2 as proved in Proposition 1.7 below. However the point is not unique and we fix an arbitrary choice: it turns out that such arbitrariness does not play any role for our arguments. We are now ready to introduce the main parameters of the construction.

Assumption 1.5. $C_e, C_h, \beta_2, \delta_2, M_0$ are positive constants and N_0 a natural number for which we assume always

$$\beta_2 = 4\,\delta_2 = \min\left\{\frac{1}{2m}, \frac{\gamma_1}{100}\right\}, \quad \text{where } \gamma_1 \text{ is the constant of [6, Theorem 1.4]}, \tag{1.9}$$

$$M_0 \ge C(m,n,Q)\,, \quad 2^{-N_0-1} < \frac{1}{16\sqrt{m}} \quad \text{and} \quad M_0 2^{7-N_0} \le 1\,. \tag{1.10}$$

Remark 1.6 (Choice of the parameters)**.** Along the various statements of the main propositions, we will make several further assumptions upon the parameters involved. However, we stress that their choice is consistent and finally made in the following order:

- $m, n, \bar{n}, Q$ and ε_0 are given: the dependence of the constants upon these parameters will usually not be mentioned;
- β_2 and δ_2, as already seen, have an explicit dependence upon γ_1;
- M_0 is chosen large enough, so to fulfill several inequalities and, more importantly, the assumption of Proposition 3.3, which depends only on δ_2;

- N_0 is chosen so to satisfy (1.10) (and hence depends on M_0) but also so large that Proposition 3.6 holds;
- C_e is then chosen, depending upon all the previous parameters, so large that the statements of Proposition 1.7, Theorem 1.12 and Proposition 3.3 hold;
- C_h is chosen large enough, depending also on C_e, in particular such that Propositions 1.7 and 3.1 hold;
- ε_2 is the last to be chosen and will have to satisfy several smallness conditions depending upon all the other parameters.

Next we identify five families of cubes $\mathscr{S}$ and $\mathscr{W} = \mathscr{W}_e \cup \mathscr{W}_h \cup \mathscr{W}_n$, using the convention that $\mathscr{S}^j = \mathscr{S} \cap \mathscr{C}^j$ and an analogous one for all the other families. We start with $j = N_0$ and set $\mathscr{W}^{N_0-1} := \emptyset$. A cube $L \in \mathscr{C}^j$ belongs to

(EX) $\mathscr{W}_e^j$ if $\mathbf{E}(T, \mathbf{B}_L) > C_e \boldsymbol{m}_0\, \ell(L)^{2-2\delta_2}$;
(HT) $\mathscr{W}_h^j$ if $L \notin \mathscr{W}_e^j$ and $\mathbf{h}(T, \mathbf{B}_L) > C_h \boldsymbol{m}_0^{1/2m} \ell(L)^{1+\beta_2}$;
(NN) $\mathscr{W}_n^j$ if $L \notin \mathscr{W}_e^j \cup \mathscr{W}_h^j$ but it intersects an element of $\mathscr{W}^{j-1}$;
(S) $\mathscr{S}^j$ if none of the above occurs.

We then subdivide each element of $\mathscr{S}^j$ into 2^m cubes of equal side-length, clearly belonging to $\mathscr{C}^{j+1}$, and we iterate the selection procedure explained above for each of them. We finally set

$$\boldsymbol{\Gamma} := [-4,4]^m \setminus \bigcup_{L \in \mathscr{W}} L = \bigcap_{j \geq N_0} \bigcup_{L \in \mathscr{S}^j} L.$$

Proposition 1.7 (Whitney decomposition)**.** *Let Assumptions 1.2 and 1.5 hold. If ε_2 is sufficiently small, the balls $\mathbf{B}_L$ are well-defined and $(\boldsymbol{\Gamma}, \mathscr{W})$ is a Whitney decomposition of π_0. Moreover, for any fixed M_0 and N_0, there is $C^\star := C^\star(M_0, N_0)$ such that, if $C_h \geq C^\star C_e \geq (C^\star)^2$, then $\mathscr{W}^j = \emptyset$ for all $j \leq N_0 + 6$. As a consequence, under such assumption the following estimates hold:*

$$\mathbf{E}(T, \mathbf{B}_J) \leq C_e \boldsymbol{m}_0\, \ell(J)^{2-2\delta_2} \quad \textit{and} \quad \mathbf{h}(T, \mathbf{B}_J) \leq C_h \boldsymbol{m}_0^{1/2m} \ell(J)^{1+\beta_2} \quad \forall J \in \mathscr{S}\,, \tag{1.11}$$

$$\mathbf{E}(T, \mathbf{B}_L) \leq C\, \boldsymbol{m}_0\, \ell(L)^{2-2\delta_2} \quad \textit{and} \quad \mathbf{h}(T, \mathbf{B}_L) \leq C\, \boldsymbol{m}_0^{1/2m} \ell(L)^{1+\beta_2} \quad \forall L \in \mathscr{W}\,, \tag{1.12}$$

where the latter constant C depends only on $M_0, N_0, \beta_2, \delta_2, C_e, C_h$.

1.2. **Construction algorithm.** We fix next two important functions $\vartheta, \varrho : \mathbb{R}^m \to \mathbb{R}$, and set, as usual, $\varrho_\lambda := \lambda^{-m} \varrho(\frac{x}{\lambda})$.

Assumption 1.8. $\varrho \in C_c^\infty(B_1)$ is radial, $\int \varrho = 1$ and $\int |x|^2 \varrho(x)\, dx = 0$.
$\vartheta \in C_c^\infty\big([-\frac{17}{16}, \frac{17}{16}]^m, [0,1]\big)$ is identically 1 on $[-1,1]^m$.

Definition 1.9 (π-approximations)**.** Fix next any $L \in \mathscr{S}^j \cup \mathscr{W}^j$ and an m-plane π.

(a1) If [6, Theorem 1.4] can be applied to T in the cylinder $\mathbf{C}_{32 r_L}(p_L, \pi)$, we then call the resulting map $f : B_{8 r_L}(p_L, \pi) \to \mathcal{A}_Q(\pi^\perp)$ the *π-approximation of T in $\mathbf{C}_{8 r_L}(p_L, \pi)$*.
(a2) The map $h : B_{7 r_L}(p_L, \pi) \to \pi^\perp$ given by $h := (\boldsymbol{\eta} \circ f) * \varrho_{\ell(L)}$ will be called the *smoothed average of the π-approximation.*

Definition 1.10 (Interpolating functions)**.** For each L as in Definition 1.9, we select an optimal plane $\hat\pi_L$ in $\mathbf{B}_L$. We then denote by π_L a m-dimensional plane contained in $T_{p_L}\Sigma$ which minimizes $|\hat\pi_L - \pi_L|$. The π_L-approximation (if it exists) is denoted by f_L. If h is its smoothed average and $\bar h := \mathbf{p}_{T_{p_L}\Sigma}(h)$, then the map $x \mapsto h_L(x) := \Psi_{p_L}(x, \bar h(x))$ is called the *tilted interpolating function relative to* L. Moreover, if there is a map $g_L : B_{4r_L}(p_L, \pi_0) \to \pi_0^\perp$ such that $\mathbf{G}_{g_L} = \mathbf{G}_{h_L} \llcorner \mathbf{C}_{4r_L}(p_L, \pi_0)$ (notation as in [5]), then g_L is the *interpolating function relative to* L.

Definition 1.11 (Glued interpolations)**.** For each j consider the family of cubes $\mathscr{P}^j := \mathscr{S}^j \cup \bigcup_{i=N_0}^j \mathscr{W}^i$. For each $L \in \mathscr{P}^j$, define $\vartheta_L(y) := \vartheta(\frac{y-x_L}{\ell(L)})$ and set

$$\hat\varphi_j := \frac{\sum_{L\in\mathscr{P}^j} \vartheta_L g_L}{\sum_{L\in\mathscr{P}^j} \vartheta_L}\, . \tag{1.13}$$

We denote by $\bar\varphi_j(y)$ the first $\bar n$ components of $\hat\varphi_j(y)$ and set $\varphi_j(y) = \big(\bar\varphi_j(y), \Psi(y, \bar\varphi_j(y))\big)$. This latter map will be called *glued interpolation.*

Theorem 1.12 (Existence of the center manifold)**.** *Assume that the hypotheses of Proposition 1.7, Assumptions 1.2 and 1.5 hold. Let $\kappa := \min\{\varepsilon_0/2, \beta_2/4\}$. If ε_2 is sufficiently small (depending on all the other parameters), then*

(i) *φ_j is well-defined, $\|D\varphi_j\|_{C^{2,\kappa}} \le C\boldsymbol{m}_0^{1/2}$ and $\|\varphi_j\|_{C^0} \le C\boldsymbol{m}_0^{1/2m}$.*
(ii) *$\{\varphi_j\}$ is a stabilizing sequence: i.e., if $L \in \mathscr{W}^i$ and H is a cube concentric to L with $\ell(H) = \frac{9}{8}\ell(L)$, then $\varphi_j = \varphi_k$ on H for any $j, k \ge i+2$.*
(iii) *φ_j converges to a map $\boldsymbol{\varphi}$ and $\mathcal{M} := \mathrm{Gr}(\boldsymbol{\varphi}|_{]-4,4[^m})$ is a $C^{3,\kappa}$ submanifold of Σ.*

The constant C in (i) depends only upon $M_0, N_0, \beta_2, \delta_2, \gamma_1, C_e$ and C_h.

Definition 1.13 (Whitney regions)**.** The manifold $\mathcal{M}$ in Theorem 1.12 is called *a center manifold of T relative to* π_0 and $(\mathbf{\Gamma}, \mathscr{W})$ the *Whitney decomposition associated to* $\mathcal{M}$. Setting $\mathbf{\Phi}(y) := (y, \boldsymbol{\varphi}(y))$, we call $\mathbf{\Phi}(\mathbf{\Gamma})$ the *contact set.* Moreover, to each $L \in \mathscr{W}$ we associate a *Whitney region* $\mathcal{L}$ on $\mathcal{M}$ as follows:

(WR) $\mathcal{L} := \mathbf{\Phi}(H \cap [-\frac{7}{2}, \frac{7}{2}]^m)$, where H is the cube concentric to L with $\ell(H) = \frac{17}{16}\ell(L)$.

2. The $\mathcal{M}$-normal approximation and related estimates

In what follows we assume that the hypotheses of Theorem 1.12 are fulfilled. In order to simplify the notation, for any $\mathcal{V} \subset \mathcal{M}$ we will denote by $|\mathcal{V}|$ its $\mathcal{H}^m$-measure and we write $\int_{\mathcal{V}} f$ for the integration with respect to $\mathcal{H}^m$. $\mathcal{B}_r(q)$ denotes the geodesic balls in $\mathcal{M}$. Moreover, we refer to [5] for all the relevant notation pertaining the differentiation of (multiple valued) maps defined on $\mathcal{M}$, induced currents, differential-geometric tensors and so on.

Assumption 2.1. We fix the following notation and assumptions.

(U) $\mathbf{U} := \big\{x \in \mathbb{R}^{m+n} : \exists!\, y \in \mathcal{M} \text{ with } |x-y| < 1 \text{ and } (x-y) \perp \mathcal{M}\big\}$.
(P) $\mathbf{p} : \mathbf{U} \to \mathcal{M}$ is the map such that $\mathbf{p}(x)$ is the point y in (U).

(R) Having fixed all the other parameters, we assume ε_2 to be so small that $\mathbf{p}$ extends to $C^{2,\kappa}(\bar{\mathbf{U}})$ and $\mathbf{p}^{-1}(y) = y + \overline{B_1(0, \varkappa_y)}$ for every $y \in \mathcal{M}$, where $\varkappa_y$ is the n-plane perpendicular to $T_y\mathcal{M}$.

(L) We denote by $\partial_l \mathbf{U} := \mathbf{p}^{-1}(\partial\mathcal{M})$ the *lateral boundary* of $\mathbf{U}$.

The following is then a corollary of Theorem 1.12 and the construction algorithm.

Corollary 2.2. *Under the hypotheses of Theorem 1.12 and of Assumption 2.1 we have:*

(i) $\mathrm{spt}(\partial(T\llcorner\mathbf{U})) \subset \partial_l\mathbf{U}$, $\mathrm{spt}(T\llcorner[-\frac{7}{2},\frac{7}{2}]^m \times \mathbb{R}^n) \subset \mathbf{U}$ *and* $\mathbf{p}_\sharp(T\llcorner\mathbf{U}) = Q\,[\![\mathcal{M}]\!]$*;*

(ii) $\mathrm{spt}(\langle T, \mathbf{p}, \mathbf{\Phi}(q)\rangle) \subset \big\{y : |\mathbf{\Phi}(q) - y| \leq C\boldsymbol{m}_0^{1/2m}\ell(L)^{1+\beta_2}\big\}$ *for every* $q \in L \in \mathscr{W}$*;*

(iii) $\langle T, \mathbf{p}, p\rangle = Q\,[\![p]\!]$ *for every* $p \in \mathbf{\Phi}(\mathbf{\Gamma})$*.*

The main goal of this paper is to couple the center manifold of Theorem 1.12 with a good approximating map defined on it.

Definition 2.3 ($\mathcal{M}$-normal approximation)**.** An $\mathcal{M}$*-normal approximation* of T is given by a pair $(\mathcal{K}, F)$ such that

(A1) $F : \mathcal{M} \to \mathcal{A}_Q(\mathbf{U})$ is Lipschitz and takes the special form $F(x) = \sum_i [\![x + N_i(x)]\!]$, with $N_i(x) \perp T_x\mathcal{M}$ and $x + N_i(x) \in \Sigma$ for every x and i.

(A2) $\mathcal{K} \subset \mathcal{M}$ is closed, contains $\mathbf{\Phi}\big(\mathbf{\Gamma} \cap [-\frac{7}{2},\frac{7}{2}]^m\big)$ and $\mathbf{T}_F\llcorner\mathbf{p}^{-1}(\mathcal{K}) = T\llcorner\mathbf{p}^{-1}(\mathcal{K})$.

The map $N = \sum_i [\![N_i]\!] : \mathcal{M} \to \mathcal{A}_Q(\mathbf{U})$ is *the normal part* of F.

In the definition above it is not required that the map F approximates efficiently the current outside the set $\mathbf{\Phi}\big(\mathbf{\Gamma} \cap [-\frac{7}{2},\frac{7}{2}]^m\big)$. However, all the maps constructed in this paper and used in the subsequent note [5] will approximate T with a high degree of accuracy in each Whitney region. Since the corresponding detailed estimates are rather long, we prefer to state them in the following theorem. In order to simplify the notation, we will use $\|N|_{\mathcal{V}}\|_0$ to denote the number $\sup_{x\in\mathcal{V}} \mathcal{G}(N(x), Q\,[\![0]\!])$.

Theorem 2.4 (Local estimates for the $\mathcal{M}$-normal approximation)**.** *Let* $\gamma_2 := \frac{\gamma_1}{4}$*, with* γ_1 *the constant of* [6, Theorem 1.4]*. Under the hypotheses of Theorem 1.12 and Assumption 2.1, if* ε_2 *is suitably small, then there is an* $\mathcal{M}$*-normal approximation* $(\mathcal{K}, F)$ *such that the following estimates hold on every Whitney region* $\mathcal{L}$ *associated to a cube* $L \in \mathscr{W}$*:*

$$\mathrm{Lip}(N|_{\mathcal{L}}) \leq C\boldsymbol{m}_0^{\gamma_2}\ell(L)^{\gamma_2} \quad \text{and} \quad \|N|_{\mathcal{L}}\|_{C^0} \leq C\boldsymbol{m}_0^{1/2m}\ell(L)^{1+\beta_2}, \tag{2.1}$$

$$|\mathcal{L}\setminus\mathcal{K}| + \|\mathbf{T}_F - T\|(\mathbf{p}^{-1}(\mathcal{L})) \leq C\boldsymbol{m}_0^{1+\gamma_2}\ell(L)^{m+2+\gamma_2}, \tag{2.2}$$

$$\int_{\mathcal{L}} |DN|^2 \leq C\boldsymbol{m}_0\,\ell(L)^{m+2-2\delta_2}\,. \tag{2.3}$$

Moreover, for every $a > 0$ *and every Borel* $\mathcal{V} \subset \mathcal{L}$*, we have*

$$\int_{\mathcal{V}} |\boldsymbol{\eta}\circ N| \leq C\boldsymbol{m}_0\left(\ell(L)^{3+\beta_2/3} + a\,\ell(L)^{2+\gamma_2}\right)|\mathcal{V}| + \frac{C}{a}\int_{\mathcal{V}} \mathcal{G}\big(N, Q\,[\![\boldsymbol{\eta}\circ N]\!]\big)^{2+\gamma_2}\,. \tag{2.4}$$

The constant C *depends on all the parameters introduced so far except* ε_2*.*

Remark 2.5 (Global estimates)**.** As a simple consequence of (2.1) - (2.3) and the structure of the Whitney decomposition, if we denote by $\mathcal{M}'$ the domain $\mathbf{\Phi}\big([-\frac{7}{2},\frac{7}{2}]^m\big)$, then we obtain the following global estimates

$$\mathrm{Lip}(N|_{\mathcal{M}'}) \leq C\boldsymbol{m}_0^{\gamma_2} \quad \text{and} \quad \|N|_{\mathcal{M}'}\|_{C^0} \leq C\boldsymbol{m}_0^{1/2m}, \tag{2.5}$$

$$|\mathcal{M}'\setminus\mathcal{K}| + \|\mathbf{T}_F - T\|(\mathbf{p}^{-1}(\mathcal{M}')) \leq C\boldsymbol{m}_0^{1+\gamma_2}, \tag{2.6}$$

$$\int_{\mathcal{M}'} |DN|^2 \leq C\boldsymbol{m}_0 \,. \tag{2.7}$$

Observe that $N\equiv 0$ over $\mathbf{\Phi}(\mathbf{\Gamma})$ and thus the second inequality in (2.5) follows easily from the second inequality of (2.1), recalling that $\ell(L)\leq 1$ for any cube $L\in\mathscr{W}$. For the same reasons, from (2.3) we conclude

$$\int_{\mathcal{M}'} |DN|^2 \leq C\boldsymbol{m}_0 \sum_{L\in\mathscr{W}} \ell(L)^{m+2-2\delta_2} \leq C\boldsymbol{m}_0 \sum_{L\in\mathscr{W}} \ell(L)^m \leq C\boldsymbol{m}_0\,,$$

where in the latter inequality we have used that the interiors of the L's are pairwise disjoint and all contained in $[-4,4]^m$. (2.6) follows from (2.2) with similar considerations. Coming to the first inequality in (2.5) fix any two points $p=\mathbf{\Phi}(x), q=\mathbf{\Phi}(y)\in\mathcal{M}'$. Observe that the length of the geodesic segment joining p and q is comparable, up to constants, to $|x-y|$. If $x,y\in\mathbf{\Gamma}$, then $N(p)=N(q)=Q\,[\![0]\!]$ and so $\mathcal{G}(N(p),N(q))=0$. If $x\in\mathbf{\Gamma}$ and $y\not\in\mathbf{\Gamma}$, then y belongs to some $L\in\mathscr{W}$ and, by the properties of the Whitney decomposition, $\ell(L)\leq C|x-y|$, where C is a geometric constant. Thus, using the second inequality in (2.1) we conclude $\mathcal{G}(N(q),N(p)) = \mathcal{G}(N(q),Q\,[\![0]\!]) \leq \|N|_{\mathcal{L}}\|_{C^0} \leq C\boldsymbol{m}_0^{1/2m}\ell(L)^{1+\beta_2} \leq C\boldsymbol{m}_0^{1/2m}|x-y|$. Finally, if $x,y\not\in\mathbf{\Gamma}$ we analyse two cases. If the geodesic segment $[x,y]$ intersects $\mathbf{\Gamma}$, then we conclude the same inequality as above. Otherwise there are points $x=z_0,z_1,\ldots,z_N=y$ in $[x,y]$ such that each segment $[z_{i-1},z_i]$ is contained in some single $L_i\in\mathscr{W}$ and $\sum_i|z_i-z_{i-1}|=|x-y|$. It then follows from the first bound in (2.1) that

$$\mathcal{G}(N(p),N(q)) \leq \sum_i \mathcal{G}(N(\mathbf{\Phi}(z_i),N(\mathbf{\Phi}(z_{i-1}))) \leq C\boldsymbol{m}_0^{\gamma_2}\sum_i |z_i-z_{i-1}| = C\boldsymbol{m}_0^{\gamma_2}|x-y|\,.$$

Recalling that $\gamma_2\leq\frac{1}{2m}$, all the cases examined prove the first inequality in (2.5).

3. Additional conclusions upon $\mathcal{M}$ and the $\mathcal{M}$-normal approximation

3.1. **Height bound and separation.** We now analyze more in detail the consequences of the various stopping conditions for the cubes in $\mathscr{W}$. We first deal with $L\in\mathscr{W}_h$.

Proposition 3.1 (Separation)**.** *There is a dimensional constant $C^\sharp>0$ with the following property. Assume the parameters $\beta_2,\delta_2,N_0,M_0,C_e$ and C_h fulfill the assumptions of Theorems 1.12 and 2.4, and in addition $C_h^{2m}\geq C^\sharp C_e$. If ε_2 is sufficiently small, then the following conclusions hold for every $L\in\mathscr{W}_h$:*

(S1) $\Theta(T,p)\leq Q-\frac{1}{2}$ *for every* $p\in\mathbf{B}_{16r_L}(p_L)$.
(S2) $L\cap H=\emptyset$ *for every* $H\in\mathscr{W}_n$ *with* $\ell(H)\leq\frac{1}{2}\ell(L)$;
(S3) $\mathcal{G}\big(N(x),Q\,[\![\boldsymbol{\eta}\circ N(x)]\!]\big)\geq\frac{1}{4}C_h\boldsymbol{m}_0^{1/2m}\ell(L)^{1+\beta_2}$ *for every* $x\in\mathcal{L}$;

A simple corollary of the previous proposition is the following.

Corollary 3.2 (Domains of influence)**.** *The cubes in $\mathscr{W}_n$ con be partitioned in disjoint families $\mathscr{W}_n(L)$, where L is running in $\mathscr{W}_e$. More precisely, for such H and L there is a chain $L = L_0, L_1, \ldots, L_j$ such that $L_i \in \mathscr{W}_n$, $L_i \cap L_{i-1} \neq \emptyset$, $\ell(L_i) = 2\ell(L_{i-1})$ and $L_j \cap H \neq \emptyset$. In particular, $H \subset B_{3\sqrt{m}\ell(L)}(x_L)$ for every $H \in \mathscr{W}_n(L)$.*

3.2. **Splitting before tilting I.** The following is the main consequence of the splitting before tilting phenomenon. We refer the reader to the discussion in the introduction for the reasons behind this terminology. Observe that the key assumption of the theorem (i.e. $L \in \mathscr{W}_e$) is that the excess does not decay at some given scale (hence the tangent planes "tilt") and the main conclusion (3.2) implies a certain amount of separation between the sheets of the current (hence some "splitting").

Proposition 3.3. *(Splitting I) Assume the hypotheses of Theorems 1.12 and 2.4 hold. If $M_0 \geq C(\delta_2)$, $C_e \geq C(M_0, \delta_2)$ and ε_2 is chosen sufficiently small, then the following estimates hold for every $L \in \mathscr{W}_e$ ($\mathcal{L}$ its associated Whitney region) and every set $\Omega := \mathbf{\Phi}(B_{\ell(L)/4}(q, \pi_0))$, where $q \in \pi_0$ is an arbitrary point with* $\mathrm{dist}(L, q) \leq 4\sqrt{m}\,\ell(L)$*:*

$$C_e \boldsymbol{m}_0 \ell(L)^{m+2-2\delta_2} \leq \ell(L)^m \mathbf{E}(T, \mathbf{B}_L) \leq C \int_\Omega |DN|^2 \,, \tag{3.1}$$

$$\int_{\mathcal{L}} |DN|^2 \leq C \ell(L)^m \mathbf{E}(T, \mathbf{B}_L) \leq C \ell(L)^{-2} \int_\Omega |N|^2 \,. \tag{3.2}$$

The constant C depends on all the parameters involved except ε_2.

3.3. **Persistence of Q points.** We next state two important properties triggered by the existence of $p \in \mathrm{spt}(T)$ with $\Theta(p, T) = Q$, both related to the splitting before tilting.

Proposition 3.4. *(Splitting II) Assume the hypotheses of Theorem 1.12 hold. For every $\alpha, \bar\alpha, \hat\alpha > 0$, there exists $\varepsilon_3 > 0$ (depending upon all the parameters $\beta_2, \delta_2, M_0, N_0, C_e, C_h$ and also $\alpha, \bar\alpha$ and $\hat\alpha$) such that the following holds. Assume $s \in]0,1[$,*

$$\sup\left\{\ell(L) : L \in \mathscr{W}, L \cap B_{3s}(0, \pi_0) \neq \emptyset\right\} \leq s \,, \tag{3.3}$$

$$\mathcal{H}^{m-2+\alpha}_\infty\left(\{\Theta(T, \cdot) = Q\} \cap \mathbf{B}_{3s}\right) \geq \bar\alpha s^{m-2+\alpha}, \tag{3.4}$$

and $\min\left\{s, \boldsymbol{m}_0\right\} \leq \varepsilon_3$*. Then,*

$$\sup\left\{\ell(L) : L \in \mathscr{W}_e \text{ and } L \cap B_{3s}(0, \pi_0) \neq \emptyset\right\} \leq \hat\alpha s \,.$$

Proposition 3.5. *(Persistence of Q-points) Assume the hypotheses of the Theorems 1.12, 2.4 and Proposition 3.3 hold. For every $\eta_2 > 0$ there is $\bar s, \bar\ell > 0$, with the following property. If $L \in \mathscr{W}_e, \ell(L) \leq \bar\ell$, $\Theta(T, p) = Q$ and $\mathrm{dist}(\mathbf{p}_{\pi_0}(\mathbf{p}(p)), L) \leq 4\sqrt{m}\,\ell(L)$, then*

$$\fint_{\mathcal{B}_{\bar s\ell(L)}(\mathbf{p}(p))} \mathcal{G}^2\big(N, Q\,[\![\boldsymbol{\eta}\circ N]\!]\big) \leq \frac{\eta_2}{\omega_m \ell(L)^{m-2}} \int_{\mathcal{B}_{\ell(L)}(\mathbf{p}(p))} |DN|^2 \,. \tag{3.5}$$

3.4. **Comparison between different center manifolds.** We list here a final key property of center manifolds and $\mathcal{M}$-normal approximations. Once again this is also a consequence of the splitting before tilting phenomenon. In what follows we use the notation $\iota_{0,r}$ for the map $z \mapsto \frac{z}{r}$.

Proposition 3.6 (Comparing center manifolds)**.** *Assume the hypotheses of Theorems 1.12, 2.4 and Proposition 3.3. If N_0 is larger than a geometric constant, there is $\bar{c}_s > 0$ (which depends on all the parameters of Assumption 1.5* except for ε_2*) with the following property. If ε_2 is sufficiently small, $c_s := \frac{1}{16\sqrt{m}}$ and $r \in]0,1[$ is a radius such that:*

(a) *$\ell(L) \leq c_s\rho$ for every $\rho > r$ and every $L \in \mathscr{W}$ with $L \cap B_\rho(0,\pi_0) \neq \emptyset$;*
(b) *$\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}\rho}) < \varepsilon_2$ for every $\rho > r$;*
(c) *there is $L \in \mathscr{W}$ such that $\ell(L) \geq c_s r$ and $L \cap B_r(0,\pi_0) \neq \emptyset$;*

then

(i) *the current $T' := (\iota_{0,r})\sharp T \llcorner \mathbf{B}_{6\sqrt{m}}$ and the submanifold $\Sigma' := \iota_{0,r}(\Sigma)$ satisfy the assumptions of Theorems 1.12 and 2.4 for some plane π in place of π_0;*
(ii) *for the center manifold $\mathcal{M}'$ of T' relative to π and the $\mathcal{M}'$-normal approximation N' as in Theorem 2.4, we have*

$$\int_{\mathcal{M}'\cap\mathbf{B}_2} |N'|^2 \geq \bar{c}_s \max\left\{\mathbf{E}(T', \mathbf{B}_{6\sqrt{m}}), \mathbf{c}(\Sigma')^2\right\}. \tag{3.6}$$

4. Center manifold's construction

In this section we lay down the technical preliminaries to prove Theorem 1.12, state the related fundamental estimates and show how the theorem follows from them.

4.1. **Consistency of the construction algorithm.** The main preliminary technical work consists in ensuring that all the building blocks of the construction algorithm are in fact well-defined. Along the way we also conclude some pieces of information which settle some of the estimates in Theorem 2.4.

Lemma 4.1 (Well-definition of $\mathbf{B}_L$ and $\mathscr{W}$)**.** *Let Assumptions 1.2 and 1.5 hold. If ε_2 is sufficiently small, the balls $\mathbf{B}_L$ are well-defined and $(\mathbf{\Gamma}, \mathscr{W})$ is a Whitney decomposition of π_0. Moreover, for any fixed M_0 and N_0 there is $C^\star := C^\star(M_0, N_0)$ such that, if $C_h \geq C^\star C_e \geq (C^\star)^2$, then $\mathscr{W}^j = \emptyset$ for all $j \leq N_0 + 6$.*

Proof. Recalling that $T := T^0 \llcorner \mathbf{B}_{23\sqrt{m}/4}$, we start noticing that, from Assumption 1.2, it follows that

$$(\mathbf{p}_{\pi_0})_\sharp T \llcorner \mathbf{C}_{11\sqrt{m}/2} = Q\,[\![B_{11\sqrt{m}/2}]\!] \quad \text{and} \quad \mathbf{h}(T, \mathbf{C}_{5\sqrt{m}}, \pi_0) \leq C_0 \boldsymbol{m}_0^{1/2m}. \tag{4.1}$$

To this regard, we can argue by contradiction: if the first statement were false, we would have a sequence of currents T^0_k in $\mathbf{B}_{6\sqrt{m}}$ and of submanifolds Σ_k satisfying Assumption 1.2 with $\varepsilon_2(k) \downarrow 0$ and $(\mathbf{p}_{\pi_0})_\sharp T^0_k \llcorner (\mathbf{C}_{11\sqrt{m}/2} \cap \mathbf{B}_{23\sqrt{m}/4}) \neq Q\,[\![B_{11\sqrt{m}/2}]\!]$. It is then easy to see that

$$T^0_k \rightharpoonup T_\infty := Q\,[\![B_{6\sqrt{m}}]\!] \quad \text{and} \quad \mathrm{spt}(T^0_k) \to \mathrm{spt}(T_\infty) \ \text{ locally in the Hausdorff sense.}$$

Since ∂T^0_k vanishes in $\mathbf{B}_{6\sqrt{m}}$, $T^0_k \llcorner (\mathbf{C}_{11\sqrt{m}/2} \cap \mathbf{B}_{23\sqrt{m}/4})$ has no boundary in $\mathbf{C}_{11\sqrt{m}/2}$ for k large enough, thereby implying that $(\mathbf{p}_{\pi_0})_\sharp T^0_k \llcorner (\mathbf{C}_{11\sqrt{m}/2} \cap \mathbf{B}_{23\sqrt{m}/4}) = Q_k \, [\![B_{11\sqrt{m}/2}]\!]$ for some integer Q_k. Since $T^0_k \rightharpoonup T_\infty$, we deduce that $Q_k = Q$ for k large enough, giving the desired contradiction. Having shown the first claim in (4.1), the height bound now follows easily from Theorem A.1 because of $(\mathbf{p}_{\pi_0})_\sharp T^0 \llcorner (\mathbf{C}_{11\sqrt{m}/2} \cap \mathbf{B}_{23\sqrt{m}/4}) = Q \, [\![B_{11\sqrt{m}/2}]\!]$ and $\Theta(T^0, 0) = Q$: in particular, the latter assumption together with Theorem A.1(iii) implies that there is one single open set $\mathbf{S}_1$ as in Theorem A.1(i), which in turn must contain the origin.

By the slicing theory of currents (see [12, Section 28] or [8, 4.3.8]) and by (4.1), there is a set $A \subset B_{5\sqrt{m}}$ of full measure such that, for every $x \in A$, there exist $k_i(x) \in \mathbb{N}$ with $\sum_i k_i = Q$, and points $(x, y_i(x)) \in \mathrm{spt}(T)$ with $|y_i(x)| \leq C_0 \boldsymbol{m}_0^{1/2m}$, for which we have

$$\langle T, \mathbf{p}_{\pi_0}, x \rangle = \sum_{i=1}^{N(x)} k_i(x) \delta_{(x, y_i(x))} \quad \forall \, x \in A.$$

By the monotonicity formula and the density of A in $B_{5\sqrt{m}}$, we conclude that $\mathrm{spt}(T) \cap (x + \pi_0^\perp) \neq \emptyset$ for every $x \in \overline{B}_{5\sqrt{m}}$. Hence, $\mathbf{B}_L$ is well-defined for every cube $L \in \mathscr{C}$ and its center p_L satisfies $|p_L| \leq 4\sqrt{m} + C_0 \boldsymbol{m}_0^{1/2m}$.

Fix now $L \in \mathscr{W}^j$ with $N_0 \leq j \leq N_0 + 6$. Since $r_L \leq 2^{-7}\sqrt{m}$ by Assumption 1.5, we have that $\mathbf{B}_L \subset \mathbf{B}_{5\sqrt{m}}$ if ε_2 is small enough, and

$$\mathbf{E}(T, \mathbf{B}_L, \pi_0) \leq \frac{6^m}{(64 M_0 2^{-N_0-6})^m} \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}}, \pi_0) \leq \frac{6^m}{(64 M_0)^m 2^{-(N_0+6)m}} \, \boldsymbol{m}_0 \, .$$

For a suitable $C^\star(M_0, N_0)$ the inequality $C_e \geq C^\star$ implies

$$\mathbf{E}(T, \mathbf{B}_L) \leq \mathbf{E}(T, \mathbf{B}_L, \pi_0) \leq C_e \boldsymbol{m}_0 \, \ell(L)^{2-2\delta_2} \, .$$

Let now $\hat{\pi}_L$ be an optimal plane in $\mathbf{B}_L$: since the center p_L of $\mathbf{B}_L$ belongs to $\mathrm{spt}(T)$, the monotonicity formula guarantees $\|T\|(\mathbf{B}_L) \geq c_0 r_L^m$ (cp. [12, Section 17] or [6, Appendix A]). We then conclude

$$|\hat{\pi}_L - \pi_0|^2 \leq C \big(\mathbf{E}(T, \mathbf{B}_L, \pi_0) + \mathbf{E}(T, \mathbf{B}_L, \hat{\pi}_L) \big) \leq C_0 C_e \boldsymbol{m}_0 \, \ell(L)^{2-2\delta_2} \, , \tag{4.2}$$

where C_0 is a geometric constant. This in turn implies that

$$\begin{aligned} \mathbf{h}(T, \mathbf{B}_L) &\leq C \, |\hat{\pi}_L - \pi_0| \, \ell(L) + \mathbf{h}(T, \mathbf{B}_L, \pi_0) \leq C \, C_e^{1/2} \boldsymbol{m}_0^{1/2} \ell(L)^{2-\delta_2} + \mathbf{h}(T, \mathbf{C}_{5\sqrt{m}}, \pi_0) \\ &\stackrel{(4.1)}{\leq} C(N_0)(C_e^{1/2} + 1) \boldsymbol{m}_0^{1/2m} \ell(L)^{1+\beta_2} \, . \end{aligned}$$

Thus, if $C^\star(M_0, N_0)$ is chosen sufficiently large and $C_h \geq C^\star C_e \geq (C^\star)^2$, neither the condition (EX) nor (HT) apply to L. Therefore, $\mathscr{W}^{N_0} = \emptyset$. Similarly, at the successive step, none of the cube in $\mathscr{S}^{N_0+1}$ satisfies the conditions (EX), (HT) or (NN), because $\mathscr{W}^{N_0} = \emptyset$. Proceeding in this way, we conclude that $\mathscr{W}^j = \emptyset$ for every $j \leq N_0 + 6$. $\square$

Next we show some basic estimates for the tilting of the optimal planes and the height functions for cubes in $\mathscr{W} \cup \mathscr{S}$.

Proposition 4.2 (Tilting of optimal planes). *Assume the hypotheses of Lemma 4.1 hold and fix the parameters $M_0, N_0, \beta_2, \delta_2, C_e$ and C_h. If ε_2 is sufficiently small, then the following holds for every $H, L \in \mathscr{W} \cup \mathscr{S}$ with $H \subset L$:*

(i) $\mathbf{B}_H \subset \mathbf{B}_L$;
(ii) $|\hat{\pi}_L - \pi_L| \leq \bar{C} \boldsymbol{m}_0^{1/2} \ell(L)^{1-\delta_2}$;
(iii) $|\pi_H - \pi_L| \leq \bar{C} \boldsymbol{m}_0^{1/2} \ell(L)^{1-\delta_2}$;
(iv) $|\pi_L - \pi_0| \leq \bar{C} \boldsymbol{m}_0^{1/2}$;
(v) $\mathbf{h}(T, \mathbf{C}_{36r_L}(p_L, \pi_0)) \leq C \boldsymbol{m}_0^{1/2m} \ell(L)$ *and* $\mathrm{spt}(T) \cap \mathbf{C}_{36r_L}(p_L, \pi_0) \subset \mathbf{B}_L$;
(vi) $\mathbf{h}(T, \mathbf{C}_{36r_L}(p_L, \pi_H)) \leq C \boldsymbol{m}_0^{1/2m} \ell(L)^{1+\beta_2}$ *and* $\mathrm{spt}(T) \cap \mathbf{C}_{36r_L}(p_L, \pi_H) \subset \mathbf{B}_L$.

where $\bar{C}$ is given by $C_e^{1/2}$ times a geometric constant. In addition, (iii) and (vi) hold true also for $H, L \in \mathscr{W}^j \cup \mathscr{S}^j$ with $H \cap L \neq \emptyset$ and, in particular, Proposition 1.7 follows.

Proof. We prove that, for every $i \geq N_0$, given a sequence of cubes $H_i \subset \ldots \subset H_{N_0}$, with $H_j \in \mathscr{W}^j \cup \mathscr{S}^j$, then (i) - (vi) hold for every $H = H_j$ and $L = H_k$ with $j \geq k$ and $j, k \in \{N_0, \ldots, i\}$. We proceed by induction on i. The basic step $i = N_0$ follows easily from Lemma 4.1 and the definition of the Whitney Decomposition. We start observing that for (i) and (iii) there is nothing to prove, since the only case is $H = L = H_{N_0}$. Next, since $\mathscr{W}^{N_0} = \emptyset$ by Lemma 4.1, the cube H_{N_0} does not satisfy condition (EX). Therefore, using the monotonicity formula, $\|T\|(\mathbf{B}_{H_{N_0}}) \geq c_0 \, r_{H_{N_0}}^m$ and there exists at least a point $p \in \mathrm{spt}(T) \cap \mathbf{B}_{H_{N_0}}$ such that

$$|\vec{T}(p) - \hat{\pi}_{H_{N_0}}|^2 \leq \mathbf{E}(T, \mathbf{B}_{H_{N_0}}) \frac{C \, r_{H_{N_0}}^m}{\|T\|(\mathbf{B}_{H_{N_0}})} \leq C \boldsymbol{m}_0 \, \ell(H_{N_0})^{2-2\delta_2} \,. \tag{4.3}$$

Since $\vec{T}(p)$ is an m-vector of $T_p\Sigma$, this implies that $|\mathbf{p}_{T_p\Sigma}(\hat{\pi}_{H_{N_0}}) - \hat{\pi}_{H_{N_0}}| \leq C \boldsymbol{m}_0^{1/2} \ell(H_{N_0})^{1-\delta_2}$. Recalling that $|\mathbf{p}_{T_{p_{H_{N_0}}}\Sigma}(\hat{\pi}_{H_{N_0}}) - \mathbf{p}_{T_p\Sigma}(\hat{\pi}_{H_{N_0}})| \leq C r_{H_{N_0}} \mathbf{A} \leq C \boldsymbol{m}_0^{1/2} \ell(H_{N_0})$, we conclude (ii). Next, (iv) follows simply from (4.2) and (ii), while (v) follows from the inclusion $\mathrm{spt}(T) \cap \mathbf{C}_{36 r_{H_{N_0}}}(p_{H_{N_0}}, \pi_0) \subset \mathbf{C}_{5\sqrt{m}}$ and (4.1). Finally, for what concerns (vi), we notice that by (ii), (iii) and (v), if ε_2 is sufficiently small, then $\mathrm{spt}(T) \cap \mathbf{C}_{36 r_{H_{N_0}}}(p_{H_{N_0}}, \pi_{H_{N_0}}) \subset \mathbf{C}_{5\sqrt{m}}$. From (4.1) it follows then that $\mathrm{spt}(T) \cap \mathbf{C}_{36 r_{H_{N_0}}}(p_{H_{N_0}}, \pi_{H_{N_0}}) \subset \mathbf{B}_{H_{N_0}}$. Since $H_{N_0} \notin \mathscr{W}$, we then conclude

$$\begin{aligned}\mathbf{h}(T, \mathbf{C}_{36 r_{H_{N_0}}}(p_{H_{N_0}}, \pi_{H_{N_0}}), \pi_{H_{N_0}} &\leq \mathbf{h}(T, \mathbf{B}_{H_{N_0}}) + C \ell(H_{N_0}) |\pi_{H_{N_0}} - \hat{\pi}_{H_{N_0}}| \\ &\leq C \boldsymbol{m}_0^{1/2m} \ell(H_{N_0})^{1+\beta_2} + C \boldsymbol{m}_0^{1/2} \ell(H_{N_0})^{2-\delta_2} \,.\end{aligned}$$

Now we pass to the inductive step: we assume to have proved the conclusions for i and show that they hold also for $i+1$. For what concerns (i), it is enough to prove that $\mathbf{B}_{H_{i+1}} \subset \mathbf{B}_{H_i}$, because the other inclusions follows by the inductive hypothesis. To this aim, we notice that by (iv) applied to H_i and $|x_{H_i} - x_{H_{i+1}}| \leq \sqrt{m} \, \ell(H_i)$, we deduce that $|p_L - p_H| \leq C \, \ell(L)$ for some geometric constant $C > 0$. Therefore, if M_0 is taken sufficiently large, we infer $\mathbf{B}_{H_{i+1}} \subset \mathbf{B}_{H_i}$. We show now (ii) for $L = H_{i+1}$. Note that, by (i),

$$\mathbf{E}(T, \mathbf{B}_{H_{i+1}}) \leq C\, \mathbf{E}(T, \mathbf{B}_{H_i}) \leq C\, C_e\, \boldsymbol{m}_0\, \ell(H_{i+1})^{2-2\delta_2}, \tag{4.4}$$

for some geometric constant $C > 0$. Therefore, we can argue as for H_{N_0} and conclude as above. For (iii) and (iv), we start considering the case $H = H_l$ and $L = H_{l-1}$, for some $l \in \{N_0, \ldots, i+1\}$. Note that, by the inclusion in (i), we can argue again by monotonicity formula (used to estimate $\|T\|(\mathbf{B}_{H_l})$ and $\|T\|(\mathbf{B}_{H_{l-1}})$ from below) and infer that

$$|\hat{\pi}_{H_{l-1}} - \hat{\pi}_{H_l}|^2 \leq \big(\mathbf{E}(T, \mathbf{B}_{H_{l-1}}) + \mathbf{E}(T, \mathbf{B}_{H_l})\big) \frac{C\, r^m_{H_{l-1}}}{\|T\|(\mathbf{B}_{H_l})} \leq C\, C_e\, \boldsymbol{m}_0\, \ell(H_l)^{2-2\delta_2}\,. \tag{4.5}$$

Therefore, using (ii) we conclude (iii) for generic H and L by the estimate $\sum_{l=j}^{\infty} \ell(H_l)^{1-\delta_2} \leq C\, \ell(H_j)^{1-\delta_2}$. As for (iv) it follows from (iii) and the case $|\pi_{H_{N_0}} - \pi_0| \leq \bar{C} \boldsymbol{m}_0^{1/2}$. Coming now to (v), by inductive hypothesis it is enough to show it for $L = H_{i+1}$. To this aim, notice that, by (v) for H_i, we conclude $\mathrm{spt}(T) \cap \mathbf{C}_{36 r_{H_i}}(p_{H_i}, \pi_0) \subset \mathbf{B}_{H_i}$. Next, since $|x_{H_{i+1}} - x_{H_i}| \leq \sqrt{m}\, \ell(H_i)$ and $r_{H_{i+1}} = \frac{1}{2} r_{H_i}$, we obviously have $\mathbf{C}_{36 r_{H_{i+1}}}(p_{H_{i+1}}, \pi_0) \subset \mathbf{C}_{36 r_{H_i}}(p_{H_i}, \pi_0)$, provided M_0 is larger than a geometric constant. Thus:

$$\begin{aligned}\mathbf{h}(T, \mathbf{C}_{36 r_{H_{i+1}}}(p_{H_{i+1}}, \pi_0)) &\leq \mathbf{h}(T, \mathbf{B}_{H_i}) + C\, r_{H_i} |\hat{\pi}_{H_i} - \pi_0| \\ &\stackrel{\text{(iv)}}{\leq} C_h\, \boldsymbol{m}_0^{1/2m} \ell(H_i)^{1+\beta_2} + C\, \boldsymbol{m}_0^{1/2} \ell(H_i) \leq C\, \boldsymbol{m}_0^{1/2m} \ell(H_i),\end{aligned}$$

where we used $H_i \in \mathscr{S}^i$. The inclusion $\mathrm{spt}(T) \cap \mathbf{C}_{36 r_{H_{i+1}}}(p_{H_{i+1}}, \pi_0) \subset \mathbf{B}_{H_{i+1}}$ is an obvious corollary of the bound and of the fact that the center of the ball $\mathbf{B}_{H_{i+1}}$ (i.e. the point $p_{H_{i+1}}$) belongs to $\mathrm{spt}(T) \cap \mathbf{C}_{36 r_{H_{i+1}}}(p_{H_{i+1}}, \pi_0)$.

Next we show (vi) for $H = H_{i+1}$ and L an ancestor of H (included the case $L = H_{i+1}$). First we consider the case $L = H_{N_0}$. By $|\pi_{H_{i+1}} - \pi_0| \leq \bar{C} \boldsymbol{m}_0^{1/2}$ and a simple geometric argument, it is easy to see that, provided ε_2 is sufficiently small, $\mathbf{C}_{36 r_L}(p_L, \pi_H) \cap \mathbf{B}_{6\sqrt{m}} \cap \mathbf{C}_{5\sqrt{m}}$ and thus, by (4.1) we conclude

$$\mathbf{h}(T, \mathbf{C}_{36 r_L}(p_L, \pi_H)) \leq \mathbf{h}(T, \mathbf{C}_{5\sqrt{m}}, \pi_0) + C r_L |\pi_H - \pi_0| \leq C \boldsymbol{m}_0^{1/2m} \ell(L)^{1+\beta_2}$$

(recalling that $\ell(L) = 2^{-N_0}$!).

Otherwise we have $L = H_l$ with $l > N_0$ and we can set $J := H_{l+1}$. We have already observed that $|p_J - p_L| \leq C \ell(J)$ for a geometric constant. Moreover we have $|\pi_H - \pi_J| \leq C \boldsymbol{m}_0^{1/2} \ell(J)^{1-\delta_2}$. If ε_2 is sufficiently small, a simple geometric argument shows that

$$\mathbf{C}_{36 r_L}(p_L, \pi_H) \cap \mathbf{B}_{6\sqrt{m}} \subset \mathbf{C}_{36 r_J}(p_J, \pi_J)\,.$$

On the other hand by inductive hypothesis we have $\mathbf{C}_{36 r_J}(p_J, \pi_J) \cap \mathrm{spt}(T) \subset \mathbf{B}_J$ and, since $J \notin \mathscr{W}$, we easily conclude

$$\mathbf{h}(T, \mathbf{C}_{36 r_L}(p_L, \pi_H)) \leq \mathbf{h}(T, \mathbf{B}_J) + C r_J \left(|\hat{\pi}_J - \pi_J| + |\pi_J - \pi_L|\right) \leq C \boldsymbol{m}_0^{1/2} \ell(J)^{1+\beta_2}\,,$$

where the constant C depends only on $\bar{C}$ and C_h. Since $\ell(J) = 2\ell(L)$, this concludes the proof of the bound. As above, the inclusion $\mathrm{spt}(T) \cap \mathbf{C}_{36 r_L}(p_L, \pi_H) \subset \mathbf{B}_L$ follows from the bound.

We pass to the last claim of the proposition. If $H, L \in \mathscr{W}^j \cup \mathscr{S}^j$ are such that $H \cap L \neq \emptyset$, then $|x_L - x_H| \leq 2\sqrt{m}\,\ell(H)$ and by (v) it follows that $|p_L - p_H| \leq C\,\ell(H)$ for some geometric constant $C > 0$. This in turn implies that $\mathbf{B}_{36 r_L}(p_L) \subset \mathbf{B}_H \cap \mathbf{B}_L$ and therefore, by the monotonicity formula, we conclude (iii):

$$|\hat{\pi}_H - \hat{\pi}_L|^2 \leq \big(\mathbf{E}(T, \mathbf{B}_H) + \mathbf{E}(T, \mathbf{B}_L)\big)\frac{C\, r_H^m}{\|T\|(\mathbf{B}_{36 r_L}(p_L))} \leq C\boldsymbol{m}_0\, \ell(H)^{2-2\delta_2}.$$

For (vi), assume first $L \in \mathscr{C}^j$ with $j > N_0$. Note that by (iii) we can argue as above and conclude that $\mathrm{spt}(T) \cap \mathbf{C}_{36 r_L}(p_L, \pi_H) \subset \mathbf{B}_J$ for the father J of L. The proof of (vi) follows then the same pattern. When $L \in \mathscr{C}^{N_0}$ the argument is entirely equal to the one above and we leave it to the reader.

Thanks to Lemma 4.1, Proposition 1.7 now follows straightforwardly: indeed, (1.11) is an immediate consequence of the definition and (1.12) follows from (i) and (iii). $\square$

Next, we prove that the building blocks for the construction of the center manifold are well-defined. For later purposes, we introduce the following notation.

Definition 4.3. Let $H \in \mathscr{W}^j \cup \mathscr{S}^j$ and let L be either an ancestor of H (including H itself) or an element of $\mathscr{W}^j \cup \mathscr{S}^j$ with $H \cap L \neq \emptyset$. The π_H-approximation of T in the cylinder $\mathbf{C}_{32 r_L}(p_L, \pi_H)$, derived by [6, Theorem 1.4] (if it can be applied) will be denoted by f_{HL}. Similarly, setting $\bar{h} := \mathbf{p}_{T_{p_L}\Sigma}\big((\boldsymbol{\eta} \circ f_{HL}) * \rho_{\ell(L)}\big)$, then the map $x \mapsto h_{HL}(x) := \Psi_{p_H}(x, \bar{h}(x))$ is called the tilted interpolating function and, if it exists, the map $g_{HL} : B_{4 r_L}(p_L, \pi_0) \to \pi_0^\perp$ such that $\mathbf{G}_{g_{HL}} = \mathbf{G}_{h_{HL}} \llcorner \mathbf{C}_{4 r_L}(p_L, \pi_0)$ is the interpolating function relative to the cubes H and L.

If $H = L$, then $h_{HL} = h_H$ and $g_{HL} = g_H$ are the interpolating functions of Definition 1.10.

Proposition 4.4 (Consistency of the center manifold algorithm)**.** *Assume the hypotheses of Proposition 4.2 hold and fix M_0, N_0, β_2, δ_2, C_e and C_h. The following facts are true provided ε_2 is sufficiently small. Let $H \in \mathscr{W}^j \cup \mathscr{S}^j$ and let L be either an ancestor of H (including H itself) or an element of $\mathscr{W}^j \cup \mathscr{S}^j$ with $H \cap L \neq \emptyset$. Then,*

(i) $(\mathbf{p}_{\pi_H})_\sharp(T \llcorner \mathbf{C}_{34 r_L}(p_L, \pi_H)) = Q\,[\![B_{34 r_L}(\mathbf{p}_{\pi_H}(p_L), \pi_H)]\!]$;

(ii) *the π_H-approximation f_{HL} and the interpolating function g_{HL} are well-defined.*

Proof. To prove (i), we join $\pi_H =: \pi(1)$ and $\pi_0 =: \pi(0)$ with a continuous one-parameter family of planes $\pi(t)$ with the property that $|\pi(t) - \pi_0| \leq C|\pi_H - \pi_0|$, with $C > 0$ some geometric constant. If ε_2 is suitably small, by Proposition 4.2 we have $\mathrm{spt}(T) \cap \mathbf{C}_{34 r_L}(p_L, \pi_t) \subset \mathbf{C}_{36 r_L}(p_L, \pi_0)$ for every $t \in [0,1]$. We consider then the currents $S(t) := (\mathbf{p}_{\pi(t)})_\sharp T \llcorner \mathbf{C}_{34 r_L}(p_L, \pi(t))$ and note that $S(t) = Q(t)\,\big[\!\big[B_{34 r_L}(\mathbf{p}_{\pi(t)}(p_L), \pi(t))\big]\!\big]$, where $Q(t)$ is an integer for every t by the Constancy Theorem. On the other hand $t \mapsto S(t)$ is weakly continuous in the space of currents and thus $Q(t)$ must be constant. Since $Q(0) = Q$ by Lemma 4.1, this proves the desired claim.

For what concerns (ii), by Proposition 4.2 it follows that, for ε_2 smaller than a geometric constant,

$$\mathrm{spt}(T \llcorner \mathbf{C}_{32 r_L}(p_L, \pi_H)) \subset \mathbf{B}_L \subset \mathbf{B}_{5\sqrt{m}}. \tag{4.6}$$

We then conclude

$$\mathbf{E}(T,\mathbf{C}_{32r_L}(p_L,\pi_H))\le C\mathbf{E}(T,\mathbf{B}_L,\pi_H)\le C\mathbf{E}(T,\mathbf{B}_L)+C|\pi_H-\hat\pi_L|^2\le C\boldsymbol{m}_0\,\ell(L)^{2-2\delta_2}\,.$$

If ε_2 is sufficiently small, then $\mathbf{E}(T,\mathbf{C}_{32r_L}(p_L,\pi_H))<\varepsilon_1$, where ε_1 is the constant of [6, Theorem 1.4]. On the other hand (4.6) implies also that $\partial(T\llcorner\mathbf{C}_{32r_L}(p_L,\pi_H))$ vanishes in $\mathbf{C}_{32r_L}(p_L,\pi_H))$. Therefore, the current $T\llcorner\mathbf{C}_{32r_L}(p_L,\pi_H)$ and the submanifold Σ satisfy all the assumptions of [6, Theorem 1.4] in the cylinder $\mathbf{C}_{32r_L}(p_L,\pi_H)$ and therefore the approximation f_{HL} is well-defined. By [6, Theorem 1.4] and the properties of Ψ_{p_H}, we have

$$\mathrm{Lip}(h_{HL})\le C\mathrm{Lip}(\boldsymbol{\eta}\circ f_{HL})\le C\left(\mathbf{E}(T\llcorner\mathbf{C}_{32r_L}(p_L,\pi_H))\right)^{\gamma_1}\le C\boldsymbol{m}_0^{\gamma_1}\ell(L)^{\gamma_1},$$

and

$$\begin{aligned}|h_{HL}-\mathbf{p}_{\pi_H^\perp}(p_L)|&\le C|\boldsymbol{\eta}\circ f_{HL}-\mathbf{p}_{\pi_H^\perp}(p_L)|\le C\mathcal{G}\big(f_{HL},Q\,[\![p_{\pi_H^\perp}(p_L)]\!]\big)\\&\le\ C\mathbf{h}(T,\mathbf{C}_{32r_L}(p_L,\pi_H))+\big(\mathbf{E}(T,\mathbf{C}_{32r_L}(p_L,\pi_H))^{1/2}+\mathbf{A}\,r_L\big)r_L\\&\le C\boldsymbol{m}_0^{1/2m}r_L,\end{aligned}$$

where the constant C does not depend on ε_2. If ε_2 is then smaller than a suitable constant, we can apply Lemma B.1 to conclude that the interpolating function g_{HL} is well-defined. □

4.2. **Key estimates and proof of Theorem 1.12.** We are now ready to state the key construction estimates and show how Theorem 1.12 follows easily from them.

Proposition 4.5 (Construction estimates)**.** *Assume the hypotheses of Proposition 4.4 hold and set* $\kappa=\min\{\beta_2/4,\varepsilon_0/2\}$. *Then, the following holds for any pair of cubes* $H,L\in\mathscr{P}^j$ *(cf. Definition 1.11):*

(i) $\|g_H\|_{C^0}\le C\,\boldsymbol{m}_0^{1/2m}$ *and* $\|Dg_H\|_{C^{2,\kappa}}\le C\boldsymbol{m}_0^{1/2}$*;*

(ii) *if* $H\cap L\neq\emptyset$, *then* $\|g_H-g_L\|_{C^i(B_{r_L}(x_L))}\le C\boldsymbol{m}_0^{1/2}\ell(H)^{3+\kappa-i}$ *for every* $i\in\{0,\dots,3\}$*;*

(iii) $|D^3g_H(x_H)-D^3g_L(x_L)|\le C\boldsymbol{m}_0^{1/2}|x_H-x_L|^\kappa$*;*

(iv) $\|g_H-y_H\|_{C^0}\le C\boldsymbol{m}_0^{1/2m}\ell(H)$ *and* $|\pi_H-T_{(x,g_H(x))}\mathbf{G}_{g_H}|\le C\boldsymbol{m}_0^{1/2}\ell(H)^{1-\delta_2}$ *for all* $x\in H$*;*

(v) *if* L' *is the cube concentric to* $L\in\mathscr{W}^j$ *with* $\ell(L')=\frac{9}{8}\ell(L)$, *then*

$$\|\varphi_i-g_L\|_{L^1(L')}\le C\,\boldsymbol{m}_0\,\ell(L)^{m+3+\beta_2/3}\quad\text{for all }\ i\ge j.$$

The constant C *depends upon* $\beta_2,\delta_2,M_0,N_0,C_e$ *and* C_h *but not on* ε_2.

Using the estimates in the above proposition, we can prove the main existence result for the center manifold.

Proof of Theorem 1.12. The well-definition of the glued interpolations φ_j follows from Proposition 4.4. Define $\chi_H:=\vartheta_H/(\sum_{L\in\mathscr{P}^j}\vartheta_L)$ and observe that

$$\sum\chi_H=1\quad\text{and}\quad\|\chi_H\|_{C^i}\le C(i,m,n)\,\ell(H)^{-i}\qquad\forall i\in\mathbb{N}\,.\tag{4.7}$$

Set $\mathscr{P}^j(H) := \{L \in \mathscr{P}^j : L \cap H \neq \emptyset\} \setminus \{H\}$. By construction $\ell(H) \leq 2\,\ell(L)$ for every $L \in \mathscr{P}^j$ and the cardinality of $\mathscr{P}^j(H)$ is bounded by a constant $C(m)$. The estimate $|\hat\varphi_j| \leq C\boldsymbol{m}_0^{1/2m}$ follows then easily from Proposition 4.5(i). For $i \in \{1, \dots, 3\}$ and $x \in H$, write

$$D^i\hat\varphi_j(x) = D^i\Big(g_H\chi_H + \sum_{L\in\mathscr{P}^j(H)} g_L\chi_L\Big)(x) = D^i g_H(x) + D^i \sum_{L\in\mathscr{P}^j(H)} (g_L - g_H)\chi_L\,(x)\,. \quad (4.8)$$

Using the Leibnitz rule, (4.7) and the estimates of Proposition 4.5(i) - (ii), we get

$$\|D^i\hat\varphi_j\|_{C^0(H)} \leq \|g_H\|_{C^i} + \sum_{0\leq l\leq i}\sum_{L\in\mathscr{P}^j(H)} \|g_L - g_H\|_{C^l(\mathrm{spt}(\chi_L))}\ell(L)^{l-i} \leq C\boldsymbol{m}_0^{\frac{1}{2}}\big(1 + \ell(H)^{3+\kappa-i}\big),$$

(assuming M_0 is larger than the geometric constant $2\sqrt{m}$, we have $\mathrm{spt}(\chi_L) \subset B_{r_L}(x_L)$, implying that Proposition 4.5(ii) can be applied). On the other hand observe that, by interpolation between $\|g_H - g_L\|_{C^0} \leq C\boldsymbol{m}_0^{1/2}\ell(H)^{3+\kappa}$ and $[D^3 g_H - D^3 g_L]_\kappa \leq C\boldsymbol{m}_0^{1/2}$, we obtain

$$\|g_H - g_L\|_{C^{j,\bar\kappa}} \leq C\boldsymbol{m}_0^{1/2}\ell(H)^{3+\kappa-j-\bar\kappa} \quad \text{for any } \ j + \bar\kappa \leq 3 + \kappa.$$

Thus,

$$[D^3\hat\varphi_j]_{\kappa,H} \leq \sum_{0\leq l\leq 3}\sum_{L\in\mathscr{P}^j(H)} \ell(H)^{l-3}\big(\ell(H)^{-\kappa}\|D^l(g_L - g_H)\|_{C^0(H)} + [D^l(g_L - g_H)]_{\kappa,H}\big)$$
$$+ [D^3 g_H]_{\kappa,H} \leq C\boldsymbol{m}_0^{1/2}\,.$$

Fix now $x, y \in [-4,4]^m$, let $H, L \in \mathscr{P}^j$ be such that $x \in H$ and $y \in L$. If $H \cap L \neq \emptyset$, then

$$|D^3\hat\varphi_j(x) - D^3\hat\varphi_j(y)| \leq C\big([D^3\hat\varphi_j]_{\kappa,H} + [D^3\hat\varphi_j]_{\kappa,L}\big)|x - y|^\kappa. \quad (4.9)$$

If $H \cap L = \emptyset$, say $\ell(H) \leq \ell(L)$, then

$$\max\big\{|x - x_H|, |y - x_L|\big\} \leq \ell(L) \leq 2\,|x - y|.$$

Moreover, by construction $\hat\varphi_j$ is identically equal to g_H in a neighborhood of its center x_H. Thus, we can estimate

$$\begin{aligned}|D^3\hat\varphi_j(x) - &D^3\hat\varphi_j(y)|\\ &\leq |D^3\hat\varphi_j(x) - D^3\hat\varphi_j(x_H)| + |D^3 g_H(x_H) - D^3 g_L(x_L) + |D^3\hat\varphi_j(x_L) - D^3\hat\varphi_j(y)||\\ &\leq C\boldsymbol{m}_0^{1/2}\,(|x - x_H|^\kappa + |x_H - x_L|^\kappa + |y - x_L|^\kappa) \leq C\boldsymbol{m}_0^{1/2}|x - y|^\kappa\,,\end{aligned} \quad (4.10)$$

where we used (4.9) and Proposition 4.5(iii).

We have then proved that $\|D^3\hat\varphi_j\|_{C^{2,\kappa}} \leq C\boldsymbol{m}_0^{1/2}$. Since $\varphi_j(x) = (\bar\varphi_j(x), \Psi(x, \bar\varphi_j(x)))$, where $\bar\varphi_j(x)$ denote the first $\bar n$ components of $\hat\varphi_j(x)$, Theorem 1.12(i) follows easily from the chain rule.

Let $L \in \mathscr{W}^i$ and fix $j \geq i+2$. Observe that, by the inductive procedure defining $\mathscr{S}^j \cup \mathscr{W}^j$, we have $\mathscr{P}^j(H) = \mathscr{P}^{i+2}(H) \subset \mathscr{W}$. Moreover, by Assumption 1.8, $\mathrm{spt}(\vartheta_L) \cap \mathrm{spt}(\vartheta_H) = \emptyset$ $\forall L \notin \mathscr{P}^j(H)$. Thus, Theorem 1.12(ii) follows.

Finally, to prove (iii), it suffices to show that $\{\varphi_j\}$ is a Cauchy sequence in C^0 (the convergence up to subsequence follows straightforwardly from (i)). To this aim, let $x \in$

$[-4,4]^m$ and assume that $x \in L \cap H$ with $L \in \mathscr{P}^j$ and $H \in \mathscr{P}^{j+1}$. Without loss of generality, we can make the choice of H and L in such a way that either $H = L$ or H is a son for L. Now, if $\ell(L) \geq 2^{-j+2}$, then by (ii) we have $\varphi_j(x) = \varphi_{j+1}(x)$. Otherwise, from (i) and Proposition 4.5(iv), we can conclude that:

$$\begin{aligned} |\hat\varphi_j(x) - \hat\varphi_{j+1}(x)| &\leq |\hat\varphi_j(x) - \hat\varphi_j(x_H)| + |g_H(x_H) - g_L(x_L)| + |\hat\varphi_{j+1}(x) - \hat\varphi_{j+1}(x_L)| \\ &\leq C\big(\|\hat\varphi_j\|_{C^1} + \|\hat\varphi_{j+1}\|_{C^1}\big)2^{-j} + \|g_H - y_H\|_{C^0} + \|g_L - y_L\|_{C^0} + |y_H - y_L| \\ &\leq C\boldsymbol{m}_0^{1/2m} 2^{-j} + |p_H - p_L| \,. \end{aligned} \tag{4.11}$$

Since $\mathbf{B}_H \subset \mathbf{B}_L$ by Proposition 4.2(ii), we conclude $|\hat\varphi_j(x) - \hat\varphi_{j+1}(x)| \leq C2^{-j}$, where the constant C depends upon the various parameters, but not on j. Given that Ψ is Lipschitz, we get $\|\varphi_j - \varphi_{j+1}\|_{C^0} \leq C\, 2^{-j}$ and conclude. □

5. Proof of the three key construction estimates

5.1. **Elliptic PDE for the average.** This section contains the most important computation, namely the derivation via a first variation argument of a suitable elliptic system for the average of the π-approximations. In order to simplify the notation we introduce the following definition.

Definition 5.1 (Tangential parts)**.** Having fixed $H \in \mathscr{P}^j$ and $\pi := \pi_H \subset T_{p_H}\Sigma$, we let $\varkappa$ be the orthogonal complement of π in $T_{p_H}\Sigma$. For any given point $q \in \mathbb{R}^{m+n}$, any set $\Omega \subset \pi$ and any map $\xi : q + \Omega \to \pi^\perp$, we denote by $\bar\xi$ the map $\mathbf{p}_\varkappa \circ \xi$, and call it the *tangential part of* ξ. Analogous notation will be used for multiple-valued maps.

Proposition 5.2 (Elliptic system)**.** *Assume the hypotheses of Proposition 4.5 hold. Let $H \in \mathscr{W}^j \cup \mathscr{S}^j$ and L be either an ancestor of H or another element $L \in \mathscr{W}^j \cap \mathscr{S}^j$ with $H \cap L \neq \emptyset$ (possibly also H itself). Set $\pi := \pi_H$, $r := r_L$, $p := p_L$, $B := B_{8r}(p,\pi)$. Let $f : B \to \mathcal{A}_Q(\pi^\perp)$ be the π-approximation of T in $\mathbf{C}_{8r}(p,\pi)$ and h its smoothed average, according to Definition 1.9. Then, there is a matrix $\mathbf{L}$, which depends on Σ and H but not on L, such that $|\mathbf{L}| \leq C\mathbf{A}^2 \leq C\boldsymbol{m}_0$ and*

$$\left| \int \big(D(\boldsymbol{\eta}\circ\bar f) : D\zeta + (\mathbf{p}_\pi(x - p_H))^t \cdot \mathbf{L}\cdot\zeta\big)\right| \leq C\boldsymbol{m}_0\, r^{m+1+\beta_2}\big(r\,\|\zeta\|_{C^1} + \|\zeta\|_{C^0}\big)\,, \tag{5.1}$$

for every test function $\zeta \in C^\infty_c(B,\varkappa)$. Moreover,

$$\|\bar h - \boldsymbol{\eta}\circ\bar f\|_{L^1(B_{7r}(p,\pi))} \leq C\boldsymbol{m}_0\, r^{m+3+\beta_2}\,. \tag{5.2}$$

The constant C depends on all parameters except ε_2 (in particular it does not depend on H and L).

Proof. We fix a system of coordinates $(x,y,z) \in \pi \times \varkappa \times (T_{p_H}\Sigma)^\perp$ so that $p_H = (0,0,0)$. Also, in order to simplify the notation, although the domains of the various maps are subsets of $p_L + \pi$, we will from now on consider them as functions of x (i.e. we shift their domains to $\mathbf{p}_\pi(\Omega)$). We also drop the subscript p_H for the map Ψ_{p_H} of Assumption 1.2. Recall that $\Psi(0,0) = 0$, $D\Psi(0,0) = 0$ and $\|\Psi\|_{C^{3,\varepsilon_0}} \leq C\boldsymbol{m}_0^{1/2}$.

Given a test function ζ and any point $q=(x,y,z)\in\Sigma$, we consider the vector field $\chi(q)=(0,\zeta(x),D_y\Psi(x,y)\cdot\zeta(x))$. Observe that χ is tangent to Σ and therefore $\delta T(\chi)=0$ Thus,

$$|\delta\mathbf{G}_f(\chi)|\leq|\delta\mathbf{G}_f(\chi)-\delta T(\chi)|\leq C\int_{\mathbf{C}_{8r}(p_L,\pi)}|D\chi|\,d\|\mathbf{G}_f-T\|\,. \tag{5.3}$$

Observe also that $|\chi|\leq C|\zeta|$ and $|D\chi|\leq C|\zeta|+C|D\zeta|$. Set now $E:=\mathbf{E}\big(T,\mathbf{C}_{32r}(p_L,\pi)\big)$ and apply [6, Theorem 1.4] to conclude that

$$|Df|\leq CE^{\gamma_1}\leq C\boldsymbol{m}_0^{\gamma_1}r^{\gamma_1}, \tag{5.4}$$

$$|f|\leq C\mathbf{h}(T,\mathbf{C}_{32r}(p_L,\pi))+(E^{1/2}+r\mathbf{A})r\leq C\boldsymbol{m}_0^{1/2m}r^{1+\beta_2}, \tag{5.5}$$

$$\int_B|Df|^2\leq C\,r^mE\leq C\boldsymbol{m}_0\,r^{m+2-2\delta_2}. \tag{5.6}$$

Concerning (5.5) observe that the statement of [6, Theorem 1.4] bounds indeed $\operatorname{osc}(f)$. However, in our case we have $p_H=(0,0,0)\in\operatorname{spt}(T)$ and $\operatorname{spt}(T)\cap\operatorname{Gr}(f)\neq\emptyset$. Thus we conclude $|f|\leq\operatorname{osc}(f)+\mathbf{h}(T,\mathbf{C}_{32r}(p_L,\pi))$.

Writing $f=\sum_i[\![f_i]\!]$ and $\bar f=\sum_i[\![\bar f_i]\!]$, since $\operatorname{Gr}(f)\subset\Sigma$, we have $f=\sum_i[\![(\bar f_i,\Psi(x,\bar f_i))]\!]$. From [5, Theorem 4.1] we can infer that

$$\begin{aligned}\delta\mathbf{G}_f=\int_B\sum_i\Big(&\underbrace{D_{xy}\Psi(x,\bar f_i)\cdot\zeta}_{(A)}+\underbrace{(D_{yy}\Psi(x,\bar f_i)\cdot D\bar f_i)\cdot\zeta}_{(B)}+\underbrace{D_y\Psi(x,\bar f_i)\cdot D_x\zeta}_{(C)}\Big)\\ &:\Big(\underbrace{D_x\Psi(x,\bar f_i)}_{(D)}+\underbrace{D_y\Psi(x,\bar f_i)\cdot D\bar f_i}_{(E)}\Big)+\int_B\sum_i D\zeta:D\bar f_i+\mathrm{Err}\,.\end{aligned} \tag{5.7}$$

To avoid cumbersome notation we use $\|\cdot\|_0$ for $\|\cdot\|_{C^0}$ and $\|\cdot\|_1$ for $\|\cdot\|_{C^1}$. Recalling [5, Theorem 4.1], the error term Err in (5.7) satisfies the inequality

$$|\mathrm{Err}|\leq C\int|D\chi||Df|^3\leq\|\zeta\|_1\int|Df|^3\leq C\|\zeta\|_1\boldsymbol{m}_0^{1+\gamma_1}r^{m+2-2\delta_2+\gamma_1}\,. \tag{5.8}$$

The second integral in (5.7) is obviously $Q\int_B D\zeta:D(\boldsymbol{\eta}\circ\bar f)$. We therefore expand the product in the first integral and estimate all terms separately. We will greatly profit from the Taylor expansion $D\Psi(x,y)=D_xD\Psi(0,0)\cdot x+D_yD\Psi(0,0)\cdot y+O\big(\boldsymbol{m}_0^{1/2}(|x|^2+|y|^2)\big)$. In particular we gather the following estimates:

$$|D\Psi(x,\bar f_i)|\leq C\boldsymbol{m}_0^{1/2}r\quad\text{and}\quad D\Psi(x,\bar f_i)=D_xD\Psi(0,0)\cdot x+O\big(\boldsymbol{m}_0^{1/2+1/2m}r^{1+\beta_2}\big),$$
$$|D^2\Psi(x,\bar f_i)|\leq C\boldsymbol{m}_0^{1/2}\quad\text{and}\quad D^2\Psi(x,\bar f_i)=D^2\Psi(0,0)+O\big(\boldsymbol{m}_0^{1/2}r\big)\,.$$

We are now ready to compute

$$\int \sum_i (A) : (D) = \int \sum_i (D_{xy}\Psi(0,0)\cdot\zeta) : D_x\Psi(x,\bar f_i) + O\Big(\boldsymbol{m}_0\, r^2 \int |\zeta|\Big)$$
$$= \int \sum_i (D_{xy}\Psi(0,0)\cdot\zeta : D_{xx}\Psi(0,0)\cdot x + O\Big(\boldsymbol{m}_0\, r^{1+\beta_2}\int |\zeta|\Big)\,. \qquad (5.9)$$

Obviously the first integral in (5.9) has the form $\int x^t \cdot \mathbf{L}_{AD}\cdot\zeta$. Next, we estimate

$$\int \sum_i (A) : (E) = O\Big(\boldsymbol{m}_0^{1+\gamma_1} r^{1+\gamma_1}\int |\zeta|\Big), \qquad (5.10)$$

$$\int \sum_i (B) : ((D)+(E)) = \Big(\boldsymbol{m}_0^{1+\gamma_1} r^{1+\gamma_1}\int |\zeta|\Big), \qquad (5.11)$$

$$\int \sum_i (C) : (E) = O\Big(\boldsymbol{m}_0^{1+\gamma_1} r^{2+\gamma_1}\int |D\zeta|\Big). \qquad (5.12)$$

Finally we compute

$$\int \sum_i (C) : (D) = \int \sum_i ((D_{xy}\Psi(0,0)\cdot x)\cdot D_x\zeta) : D_x\Psi(x,\bar f_i) + O\Big(\boldsymbol{m}_0\, r^{2+\beta_2}\int |D\zeta|\Big)$$
$$= \int \sum_i (D_{xy}\Psi(0,0)\cdot x)\cdot D_x\zeta) : (D_{xx}\Psi(0,0)\cdot x) + O\Big(\boldsymbol{m}_0\, r^{2+\beta_2}\int |D\zeta|\Big)\,.$$

Integrating by parts in the last integral we reach

$$\int \sum_i (C) : (D) = \int x^t\cdot \mathbf{L}_{CD}\cdot\zeta + O\Big(\boldsymbol{m}_0\, r^{2+\beta_2}\int |D\zeta|\Big)\,. \qquad (5.13)$$

Set next $\mathbf{L} := \mathbf{L}_{AD} + \mathbf{L}_{CD}$. Clearly $\mathbf{L}$ is a quadratic function of $D^2\Psi(0,0)$, i.e. a quadratic function of the tensor A_Σ at the point p_H. In order to summarize all our estimates we introduce some simpler notation. We define $\mathbf{f} = \boldsymbol{\eta}\circ\bar f$, $\ell := \ell(L)$ and recall that K is the closed set of [6, Theorem 1.4], on which $\mathbf{G}_f$ and T coincide: $\mathbf{G}_f \llcorner (K\times\pi^\perp) = T\llcorner(K\times\pi^\perp)$. Let μ be the measure on B given by

$$\mu(E) := |E\setminus K| + \|T\|((E\setminus K)\times\mathbb{R}^n)\,.$$

We can then summarize (5.3) and (5.7) - (5.13) into the following estimate:

$$\Big|\int \big(D\mathbf{f} : D\zeta + x^t\cdot\mathbf{L}\cdot\zeta\big)\Big| \leq C\boldsymbol{m}_0\, r^{1+\beta_2}\int \big(r|D\zeta(x)| + |\zeta(x)|\big)\,dx$$
$$+ C\int \big(|D\zeta(x)| + |\zeta(x)|\big)\big(|Df(x)|^3 dx + d\mu(x)\big)\,. \qquad (5.14)$$

From (5.4) and (5.6), we infer that

$$\int |Df|^3 \leq C r^m \mathrm{Lip}(f) E \leq C\, \boldsymbol{m}_0^{1+\gamma_1} r^{m+2-2\delta_2+\gamma_1}, \tag{5.15}$$

$$\mu(B) \leq C E^{\gamma_1}(E + r^2 \mathbf{A}^2) r^m \leq C\, \boldsymbol{m}_0\, r^{m+2-2\delta_2+\gamma_1}\,. \tag{5.16}$$

Therefore (5.1) follows from (5.14) and our choice of the parameters in Assumption 1.5.

We next come to (5.2). Fix a smooth radial test function ς and set $\zeta(\cdot) := \varsigma(z-\cdot)e_i$, where $e_{m+1},\ldots,e_{m+\bar n}$ is on orthonormal base of $\varkappa$. Observe that, if in addition we assume $\int \varsigma = 0$, then $\int y_i \varsigma(z-y_i)dx = 0$. Under these assumptions, from (5.14) we get

$$\left|\int \langle D\mathbf{f}^i(y), D\varsigma(z-y)\rangle\, dy\right| \leq C \int |Df|^3(y)(|D\varsigma| + |\varsigma|)(z-y)\, dy$$
$$+ C \int (|D\varsigma| + |\varsigma|)(z-y)\, d\mu(y) + C\boldsymbol{m}_0 r^{1+\beta_2} \int (r|D\varsigma| + |\varsigma|). \tag{5.17}$$

Recall the standard estimate on convolutions $\|a * \mu\|_{L^1} \leq \|a\|_{L^1}\mu(B)$, and integrate (5.17) in z: by (5.15) and (5.16) we reach

$$\|D\mathbf{f}^i * D\varsigma\|_{L^1} \leq C\boldsymbol{m}_0\, r^{m+1+\beta_1} \int (r|D\varsigma| + |\varsigma|) \quad \forall \varsigma \in C^\infty_c(B_\ell) \text{ with } \int \varsigma = 0\,. \tag{5.18}$$

By a simple density argument, (5.18) holds also when $\varsigma \in W^{1,1}$ is supported in B_ℓ and $\int \varsigma = 0$. Observe next

$$\bar h(x) - \mathbf{f}(x) = \int \varrho_\ell(y)(\mathbf{f}(x-y) - \mathbf{f}(x))\, dy = \int \varrho_\ell(y) \int_0^1 D\mathbf{f}(x-\sigma y)\cdot(-y)\, d\sigma\, dy$$
$$= \int\int_0^1 \varrho_\ell\left(\tfrac{w}{\sigma}\right) D\mathbf{f}(x-w)\cdot \frac{-w}{\sigma^{m+1}}\, dw = \int D\mathbf{f}(x-w)\cdot \underbrace{(-w)\int_0^1 \varrho_\ell\left(\tfrac{w}{\sigma}\right)\sigma^{-m-1}\, d\sigma}_{=:\Upsilon(w)}\, dw\,.$$

Note that Υ is smooth on $\mathbb{R}^m \setminus \{0\}$ and unbounded in a neighborhood of 0. However,

$$\|\Upsilon\|_{L^1} = \int\int_0^1 |w|\left|\varrho\left(\tfrac{w}{\ell\sigma}\right)\right| \ell^{-m}\sigma^{-m-1}\, d\sigma\, dw = \ell \int\int_0^1 |u||\varrho(u)|\, d\sigma\, du \leq Cr\,. \tag{5.19}$$

Observe also that $\Upsilon(w) = w\,\psi(|w|)$. Therefore Υ is a gradient. Since $\Upsilon(w)$ vanishes outside a compact set, integrating along rays from ∞, we can compute a potential for it:

$$\varsigma(w) = \int_{|w|}^\infty \tau \int_0^1 \varrho_\ell\left(\tfrac{w\tau}{|w|\sigma}\right)\sigma^{-m-1}\, d\sigma\, d\tau = |w|^2 \int_1^\infty t \int_0^1 \varrho_\ell\left(\tfrac{wt}{\sigma}\right)\sigma^{-m-1}\, d\sigma\, dt\,. \tag{5.20}$$

Then, ς is a $W^{1,1}$ function, supported in $B_\ell(0)$, $\int \varsigma = 0$ by Assumption 1.8 and (5.17). Summarizing, $\hat{h}^i - \mathbf{f}^i = (D\mathbf{f}^i) * D\varsigma$ for a convolution kernel for which (5.18) holds. Since

$$\begin{aligned}\|\varsigma\|_{L^1} &\leq \int \int_1^\infty \int_0^1 t|w|^2 \left|\varrho\left(\tfrac{wt}{\ell\sigma}\right)\right| \ell^{-m}\sigma^{-m-1} d\sigma\, dt\, dw\\ &= \ell^2 \int_1^\infty \int_0^1 \int |u|^2 |\rho(u)| du\, \sigma d\sigma\, t^{-m-1} dt \leq C r^2\,, \end{aligned} \tag{5.21}$$

we then conclude from (5.17) and (5.18)

$$\int |\bar{h} - \mathbf{f}| \leq C \boldsymbol{m}_0\, r^{m+1+\beta_2} \int (r|D\varsigma| + |\varsigma|) \leq C\boldsymbol{m}_0\, r^{m+3+\beta_2}. \qquad \square$$

5.2. C^k **estimates for** h_{HL} **and** g_{HL}**.** We fix cubes H, L as in Proposition 5.2 and the maps h_{HL} and g_{HL} of Definition 4.3.

Lemma 5.3. *Assume that H and L are as in Proposition 5.2 and the hypotheses of Proposition 4.5 hold. Set $B' := B_{5r_H}(p_H, \pi_H)$ and $B := B_{4r_H}(p_H, \pi_0)$. Then,*

$$\|h_{HL} - h_H\|_{C^j(B')} + \|g_{HL} - g_H\|_{C^j(B)} \leq C\boldsymbol{m}_0\, \ell(L)^{3+2\kappa-j} \qquad \forall j \in \{0, \ldots, 3\}\,, \tag{5.22}$$

$$\|h_{HL} - h_H\|_{C^{3,\kappa}(B')} + \|g_{HL} - g_H\|_{C^{3,\kappa}(B)} \leq C\boldsymbol{m}_0\, \ell(L)^{\kappa}\,. \tag{5.23}$$

As a consequence Proposition 4.5(i) and (iv) hold.

Proof. Consider a triple of cubes H, J and L where:

(a) either $J = H$ and L is the father of J;
(b) or $J = H$, and $L \in \mathscr{S}^j \cup \mathscr{W}^j$ adjacent to H;
(c) or J is an ancestor of H and L the father of J.

In order to simplify the notation let $\pi := \pi_H$ and $r := r_J$. By Proposition 4.2(i), up to choose M_0 larger than a geometric constant, we can assume that $B^\flat := B_{6r}(p_J, \pi) \subset B^\sharp = B_{13r/2}(p_J, \pi) \subset \bar{B} := B_{7r_L}(p_L, \pi)$. Consider the π-approximations f_{HL} and f_{HJ}, respectively in $\mathbf{C}_{8r}(p_L, \pi)$ and $\mathbf{C}_{8r_J}(p_J, \pi)$, and introduce the corresponding maps

$$\begin{aligned} \bar{\mathbf{f}}_L &:= \mathbf{p}_\varkappa(\boldsymbol{\eta} \circ f_{HL}) \quad \text{and} \quad \bar{\mathbf{f}}_J := \mathbf{p}_\varkappa(\boldsymbol{\eta} \circ f_{HJ}),\\ \bar{h}_{HL} &:= \bar{\mathbf{f}}_L * \varrho_{\ell(L)} \quad \text{and} \quad \bar{h}_{HJ} = \bar{\mathbf{f}}_J * \varrho_{\ell(J)}\,. \end{aligned}$$

If $\mathbf{l}$ is an affine function on $\mathbb{R}^m$ and ς a radial convolution kernel, then $\varsigma * \mathbf{l} = \left(\int \varsigma\right) \mathbf{l}$ because $\mathbf{l}$ is an harmonic function. This means that $\int \langle (\zeta * \varrho), \mathbf{l}\rangle = \int \langle \zeta, \mathbf{l}\rangle$ for any test function ζ and any radial convolution kernel ϱ with integral 1. Similarly $\int \langle (\zeta * \partial^I \varrho), \mathbf{l}\rangle = \int \langle \zeta, \partial^I \mathbf{l}\rangle$ for any partial derivative ∂^I of any order. Consider now a ball $\hat{B}$ concentric to $B^\flat$ and contained in $B^\sharp$ in such a way that, if $\zeta \in C^\infty_c(\hat{B})$, then $\zeta * \varrho_{\ell(L)}$ and $\zeta * \varrho_{\ell(J)}$ are both supported in $B^\sharp$. Set $\xi := \bar{h}_{HL} - \bar{h}_{HJ}$ and (assuming $\mathbf{p}_\pi(x_H)$ is the origin of our system of coordinates) compute:

$$\int \langle \zeta, \Delta\xi\rangle = -\int D(\bar{h}_{HL} - \bar{h}_{HJ}) : D\zeta = \int D\mathbf{f}_J : D(\zeta * \varrho_{\ell(J)}) - \int D\mathbf{f}_L : D(\zeta * \varrho_{\ell(L)})$$

$$= \int \big(D\mathbf{f}_J : D(\zeta * \varrho_{\ell(J)}) + x^t \cdot \mathbf{L} \cdot (\zeta * \varrho_{\ell(J)})\big) - \int \big(D\mathbf{f}_L : D(\zeta * \varrho_{\ell(L)}) + x^t \cdot \mathbf{L} \cdot (\zeta * \varrho_{\ell(L)})\big)\,,$$

where the last line holds for any matrix $\mathbf{L}$ because $x \mapsto x^t \cdot \mathbf{L}$ is a linear function. In particular, we can use the matrix of Proposition 5.2 to achieve

$$\int \langle \zeta, \Delta\xi\rangle \le C\boldsymbol{m}_0\, r^{m+1+\beta_2}\Big(r\|\zeta * \varrho_{\ell(L)}\|_1 + r\|\zeta * \varrho_{\ell(J)}\|_1 + \|\zeta * \varrho_{\ell(J)}\|_0 + \|\zeta * \varrho_{\ell(L)}\|_0\Big)\,,$$

where $\|\cdot\|_0$ and $\|\cdot\|_1$ denote the C^0 and C^1 norms respectively. Recalling the inequality $\|\psi * \zeta\|_0 \le \|\psi\|_\infty \|\zeta\|_{L^1}$ and taking into account that $\ell(L)$ and $\ell(J)$ are both comparable to r (up to a constant depending only on M_0 and m), we achieve $\int\langle \zeta, \Delta\xi\rangle \le C\boldsymbol{m}_0\, r^{1+\beta_2}\|\zeta\|_{L^1}$. Taking the supremum over all possible test functions with $\|\zeta\|_{L^1} \le 1$, we obviously conclude $\|\Delta\xi\|_{L^\infty(\hat{B})} \le C\boldsymbol{m}_0\, r^{1+\beta_2}$. Observe that a similar estimate could be achieved for any partial derivative $D^k\xi$ simply using the identity

$$\int D(D^k(a * \varsigma)) : Db = -\int Da : (Db * D^k\varsigma)\,.$$

Summarizing we conclude

$$\|\Delta D^k(f_{HL} - f_{HJ})\|_{C^0(\hat{B})} \le \|\Delta D^k\xi\|_\infty \le C\boldsymbol{m}_0 r^{1+\beta_2-k}\,, \tag{5.24}$$

where the constant C depends upon all the parameters and on $k \in \mathbb{N}$, but not on ε_2, $\boldsymbol{m}_0$, H, J or L. By [6, Theorem 1.4] (cp. also the proof of Proposition 4.4), we have $\mathrm{osc}(f_{HL}) + \mathrm{osc}(f_{HJ}) \le C\,\boldsymbol{m}_0^{1/2m} r$ and

$$\mathcal{H}^m(\{f_{HL} \ne f_{HJ}\}) \le C\,\mathbf{E}(T, \mathbf{C}_{32r_L}(p_L, \pi_H))\, r^m \le C\,\boldsymbol{m}_0^{1+\gamma_1} r^{m+2+\gamma_1/2}.$$

Therefore, taking into account (5.2), we conclude $\|\bar{h}_{HL} - \bar{h}_{HJ}\|_{L^1} \le C\,\boldsymbol{m}_0\, r^{m+3+\beta_2}$. Thus, we appeal to Lemma C.1 and use the latter estimate together with (5.24) (in the case $k=0$) to get $\|\bar{h}_{HL} - \bar{h}_{HJ}\|_{C^k(B')} \le C\boldsymbol{m}_0 r^{3+\beta_2-k}$ for $k = \{0,1\}$ and for every concentric smaller ball $B' \subset \hat{B}$ (where the constant depends also on the ratio between the corresponding radii). This implies $\|D(\bar{h}_{HL} - \bar{h}_{HJ})\|_{L^1(B')} \le C\boldsymbol{m}_0 r^{m+2+\beta_2}$ and hence we can use again Lemma C.1 (based on the case $k=1$ of (5.24)) to conclude $\|\bar{h}_{HL} - \bar{h}_{HJ}\|_{C^2(B'')} \le C\boldsymbol{m}_0 r^{1+\beta_2}$. Iterating another two times we can then conclude $\|\bar{h}_{HL} - \bar{h}_{HJ}\|_{C^k(B^\sharp)} \le C\boldsymbol{m}_0 r^{3+\beta_2-k}$ for $k \in \{0,1,2,3,4\}$. By interpolation, since $\kappa \le \beta_2/4$, $\|\bar{h}_{HL} - \bar{h}_{HJ}\|_{C^{3+\kappa}} \le C\boldsymbol{m}_0\,\ell(L)^{3\kappa}$.

Fix now $L =: L_j \subset L_{j-1} \subset \ldots \subset L_{N_0}$ be the chain of fathers with $L_i \in \mathscr{S}^i$. Summing the corresponding estimates, we get

$$\|\bar{h}_{HL} - \bar{h}_{HL_{N_0}}\|_{C^{3,\kappa}} \le C\sum_{i=N_0}^{j-1} \|\bar{h}_{HL_{i+1}} - \bar{h}_{HL_i}\|_{C^{3,\kappa}} \le C\boldsymbol{m}_0 \sum_i 2^{-3\kappa i} \le C\boldsymbol{m}_0\,. \tag{5.25}$$

Observe next that $\bar h_{HL_{N_0}} = \mathbf{f}_{HL_{N_0}} * \varrho_{2^{-N_0}}$ and that

$$\|D\mathbf{f}_{HL_{N_0}}\|^2_{L^2} \leq \mathrm{Dir}(f_{HL_{N_0}}) \leq C\mathbf{E}(T, \mathbf{C}_{32r_{L_{N_0}}}(p_{L_{N_0}}, \pi_H)) \leq C\boldsymbol{m}_0 + C|\pi_H - \pi_0|^2 \leq C\boldsymbol{m}_0\,.$$

Thus, by standard convolution estimates, $\|D\bar h_{L_{N_0}}\|_{C^k} \leq C\boldsymbol{m}_0^{1/2}$ (where the constant C depends on $k \in \mathbb{N}$ and on he various parameters). The latter estimate combined with (5.25) leads to $\|D\bar h_{HL}\|_{C^{2,\kappa}} \leq C\boldsymbol{m}_0^{1/2}$. Moreover, we infer $\|\bar h_{HL}\|_{C^0} \leq C\boldsymbol{m}_0^{1/2m}$, appealing again to (5.25) and using this time $\|\bar h_{HL_{N_0}}\| \leq C\boldsymbol{m}_0^{1/2m}$. Since $h_{HL} = \Psi(x, \bar h_{HL})$ and $h_{HJ} = \Psi(x, \bar h_{HJ})$, we deduce the corresponding estimates for h_{HL} and h_{HJ} from the chain rule.

Now we pass to prove Proposition 4.5(i) and (iv). Since $h_{HH} = h_H$ and $g_{HH} = g_H$, the first claim of (i) follows then from Lemma B.1. Coming to (iv), the estimate on $g_H - y_H$ is a straightforward consequence of the height bound, [6, Theorem 1.4] and Lemma B.1 (applied to h_H). Note that, together with (4.1), this implies the second claim in (i). Next, observe that

$$\|Dh_H\|^2_{L^2} \leq C\|D(\boldsymbol{\eta}\circ f_H)\|^2_{L^2} \leq C\mathrm{Dir}(f_H) \leq C\boldsymbol{m}_0\,\ell(H)^{2-2\delta_2}.$$

Thus, there is at least one point $q \in \mathrm{Gr}(h_H)$ such that $|T_q\mathbf{G}_{h_H} - \pi_H| \leq C\boldsymbol{m}_0^{1/2}\ell(H)^{1-\delta_2}$. Since $\|D^2h_H\|_0 \leq C\boldsymbol{m}_0^{1/2}$, we then conclude that $|T_{q'}\mathbf{G}_{h_H} - \pi_H| \leq C\boldsymbol{m}_0^{1/2}\ell(H)^{1-\delta_2}$ holds indeed for any point $q' \in \mathrm{Gr}(h_H)$. Since $\mathrm{Gr}(g_H)$ is a subset of $\mathrm{Gr}(h_H)$ (with the same orientation!), the second inequality of Proposition 4.5(iv) follows. □

5.3. **Tilted L^1 estimate.** In order to achieve Proposition 4.5(ii) and (iii), we need to compare tilted interpolating functions coming from different coordinates. To this aim, we set the following terminology.

Definition 5.4 (Distant relation)**.** Four cubes H, J, L, M make a distant relation between H and L if $J, M \in \mathscr{S}^j \cup \mathscr{W}^j$ have nonempty intersection, H is a descendant of J (or J itself) and L a descendant of M (or M itself).

Lemma 5.5 (Tilted L^1 estimate)**.** *Assume the hypotheses of Proposition 4.5 hold. Let H, J, L and M be a distant relation between H and L, and let h_{HJ}, h_{LM} be the maps given in Definition 4.3. Consider the map $\hat h_{LM} : B_{4r_J}(p_J, \pi_H) \to \pi_H^\perp$ such that $\mathbf{G}_{\hat h_{LM}} = \mathbf{G}_{h_{LM}} \llcorner \mathbf{C}_{4r_J}(p_J, \pi_H)$ (the existence is ensured by Lemma B.1). Then,*

$$\|h_{HJ} - \hat h_{LM}\|_{L^1(B_{2r_J}(p_J,\pi_H))} \leq C\boldsymbol{m}_0\,\ell(J)^{m+3+\beta_2/3}\,. \tag{5.26}$$

Proof. First observe that Lemma B.1 can be applied because

$$|\pi_H - \pi_L| \leq |\pi_H - \pi_J| + |\pi_J - \pi_M| + |\pi_M - \pi_L| \leq C\boldsymbol{m}_0^{1/2}\ell(J)^{1-\delta_2}.$$

Set $\pi := \pi_H$ and $\varkappa$ for its orthogonal complement in $T_{p_H}\Sigma$, and similarly $\bar\pi = \pi_L$ and $\bar\varkappa$ its orthogonal in $T_{p_L}\Sigma$. After a translation we also assume $p_J = 0$, and write $r = r_J = r_M$, $\ell = \ell(J) = \ell(M)$ and $E := \mathbf{E}(T, \mathbf{C}_{32r}(0, \pi))$, $\bar E := \mathbf{E}(T, \mathbf{C}_{32r}(p_M, \bar\pi))$. Recall that $\max\{E, \bar E\} \leq C\boldsymbol{m}_0\,\ell^{2-2\delta_2}$. We fix also the maps $\Psi : T_0\Sigma \to T_0\Sigma^\perp$ and $\bar\Psi : T_{p_L}\Sigma \to T_{p_L}\Sigma^\perp$

whose graphs coincide with the submanifold Σ. Observe that $|\pi-\bar\pi|+|\varkappa_0-\bar\varkappa| \le C\boldsymbol{m}_0^{1/2}\ell^{1-\delta_2}$, $\|\Psi\|_{C^{3,\varepsilon_0}} + \|\bar\Psi\|_{C^{3,\varepsilon_0}} \le C\boldsymbol{m}_0^{1/2}$ and

$$\|D\Psi\|_{C^0(B_{8r})} + \|D\bar\Psi\|_{C^0(B_{8r})} \le C\boldsymbol{m}_0^{1/2}\ell^{1-\delta_2}.$$

Consider the map $\hat f : B_{4r}(0,\pi) \to \mathcal{A}_Q(\pi^\perp)$ such that $\mathbf{G}_{\hat f} = \mathbf{G}_{f_{LM}} \llcorner \mathbf{C}_{4r}(0,\pi)$, which exists by [5, Proposition 5.2]. Recalling the estimates therein and those of [6, Theorem 1.4], if we set $f = f_{HJ}$ we have

$$\mathrm{Lip}(f) + \mathrm{Lip}(\hat f) \le C\boldsymbol{m}_0^{\gamma_1}\ell^{\gamma_1} \quad \text{and} \quad |f| + |\hat f| \le C\boldsymbol{m}_0^{1/2m}\ell^{1+\beta_2}, \tag{5.27}$$

$$\mathrm{Dir}(f) + \mathrm{Dir}(\hat f) \le C\,\boldsymbol{m}_0\,\ell^{m+2-2\delta_2}\,. \tag{5.28}$$

Consider next the projections A and $\hat A$ onto π of the closed sets $\mathrm{Gr}(f) \setminus \mathrm{spt}(T)$ and $\mathrm{Gr}(\hat f) \setminus \mathrm{spt}(T)$. We know from [6, Theorem 1.4] that

$$|A \cup \hat A| \le C\left[\|\mathbf{G}_f - T\|(\mathbf{C}_{32}(0,\pi)) + \|\mathbf{G}_{\hat f} - T\|(\mathbf{C}_{32}(p_M,\bar\pi))\right] \le C\boldsymbol{m}_0\,\ell^{m+2+\gamma_1}\,. \tag{5.29}$$

Define next $\mathbf{f} = \Psi(x, \mathbf{p}_\varkappa(\boldsymbol{\eta}\circ f))$, $h := h_{HJ} = \Psi(x, \mathbf{p}_\varkappa((\boldsymbol{\eta}\circ f) * \varrho_\ell))$, $\mathbf{f}_M = \bar\Psi(x, \mathbf{p}_{\bar\varkappa}(\boldsymbol{\eta}\circ f_{LM}))$ and $h_{LM} = \bar\Psi(x, \mathbf{p}_{\bar\varkappa}((\boldsymbol{\eta}\circ f_{LM}) * \varrho_\ell))$. We define that $\hat h : B_{4r}(0,\pi) \to \pi^\perp$ such that $\mathbf{G}_{\hat h} = \mathbf{G}_{h_{LM}} \llcorner \mathbf{C}_{4r}(0,\pi)$ and $\hat{\mathbf{f}}$ such that $\mathbf{G}_{\hat{\mathbf{f}}} = \mathbf{G}_{\mathbf{f}_M} \llcorner \mathbf{C}_{4r}(0,\pi)$. We use Proposition 5.2, the Lipschitz regularity of $\bar\Psi$ and Lemma B.1 to conclude

$$\|\hat h - \hat{\mathbf{f}}\|_{L^1} \le C\|h_{LM} - \mathbf{f}_M\|_{L^1} \le C\boldsymbol{m}_0\,r^{m+3+\beta_2}.$$

Likewise $\|h - \mathbf{f}\|_{L^1} \le C\boldsymbol{m}_0 r^{m+3+\beta_2}$. We therefore need to estimate $\|\mathbf{f} - \hat{\mathbf{f}}\|_{L^1}$. Define next the map $\mathbf{g} = \Psi(x, \mathbf{p}_\varkappa(\boldsymbol{\eta}\circ\hat f))$ and observe that the $\|\mathbf{g} - \mathbf{f}\|_{L^1} \le C\|\boldsymbol{\eta}\circ\hat f - \boldsymbol{\eta}\circ f\|_{L^1}$. On the other hand, since the two maps $\hat f$ and f differ only on $A \cup \bar A$, we can estimate the latter with $C|A\cup\bar A|(|f| + |\bar f|) \le C\boldsymbol{m}_0^{1+1/2m}\ell^{3+m+\gamma_1+\beta_2}$. It thus suffices to estimate $\|\mathbf{g} - \hat{\mathbf{f}}\|_{L^1}$. This estimate is indeed independent of the rest and we prove it in the next lemma. $\square$

Lemma 5.6. *Consider two triples of planes $(\pi, \varkappa, \varpi)$ and $(\tilde\pi, \tilde\varkappa, \tilde\varpi)$, where*

- *$\pi, \tilde\pi$ are m-dimensional;*
- *$\varkappa$ and $\tilde\varkappa$ are $\bar n$-dimensional and orthogonal, respectively, to π and $\tilde\pi$;*
- *ϖ and $\tilde\varpi$ l-dimensional and orthogonal, respectively, to $\pi\times\varkappa$ and $\tilde\pi\times\tilde\varkappa$.*

Assume $|\pi-\tilde\pi|, |\varkappa-\tilde\varkappa| \le C\boldsymbol{m}_0^{1/2}r^{1-\delta_2}$ and let $\Psi : \pi\times\varkappa \to \varpi$, $\tilde\Psi : \tilde\pi\times\tilde\varkappa \to \tilde\varpi$ be two maps whose graphs coincide and such that $\|\Psi\|_{C^{3,\varepsilon_0}} + \|\tilde\Psi\|_{C^{3,\varepsilon_0}} \le C\boldsymbol{m}_0^{1/2}$, $|\Psi(0)| + |\tilde\Psi(0)| \le C\boldsymbol{m}_0^{1/2}r$ and

$$\|D\Psi\|_{C^0(B_{8r})} + \|D\tilde\Psi\|_{C^0(B_{8r})} \le C\boldsymbol{m}_0^{1/2}r^{1-\delta_2}.$$

Let $\tilde u : B_{8r}(0,\tilde\pi) \to \mathcal{A}_Q(\tilde\varkappa)$ be a map with

$$\mathrm{Lip}(\tilde u) \le C\boldsymbol{m}_0^{\gamma_1}r^{\gamma_1}, \quad \|\tilde u\|_0 \le C\boldsymbol{m}_0^{1/2m}r^{1+\beta_2} \quad \text{and} \quad \mathrm{Dir}(\tilde u) \le C\boldsymbol{m}_0 r^{m+2+\gamma_1}. \tag{5.30}$$

Consider the maps $\tilde g(x) = \sum_i [\![(\tilde u_i(x), \tilde\Psi(x, \tilde u_i(x)))]\!]$, $\tilde{\mathbf{g}}(x) = (\boldsymbol{\eta}\circ\tilde u(x), \tilde\Psi(\boldsymbol{\eta}\circ\tilde u(x)))$. Let $u : B_{4r}(0,\pi) \to \mathcal{A}_Q(\varkappa)$ be such that the map $g(x) := \sum_i [\![(u_i(x), \Psi(x, u_i(x)))]\!]$ satisfies

$\mathbf{G}_g = \mathbf{G}_{\tilde g} \llcorner \mathbf{C}_{4r}(0,\pi)$ *and* $\hat{\mathbf{g}} : B_{4r}(0,\pi) \to \varkappa \times \varpi$ *be such that* $\mathbf{G}_{\hat{\mathbf{g}}} = \mathbf{G}_{\tilde{\mathbf{g}}} \llcorner \mathbf{C}_{4r}(0,\pi)$. *If* $\mathbf{g}(x) := (\boldsymbol{\eta}\circ u(x), \Psi(x, \boldsymbol{\eta}\circ u(x)))$, *then*

$$\|\mathbf{g} - \hat{\mathbf{g}}\|_{L^1} \leq C\boldsymbol{m}_0\, r^{m+3+\beta_2/3}\,. \tag{5.31}$$

Proof. We start fixing the following terminology: we say that $R \in SO(m+\bar n+l)$ is a *small 2d-rotation* if there are two orthonormal vectors e_1, e_2 and an angle $|\theta| \leq C(|\pi-\tilde\pi|+|\varkappa-\tilde\varkappa|)$ such that $R(e_1) = \cos\theta\, e_1 + \sin\theta\, e_2$, $R(e_2) = \cos\theta\, e_2 - \sin\theta\, e_1$ and $R(v) = v$ for every $v \perp \mathrm{span}\,(e_1, e_2)$. We then say that:

- R is of type A with respect to $(\pi, \varkappa, \varpi)$ if $e_1 \in \varkappa$ and $e_2 \in \varpi$;
- R is of type B with respect to $(\pi, \varkappa, \varpi)$ if $e_1 \in \pi$ and $e_2 \in \varkappa$;
- R is of type C with respect to $(\pi, \varkappa, \varpi)$ if $e_1 \in \pi$ and $e_2 \in \varpi$.

The lemma is based on the following claim, whose proof is postponed to the end.

Claim. *There is a number N depending only on $(m, \bar n, l)$ and a constant C such that $(\pi, \varkappa, \varpi) =: (\pi_0, \varkappa_0, \varpi_0)$ and $(\tilde\pi, \tilde\varkappa, \tilde\varpi) = (\pi_{\bar N}, \varkappa_{\bar N}, \varpi_{\bar N})$ can be joined by a chain of triples $(\pi_j, \varkappa_j, \varpi_j)$ of length $\bar N \leq N$ such that each $(\pi_j, \varkappa_j, \varpi_j)$ is the image of $(\pi_{j-1}, \varkappa_{j-1}, \varpi_{j-1})$ under a small 2d-rotation of type A, B or C with respect to the latter triple.*

For the rest of the proof we will then focus on proving (5.31) under the assumption that the triple $(\pi, \varkappa, \varpi)$ is obtained from $(\tilde\pi, \tilde\varkappa, \tilde\varpi)$ applying a small $2d$-rotation of type A, B or C. We then iterate the estimate N times and achieve a slight variant of (5.31) in the case of two general triples:

$$\|\mathbf{g} - \hat{\mathbf{g}}\|_{L^1(B_{2^{-N}8r})} \leq C\boldsymbol{m}_0 r^{m+3+\beta_2/3}\,. \tag{5.32}$$

Since N is just a geometric constant, a simple covering argument will then conclude (5.31).

Type A. In this case we show the stronger bound $\|\mathbf{g} - \hat{\mathbf{g}}\|_{C^0} \leq C\boldsymbol{m}_0\, r^{3+\beta_2/2}$. Use the notation $(z, w) \in \varkappa \times \varpi$ and $(\tilde z, \tilde w) \in \tilde\varkappa \times \tilde\varpi$ for the same point. In what follows we will drop the $\cdot$ when writing the usual products between matrices. We then have $\tilde z = Uz + Vw$ and $\tilde w = Wz + Zw$, where the orthogonal matrix

$$L := \begin{pmatrix} U & V \\ W & Z \end{pmatrix}$$

has the property that $|L - \mathrm{Id}| \leq C\boldsymbol{m}_0^{1/2} r^{1-\delta_2}$. Clearly, Ψ and $\tilde\Psi$ are related by the identity

$$Wz + Z\Psi(x,z) = \tilde\Psi(x, Uz + V\Psi(x,z))\,. \tag{5.33}$$

Fix x and $g(x) = \sum_i [\![(u_i(x), \Psi(x, u_i(x)))]\!] =: \sum_i [\![(z_i, \Psi(x, z_i))]\!]$. We then have

$$\mathbf{g}(x) = (a,b) := \Big(\frac{1}{Q}\sum z_i, \Psi\Big(x, \frac{1}{Q}\sum z_i\Big)\Big) \quad \text{in } \varkappa\times\varpi,$$

and

$$\hat{\mathbf{g}}(x) = L^{-1}\left(U\tfrac{1}{Q}\sum z_i + V\tfrac{1}{Q}\sum \Psi(x,z_i), \tilde\Psi\left(x, U\tfrac{1}{Q}\sum z_i + V\tfrac{1}{Q}\sum\Psi(x,z_i)\right)\right) =: L^{-1}(c,d)\,.$$

Since L is orthogonal, we have

$$
\begin{aligned}
&|\hat{\mathbf{g}}(x) - \mathbf{g}(x)| = |L(a,b) - (c,d)| \\
&= \Big| \Big(V \Big(\Psi \Big(x, \tfrac{1}{Q} \sum z_i \Big) - \tfrac{1}{Q} \sum_i \Psi(x, z_i) \Big), W \tfrac{1}{Q} \sum_i z_i + Z \Psi \Big(x, \tfrac{1}{Q} \sum_i z_i \Big) \\
&\qquad\qquad - \tilde{\Psi} \Big(x, U \tfrac{1}{Q} \sum z_i + V \tfrac{1}{Q} \sum \Psi(x, z_i) \Big) \Big) \Big| \\
&\overset{(5.33)}{=} \Big| \Big(V \Big(\Psi \Big(x, \tfrac{1}{Q} \sum z_i \Big) - \tfrac{1}{Q} \sum_i \Psi(x, z_i) \Big), \\
&\qquad \tilde{\Psi} \Big(x, U \tfrac{1}{Q} \sum z_i + V \Psi \Big(\tfrac{1}{Q} \sum z_i \Big) \Big) - \tilde{\Psi} \Big(x, U \tfrac{1}{Q} \sum z_i + V \tfrac{1}{Q} \sum \Psi(x, z_i) \Big) \Big) \Big| .
\end{aligned}
$$

Thus,

$$
|\hat{\mathbf{g}}(x) - \mathbf{g}(x)| \leq \Big(1 + \mathrm{Lip}(\tilde{\Psi}) \Big) |V| \Big| \tfrac{1}{Q} \sum \Psi(x, z_i) - \Psi \Big(x, \tfrac{1}{Q} \sum z_i \Big) \Big| .
$$

Observe that $|V| \leq |L - \mathrm{Id}| \leq C \boldsymbol{m}_0^{1/2} r^{1-\delta_2}$. On the other hand, with a simple Taylor expansion around the point $(x, \frac{1}{Q} \sum z_i)$ we easily achieve

$$
\Big| \tfrac{1}{Q} \sum \Psi(x, z_i) - \Psi \Big(x, \tfrac{1}{Q} \sum z_i \Big) \Big| \leq C \|D\Psi\|_0 \sum_i \Big| z_i - \tfrac{1}{Q} \sum z_i \Big| \leq C \boldsymbol{m}_0^{1/2 + 1/2m} r^{2+\beta_2-\delta_2} .
$$

Type B. In this case $\Psi = \tilde{\Psi}$ and, given its Lipschitz regularity, it suffices to estimate $\|\boldsymbol{\eta} \circ u - \mathbf{p}_\varkappa(\hat{\mathbf{g}})\|_{L^1}$. We fix an orthonormal base $e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+\bar{n}}$, where the first m vectors span π and the remaining span $\varkappa$. We also assume that the rotation R acts on the plane spanned by $\{e_m, e_{m+1}\}$ and set $v = R(e_m) = a\, e_m + b\, e_{m+1}$ and $v_{m+1} = R(e_{m+1})$. We then define two systems of coordinates: given $q \in \mathbb{R}^m \times R^{\bar{n}}$, we write

$$
\begin{aligned}
q &= \sum_i z_i(q) e_i + t(q) e_m + \tau(q) e_{m+1} + \sum_j y^j(q) e_{j+m} \\
&= \sum_i z_i(q) e_i + s(q) v_m + \sigma(q) v_{m+1} + \sum_j y^j(q) e_{j+m} .
\end{aligned}
$$

The first will be called (t,τ)-coordinates and the second (s,σ)-coordinates.

We fix for the moment $x \in \mathbb{R}^{m-1}$ with $|x| \leq 4r$ and focus our attention on the interval $\tilde{I}_x = \{s : |(x,s)| \leq 6r\}$. We restrict the map $\tilde{u}$ to this interval and, by [4, Proposition 1.2] we know that there is a Lipschitz selection such that $\tilde{u}(x,s) = \sum_i [\![\theta_i(s)]\!]$. In the (s,σ)-coordinates: $\mathrm{Gr}(\theta_i) = \{(x, s, \theta_i^1(s), \ldots, \theta_i^{\bar{n}}(s)) : s \in \tilde{I}_x\}$. In the (t,τ) coordinates we can choose functions ϑ_i, also defined on an appropriate interval, whose graphs coincide with the ones of the θ_i. We then obviously must have $u(x,t) = \sum_i [\![\vartheta_i(t)]\!]$ on the domain of definition of g. The coordinate functions θ_i^j and ϑ_i^j are linked by the following relations

$$
\begin{cases}
\Phi_i(t) = a\, t + b\, \vartheta_i^1(t), \\
\theta_i^1(\Phi_i(t)) = -b\, t + a\, \vartheta_i^1(t), \\
\theta_i^l(\Phi_i(t)) = \vartheta_i^l(t), & \text{for } l = 2, \ldots, \bar{n}.
\end{cases}
\tag{5.34}
$$

For Φ_i holds $\mathrm{Lip}(\Phi_i) \leq (1 + C\boldsymbol{m}_0^{1/2} r^{1-\delta_2}) \leq 2$. Likewise we can assume that $\mathrm{Lip}(\Phi_i^{-1}) \leq 2$. Consider now $\tilde{v}(s) = \boldsymbol{\eta} \circ \tilde{u}(x,s) = \frac{1}{Q}\sum_i \theta_i(s)$ and the corresponding $t \mapsto \hat{v}(t) = \mathbf{p}_\varkappa \circ \hat{\mathbf{g}}(x,t)$, linked to $\boldsymbol{\eta} \circ \tilde{u}(x,\cdot)$ through a relation as in (5.34) with a corresponding map Φ:

$$\begin{cases} \Phi(t) = a\,t + b\,\hat{v}^1(t), \\ \frac{1}{Q}\sum_i \theta_i^1(\Phi(t)) = \tilde{v}^1(\Phi(t)) = -b\,t + a\,\hat{v}_i^1(t), \\ \frac{1}{Q}\sum_i \theta_i^l(\Phi(t)) = \tilde{v}^l(\Phi(t)) = \hat{v}^l(t), \qquad \text{for} \quad l = 2, \ldots, \bar{n}. \end{cases} \tag{5.35}$$

Moreover, write $v(t) = \frac{1}{Q}\sum_i \vartheta(t) = \boldsymbol{\eta} \circ u(x,t)$. We can then write

$$\begin{aligned} \boldsymbol{\eta} \circ u(x,t) - \mathbf{p}_\varkappa(\mathbf{g}(x,t)) &= v(t) - \hat{v}(t) = Q^{-1}\sum_i (\vartheta_i(t) - \hat{v}(t)) \\ =& Q^{-1}\sum_i \Big(\underbrace{a^{-1}\theta_i^1(\Phi_i(t)) - a^{-1}\theta_i^1(\Phi(t))}_{1^{\text{st}}\text{ component}}, \ldots, \underbrace{\theta_i^l(\Phi_i(t)) - \theta_i^l(\Phi(t))}_{l^{\text{th}}\text{ component}}, \ldots \Big). \end{aligned} \tag{5.36}$$

This implies that

$$|\boldsymbol{\eta} \circ u(x,t) - \mathbf{p}_\varkappa(\mathbf{g}(x,t))| = |v(t) - \hat{v}(t)| \leq C \sum_i \left| \int_{\Phi(t)}^{\Phi_i(t)} D\theta(\tau)\, d\tau \right|. \tag{5.37}$$

Next we compute

$$\Phi_i(t) - \Phi(t) = b\left(\vartheta_i^1(t) - \hat{v}^1(t)\right) = b\big(\vartheta_i^1(t) - v^1(t)\big) + b\big(v^1(t) - \hat{v}^1(t)\big). \tag{5.38}$$

SInce $|b| \leq C\boldsymbol{m}_0^{1/2}\ell^{1-\delta_2}$, the terms in (5.38) can be estimated respectively as follows:

$$\begin{gathered} |b||\vartheta_i^1(t) - v^1(t)| = |b||u_i^1(x,t) - (\boldsymbol{\eta} \circ u)^1(t)| \leq C\boldsymbol{m}_0^{1/2 + 1/2m} r^{2-\delta_2+\beta_2} \leq C\boldsymbol{m}_0^{1/2+1/2m} r^{2+2\beta_2/3}, \\ |v^1(t) - \hat{v}^1(t)| \stackrel{(5.37)}{\leq} \|D\theta\|_{L^\infty} \sum_{i=1}^Q |\Phi_i(t) - \Phi(t)| \leq C\,\boldsymbol{m}_0^{\gamma_1} r^{\gamma_1} \sum_{i=1}^Q |\Phi_i(t) - \Phi(t)|. \end{gathered}$$

Combining the last two inequalities with (5.38), we therefore conclude, for ε_2 small enough,

$$\sum_{i=1}^Q |\Phi_i(t) - \Phi(t)| \leq C\,\boldsymbol{m}_0^{1/2+1/2m} r^{2+2\beta_2/3} =: \rho\,. \tag{5.39}$$

With this estimate at our disposal we can integrate (5.37) in t to conclude

$$\int_{I_x} |v(t) - \hat{v}(t)| \leq C \int_{I_x} \int_{\Phi(t)-C\rho}^{\Phi(t)+C\rho} |D\theta|(\tau)\, d\tau\, dt \leq C \int_{\tilde{I}_x} \int_{s-C\rho}^{s+C\rho} |D\tilde{g}|(x,\tau)\, d\tau\, ds\,,$$

where in the latter inequality we have used the change of variables $s = \Phi(t)$ and the fact that both the Lipschitz constants of Φ and its inverse are under control. Integrating over

x and recalling that $v(t)-\hat v(t)=\boldsymbol{\eta}\circ u(x,t)-\mathbf{p}_\varkappa(\hat{\mathbf{g}}(x,t))$ we achieve

$$\int_{B_{4r}}|\boldsymbol{\eta}\circ u-\mathbf{p}_\varkappa\circ\hat{\mathbf{g}}|\leq\int_{B_{4r}}\int_{-\sqrt{36r^2-|x|^2}}^{\sqrt{36r^2-|x|^2}}\int_{s-C\rho}^{s+C\rho}|D\tilde g|(x,\tau)\,d\tau\,ds\,dx\leq C\rho\int_{B_{6r+C\rho}}|D\tilde g|$$
$$\leq C\boldsymbol{m}_0^{1/2}r^{2+2\beta_2/3}r^{m/2}\left(\int_{B_{8r}}|D\tilde g|^2\right)^{1/2}\leq C\boldsymbol{m}_0\,r^{m+3+2\beta_2/3-\delta_2}\leq C\boldsymbol{m}_0\,r^{m+3+\beta/3}\,.$$

Type C. Consider $\boldsymbol{\eta}\circ\tilde g$ and the map $\xi:B_{4r}(0,\pi)\to\pi^\perp$ such that $\mathbf{G}_\xi=\mathbf{G}_{\boldsymbol{\eta}\circ\tilde g}\llcorner\mathbf{C}_{4r}(0,\pi)$. We can then apply the argument of the estimate for type B to conclude

$$\|\boldsymbol{\eta}\circ u-\mathbf{p}_\varkappa(\xi)\|_{L^1(B_{4r})}\leq\|\boldsymbol{\eta}\circ g-\xi\|_{L^1(B_{4r})}\leq C\boldsymbol{m}_0\,r^{m+3+\beta_2/3}\,. \tag{5.40}$$

We need only to estimate $\|\mathbf{p}_\varkappa(\xi)-\mathbf{p}_\varkappa(\hat{\mathbf{g}})\|_{L^1}$: since $\mathbf{g}(x)=(\boldsymbol{\eta}\circ u(x),\Psi(x,\boldsymbol{\eta}\circ u(x)))$ and $\hat{\mathbf{g}}(x)=(\mathbf{p}_\varkappa(\hat{\mathbf{g}}(x)),\Psi(x,\mathbf{p}_\varkappa(\hat{\mathbf{g}}(x)))$, the claims then follows from the Lipschitz regularity of Ψ. Define the maps v,w and w' as follows:

$$\tilde{\mathbf{g}}(\tilde x)=\Big(\boldsymbol{\eta}\circ\tilde u(\tilde x),\tilde\Psi\left(\boldsymbol{\eta}\circ\tilde u(\tilde x)\right)\Big)=:(v(\tilde x),w(\tilde x)),$$
$$\boldsymbol{\eta}\circ\tilde g(\tilde x)=\Big(\boldsymbol{\eta}\circ\tilde u(\tilde x),\tfrac{1}{Q}\textstyle\sum_i\tilde\Psi(\tilde x,\tilde u_i(\tilde x))\Big)=:(v(\tilde x),w'(\tilde x))\,.$$

Using a Taylor expansion for $\tilde\Psi$ we conclude

$$\|\tilde{\mathbf{g}}-\boldsymbol{\eta}\circ\tilde g\|_0=\|w-w'\|_0\leq C\|D\tilde\Psi\|_0\sum_i|\tilde u_i-\boldsymbol{\eta}\circ\tilde u|\leq\boldsymbol{m}_0^{1/2+1/2m}r^{2+\beta_2-\delta_2}\,. \tag{5.41}$$

Consider an orthogonal transformation

$$L=\begin{pmatrix}U & V\\ W & Z\end{pmatrix}$$

with the properties that $(\tilde x,\tilde z)\in\tilde\pi\times\tilde\varpi$ corresponds to $(U\tilde x+V\tilde z,W\tilde x+Z\tilde z)\in\pi\times\varpi$ and $|L-\mathrm{Id}|\leq C\boldsymbol{m}_0^{1/2}r^{1-\delta_2}$. We then have the following relations: $\mathbf{p}_\varkappa(\hat{\mathbf{g}}(x))=v(\Phi^{-1}(x))$ and $\mathbf{p}_\varkappa(\xi(x))=v((\Phi')^{-1}(x))$, where Φ^{-1} and $(\Phi')^{-1}$ are the inverse, respectively, of the maps $\Phi(\tilde x)=U\tilde x+Vw(\tilde x)$ and $\Phi'(\tilde x)=U\tilde x+Vw'(\tilde x)$. Recalling that $|V|\leq|L|\leq C\boldsymbol{m}_0^{1/2}r^{1-\delta_2}$, we conclude that

$$|\Phi'(\tilde x)-\Phi(\tilde x)|\leq|V|\,|w(\tilde x)-w'(\tilde x)|\leq C\boldsymbol{m}_0^{1+1/2m}r^{3+\beta_2-\delta_2}\quad\text{for every }\tilde x.$$

On the other hand we also know that Φ^{-1} has Lipschitz constant at most 2 and so we achieve $|\Phi^{-1}(\Phi'(\tilde x))-\tilde x|\leq C\boldsymbol{m}_0^{1+1/2m}r^{3+\beta_2-\delta_2}$. Being valid for any $\tilde x$ we can apply it to $\tilde x=(\Phi')^{-1}(x)$ to conclude $|\Phi^{-1}(x)-(\Phi')^{-1}(x)|\leq C\boldsymbol{m}_0^{1+1/2m}r^{3+\beta_2-\delta_2}$. Using then $\mathrm{Lip}(v)\leq\mathrm{Lip}(\tilde u)\leq C\boldsymbol{m}_0^{\gamma_1}r^{\gamma_1}$, we conclude the pointwise bound

$$|\mathbf{p}_\varkappa(\hat{\mathbf{g}}(x))-\mathbf{p}_\varkappa(\xi(x))|=|v(\Phi^{-1}(x))-v((\Phi')^{-1}(x))|\leq C\boldsymbol{m}_0^{1+\gamma_1+1/2m}r^{3+\beta_2+\gamma_1-\delta_2}\,.$$

Proof of the Claim. We first show that, if $\varpi=\tilde\varpi$, or $\varkappa=\tilde\varkappa$ or $\pi=\tilde\pi$, then the claim can be achieved with small $2d$-rotations all of the same type, namely of type B, C and A, respectively. Assume for instance that $\varpi=\tilde\varpi$. Let ω be the intersection of π and $\tilde\pi$ and ω' be the intersection of $\varkappa$ and $\tilde\varkappa$. Pick a vector $e\in\pi$ which is not contained in $\tilde\pi$ and is orthogonal to ω. Let $\tilde e := \frac{\mathbf{p}_{\tilde\pi}(e)}{|\mathbf{p}_{\tilde\pi}(e)|}$. Then, $\tilde e$ is necessarily orthogonal to ω and the angle between $\tilde e$ and e is controlled by $|\pi-\tilde\pi|$. There is therefore a small $2d$-rotation R such that $R(e)=\tilde e$. It turns out that R keeps ϖ and ω fixed. So the new triple $(R(\pi), R(\varkappa), R(\varpi))$ has the property that $R(\varpi)=\varpi=\tilde\varpi$ and the dimension of $R(\pi)\cap\tilde\pi$ is larger than that of $\pi\cap\tilde\pi$. This procedure can be repeated and after $N\leq m$ times it leads to a triple of planes $(\pi_N,\varkappa_N,\varpi_N)$ with $\varpi_N=\tilde\varpi$ and $\pi_N=\tilde\pi$. This however implies necessarily $\tilde\varkappa=\varkappa_N$.

Assume therefore that ϖ and $\tilde\varpi$ do not coincide. Let $\omega := (\varkappa\times\pi)\cap(\tilde\varkappa\times\tilde\pi)$. There is then a unit vector $\tilde e\in\tilde\varkappa$ or a unit vector $\tilde e\in\tilde\pi$ which does not belong to $\pi\times\varkappa$ and which is orthogonal to ω. Assume for the moment that we are in the first case, and consider the vector $e := \frac{\mathbf{p}_{\pi\times\varkappa}(\tilde e)}{|\mathbf{p}_{\pi\times\varkappa}(\tilde e)|}$. The vector e forms an angle with the plane $\varkappa$ bounded by $C|\varkappa-\tilde\varkappa|$. Therefore there is a rotation R with $|R-Id|\leq C|\varkappa-\tilde\varkappa|$ of the plane $\pi\times\varkappa$ with the property that $R(\varkappa)$ contains e and keep fixed ω, which is orthogonal to e. By the previous step, R can be written as composition $R_{N'}\circ\ldots\circ R_1$ of small $2d$-rotations of type B keeping ϖ fixed. Since $e\perp R(\pi)$, we can then find a small $2d$-rotation S of type A with respect to $(R(\pi), R(\varkappa), \varpi)$ acting on the plane $\mathrm{span}\,(\tilde e, e)$ for which $S(R(\varkappa))\ni\tilde e$. S keeps then ω fixed. An analogous argument works if the vector $e\in\tilde\pi$. We therefore conclude that, after applying a finite number of rotations $R_1,\ldots,R_{N'},R_{N'+1}$ of the three types above, the dimension of $R_{N'+1}\circ R_{N'}\circ\ldots\circ R_1(\pi\times\varkappa)\cap\tilde\pi\times\tilde\varkappa$ is larger than that $\pi\times\varkappa\cap\tilde\pi\times\tilde\varkappa$ (where the number N' is smaller than a geometric constant depending only on m and $\bar n$). Obviously, after at most $m+\bar n$ iterations of this argument, we are reduced to the situation $\pi\times\varkappa=\tilde\pi\times\tilde\varkappa$. □

5.4. **Proof of Proposition 4.5.** We are finally ready to complete the proof of Proposition 4.5. Recall that (i) and (iv) have already been shown in Lemma 5.3. In order to show (ii) fix two cubes $H, L\in\mathscr{P}^j$ with nonempty intersection. If $\ell(H)=\ell(L)$, then we can apply Lemma 5.5 to conclude

$$\|h_H-\hat h_L\|_{L^1(B_{2r_H}(p_H,\pi_H))}\leq C\boldsymbol{m}_0\,\ell(H)^{m+3+\beta_2/3}\leq C\boldsymbol{m}_0\,\ell(H)^{m+3+\kappa}\,. \tag{5.42}$$

If $\ell(H)=\frac{1}{2}\ell(L)$, then let J be the father of H. Obviously, $J\cap L\neq\emptyset$. You can therefore apply Lemma 5.5 above to infer $\|h_{HJ}-\hat h_L\|_{L^1(B_{2r_J}(p_J,\pi_H)}\leq C\boldsymbol{m}_0\,\ell(J)^{m+3+\beta_2/3}$. On the other hand, by Lemma 5.3, $\|h_H-h_{HJ}\|_{L^1(B_{2r_H}(p_H,\pi_H))}\leq Cr^m\|h_H-h_{HJ}\|_0\leq C\boldsymbol{m}_0\,\ell(J)^{m+3+\kappa}$. Thus we conclude (5.42) as well. Note that $\mathbf{G}_{g_L}\llcorner\mathbf{C}_{r_H}(x_H,\pi_0)=\mathbf{G}_{\hat h_L}\llcorner\mathbf{C}_{r_H}(x_H,\pi_0)$ and that the same property holds with g_H and h_H. We can thus appeal to Lemma B.1 to conclude

$$\|g_H-g_L\|_{L^1(B_{r_H}(p_H,\pi_0))}\leq C\boldsymbol{m}_0\,\ell(H)^{m+3+\kappa}\,. \tag{5.43}$$

However, recall also that $[D^3(g_H-g_L)]_\kappa\leq C\boldsymbol{m}_0^{1/2}$. We can then apply Lemma C.2 to conclude (ii).

Now, if $L\in\mathscr{W}^j$ and $i\geq j$, consider the subset $\mathscr{P}^i(L)$ of all cubes in $\mathscr{P}^i$ which intersect L. If L' is the cube concentric to L with $\ell(L')=\frac{9}{8}\ell(L)$, we then have by definition of φ_j:

$$\|\varphi_i-g_L\|_{L^1(L')}\leq C\sum_{H\in\mathscr{P}^i(L)}\|g_H-g_L\|_{L^1(B_{r_L}(p_L,\pi_0))}\leq C\boldsymbol{m}_0\,\ell(H)^{m+3+\kappa}\,, \tag{5.44}$$

which is the claim of (v).

As for (iii), observe first that the argument above applies also when L is the father of H. Iterating then the corresponding estimates, it is easy to see that

$$|D^3 g_H(x_H) - D^3 g_J(x_J)| \leq C\boldsymbol{m}_0^{1/2}\ell(J)^\kappa \quad \text{for any ancestor } J \text{ of } H\,. \tag{5.45}$$

Fix now any pair $H, L \in \mathscr{P}^j$. Let H_i, L_i be the "first ancestors" of H and L which are adjacent, i.e. among all ancestors of H and L with the same side-length $\ell(H_i) = \ell(L_i) =: \ell$ and nonempty intersection, we assume the side-length is the smallest possible. We can therefore use the estimates obtained so far to conclude

$$\begin{aligned}|D^3 g_H(x_H) - D^3 g_L(x_L)| &\leq |D^3 g_H(x_H) - D^3 g_{H_i}(x_{H_i})| + |D^3 g_{H_i}(x_{H_i}) - D^3 g_{L_i}(x_{L_i})|\\ &\quad + |D^3 g_{L_i}(x_{L_i}) - D^3 g_L(x_L)| \leq C\boldsymbol{m}_0^{1/2}\ell^\kappa.\end{aligned}$$

A simple geometric consideration shows that $|x_L - x_H| \geq c_0\ell$, where c_0 is a dimensional constant, thus completing the proof.

6. Existence and estimates for the $\mathcal{M}$-normal approximation

We start proving the corollary of Theorem 2.4.

6.1. **Proof of Corollary 2.2.** The first two statements of (i) follow immediately from Theorem 1.12(i) and Proposition 4.2(v). Coming to the third claim of (i), we extend the function $\boldsymbol{\varphi}$ to the entire plane π_0 by increasing its $C^{3,\kappa}$ norm by a constant geometric factor. Let $\boldsymbol{\varphi}_t(x) := t\boldsymbol{\varphi}(x)$ for $t \in [0,1]$, $\mathcal{M}_t := \mathrm{Gr}(\boldsymbol{\varphi}_t|_{]-4,4[^m})$ and set

$$\mathbf{U}_t := \{x + y : x \in \mathcal{M}_t, y \perp T_x\mathcal{M}_t, |y| < 1\}\,.$$

For ε_2 sufficiently small the orthogonal projection $\mathbf{p}_t : \mathbf{U}_t \to \mathcal{M}_t$ is a well-defined $C^{2,\kappa}$ map for every $t \in [0,1]$, which depends smoothly on t. It is also easy to see that $\partial T \llcorner \mathbf{U}_t = \emptyset$. Thus, $(\mathbf{p}_t)_\sharp(T \llcorner \mathbf{U}_t) = Q(t)\,[\![\mathcal{M}_t]\!]$ for some integer $Q(t)$. On the other hand these currents depend continuously on t and therefore $Q(t)$ must be a constant. Since $\mathcal{M}_0 =]-4,4[^m \times \{0\} \subset \pi_0$ and $\mathbf{p}_0 = \mathbf{p}_{\pi_0}$, we conclude $Q(0) = Q$.

For what concerns (ii), consider $q \in L \in \mathscr{W}$, set $p := \boldsymbol{\Phi}(q)$ and $\pi := T_p\mathcal{M}$, whereas π_L is as in Definition 1.10. Let J be the cube concentric to L and with side-length $\frac{17}{16}\ell(L)$. By the definition of $\boldsymbol{\varphi}$, Theorem 1.12(ii) and Proposition 4.5, we have that, denoting by $\bar{\boldsymbol{\varphi}}$ and $\bar{g}_L$ the first $\bar{n}$ components of the corresponding maps,

$$\|\bar{\boldsymbol{\varphi}} - \bar{g}_L\|_{C^0(J)} \leq C \sum_{H \in \mathscr{W}, H \cap L \neq \emptyset} \|g_L - g_H\|_{C^0} \leq C\boldsymbol{m}_0^{1/2}\ell(L)^{3+\kappa}\,.$$

So, since $\boldsymbol{\varphi} = (\bar{\boldsymbol{\varphi}}, \Psi(x, \bar{\boldsymbol{\varphi}}))$ and $g_H = (\bar{g}_H, \Psi(x, \bar{g}_H))$, we conclude $\|g_L - \boldsymbol{\varphi}\|_{C^0(J)} \leq C\boldsymbol{m}_0^{1/2}\ell(L)^{3+\kappa}$. On the other the graph of g_L coincides with the graph of the tilted interpolating function h_L. Consider in $\mathbf{C} := \mathbf{C}_{8r_L}(p_L, \pi_L)$ the π_L-approximation f_L used in the construction algorithm and recall that, by [6, Theorem 1.4].

$$\begin{aligned}\mathrm{osc}\,(f_L) &\leq C\left(\mathbf{h}(T, \mathbf{C}_{32r_L}(p_L, \pi_L), \pi_L) + ((\mathbf{E}(T, \mathbf{C}_{32r_L}(p_L, \pi_L)))^{1/2} + r_L\mathbf{A})r_L\right)\\ &\leq C\boldsymbol{m}_0^{1/2m}\ell(L)^{1+\beta_2}\,.\end{aligned}$$

Setting $p_L = (z_L, w_L) \in \pi_L \times \pi_L^\perp$ and recalling that $p_L \in \mathrm{spt}(T)$, we easily conclude that $\|\boldsymbol{\eta}\circ f_L - w_L\|_{C^0} \leq C\boldsymbol{m}_0^{1/2m}\ell(L)^{1+\beta_2}$. This implies $\|h_L - w_L\|_{C^0} \leq C\boldsymbol{m}_0^{1/2m}\ell(L)^{1+\beta_2}$. Putting all these estimates together, we easily conclude that, for any point p in $\mathrm{spt}(T)\cap\mathbf{C}_{7r_L}(p_L,\pi_L)$ the distance to the graph of h_L is at most $C\boldsymbol{m}_0^{1/2m}\ell(L)^{1+\beta_2}$.

Finally, we show (iii). Fix a point $p \in \boldsymbol{\Gamma}$. By construction, there is an infinite chain $L_{N_0} \supset L_{N_0+1} \supset \ldots \supset L_j \supset \ldots$ of cubes $L_j \in \mathscr{S}^j$ such that $p = \bigcap_j L_j$. Set $\pi_j := \pi_{L_j}$. From Proposition 4.2 we infer that the planes π_j converge to a plane π with a rate $|\pi_j - \pi| \leq C\boldsymbol{m}_0^{1/2}2^{-j(1-\delta_2)}$. Moreover, the rescaled currents $(\iota_{p_{L_j},2^{-j}})_\sharp T$ (where the map $\iota_{q,r}$ is given by $\iota_{q,r}(z) = \frac{z-q}{r}$) converge to $Q\,[\![\pi]\!]$. Since $|\boldsymbol{\Phi}(p) - p_{L_j}| \leq C\sqrt{m}\,2^{-j}$ for some constant C independent of j, we easily conclude that $\Theta(T,\boldsymbol{\Phi}(p)) = Q$ and $Q\,[\![\pi]\!]$ is the unique tangent cone to T at $\boldsymbol{\Phi}(p)$. We next show that $\mathbf{p}^{-1}(\boldsymbol{\Phi}(p)) = \{\boldsymbol{\Phi}(p)\}$. Indeed, assume there were $q \neq \boldsymbol{\Phi}(p)$ such that $\mathbf{p}(q) = \boldsymbol{\Phi}(p)$ and let j be such that $2^{-j-1} \leq |\boldsymbol{\Phi}(p) - q| \leq 2^{-j}$. Provided ε_2 is sufficiently small, Proposition 4.2(v) guarantees that $j \geq N_0$. Consider the cube L_j in the chain above and recall that $\mathbf{h}(T,\mathbf{C}_{32r_{L_j}}(p_{L_j},\pi_j)) \leq C\boldsymbol{m}_0^{1/2m}2^{-j(1+\beta_2)}$. Hence,

$$
\begin{aligned}
2^{-j-1} &\leq |q - \boldsymbol{\Phi}(p)| = |\mathbf{p}_\pi(q - \boldsymbol{\Phi}(p))| \leq C|q-\boldsymbol{\Phi}(p)||\pi - \pi_j| + \mathbf{h}(T,\mathbf{C}_{32r_{L_j}}(p_{L_j},\pi_j))\\
&\leq C\boldsymbol{m}_0^{1/2}2^{-j(1-\delta_2)}2^{-j} + C\boldsymbol{m}_0^{1/2m}2^{-j(1+\beta_2)} \leq C\varepsilon_2^{1/2m}2^{-j}\,,
\end{aligned}
$$

which, for an appropriate choice of ε_2 (depending only on the various other parameters $\beta_2, \delta_2, \gamma_1, C_e, C_h, M_0, N_0$) is a contradiction.

6.2. **Construction of the $\mathcal{M}$-normal approximation and first estimates.** We set $F(p) = Q\,[\![p]\!]$ for $p \in \boldsymbol{\Phi}(\boldsymbol{\Gamma})$. For every $L \in \mathscr{W}^j$ consider the π_L-approximating function $f_L : \mathbf{C}_{8r_L}(p_L,\pi_L) \to \mathcal{A}_Q(\pi_L^\perp)$ of Definition 1.9 and $K_L \subset B_{8r_L}(p_L,\pi_L)$ the maximal (closed) set such that $\mathbf{G}_{f_L|_{K_L}} = T\llcorner(K_L\times\pi_L^\perp)$. We then denote by $\mathscr{D}(L)$ the portions of the supports of T and $\mathrm{Gr}(f_L)$ which differ:

$$
\mathscr{D}(L) := (\mathrm{spt}(T)\cup\mathrm{Gr}(f_L))\cap\left[(B_{8r_L}(p_L,\pi_L)\setminus K_L)\times\pi_L^\perp\right].
$$

Observe that, by [6, Theorem 1.4] and Assumption 1.5, we have

$$
\mathcal{H}^m(\mathscr{D}(L)) \leq CE^{\gamma_1}(E + \ell(L)^2\mathbf{A}^2)\ell(L)^m \leq C\boldsymbol{m}_0^{1+\gamma_2}\ell(L)^{m+2+\gamma_2}\,, \tag{6.1}
$$

where $E = \mathbf{E}(T,\mathbf{C}_{32r_L}(p_L,\pi_L))$. Let $\mathcal{L}$ be the Whitney region in Definition 1.13 and set $\mathcal{L}' := \boldsymbol{\Phi}(J)$ where J is the cube concentric to L with $\ell(J) = \frac{9}{8}\ell(L)$. Observe that our choice of the constants is done in such a way that,

$$
L\cap H = \emptyset \quad\Longleftrightarrow\quad \mathcal{L}'\cap\mathcal{H}' = \emptyset \qquad \forall H, L \in \mathscr{W}\,, \tag{6.2}
$$

$$
\boldsymbol{\Phi}(\boldsymbol{\Gamma})\cap\mathcal{L}' = \emptyset \qquad \forall L \in \mathscr{W}\,. \tag{6.3}
$$

We then apply [5, Theorem 5.1] to obtain maps $F_L, N_L : \mathcal{L}' \to \mathcal{A}_Q(\mathbf{U})$ with the following poperties:

- $F_L(p) = \sum_i [\![p + (N_L)_i(p)]\!]$,
- $(N_L)_i(p) \perp T_p\mathcal{M}$ for every $p \in \mathcal{L}'$
- and $\mathbf{G}_{f_L}\llcorner(\mathbf{p}^{-1}(\mathcal{L}')) = \mathbf{T}_{F_L}\llcorner(\mathbf{p}^{-1}(\mathcal{L}'))$.

For each L consider the set $\mathscr{W}(L)$ of elements in $\mathscr{W}$ which have a nonempty intersection with L. We then define the set $\mathcal{K}$ in the following way:

$$\mathcal{K} = \mathcal{M} \setminus \Big(\bigcup_{L\in\mathscr{W}} \Big(\mathcal{L}' \cap \bigcup_{M\in\mathscr{W}(L)} \mathbf{p}(\mathscr{D}(M))\Big)\Big) . \tag{6.4}$$

In other words $\mathcal{K}$ is obtained from $\mathcal{M}$ by removing in each $\mathcal{L}'$ those points for which there is a neighboring cube M such that the slice of $\mathbf{F}_M$ at x (relative to the projection $\mathbf{p}$) does not coincide with the slice of T. Observe that, by (6.3), $\mathcal{K}$ contains necessarily $\mathbf{\Gamma}$. Moreover, recall that $\mathrm{Lip}(\mathbf{p}) \leq C$, that the cardinality $\mathscr{W}(L)$ is bounded by a geometric constant and that each element of $\mathscr{W}(L)$ has side-length at most twice that of L. Thus (6.1) implies

$$|\mathcal{L}\setminus\mathcal{K}| \leq |\mathcal{L}'\setminus\mathcal{K}| \leq \sum_{H\in\mathscr{W}(J)}\sum_{J\in\mathscr{W}(L)} \mathbf{p}(\mathscr{D}(H)) \leq C\boldsymbol{m}_0^{1+\gamma_2}\ell(L)^{m+2+\gamma_2} . \tag{6.5}$$

On $\mathbf{\Gamma}$ we define $F(p) = Q\,[\![p]\!]$. By (6.2), if J and L are such that $\mathcal{J}'\cap\mathcal{L}'\neq\emptyset$, then $J\in\mathscr{W}(L)$ and therefore $F_L = F_J$ on $\mathcal{K}\cap(\mathcal{J}'\cap\mathcal{L}')$. We can therefore define a unique map on $\mathcal{K}$ by simply setting $F(p) = F_L(p)$ if $p\in\mathcal{K}\cap\mathcal{L}'$. Our resulting map has obviously the Lipschitz bound of (2.1) in each $\mathcal{L}\cap\mathcal{K}$. Moreover, $\mathbf{T}_F = T\llcorner\mathbf{p}^{-1}(\mathcal{K})$, which implies two facts. First, by Corollary 2.2(ii) we also have that $N(p) := \sum_i [\![F_i(p) - p]\!]$ enjoys the bound $\|N|_{\mathcal{L}\cap\mathcal{K}}\|_{C^0} \leq C\boldsymbol{m}_0^{1/2m}\ell(L)^{1+\beta_2}$. Secondly,

$$\|T\|(\mathbf{p}^{-1}(\mathcal{L}\setminus\mathcal{K})) \leq Q\sum_{M\in\mathscr{W}(L)}\sum_{H\in\mathscr{W}(M)} \mathcal{H}^m(\mathscr{D}(H)) \leq C\boldsymbol{m}_0^{1+\gamma_2}\ell(L)^{m+2+\gamma_2} . \tag{6.6}$$

Hence, F and N satisfy the bounds (2.1) on $\mathcal{K}$. We next extend them to the whole center manifold and conclude (2.2) from (6.6) and (6.5). The extension is achieved in three steps:

- we first extend the map F to a map $\tilde{F}$ taking values in $\mathcal{A}_Q(\mathbf{U})$;
- we then modify $\tilde{F}$ to achieve the form $\hat{F}(x) = \sum_i [\![x + \hat{N}_i(x)]\!]$ with $\hat{N}_i(x) \perp T_x\mathcal{M}$ for every x;
- we finally modify $\hat{F}$ to reach the desired extension $F(x) = \sum_i [\![x + N_i(x)]\!]$, with $N_i(x)\perp T_x\mathcal{M}$ and $x + N_i(x)\in\Sigma$ for every x.

First extension. We use on $\mathcal{M}$ the coordinates induced by its graphical structure, i.e. we work with variables in flat domains. Note that the domain parametrizing the Whitney region for $L\in\mathscr{W}$ is then the cube concentric to L and with side-length $\frac{17}{16}\ell(L)$. The multi-valued map N is extended to a multi-valued $\tilde{N}$ inductively to appropriate neighborhoods of the skeleta of the Whitney decomposition (a similar argument has been used in [4, Section 1.2.2]). The extension of F will obviously be $\tilde{F}(x) = \sum_i [\![\tilde{N}_i(x) + x]\!]$. The neighborhoods of the skeleta are defined in this way:

(1) if p belongs to the 0-skeleton, we let $L\in\mathscr{W}$ be (one of) the smallest cubes containing it and define $U^p := B_{\ell(L)/16}(p)$;
(2) if $\sigma = [p,q]\subset L$ is the edge of a cube $L\in\mathscr{W}$, we then define U^σ to be the neighborhood of size $\frac{1}{4}\frac{\ell(L)}{16}$ of σ minus the closure of the unions of the U^r's, where r runs in the 0-skeleton;

(3) we proceed inductively till the $m-1$-skeleton: given a k-dimensional facet σ of a cube L, U^σ is its neighborhood of size $4^{-k}\frac{\ell(L)}{16}$ minus the closure of the union of all U^τ's, where τ runs among all facets of dimension at most $k-1$.

Denote by $\bar{U}$ the closure of the union of all these neighborhoods and let $\{V_i\}$ be the connected components of the complement. For each V_i there is a $L_i \in \mathscr{W}$ such that $V_i \subset L_i$. Moreover, V_i has distance $c_0\ell(L)$ from ∂L_i, where c_0 is a geometric constant. It is also clear that if τ and σ are two distinct facets of the same cube with the same dimension, then the distance between any pair of points x, y with $x \in U^\tau$ and $y \in U^\sigma$ is at least $c_0\ell(L)$. Cp. with Figure 1.

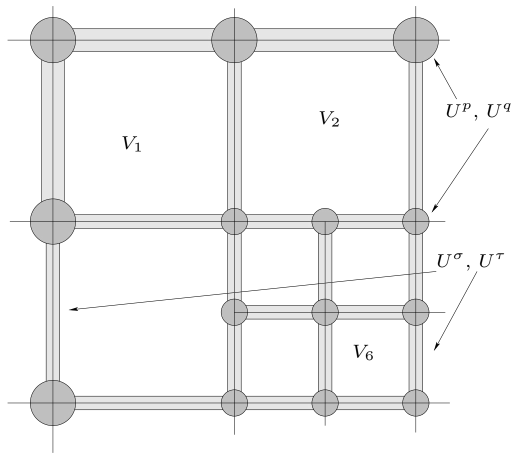

FIGURE 1. The sets U^p, U^σ and V_i.

At a first step we extend N to a new map $\tilde{N}$ separately on each U^p, where p are the points in the 0-skeleton. Fix $p \in L$ and let $\mathrm{St}(p)$ be the union of all cubes which contain p. Observe that the Lipschitz constant of $N|_{\mathcal{K}\cap \mathrm{St}(p)}$ is smaller than $C\boldsymbol{m}_0^{\gamma_2}\ell(L)^{\gamma_2}$ and that $|N| \leq C\boldsymbol{m}_0^{1/2m}\ell(L)^{1+\beta_2}$. We can therefore extend the map N to U^p at the price of slightly enlarging this Lipschitz constant and this height bound, using [4, Theorem 1.7]. Being the U^p disjoint, the resulting map, for which we use the symbol $\tilde{N}$, is well-defined.

It is obvious that this map has the desired height bound in each Whitney region. We therefore want to estimate its Lipschitz constant. Consider $L \in \mathscr{W}$ and H concentric to L with side-length $\ell(H) = \frac{17}{16}\ell(L)$. Let $x, y \in H$. If $x, y \in \mathcal{K}$, then there is nothing to check. If $y \in U^p$ for some p and $x \notin \bigcup_q U^q$, then $x \in \mathrm{St}(p)$ and $\mathcal{G}(\tilde{N}(x), \tilde{N}(y)) \leq C\boldsymbol{m}_0^{\gamma_2}\ell(L)^{\gamma_2}|x-y|$. The same holds when $x, y \in U^p$. The remaining case is $x \in U^p$ and $y \in U^q$ with $p \neq q$. Observe however that this would imply that p, q are both vertices of L. Given that $L \setminus \mathcal{K}$ has much smaller measure than L there is at least one point $z \in L \cap \mathcal{K}$. It is then obvious that

$$\mathcal{G}(\tilde{N}(x), \tilde{N}(y)) \leq \mathcal{G}(\tilde{N}(x), \tilde{N}(z)) + \mathcal{G}(\tilde{N}(z), \tilde{N}(y)) \leq C\boldsymbol{m}_0^{\gamma_2}\ell(L)^{\gamma_2}\ell(L),$$

and, since $|x-y| \geq c_0\ell(L)$, the desired bound readily follows. The map is also Lipschitz in any neighborhood of a point $x \in \mathbf{\Gamma}$. Thus we can extend it to the closure of its domain, which indeed, by the property of the Whitney decomposition, is simply the union of $\mathcal{K}$ and the closures of the U^p's.

This procedure can now be iterated over all skeleta inductively on the dimension k of the corresponding skeleton, up to $k = m-1$: in the argument above we simply replace points p with k-dimensional faces σ, defining $\mathrm{St}(\sigma)$ as the union of the cubes which contain σ. In the final step we then extend over the domains V_i's: this time $\mathrm{St}(V_i)$ will be defined as the union of the cubes which intersect the cube $L_i \supset V_i$. The correct height and Lipschitz bounds follow from the same arguments. Since the algorithm is applied $m+1$ times, the original constants have been enlarged by a geometric factor.

Second extension: orthogonality. For each $x \in \mathcal{M}$ let $\mathbf{p}^\perp(x,\cdot) : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ be the orthogonal projection on $(T_x\mathcal{M})^\perp$ and set $\hat{N}(x) = \sum_i [\![\mathbf{p}^\perp(x, \tilde{N}_i(x))]\!]$. Obviously $|\hat{N}(x)| \leq |\tilde{N}(x)|$, so the L^∞ bound is trivial. We now want to show the estimate on the Lipschitz constant. To this aim, fix two points p, q in the same Whitney region associated to L and parameterize the corresponding geodesic segment $\sigma \subset \mathcal{M}$ by arc-length $\gamma : [0, d(p,q)] \to \sigma$, where $d(p,q)$ denotes the geodesic distance on $\mathcal{M}$. Use [4, Proposition 1.2] to select Q Lipschitz functions $N_i' : \sigma \to \mathbf{U}$ such that $\tilde{N}|_\gamma = \sum [\![N_i']\!]$ and $\mathrm{Lip}(N_i') \leq \mathrm{Lip}(\tilde{N})$. Fix a frame $\nu_1, \ldots, \nu_n$ on the normal bundle of $\mathcal{M}$ with the property that $\|D\nu_i\|_{C^0} \leq C$ (which is possible since $\mathcal{M}$ is the graph of a $C^{3,\kappa}$ function, cp. [5, Appendix A]). We have $\hat{N}(\gamma(t)) = \sum_i [\![\hat{N}_i(t)]\!]$, where

$$\hat{N}_i(t) = N_i'(\gamma(t)) - \sum [\nu_j(\gamma(t)) \cdot N_i'(\gamma(t))]\, \nu_j(t).$$

Hence we can estimate

$$\left|\frac{d\hat{N}_i}{dt}\right| \leq C\mathrm{Lip}(N_i') + C\sum_j \|D\nu_j\|\|N_i'\|_{C^0} \leq C\boldsymbol{m}_0^{\gamma_2}\ell(L)^{\gamma_2} + C\boldsymbol{m}_0^{1/2m}\ell(L)^{1+\beta_2} \leq C\boldsymbol{m}_0^{\gamma_2}\ell(L)^{\gamma_2}.$$

Integrating this inequality we find

$$\mathcal{G}(\hat{N}(p), \hat{N}(q)) \leq \sum_{i=1}^Q |\hat{N}_i(d(p,q)) - \hat{N}_i(0)| \leq C\boldsymbol{m}_0^{\gamma_2}\ell(L)^{\gamma_2} d(p,q)\,.$$

Since $d(p,q)$ is comparable to $|p-q|$, we achieve the desired Lipschitz bound.

Third extension and conclusion. For each $x \in \mathcal{M} \subset \Sigma$ consider the orthogonal complement $\varkappa_x$ of $T_x\mathcal{M}$ in $T_x\Sigma$. Let $\mathcal{T}$ be the fiber bundle $\bigcup_{x\in\mathcal{M}} \varkappa_x$ and observe that, by the regularity of both $\mathcal{M}$ and Σ there is a global $C^{2,\kappa}$ trivialization (argue as in [5, Appendix A]). It is then obvious that there is a $C^{2,\kappa}$ map $\Xi : \mathcal{T} \to \mathbb{R}^{m+n}$ with the following property: for each (x,v), $q := x + \Xi(x,v)$ is the only point in Σ which is orthogonal to $T_x\mathcal{M}$ and such that $\mathbf{p}_{\varkappa_x}(q-x) = v$. We then set $N(x) = \sum_i [\![\Xi(x, \mathbf{p}_{\varkappa_x}(\hat{N}_i(x)))]\!]$. Obviously, $N(x) = \hat{N}(x)$ for $x \in \mathcal{K}$, simply because in this case $x + N_i(x)$ belongs to Σ.

In order to show the Lipschitz bound, denote by $\Omega(x,q)$ the map $\Xi(x, \mathbf{p}_{\varkappa_x}(q))$. Ω is a $C^{2,\kappa}$ map. Thus

$$|\Omega(x,q) - \Omega(x,p)| \leq C|q-p|\,. \tag{6.7}$$

Moreover, since $\Omega(x,0)=0$ for every x, we have $D_x\Omega(x,0)=0$. We therefore conclude that $|D_x\Omega(x,q)|\leq C|q|$ and hence that

$$|\Omega(x,q)-\Omega(y,q)|\leq C|q||y-x|\,. \tag{6.8}$$

Thus, fix two points $x,y\in\mathcal{L}$ and let assume that $\mathcal{G}(\hat{N}(x),\hat{N}(y))^2=\sum_i|\hat{N}_i(x)-\hat{N}_i(y)|^2$ (which can be achieved by a simple relabeling). We then conclude

$$\begin{aligned}\mathcal{G}(N(x),N(y))^2 &\leq 2\sum_i|\Omega(x,\hat{N}_i(x))-\Omega(x,\hat{N}_i(y))|^2+2\sum_i|\Omega(x,\hat{N}_i(y))-\Omega(y,\hat{N}_i(y))|^2\\ &\leq C\mathcal{G}(\hat{N}(x),\hat{N}(y))^2+C\sum_i|\hat{N}_i(y)|^2|x-y|^2\\ &\leq C\boldsymbol{m}_0^{2\gamma_2}\ell(L)^{2\gamma_2}|x-y|^2+C\boldsymbol{m}_0^{1/2m}\ell(L)^{1+\beta_2}|x-y|^2\,. \end{aligned}\tag{6.9}$$

This proves the desired Lipschitz bound. Finally, using the fact that $\Omega(x,0)=0$, we have $|\Omega(x,v)|\leq C|v|$ and the L^∞ bound readily follows.

6.3. **Estimates** (2.3) **and** (2.4)**.** First consider the cylinder $\mathbf{C}:=\mathbf{C}_{8r_L}(p_L,\pi_L)$. Denote by $\vec{\mathcal{M}}$ the unit m-vector orienting $T\mathcal{M}$ and by $\vec{\tau}$ the one orienting $T\mathbf{G}_{h_L}=T\mathbf{G}_{g_L}$. Recalling that g_L and $\boldsymbol{\varphi}$ coincide in a neighborhood of x_L, by Proposition 4.5(iv) we have

$$\sup_{p\in\mathcal{M}\cap\mathbf{C}}|\vec{\tau}(x_L,g_L(x_L))-\vec{\mathcal{M}}(p)|\leq C\|D^2\boldsymbol{\varphi}\|_{C^0}\,\ell(L)\leq C\boldsymbol{m}_0^{1/2}\ell(L).$$

Recalling moreover that $\|Dh_L\|^2_{L^2}\leq C\mathrm{Dir}(f_L)\leq C\boldsymbol{m}_0\ell(L)^{2-2\delta_2}$, we also conclude the existence of at least one point $q\in\mathbf{C}\cap\mathcal{M}$ such that $|\vec{\mathcal{M}}(q)-\pi_L|\leq C\boldsymbol{m}_0^{1/2}\ell(L)^{1-\delta_2}$. Since $\|D^2h_L\|\leq C\boldsymbol{m}_0^{1/2}$ we then conclude $|\vec{\tau}(x_L,g_L(x_L))-\tau(q)|\leq C\boldsymbol{m}_0^{1/2}\ell(L)$, which intern implies This in turn implies that $\sup_{\mathbf{C}\cap\mathcal{M}}|\vec{\mathcal{M}}-\pi_L|\leq C\boldsymbol{m}_0^{1/2}\ell(L)^{1-\delta_2}$. Therefore, we can estimate

$$\begin{aligned}&\int_{\mathbf{p}^{-1}(\mathcal{L})}|\vec{\mathbf{T}}_F(x)-\vec{\mathcal{M}}(\mathbf{p}(x))|^2d\|\mathbf{T}_F\|(x)\\ &\qquad\leq C\int_{\mathbf{p}^{-1}(\mathcal{L})}|\vec{T}(x)-\vec{\mathcal{M}}(\mathbf{p}(x))|^2d\|T\|(x)+C\boldsymbol{m}_0^{1+\gamma_2}\ell(L)^{m+2+\gamma_2}\\ &\qquad\leq \int_{\mathbf{p}^{-1}(\mathcal{L})}|\vec{T}(x)-\vec{\pi}_L|^2\|T\|(x)+C\boldsymbol{m}_0\ell(L)^{m+2-2\delta_2} \end{aligned}\tag{6.10}$$

In turn, since $\mathbf{p}^{-1}(\mathcal{L})\cap\mathrm{spt}(T)\subset\mathbf{C}$, the integral in (6.10) is smaller than $C\ell(L)^m\mathbf{E}(T,\mathbf{C},\pi_L)$. By [5, Proposition 3.4] we then conclude

$$\begin{aligned}\int_{\mathcal{L}}|DN|^2&\leq C\int_{\mathbf{p}^{-1}(\mathcal{L})}|\vec{\mathbf{T}}_F(x)-\vec{\mathcal{M}}(\mathbf{p}(x))|^2\|\mathbf{T}_F\|(x)+C\|A_{\mathcal{M}}\|^2_{C^0}\int_{\mathcal{L}}|N|^2\\ &\leq C\boldsymbol{m}_0\,\ell(L)^{m+2-2\delta_2}+C\boldsymbol{m}_0\,\ell(L)^{m+2+2\beta_2}\,,\end{aligned}$$

where we have used $\|A_{\mathcal{M}}\|_{C^0}\leq C\,\|D^2\boldsymbol{\varphi}\|_{C^0}\leq C\boldsymbol{m}_0^{1/2}$.

We finally come to (2.4). First observe that, by (2.1) and (2.2),

$$\int_{\mathcal{L}\setminus\mathcal{K}} |\boldsymbol{\eta}\circ N| \leq C\,\boldsymbol{m}_0^{1/2m}\ell(L)^{1+\beta_2}|\mathcal{L}\setminus\mathcal{K}| \leq C\boldsymbol{m}_0^{1+\gamma_2+1/2m}\ell(L)^{m+3+\beta_2+\gamma_2}\,. \tag{6.11}$$

Fix now $p\in\mathcal{K}$. Recalling that $F_L(x) = \sum_i [\![p+(N_L)_i(p)]\!]$ is given by [5, Theorem 5.1] applied to the map f_L, we can use [5, Theorem 5.1(5.4)] to conclude

$$\begin{aligned}|\boldsymbol{\eta}\circ N_L(p)| &\leq C\,|\boldsymbol{\eta}\circ f_L(\mathbf{p}_{\pi_L}(p)) - \mathbf{p}^\perp_{\pi_L}(p)| + C\,\mathrm{Lip}(N_L|_{\mathcal{L}})\,|T_p\mathcal{M}-\pi_L|\,|N_L|(p)\\ &\leq C|\boldsymbol{\eta}\circ f_L(\mathbf{p}_{\pi_L}(p)) - \mathbf{p}^\perp_{\pi_L}(p)|\\ &\qquad + C\boldsymbol{m}_0^{1/2+\gamma_2}\ell(L)^{1+\gamma_2-\delta_2}\left(\mathcal{G}(N_L(p), Q\,[\![\boldsymbol{\eta}\circ N_L(p)]\!]) + Q|\boldsymbol{\eta}\circ N_L|(p)\right)\,.\end{aligned}$$

For ε_2 sufficiently small (depending only on $\beta_2,\gamma_2,M_0,N_0,C_e,C_h$), we then conclude that

$$\begin{aligned}|\boldsymbol{\eta}\circ N_L(p)| &\leq C\,|\boldsymbol{\eta}\circ f_L(\mathbf{p}_{\pi_L}(p)) - \mathbf{p}_{\pi_L^\perp}(p)| + C\boldsymbol{m}_0^{1/2+\gamma_2}\ell(L)^{1+\gamma_2-\delta_2}\mathcal{G}(N_L(p), Q\,[\![\boldsymbol{\eta}\circ N_L(p)]\!])\\ &\leq C\,|\boldsymbol{\eta}\circ f_L(\mathbf{p}_{\pi_L}(p)) - \mathbf{p}^\perp_{\pi_L}(p)| + C\,a\,\boldsymbol{m}_0^{1+\gamma_2}\,\ell(L)^{(1+2\gamma_2-\delta_2)\frac{2+\gamma_2}{1+\gamma_2}}\\ &\qquad + \frac{C}{a}\,\mathcal{G}(N_L(p), Q\,[\![\boldsymbol{\eta}\circ N_L(p)]\!])^{2+\gamma_2}\,.\end{aligned} \tag{6.12}$$

Our choice of δ_2 makes the exponent $(1+2\gamma_2-\delta_2)\frac{2+\gamma_2}{1+\gamma_2}$ larger than $2+\gamma_2$. Let next $\varphi' : \pi_L \to \pi_L^\perp$ such that $\mathbf{G}_{\varphi'} = \mathcal{M}$. Applying Lemma B.1 we conclude that

$$\int_{\mathcal{K}\cap\mathcal{V}} |\boldsymbol{\eta}\circ f_L(\mathbf{p}_{\pi_L}(p)) - \mathbf{p}_{\pi_L^\perp}(p))| \leq \int_{\mathbf{p}_{\pi_L}(\mathcal{K}\cap\mathcal{V})} |\boldsymbol{\eta}\circ f_L(x) - \varphi'(x)| \leq C\int_H |g_L(x) - \boldsymbol{\varphi}(x)|\,,$$

where H is a cube concentric to L with side-length $\ell(H) = \frac{9}{8}\ell(L)$. From Proposition 4.5(v) we get $\|\boldsymbol{\varphi} - g_L\|_{L^1(H)} \leq C\boldsymbol{m}_0\ell(L)^{m+3+\beta_2/3}$ and (2.4) follows integrating (6.12).

7. Separation and splitting before tilting

7.1. **Vertical separation.** In this section we prove Proposition 3.1 and Corollary 3.2.

Proof of Proposition 3.1. Let J be the father of L. By Proposition 4.2, Theorem A.1 can be applied to the cylinder $\mathbf{C} := \mathbf{C}_{36r_J}(p_J,\pi_J)$. Moreover, $|p_J - p_L| \leq C\ell(J)$, where C is a geometric constant, and $r_J = 2r_L$. Thus, if M_0 is larger than a geometric constant, we have $\mathbf{B}_L \subset \mathbf{C}_{34r_J}(p_J,\pi_J)$. Denote by $\mathbf{q}_L$, $\mathbf{q}_J$ the projections $\mathbf{p}_{\hat\pi_L^\perp}$ and $\mathbf{p}_{\pi_J^\perp}$ respectively. Since $L\in\mathscr{W}_h$, there are two points $p_1,p_2\in\mathrm{spt}(T)\cap\mathbf{B}_L$ such that $|\mathbf{q}_L(p_1-p_2)| \geq C_h\boldsymbol{m}_0^{1/2m}\ell(L)^{1+\beta_2}$. On the other hand, recalling Proposition 4.2, $|\pi_J - \hat\pi_L| \leq CC_e^{1/2}\ell(L)^{1-\delta_2}$, where C is a geometric constant. Thus,

$$|\mathbf{q}_J(p_1-p_2)| \geq |\mathbf{q}_L(p_1-p_2)| - C_2|\hat\pi_L - \pi_J||p_1-p_2| \geq C_h\boldsymbol{m}_0^{1/2m}\ell(L)^{1+\beta_2} - CC_e^{1/2}\boldsymbol{m}_0^{1/2}\ell(L)^{2-\delta_2}\,,$$

where the constant C depends upon M_0,N_0 and the dimensions. Hence, if ε_2 is sufficiently small, we actually conclude

$$|\mathbf{q}_J(p_1-p_2)| \geq \frac{15}{16}C_h\boldsymbol{m}_0^{1/2m}\ell(L)^{1+\beta_2}\,. \tag{7.1}$$

Set $E := \mathbf{E}(T, \mathbf{C}_{36r_J}(p_J, \pi_J))$ and apply Theorem A.1 to $\mathbf{C}$: the union of corresponding "stripes" $\mathbf{S}_i$ the set $\mathrm{spt}(T) \cap \mathbf{C}_{36r_J(1-CE^{1/2m}|\log E|)}(p_J, \pi_J))$, where C is a geometric constant. We can therefore assume that they contain $\mathrm{spt}(T) \cap \mathbf{C}_{34r_J}(p_J, \pi_J)$. The width of these stripes is bounded as follows:

$$\sup\left\{|\mathbf{q}_J(x-y)| : x, y \in \mathbf{S}_i\right\} \leq C\, E^{1/2m} r_J \leq C\, C_e^{1/2m} \boldsymbol{m}_0^{1/2m} \ell(L)^{1+(2-2\delta_2)/2m},$$

where C is a dimensional constant. So, if $C^\sharp$ is chosen large enough, we actually conclude that p_1 and p_2 must belong to two different stripes, say $\mathbf{S}_1$ and $\mathbf{S}_2$. By Theorem A.1(iii) we conclude that all points in $\mathbf{C}_{34r_J}(p_J, \pi_J)$ have density Θ strictly smaller than $Q - \frac{1}{2}$, thereby implying (S1). Moreover, by choosing $C^\sharp$ appropriately, we achieve that

$$|\mathbf{q}_J(x-y)| \geq \frac{7}{8} C_h \boldsymbol{m}_0^{1/2m} \ell(L)^{1+\beta_2} \quad \forall x \in \mathbf{S}_1, y \in \mathbf{S}_2\,. \tag{7.2}$$

Assume next there is $H \in \mathscr{W}_n$ with $\ell(H) \leq \frac{1}{2}\ell(L)$ and $H \cap L \neq \emptyset$. From our construction it follows that $\ell(H) = \frac{1}{2}\ell(L)$, $\mathbf{B}_H \subset \mathbf{C}_{34r_J}(p_J, \pi_J)$ and $|\pi_H - \pi_J| \leq C C_e^{1/2} \boldsymbol{m}_0^{1/2} \ell(H)^{1-\delta_2}$, for a geometric constant C (see again Proposition 4.2. We then conclude

$$|\mathbf{p}_{\pi_H^\perp}(x-y)| \geq \frac{3}{4} C_h \boldsymbol{m}_0^{1/2m} \ell(L)^{1+\beta_2} \geq \frac{3}{2} C_h \boldsymbol{m}_0^{1/2m} \ell(H)^{1+\beta_2} \qquad \forall x \in \mathbf{S}_1, y \in \mathbf{S}_2\,. \tag{7.3}$$

Now, recalling Proposition 4.2, if ε_2 is sufficiently small, $\mathbf{C}_{32r_H}(p_H, \pi_H) \cap \mathrm{spt}(T) \subset \mathbf{B}_H$. Moreover, by Theorem A.1(ii) ,

$$(\mathbf{p}_{\pi_J})_\sharp(T \llcorner (\mathbf{S}_i \cap \mathbf{C}_{32r_H}(p_H, \pi_J))) = Q_i\, [\![B_{32r_H}(p_H, \pi_J)]\!] \quad \text{for } i = 1, 2, \;\; Q_i \geq 1.$$

A simple argument already used several other times allows to conclude that indeed

$$(\mathbf{p}_{\pi_H})_\sharp(T \llcorner (\mathbf{S}_i \cap \mathbf{C}_{32r_H}(p_H, \pi_H))) = Q_i\, [\![B_{32r_H}(p_H, \pi_H)]\!] \quad \text{for } i = 1, 2, \;\; Q_i \geq 1.$$

Thus, $\mathbf{B}_H$ must necessarily contain two points x, y with $|\mathbf{p}_{\pi_H^\perp}(x-y)| \geq \frac{3}{2} C_h \boldsymbol{m}_0^{1/2m} \ell(H)^{1+\beta_2}$. Given that $|\hat{\pi}_H - \pi_H| \leq C C_e^{1/2} \boldsymbol{m}_0^{1/2} \ell(H)^{1-\delta_2}$ for some dimensional constant C, we conclude that $|\mathbf{p}_{\hat{\pi}_H^\perp}(x-y)| \geq \frac{5}{4} C_h \boldsymbol{m}_0^{1/2m} \ell(H)^{1+\beta_2}$, i.e. the cube H satisfies the stopping condition (HT), which has "priority over the condition (NN)" and thus it cannot belong to $\mathscr{W}_n$. This shows (S2).

Coming to (S3), observe that, for each $p \in \mathcal{K} \cap \mathcal{L}$, the support of $p + N(p)$ must contain at least one point $p + N_1(p) \in \mathbf{S}_1$ and at least one point $p + N_2(p) \in \mathbf{S}_2$. Now,

$$|N_1(p) - N_2(p)| \geq \frac{7}{8} C_h \boldsymbol{m}_0^{1/2m} \ell(L)^{1+\beta_2} - \ell(L)\, |T_p\mathcal{M} - \pi_J|\,. \tag{7.4}$$

Recalling, however, Proposition 4.5 and that $\mathcal{M}$ and $\mathrm{Gr}(g_H)$ coincide on a nonempty open set, we easily conclude that $|T_p\mathcal{M} - \pi_J| \leq C \boldsymbol{m}_0^{1/2} \ell(L)^{1-\delta_2}$ and, via (7.4),

$$\mathcal{G}\big(N(p), Q\,[\![\boldsymbol{\eta} \circ N(p)]\!]\big) \geq \frac{1}{2}|N_1(p) - N_2(p)| \geq \frac{3}{8} C_h \boldsymbol{m}_0^{1/2m} \ell(L)^{1+\beta_2}\,.$$

Next observe that, since $|\mathcal{L} \setminus \mathcal{K}| \leq C \boldsymbol{m}_0^{1+\gamma_2} \ell(L)^{m+2+\gamma_2}$, for every point $p \in \mathcal{L}$ there exists $q \in \mathcal{K} \cap \mathcal{L}$ which has geodesic distance to p at most $C \boldsymbol{m}_0^{1/m+\gamma_2/m} \ell(L)^{1+2/m++\gamma_2/m}$. Given the

Lipschitz bound for N and the choice $\beta_2 \leq \frac{1}{2m}$, we then easily conclude (S3):

$$\mathcal{G}(N(q), Q\,[\![\boldsymbol{\eta}\circ N(q)]\!]) \geq \frac{3}{8} C_h \boldsymbol{m}_0^{1/2m} \ell(L)^{1+\beta_2} - C \boldsymbol{m}_0^{1/m} \ell(L)^{1+2/m} \geq \frac{1}{4} C_h \boldsymbol{m}_0^{1/2m} \ell(L)^{1+\beta_2}\,.$$

□

Proof of Corollary 3.2. The proof is straightforward. Consider any $H \in \mathscr{W}_n^j$. By definition it has a nonempty intersection with some cube $J \in \mathscr{W}^{j-1}$: this cube cannot belong to $\mathscr{W}_h$ by Proposition 3.1. It is then either an element of $\mathscr{W}_e$ or an element $H_{j-1} \in \mathscr{W}_n^{j-1}$. Proceeding inductively, we then find a chain $H = H_j, H_{j-1}, \ldots, H_i =: L$, where $H_{\bar l} \cap H_{\bar l - 1} \neq \emptyset$ for every $\bar l$ and $H_{\bar l} \in \mathscr{W}_n^{\bar l}$ for every $\bar l > i$ and $L = H_i \in \mathscr{W}_e^i$. This chain is not unique, however we can choose one for each element $H \in \mathscr{W}_n$. We then define $\mathscr{W}_n(L) = \{H \in \mathscr{W}_n :$ the chain of H ends with $L\}$. Observe that, if $H \in \mathscr{W}_n(L)$ and $H = H_j, H_{j-1}, \ldots, H_i = L$ is the corresponding chain, then

$$|x_H - x_L| \leq \sum_{\bar l = i}^{j-1} |x_{H_{\bar l}} - x_{H_{\bar l+1}}| \leq \sqrt{m}\,\ell(L) \sum_{\bar l = i}^{\infty} 2^{-\bar l} \leq 2\sqrt{m}\,\ell(L)\,.$$

It then follows easily that $H \subset B_{3\sqrt{m}\ell(L)}(L)$. □

7.2. **Unique continuation for Dir-minimizers.** Proposition 3.3 is based on a De Giorgi-type decay estimate for Dir-minimizing Q-valued maps which are close to a classical harmonic function with multiplicity Q. The argument involves a unique continuation-type result for Dir-minimizers.

Lemma 7.1 (Unique continuation for Dir-minimizers)**.** *For every $\eta \in (0,1)$ and $c > 0$, there exists $\gamma > 0$ with the following property. If $w : \mathbb{R}^m \supset B_{2r} \to \mathcal{A}_Q(\mathbb{R}^n)$ is Dir-minimizing,* $\mathrm{Dir}(w, B_r) \geq c$ *and* $\mathrm{Dir}(w, B_{2r}) = 1$*, then*

$$\mathrm{Dir}(w, B_s(q)) \geq \gamma \quad \textit{for every } B_s(q) \subset B_{2r} \textit{ with } s \geq \eta\, r.$$

Proof. We start showing the following claim:

(UC) if Ω is a connected open set and $w \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ is Dir-minimizing in any open $\Omega' \subset\subset \Omega$, then either w is constant or $\int_J |Dw|^2 > 0$ on any open $J \subset \Omega$.

We prove (UC) by induction on Q. If $Q = 1$, this is the classical unique continuation for harmonic functions. Assume now it holds for all $Q^* < Q$ and we prove it for Q-valued maps. Assume $w \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ and $J \subset \Omega$ is an open set on which $|Dw| \equiv 0$. Without loss of generality, we can assume J connected and $w|_J \equiv T$ for some $T \in \mathcal{A}_Q$. Let J' be the interior of $\{w = T\}$ and $K := \overline{J'} \cap \Omega$. We prove now that K is open, which in turn by connectedness of Ω concludes (UC). We distinguish two cases.

Case (a): the diameter of T is positive. Since w is continuous, for every $x \in K$ there is $B_\rho(x)$ where w separates into $[\![w_1]\!] + [\![w_2]\!]$ and each w_i is a Q_i-valued Dir-minimizer. Since $J' \cap B_\rho(x) \neq \emptyset$, each w_i is constant in a (nontrivial) open subset of $B_\rho(x)$. By inductive hypothesis each w_i is constant in $B_\rho(x)$ and therefore $w = T$ in $B_\rho(x)$, that is $B_\rho(x) \subset J' \subset K$.

Case (b): $T = Q[\![p]\!]$ for some p. Let $\tilde J$ be the interior of $\{w = Q[\![\boldsymbol{\eta}\circ w]\!]\}$ and $\tilde K := \tilde J \cap \Omega$. By [4, Definition 0.10], $\Omega\cap\partial\tilde J$ is contained in the singular set of w. By [4, Theorem 0.11], $\mathcal{H}^{m-2+\varepsilon}(\Omega\cap\partial J') = 0$ for every $\varepsilon>0$. Since $\tilde J$ is an open set, either $\Omega\cap\partial\tilde J$ is empty or it has positive $\mathcal{H}^{m-1}$ measure. We hence conclude that $\Omega\cap\partial\tilde J=\emptyset$, i.e. $\tilde J=\Omega$. This implies $w = Q[\![\boldsymbol{\eta}\circ w]\!]$, with $\boldsymbol{\eta}\circ w$ harmonic function (cf. [4, Lemma 3.23]). Being $\boldsymbol{\eta}\circ w|_{J'}\equiv p$, by the classical unique continuation $\boldsymbol{\eta}\circ w\equiv p$ on Ω.

We now come to proof of the proposition. Without loss of generality, we can assume $r=1$. Arguing by contradiction, there exists sequences $\{w_k\}_{k\in\mathbb{N}}\subset W^{1,2}(B_2,\mathcal{A}_Q(\mathbb{R}^n))$ and $\{B_{s_k}(q_k)\}_{k\in\mathbb{N}}$ with $s_k\geq\eta$ and such that $\mathrm{Dir}(w_k, B_{s_k}(q_k))\leq\frac{1}{k}$. By the compactness of Dir-minimizers (cp. [4, Proposition 3.20]), a subsequence (not relabeled) converges to $w\in W^{1,2}(B_2,\mathcal{A}_Q(\mathbb{R}^n))$ Dir-minimizing in every open $\Omega'\subset\subset B_2$. Up to subsequences, we can also assume that $q_k\to q$ and $s_k\to s\geq\eta>0$. Thus, $B_s(q)\subset B_2$ and $\mathrm{Dir}(w,B_s(q))=0$. By (UC) this implies that w is constant. On the other hand, by [4, Proposition 3.20] $\mathrm{Dir}(w,B_1)=\lim_k\mathrm{Dir}(w_k,B_1)\geq c>0$ gives the desired contradiction. □

Next we show that if the energy of a Dir-minimizer w does not decay appropriately, then w must split. In order to simplify the exposition, in the sequel we fix $\lambda>0$ such that

$$(1+\lambda)^{(m+2)} < 2^{\delta_2}\,. \tag{7.5}$$

Proposition 7.2 (Decay estimate for Dir-minimizers)**.** *For every $\eta>0$, there is $\gamma>0$ with the following property. Let $w:\mathbb{R}^m\supset B_{2r}\to\mathcal{A}_Q(\mathbb{R}^n)$ be Dir-minimizing in every $\Omega'\subset\subset B_{2r}$ such that*

$$\int_{B_{(1+\lambda)r}}\mathcal{G}\big(Dw, Q[\![D(\boldsymbol{\eta}\circ w)(0)]\!]\big)^2\geq 2^{\delta_2-m-2}\mathrm{Dir}(w,B_{2r})\,. \tag{7.6}$$

Then, if we set $\tilde w=\sum_i[\![w_i-\boldsymbol{\eta}\circ w]\!]$, the following holds:

$$\gamma\,\mathrm{Dir}(w,B_{(1+\lambda)r})\leq\mathrm{Dir}(\tilde w,B_{(1+\lambda)r})\leq\frac{1}{\gamma\, r^2}\int_{B_s(q)}|\tilde w|^2\quad\forall\, B_s(q)\subset B_{2\,r}\ \text{with}\ s\geq\eta\, r\,. \tag{7.7}$$

Proof. By a simple scaling argument we can assume $r=1$ and we argue by contradiction. Let w_k be a sequence of local Dir-minimizers which satisfy (7.6), $\mathrm{Dir}(w_k,B_2)=1$ and

(a) either $\int_{B_{s_k}(q_k)}|\tilde w_k|^2\to 0$ for some sequence of balls $B_{s_k}(q_k)\subset B_{2r}$ with $s_k\geq\eta$;
(b) or $\mathrm{Dir}(\tilde w_k, B_{1+\lambda})\to 0$.

Up to subsequences, w_k converges locally in $W^{1,2}$ to w locally Dir-minimizing. If (a) holds, we can appeal to Lemma 7.1 and conclude that $\tilde w=\sum_i[\![w_i-\boldsymbol{\eta}\circ w]\!]$ vanishes identically on B_2. This means in particular that $\mathrm{Dir}(\tilde w_k,B_{1+\lambda})\to\mathrm{Dir}(\tilde w,B_{1+\lambda})=0$, i.e. (b) holds.

Therefore, we can assume to be always in case (b). Let next $u_k:=\boldsymbol{\eta}\circ w_k$. From (7.6) we get

$$\begin{aligned}\int_{B_{1+\lambda}}Q|Du_k-Du_k(0)|^2 &= \int_{B_{1+\lambda}}\big(\mathcal{G}(Dw_k,Q[\![Du_k(0)]\!])^2-|D\tilde w_k|^2\big)\\ &\geq 2^{\delta_2-m-2}\int_{B_2}|Dw_k|^2-\int_{B_{1+\lambda}}|D\tilde w_k|^2\,.\end{aligned}\tag{7.8}$$

As $k\uparrow\infty$, by (b) and $\mathrm{Dir}(w_k,B_2)=1$, we then conclude

$$\int_{B_{1+\lambda}}|Du-Du(0)|^2\geq 2^{\delta_2-m-2}\geq 2^{\delta_2-m-2}\int_{B_2}|Du|^2\,. \tag{7.9}$$

Since $(1+\lambda)^{m+2}<2^{\delta_2}$, (7.9) violates the decay estimate for classical harmonic functions: $\int_{B_{1+\lambda}}|Du-Du(0)|^2\leq 2^{-m-2}(1+\lambda)^{m+2}\int_{B_2}|Du|^2$, thus concluding the proof. □

7.3. **Splitting before tilting I: Proof of Proposition 3.3.** Given $L\in\mathscr{W}_e^j$, let us consider its ancestors $H\in\mathscr{S}^{j-1}$ and $J\in\mathscr{S}^{j-6}$. Set $\ell=\ell(L)$, $\mathbf{C}:=\mathbf{C}_{8r_J}(p_J,\pi_H)$ and let $f:B_{8r_J}(p_J,\pi_H)\to\mathcal{A}_Q(\pi_H^\perp)$ be the π_H-approximation Proposition 4.4, and let $K\subset B_{8r_J}(p_J,\pi_H)$ denote the set such that $\mathbf{G}_{f|_K}=T\llcorner K\times\pi_H^\perp$. Observe that $\mathbf{B}_L\subset\mathbf{C}$ (provided ε_2 is sufficiently small, depending on all the other parameters). The following are simple consequences of Proposition 4.2:

$$E:=\mathbf{E}(T,\mathbf{C}_{32r_J}(p_J,\pi_H))\leq C\boldsymbol{m}_0\,\ell^{2-2\delta_2}\,, \tag{7.10}$$

$$\mathbf{h}(T,\mathbf{C},\pi_H)\leq C\,\boldsymbol{m}_0^{1/2m}\ell^{1+\beta_2},. \tag{7.11}$$

In particular the positive constant C does not depend on ε_2. Moreover, since $\mathbf{B}_L\subset\mathbf{C}$, $L\in\mathscr{W}_e$ and $r_L/r_J=2^{-6}$, we have

$$cC_e\,\boldsymbol{m}_0\,r_L^{2-2\delta}\leq E\,, \tag{7.12}$$

where c is only a geometric constant. We divide the proof of Proposition 3.3 in three steps.

Step 1: decay estimate for f. Let $2\rho:=64r_H-\bar C\boldsymbol{m}_0^{1/2m}\ell^{1+\beta_2}$: since $p_H\in\mathrm{spt}(T)$, it follows from (7.11) that, upon chosen $\bar C$ appropriately, $\mathrm{spt}(T)\cap\mathbf{C}_{2\rho}(p_H,\pi_H)\subset\mathbf{B}_H\subset\mathbf{C}$. Observe in particular that $\bar C$ does not depend on ε_2, although it depends upon the other parameters. In particular, setting $B=B_{2\rho}(x,\pi_H)$ with $x=\mathbf{p}_{\pi_H}(p_H)$, using the Taylor expansion in [5, Corollary 3.3] and the estimates in [6, Theorem 1.4], we get

$$\begin{aligned}\mathrm{Dir}(B,f)&\leq 2|B|\,\mathbf{E}(T,\mathbf{C}_{2\rho}(x_H,\pi_H))+C\boldsymbol{m}_0^{1+\gamma_1}\ell^{m+2+\gamma_1}\\&\leq 2\omega_m\rho^m\mathbf{E}(T,\mathbf{B}_H)+C\boldsymbol{m}_0^{1+\gamma_1}\ell^{m+2+\gamma_1}\,.\end{aligned} \tag{7.13}$$

Consider next the cylinder $\mathbf{C}':=\mathbf{C}_{64r_L}(p_L,\pi_H)$, $x':=\mathbf{p}_{\pi_H}(p_L)$ and $B'=B_{64r_L}(x',\pi_H)$. Set $A:=⨍_{B'}D(\boldsymbol{\eta}\circ f)$, $\bar A:\pi_H\to\pi_H^\perp$ the linear map $x\mapsto A\cdot x$ and π for the plane corresponding to $\mathbf{G}_{\bar A}$. Using [5, Theorem 3.5], we can estimate

$$\begin{aligned}\tfrac12\int_{B'}\mathcal{G}(Df,Q\,[\![A]\!])^2&\geq|B'|\,\mathbf{E}(T,\mathbf{C}',\pi)-C\boldsymbol{m}_0^{1+\gamma_1}\ell^{m+2+\gamma_1/2}\\&\geq|B'|\mathbf{E}(T,\mathbf{B}_L,\pi)-C\boldsymbol{m}_0^{1+\gamma_1}\ell^{m+2+\gamma_1/2}\\&\geq\omega_m(64\,r_L)^m\mathbf{E}(T,\mathbf{B}_L)-C\boldsymbol{m}_0^{1+\gamma_1}\ell^{m+2+\gamma_1/2}\end{aligned} \tag{7.14}$$

where we have used the obvious inclusion $\mathbf{B}_L \subset \mathbf{C}'$. Next, considering that $\mathbf{B}_H \supset \mathbf{B}_L$ and that, by $L \in \mathscr{W}_e^j$, $\mathbf{E}(T, \mathbf{B}_L) \geq C_e \boldsymbol{m}_0\, \ell^{2-2\delta_2}$, we conclude from (7.13) and (7.14) that

$$\mathrm{Dir}(B, f) \leq 2\omega_m \rho^m (1 + \boldsymbol{m}_0^{\gamma_1}) \mathbf{E}(T, \mathbf{B}_H)\,. \tag{7.15}$$

$$\int_{B'} \mathcal{G}(Df, Q\,[\![A]\!])^2 \geq 2\omega_m 64 r_L^m (1 - C\boldsymbol{m}_0^{\gamma_1/2}) \mathbf{E}(T, \mathbf{B}_L)\,. \tag{7.16}$$

Since $|x - x'| \leq |p_H - p_L| \leq C\ell(H)$, where C is a geometric constant (cp. Proposition 4.2), the ball $\hat{B} = B_\sigma(x, \pi_H)$, with radius $\sigma := 64 r_L + C\ell(H) = 32 r_H + C\ell(H)$ contains the ball B'. Moreover, if λ is the constant in (7.5) and M_0 is chosen sufficiently large (thus fixing a lower bound for M_0 which depends only on δ_2) we reach

$$\sigma \leq \left(\frac{1}{2} + \frac{\lambda}{4}\right) 64\, r_H \leq \left(1 + \frac{\lambda}{2}\right) \rho + \bar{C} \boldsymbol{m}_0^{1/2m} \ell^{1+\beta_2}\,.$$

In particular, choosing ε_2 sufficiently small we conclude $\sigma \leq (1+\lambda)\rho$. Now, recalling that $H \in \mathscr{S}$ and $L \in \mathscr{W}_e$, i.e. $\mathbf{E}(T, \mathbf{B}_H) \leq 2^{2-2\delta_2} \mathbf{E}(T, \mathbf{B}_L)$, we can then combine (7.13) and (7.14) to conclude

$$\begin{aligned} \mathrm{Dir}(f, B_{(1+\lambda)\rho}(x, \pi_H)) &\geq \int_{B_{(1+\lambda)\rho}(x,\pi_H)} |Df|^2 - Q|B_{(1+\lambda)\rho}(x, \pi_H)||A|^2 \\ \geq \int_{B_{(1+\lambda)\rho}(x,\pi_H)} \mathcal{G}(Df, Q\,[\![A]\!])^2 &\geq \left(2^{2\delta_2 - 2 - m} - C\boldsymbol{m}_0^{\gamma_1/2}\right) \int_{B_{2\rho}(x,\pi_H)} |Df|^2\,. \end{aligned} \tag{7.17}$$

Step 2: harmonic approximation. From now on, to simplify our notation, we use $B_s(y)$ in place of $B_s(y, \pi_H)$. Set $p := \mathbf{p}_{\pi_H}(p_J)$. From (7.12) we infer that $8 r_J\, \mathbf{A} \leq 8 r_J \boldsymbol{m}_0^{1/2} \leq E^{3/8}$ for ε_2 sufficiently small. Therefore, for every positive $\bar\eta$, we can apply [6, Theorem 1.6] to the cylinder $\mathbf{C}$ and achieve a map $w : B_{8r_J}(p, \pi_H) \to \mathcal{A}_Q(\pi_H^\perp)$ of the form $w = (u, \Psi(y, u))$ for a Dir-minimizer u and such that

$$(8\, r_J)^{-2} \int_{B_{8r_J}(p)} \mathcal{G}(f, w)^2 + \int_{B_{8r_J}(p)} (|Df| - |Dw|)^2 \leq \bar\eta\, E\, (8\, r_J)^m, \tag{7.18}$$

$$\int_{B_{8r_J}(p)} |D(\boldsymbol{\eta} \circ f) - D(\boldsymbol{\eta} \circ w)|^2 \leq \bar\eta\, E\, (8\, r_J)^m\,. \tag{7.19}$$

By $D(\Psi(y, u(y))) = \sum_i [\![D_x \Psi(y, u_i(y)) + D_v \Psi(y, u_i(y)) \cdot D u_i(y)]\!]$ we estimate

$$\int_{B_{(1+\lambda)\rho}(x)} |D(\Psi(y,u))|^2 \leq C\boldsymbol{m}_0 \int_{B_{(1+\lambda)\rho}(x)} |Du|^2 + \bar{C}\boldsymbol{m}_0 \rho^{2+m}\,,$$

where $\bar{C}$ is a geometric constant, i.e. independent of *any* parameter. Consider now $\tilde{A} := \fint D(\boldsymbol{\eta} \circ w)$. Using the mean-value property for harmonic functions,

$$\begin{aligned} \int_{B_{(1+\lambda)\rho}(x)} \mathcal{G}\big(Dw, Q[\![\tilde{A}]\!]\big)^2 \leq \int_{B_{(1+\lambda)\rho}(x)} |Du|^2 - Q\,|D(\boldsymbol{\eta} \circ u)(x)|^2\, |B_{(1+\lambda)\rho}(x)| \\ + \bar{C}\boldsymbol{m}_0 \rho^{m+2} + C\boldsymbol{m}_0 \int_{B_{(1+\lambda)\rho}(x)} |Du|^2\,, \end{aligned} \tag{7.20}$$

where again $\bar{C}$ is a geometric constant. On the other hand, since $L \in \mathscr{W}_e$, we have

$$\boldsymbol{m}_0\rho^2 \leq \frac{\bar{C}}{C_e}\mathbf{E}(T,\mathbf{B}_L) \stackrel{(7.16)}{\leq} \frac{\bar{C}\,\rho^{-m}}{C_e}\int_{B_{(1+\lambda)\rho}(x)} \mathcal{G}(Df, Q\,[\![A]\!])^2 \tag{7.21}$$

where again $\bar{C}$ is only a geometric constant. Therefore

$$\begin{aligned}
\int_{B_{(1+\lambda)\rho}(x)} \mathcal{G}\big(Du, Q\,[\![D(\boldsymbol{\eta}\circ u)(x)]\!]\,\big)^2 &= \int_{B_{(1+\lambda)\rho}(x)} |Du|^2 - Q\,|D(\boldsymbol{\eta}\circ u)(x)|^2\,|B_{(1+\lambda)\rho}(x)|\\
\stackrel{(7.20),(7.18),(7.19)}{\geq} \left(1-\frac{\bar{C}}{C_e}\right)&\int_{B_{(1+\lambda)\rho}(x)} \mathcal{G}(Df,Q\,[\![A]\!])^2 - C\boldsymbol{m}_0\mathrm{Dir}(B_{2\rho}(x),u) - C\bar{\eta}E\rho^m\\
\stackrel{(7.17),(7.16)}{\geq} \Big((1-\bar{C}C_e^{-1})2^{2\delta_2-2-m} &- C\varepsilon_2^{\gamma_1/2} - C\bar{\eta}\Big)\,\mathrm{Dir}(B_{2\rho},f) - C\varepsilon_2\mathrm{Dir}(B_{2\rho}(x),u)\\
\stackrel{(7.18)}{\geq} \Big((1-\bar{C}C_e^{-1})2^{2\delta_2-2-m} &- C\varepsilon_2^{\gamma_1/2} - C\bar{\eta}\Big)\,\mathrm{Dir}(B_{2\rho}(x),u)\,.
\end{aligned} \tag{7.22}$$

Observe that the constant $\bar{C}$ is only geometric. We can therefore choose C_e, depending only on δ_2, such that $(1-\bar{C}C_e^{-1})2^{2\delta_2-2-m} \geq 2^{3\delta_2/4-2-m}$. Next, the constant C depends upon the various parameters, but not upon $\bar{\eta}$ and ε_2. So, choosing ε_2 small enough (recall that $\bar{\eta}$ becomes smaller as we choose ε_2 small) we can now apply Proposition 7.2 to u and conclude

$$\hat{C}^{-1}\int_{B_{(1+\lambda)\rho}(x)} |Du|^2 \leq \int_{B_{\ell/8}(q)} \mathcal{G}(Du, Q\,[\![D(\boldsymbol{\eta}\circ u]\!])^2 \leq \hat{C}\ell^{-2}\int_{B_{\ell/8}(q)} \mathcal{G}(u, Q\,[\![\boldsymbol{\eta}\circ u]\!])^2\,,$$

for any ball $B_{\ell/8}(q) = \mathbf{B}_{\ell/8}(q,\pi_H) \subset B_{8r_J}(p,\pi_H) = B_{8r_j}(p,\pi_H)$, where $\hat{C}$ depends upon δ_2 and M_0. In particular, being these constants independent of ε_2 and C_e, we can use the previous estimates and reassorb error terms (possibly choosing ε_2 even smaller and C_e larger) to conclude

$$\begin{aligned}
\boldsymbol{m}_0\,\ell^{m+2-2\delta_2} &\leq C\ell^m\,\mathbf{E}(T,\mathbf{B}_L) \leq \tilde{C}\int_{B_{\ell/8}(q)} \mathcal{G}(Df, Q\,[\![D(\boldsymbol{\eta}\circ f)]\!])^2\\
&\leq \check{C}\ell^{-2}\int_{B_{\ell/8}(q)} \mathcal{G}(f, Q\,[\![\boldsymbol{\eta}\circ f]\!])^2,
\end{aligned} \tag{7.23}$$

where C, $\tilde{C}$ and $\check{C}$ are constants which depend upon δ_2, M_0 and C_e, but not on ε_2.

Step 3: Estimate for the $\mathcal{M}$-normal approximation. Now, consider any ball $B_{\ell/4}(q,\pi_0)$ with $\mathrm{dist}(L,q) \leq 4\sqrt{m}\,\ell$ and let $\Omega := \mathbf{\Phi}(B_{\ell/4}(q,\pi_0))$. Observe that $\mathbf{p}_{\pi_H}(\Omega)$ must contain a ball $B_{\ell/8}(q',\pi_H) \subset B_{8r_J}(p,\pi_H)$, because of the estimates on $\boldsymbol{\varphi}$ and $|\pi_0-\pi_H|$. Moreover, $\mathbf{p}^{-1}(\Omega)\cap \mathrm{spt}(T) \supset \mathbf{C}_{\ell/8}(q',\pi_H)\cap\mathrm{spt}(T)$. and, for an appropriate geometric constant C, Ω cannot intersect a Whitney region $\mathcal{L}'$ corresponding to an L' with $\ell(L') \geq C\ell(L)$. In particular, Theorem 2.4 implies that

$$\|\mathbf{T}_F - T\|(\mathbf{p}^{-1}(\Omega)) + \|\mathbf{T}_F - \mathbf{G}_f\|(\mathbf{p}^{-1}(\Omega)) \leq C\boldsymbol{m}_0^{1+\gamma_2}\ell^{m+2+\gamma_2}. \tag{7.24}$$

Let now F' be the map such that $\mathbf{T}_{F'}\llcorner(\mathbf{p}^{-1}(\Omega)) = \mathbf{G}_f\llcorner(\mathbf{p}^{-1}(\Omega))$. The region over which F and F' differ is contained in the projection onto Ω of $(\mathrm{Im}(F)\setminus\mathrm{spt}(T))\cup(\mathrm{Im}(F')\setminus\mathrm{spt}(T))$ and therefore its $\mathcal{H}^m$ measure is bounded as in (7.24). Together with the height bound on N and f ($|N|+|N'|\leq C\boldsymbol{m}_0^{1/2m}\ell^{1+\beta_2}$), this implies

$$\int_\Omega |N|^2 \geq \int_\Omega |N'|^2 - C\boldsymbol{m}_0^{1+1/m+\gamma_2}\ell^{m+4+2\beta_2+\gamma_2}\,. \tag{7.25}$$

On the other hand, let $\boldsymbol{\varphi}' : \mathbf{B}_{8r_J}(p,\pi_H)\to\pi_H^\perp$ be such that $\mathbf{G}_{\boldsymbol{\varphi}'} = [\![\mathcal{M}]\!]$ and $\boldsymbol{\Phi}'(z) = (z,\boldsymbol{\varphi}'(z))$; then, applying [5, Theorem 5.1 (5.3)], we conclude

$$N'(\boldsymbol{\varphi}'(z)) \geq \frac{1}{2\sqrt{Q}}\,\mathcal{G}(f(z),Q\,[\![\boldsymbol{\varphi}'(p)]\!]) \geq \frac{1}{4\sqrt{Q}}\,\mathcal{G}(f(z),Q\,[\![\boldsymbol{\eta}\circ f(z)]\!])\,,$$

which in turn implies

$$\begin{aligned}\boldsymbol{m}_0\,\ell^{m+2-2\delta_2} &\stackrel{(7.23)}{\leq} C\ell^{-2}\int_{B_{\ell/8}(q',\pi_H)}\mathcal{G}(f,Q\,[\![\boldsymbol{\eta}\circ f]\!])^2 \leq C\ell^{-2}\int_\Omega |N'|^2\\ &\leq C\ell^{-2}\int_\Omega |N|^2 + C\boldsymbol{m}_0^{1+\gamma_2+1/2m}\ell^{m+2+2\beta_2+\gamma_2}\,.\end{aligned} \tag{7.26}$$

For ε_2 sufficiently small, this leads to the second inequality of (3.2), while the first one comes from Theorem 2.4 and $\mathbf{E}(T,\mathbf{B}_L)\geq C_e\boldsymbol{m}_0\,\ell^{2-2\delta_2}$.

We next complete the proof showing (3.1). Since $D(\boldsymbol{\eta}\circ f)(z) = \boldsymbol{\eta}\circ Df(z)$ for a.e. z, we obviously have

$$\int_{B_{\ell/8}(q',\pi_H)}\mathcal{G}(Df,Q\,[\![D(\boldsymbol{\eta}\circ f)]\!])^2 \leq \int_{B_{\ell/8}(q',\pi_H)}\mathcal{G}(Df,Q\,[\![D\boldsymbol{\varphi}']\!])^2\,. \tag{7.27}$$

Let now $\vec{\mathbf{G}}_f$ be the orienting tangent m-vector to $\mathbf{G}_f$ and τ the one to $\mathcal{M}$. For a.e. z we have the inequality

$$2\sum_i |\vec{\mathbf{G}}_f(f_i(z)) - \vec{\tau}(\boldsymbol{\varphi}'(z))|^2 \geq 2\mathcal{G}(Df(z),Q\,[\![D\boldsymbol{\varphi}'(z)]\!])^2\,,$$

and, hence,

$$\begin{aligned}\int_{B_{\ell/8}(q',\pi_H)}\mathcal{G}(Df,Q\,[\![D\boldsymbol{\varphi}']\!])^2 &\leq C\int_{\mathbf{C}_{\ell/8}(q',\pi_H)}|\vec{\mathbf{G}}_f(z)-\vec{\tau}(\boldsymbol{\varphi}'(\mathbf{p}_{\pi_H}(z))|^2 d\|\mathbf{G}_f\|(z)\\ &\leq C\int_{\mathbf{C}_{\ell/8}(q',\pi_H)}|\vec{T}(z)-\vec{\tau}(\boldsymbol{\varphi}'(\mathbf{p}_{\pi_H}(z))|^2 d\|T\|(z) + C\boldsymbol{m}_0^{1+\gamma_1}\ell^{m+2+\gamma_1}\,.\end{aligned} \tag{7.28}$$

Now, thanks to the height bound and to the fact that $|\vec{\tau}-\pi_H|\leq C\boldsymbol{m}_0^{1/2}\ell^{1-\delta_2}$ in the cylinder $\hat{\mathbf{C}} = \mathbf{C}_{\ell/8}(q',\pi_H)$, we have the inequality

$$|\mathbf{p}(z)-\boldsymbol{\varphi}'(\mathbf{p}_{\pi_H}(z))| \leq C\boldsymbol{m}_0^{1/2m+1/2}\ell^{2+\beta_2-\delta_2} \leq C\boldsymbol{m}_0^{1/2m+1/2}\ell^{2+\beta_2/2} \qquad \forall z\in\mathrm{spt}(T)\cap\hat{\mathbf{C}}\,.$$

Using $\|\boldsymbol{\varphi}'\|_{C^2} \leq C\boldsymbol{m}_0^{1/2}$ we then easily conclude from (7.28) that

$$\int_{B_{\ell/8}(p,\pi_H)} \mathcal{G}(Df, Q\,[\![D\boldsymbol{\varphi}']\!])^2 \leq C\int_{\hat{\mathbf{C}}} |\vec{T}(z) - \vec{\tau}(\mathbf{p}(z))|^2 d\|T\|(z) + C\boldsymbol{m}_0^{1+\gamma_1}\ell^{m+2+\beta_2/2}$$
$$\leq C\int_{\mathbf{p}^{-1}(\Omega)} |\vec{\mathbf{T}}_F(z) - \tau(\mathbf{p}(z))|^2 d\|\mathbf{T}_F\|(z) + C\boldsymbol{m}_0^{1+\gamma_2}\ell^{m+2+\gamma_2},$$

where we used (7.24).

Since $|DN| \leq C\boldsymbol{m}_0^{\gamma_2}\ell^{\gamma_2}$, $|N| \leq C\boldsymbol{m}_0^{1/2m}\ell^{1+\beta_2}$ on Ω and $\|A_{\mathcal{M}}\|^2 \leq C\boldsymbol{m}_0$, applying now [5, Proposition 3.4] we conclude

$$\int_{\mathbf{p}^{-1}(\Omega)} |\vec{\mathbf{T}}_F(x) - \tau(\mathbf{p}(x))|^2 d\|\mathbf{T}_F\|(x) \leq (1 + C\boldsymbol{m}_0^{2\gamma_2}\ell^{2\gamma_2})\int_\Omega |DN|^2 + C\boldsymbol{m}_0^{1+1/m}\ell^{m+2+2\beta_2}\,.$$

Thus, putting all these estimates together we achieve

$$\boldsymbol{m}_0\,\ell^{m+2-2\delta_2} \leq C(1 + C\boldsymbol{m}_0^{2\gamma_2}\ell^{2\gamma_2})\int_\Omega |DN|^2 + C\boldsymbol{m}_0^{1+\gamma_2}\ell^{m+2+\gamma_2}\,. \tag{7.29}$$

Since the constant C might depend on the various other parameters but not on ε_2, we conclude that for a sufficiently small ε_2 we have

$$\boldsymbol{m}_0\ell^{m+2-2\delta_2} \leq C\int_\Omega |DN|^2\,. \tag{7.30}$$

But $\mathbf{E}(T, \mathbf{B}_L) \leq C\boldsymbol{m}_0\,\ell^{2-2\delta_2}$ and thus (3.1) follows.

8. Persistence of Q-points

8.1. **Proof of Proposition 3.4.** We argue by contradiction. Assuming the proposition does not hold, there are sequences T_k's and Σ_k's satisfying the Assumption 1.2 and radii s_k for which

(a) either $\boldsymbol{m}_0(k) := \max\{\mathbf{E}(T_k, \mathbf{B}_{6\sqrt{m}}, \mathbf{c}(\Sigma_k)\} \to 0$ and $\bar{s} = \lim_k s_k > 0$; or $s_k \downarrow 0$;
(b) the sets $\Lambda_k := \{\Theta(x, T_k) = Q\} \cap \mathbf{B}_{3s_k}$ satisfy $\mathcal{H}^{m-2+\alpha}_\infty(\Lambda_k) \geq \bar{\alpha}s_k^{m-2+\alpha}$;
(c) denoting by $\mathscr{W}(k)$ and $\mathscr{S}(k)$ the families of cubes in the Whitney decompositions related to T_k with respect to π_0, $\sup\big\{\ell(L) : L \in \mathscr{W}(k), L \cap B_{3s}(0, \pi_0) \neq \emptyset\big\} \leq s_k$;
(d) there exists $L_k \in \mathscr{W}^e(k)$ with $L_k \cap B_{3s}(0, \pi_0) \neq \emptyset$ and $\hat{\alpha}s_k < \ell(L_k) \leq s_k$.

It is not difficult to see that $\mathbf{E}(T_k, \mathbf{B}_{6\sqrt{m}s_k}) \leq C\boldsymbol{m}_0(k)s_k^{2-2\delta_2}$, where the constant C is independent of k. Indeed, this is obvious in the case $\lim_k s_k > 0$, and it follows in the case $s_k \downarrow 0$ from the fact that, for k large enough, there is an ancestor J_k of L_k with $\mathbf{B}_{6\sqrt{m}s_k} \subset \mathbf{B}_{J_k}$ and $\ell(J_k) \leq C\,s_k$.

Consider now the ancestors H_k and J_k of L_k , and the corresponding Lipschitz approximation f_k as in Section 7.3. By Proposition 4.2, it follows that

$$\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}s_k}, \pi_{H_k}) \leq C\boldsymbol{m}_0(k)\,s_k^{2-2\delta_2} \quad \text{and} \quad \mathbf{h}(T, \mathbf{B}_{6\sqrt{m}s_k}, \pi_{H_k}) \leq C\boldsymbol{m}_0(k)^{1/2m}s_k^{1+\beta_2}.$$

Let moreover g_k be the π_{H_k}-approximation of T_k in the cylinder $\mathbf{C}_{8s_k}(0, \pi_{H_k})$, given by [6, Theorem 1.4] applied to the cylinder $\mathbf{C}_k := \mathbf{C}_{32s_k}(0, \pi_{H_k})$ (since either $\boldsymbol{m}_0(k) \downarrow 0$ or $s_k \downarrow 0$, the latter theorem is applicable for k large enough). Observe also that $\mathbf{A}_k^2 := \|A_{\Sigma_k\cap\mathbf{C}_k}\|^2 \leq Cs_k^2\boldsymbol{m}_0(k)$ and by Proposition 3.3 (3.1) and [6, Theorem 1.4], we easily conclude

$$E_k := \mathbf{E}(T_k, \mathbf{C}_k, \pi_{H_k}) \geq c_0\mathbf{E}(T, \mathbf{B}_{L_k}) \geq c_0 C_e\boldsymbol{m}_0(k)\ell(L_k)^{2-2\delta_2} \geq c_0(\hat\alpha)\boldsymbol{m}_0(k)s_k^{2-2\delta_2}. \quad (8.1)$$

Observe now that on B_{8s_k} the two maps g_k and f_k coincide on a set K_k with the property that $|B_{8s_k}\setminus K_k| \leq C\boldsymbol{m}_0(k)^{1+\gamma_1}s_k^{m+2+\frac{\gamma_1}{2}}$. Moreover, by Proposition 3.3 (cp. Step 1 in Section 7.3), there exists a ball $B'_k \subset \pi_{H_k}$ contained in B_{4s_k} and with radius at least $\ell(L_k)/8$ such that

$$\int_{B'_k} \mathcal{G}(f_k, Q\,[\![\boldsymbol{\eta}\circ f_k]\!])^2 \geq \bar c\,\boldsymbol{m}_0(k)\,\ell(L_k)^{m+4-2\delta_2} \geq c_1(\hat\alpha)\,\boldsymbol{m}_0(k)\,s_k^{m+4-2\delta_2}\,. \quad (8.2)$$

Combining the latter inequality with (8.1) we conclude

$$\int_{B'_k} \mathcal{G}(g_k, Q\,[\![\boldsymbol{\eta}\circ g_k]\!])^2 \geq c_2(\hat\alpha)\,s_k^{m+2}E_k\,. \quad (8.3)$$

Note that by (8.1), we have that $\mathbf{A}_k^2\,s_k^2 \leq C^\star\,E_k$, for some $C^\star$ independent of k. In particular, since either $s_k\downarrow 0$ or $\boldsymbol{m}_0(k)\downarrow 0$, it turns out that, for k large enough, $\mathbf{A}_k s_k \leq E_k^{3/8}$. We can then apply [6, Theorem 1.6] to find a sequence of functions $w_k = (u_k, \Psi(x,u_k))$ of maps on B_{4s_k} each u_k is Dir-minimizing and such that

$$(4\,s_k)^{-2}\int_{B_{4s_k}} \mathcal{G}(g_k, w_k)^2 + \int_{B_{4s_k}} (|Dg_k| - |Dw_k|)^2 = o(E_k)s_k^m\,. \quad (8.4)$$

Up to rotations (so to get $\pi_{H_k} = \pi_0$) and dilations (of a factor s_k) of the system of coordinates, we then end up with a sequence of C^{3,ε_0} $(m+\bar n)$-dimensional submanifolds Γ_k of $\mathbb{R}^{m+n}$, area-minimizing currents S_k in Γ_k, functions h_k and $\bar w_k$ with the following properties:

(1) the excess $E_k := \mathbf{E}(S_k, \mathbf{C}_{32}(0,\pi_0))$ and the height $\mathbf{h}(S_k, \mathbf{C}_{32}(0,\pi_0),\pi_0)$ converge to 0;
(2) $\mathbf{A}_k^2 := \|A_{\Gamma_k}\|^2 \leq C^\star E_k$ and hence it also converges to 0;
(3) $\mathrm{Lip}(h_k) \leq CE_k^{\gamma_1}$;
(4) $\|\mathbf{G}_{h_k} - T_k\|(\mathbf{C}_8(0,\pi_0)) \leq CE_k^{1+\gamma_1}$;
(5) $\bar w_k = (\bar u_k, \Psi(x,\bar u_k))$ for some Dir-minimizing $\bar u_k$ in $B_4(0,\pi_0)$ and

$$\int \left((|Dh_k| - |D\bar w_k|)^2 + \mathcal{G}(h_k, \bar w_k)^2\right) = o(E_k)\,. \quad (8.5)$$

(6) for some positive constant c independent of k we have

$$\int_{B_3} \mathcal{G}(h_k, \boldsymbol{\eta}\circ h_k)^2 \geq cE_k\,; \quad (8.6)$$

(7) $\Xi_k := \{\Theta(S_k, y) = Q\}\cap\mathbf{B}_3$ has the property that $\mathcal{H}^{m-2+\alpha}_\infty(\Xi_k) \geq \bar\alpha > 0$ and $0\in\Xi_k$.

Consider the projections $\bar\Xi_k := \mathbf{p}_{\pi_0}(\Xi_k)$. We are therefore in the position of applying [6, Theorem 1.7] to conclude that, for every fixed $\rho \in (0, \frac{1}{2})$,

$$\max_{x\in\bar\Xi_k} \int_{B_\rho(x)} \mathcal{G}(h_k, Q\,[\![\boldsymbol{\eta}\circ h_k]\!])^2 = o(E_k) \quad \text{for } \; k\to+\infty.$$

Up to subsequences we can assume that $\bar\Xi_k$ (and hence also Ξ_k) converges, in the Hausdorff sense, to a compact set $\bar\Xi$, which is nonempty. Moreover, the maps $x \mapsto \hat u_k(x) = E_k^{-1/2}\sum_i [\![(\bar u_k)_i(x) - \boldsymbol{\eta}\circ\bar u_k(x)]\!]$ are easily recognized to converge, strongly in L^2, to a Dir-minimizing function u with $\boldsymbol{\eta}\circ u = 0$, vanishing identically on $\bar\Xi$. Observe that $\mathcal{G}(\Psi(x,\bar u_k), Q\,[\![\boldsymbol{\eta}\circ\Psi(x,\bar u_k)]\!])^2 \le C\mathrm{Lip}(\Psi)^2\mathcal{G}(\bar u_k, Q\,[\![\boldsymbol{\eta}\circ\bar u_k]\!])^2 \le CE_k^2$. Thus (8.5) and (8.6) easily imply that

$$\liminf_k \int_{B_3} \mathcal{G}(\hat u_k, Q\,[\![0]\!])^2 \ge \liminf E_k^{-1}\int_{B_3}\mathcal{G}(\bar u_k, \boldsymbol{\eta}\circ\bar u_k)^2 \ge c > 0\,. \tag{8.7}$$

By the strong convergence of $\hat u_k$ we then conclude that u does not vanish identically. Then, by [4, Theorem 0.11] we conclude that $\mathcal{H}^{m-2+\alpha}(\bar\Xi) = 0$, otherwise $\bar\Xi$ must have nonempty interior, which together with Lemma 7.1 implies $\bar\Xi = B_3$ and contradicts $u \not\equiv 0$. On the other hand, $\mathcal{H}^{m-2+\alpha}_\infty(\bar\Xi) \ge \limsup_k \mathcal{H}^{m-2+\alpha}_\infty(\Xi_k) \ge \bar\alpha > 0$ gives the desired contradiction.

8.2. **Proof of Proposition 3.5.** We fix the notation as in Section 7.3 and notice that

$$E := \mathbf{E}(T, \mathbf{C}_{32r_J}(p_J, \pi_H) \le C\boldsymbol{m}_0\ell(L)^{2-2\delta_2} \le C\boldsymbol{m}_0\bar\ell^{2-2\delta_2}.$$

By Proposition 3.3 we have

$$\int_{\mathcal{B}_{\ell(L)}(\mathbf{p}(p))} |DN|^2 \ge \bar c_1\,\boldsymbol{m}_0\,\ell(L)^{m+2-2\delta_2}\,. \tag{8.8}$$

Next, let $p := (x,y) \in \pi_H\times\pi_H^\perp$, fix a $\bar\eta > 0$, to be chosen later, and note that (7.12) allows us to apply [6, Theorem 1.7]: there exists then $\bar s > 0$ such that

$$\int_{B_{2\bar s\ell(L)}(x,\pi_H)} \mathcal{G}(f, Q\,[\![\boldsymbol{\eta}\circ f]\!])^2 \le \bar\eta\,\bar s^m\,\ell(L)^{m+2}E\,. \tag{8.9}$$

Observe that, no matter how small $\bar\eta$ is chosen, such estimate holds when $\bar s$ and E are appropriately small: the smallness of E is then achieved choosing $\bar\ell$ as small as needed.

Now consider the graph $\mathrm{Gr}(\boldsymbol{\eta}\circ f)\llcorner\mathbf{C}_{2\bar s\ell(L)}(x,\pi_H)$ and project it down onto $\mathcal{M}$. Since $\mathcal{M}$ is a graph over π_H of a function $\hat{\boldsymbol{\varphi}}$ with $\|D\hat{\boldsymbol{\varphi}}\|_{C^{2+\kappa}} \le C\boldsymbol{m}_0^{1/2}$ and since the Lipschitz constant of $\boldsymbol{\eta}\circ f$ is controlled by $C\boldsymbol{m}_0^{\gamma_1}$, provided ε_2 is smaller than a geometric constant we have that $\Omega := \mathbf{p}\left(\mathrm{Gr}(\boldsymbol{\eta}\circ f)\llcorner\mathbf{C}_{2\bar s\ell(L)}(x,\pi_H)\right)$ contains a ball $\mathcal{B}_{\bar s\ell(L)}(\mathbf{p}(p))$.

Consider now the map $F'(q) = \sum_i [\![q + N'_i(q)]\!]$ such that $\mathbf{T}_{F'}\llcorner\mathbf{p}^{-1}(\Omega) = \mathbf{G}_f\llcorner\mathbf{p}^{-1}(\Omega)$ given by [5, Theorem 5.1]. Consider also the map $\xi : B_{2\bar s\ell(L)}(x,\pi_H) \ni z \mapsto \mathbf{p}((z, \boldsymbol{\eta}\circ f(z)) \in \Omega$. This map is biLipschitz with controlled constant, again assuming that ε_2 is sufficiently small. Let now $\hat n : \Omega \to \mathbb{R}^{m+n}$ with the property that $\hat n(q) \perp T_q\mathcal{M}$ and $\xi(x) + \hat n(\xi(x)) = (x, \boldsymbol{\eta}\circ f(x))$. Applying the estimate of [5, Theorem 5.1 (5.5)] we then get

$$\mathcal{G}(N'(\xi(x)), Q\,[\![\boldsymbol{\eta}\circ N'(\xi(x))]\!]) \le \mathcal{G}(N'(\xi(x)), Q\,[\![\hat n(\xi(x))]\!]) \le 2\sqrt{Q}\,\mathcal{G}(f(x), Q\,[\![\boldsymbol{\eta}\circ f(x)]\!])\,.$$

Integrating the latter inequality, changing variable and using $\mathbf{B}_{\bar{s}\ell(L)}(\mathbf{p}(p)) \subset \Omega$, we then obtain

$$\int_{\mathbf{B}_{\bar{s}\ell(L)}(\mathbf{p}(p))} \mathcal{G}(N', Q\,[\![\boldsymbol{\eta}\circ N']\!])^2 \leq C\,\bar{\eta}\,\bar{s}^m\,\ell(L)^{m+2}\,E \leq C\,\bar{\eta}\,\boldsymbol{m}_0\,\bar{s}^m\ell(L)^{m+4-2\delta_2}\,.$$

Next, recalling the height bound and the fact that N and N' coincide outside a set of measure $\boldsymbol{m}_0^{1+\gamma_1}\ell(L)^{m+2+\gamma_2}$, we infer

$$\int_{\mathbf{B}_{\bar{s}\ell(L)}(\mathbf{p}(p))} \mathcal{G}(N, Q\,[\![\boldsymbol{\eta}\circ N]\!])^2 \leq C_1\,\bar{\eta}\,\boldsymbol{m}_0\,\bar{s}^m\ell(L)^{m+4-2\delta_2} + C_2\boldsymbol{m}_0^{1+\gamma_1}\ell(L)^{m+4+\gamma_2+2\beta_2}\,. \tag{8.10}$$

Since the constants $\bar{c}_1$, C_1 and C_2 in (8.8) and (8.10) are independent of $\ell(L)$ and $\bar{\eta}$, we fix $\bar{\eta}$ (and consequently $\bar{s}$) so small that $C_1\bar{\eta} \leq \bar{c}_1\frac{\eta_2}{2}$. We therefore achieve from (8.10)

$$⨍_{\mathbf{B}_{\bar{s}\ell(L)}(\mathbf{p}(p))} \mathcal{G}(N, Q\,[\![\boldsymbol{\eta}\circ N]\!])^2 \leq \frac{\bar{c}_1}{2}\,\eta_2\,\boldsymbol{m}_0\,\ell(L)^{4-2\delta_2} + C_2\,\boldsymbol{m}_0^{1+\gamma_1}\,\bar{s}^{-m}\ell(L)^{4+\gamma_2+2\beta_2}\,. \tag{8.11}$$

Having now fixed $\bar{s}$ we choose $\bar{\ell}$ so small that $C_2\bar{s}^{-m}\bar{\ell}^{2\delta_2+\gamma_2+2\beta_2} \leq \bar{c}_1\eta_2/2$. For these choices of the parameters, under the assumptions of the proposition we then infer

$$⨍_{\mathbf{B}_{\bar{s}\ell(L)}(\mathbf{p}(p))} \mathcal{G}(N, Q\,[\![\boldsymbol{\eta}\circ N]\!])^2 \leq \eta_2\,\bar{c}_1\,\boldsymbol{m}_0\,\ell(L)^{4-2\delta_2}\,. \tag{8.12}$$

The latter estimate combined with (8.8) gives the desired conclusion.

9. Comparison between different center manifolds

Proof of Proposition 3.6. We first verify (i). Observe that

$$\mathbf{E}(T', \mathbf{B}_{6\sqrt{m}}) = \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r}) \leq \liminf_{\rho\downarrow r} \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}\rho}) \leq \varepsilon_2.$$

Moreover, since Σ' is a rescaling of Σ, $\mathbf{c}(\Sigma')^2 = r^2\mathbf{c}(\Sigma)^2 \leq r^2\boldsymbol{m}_0$. Therefore, (1.7) is fulfilled by Σ' and T' as well; (1.6) follows trivially upon substituting π_0 with an optimal π for T' in $\mathbf{B}_{6\sqrt{m}}$ (which is an optimal plane for T in $\mathbf{B}_{6\sqrt{m}r}$ by a trivial scaling argument); (1.4) is scaling invariant; whereas $\partial T' \llcorner \mathbf{B}_{6\sqrt{m}} = (\iota_{0,r})_\sharp(\partial T \llcorner \mathbf{B}_{6\sqrt{m}r}) = 0$.

We now come to (ii). From now we assume N_0 to be so large that 2^{-N_0} is much smaller than c_s. In this way we know that r must be much smaller than 1. We have that $\ell(L) = c_s r$, otherwise condition (a) would be violated. Moreover, we can exclude that $L \in \mathscr{W}_n$. Indeed, in this case there must be a cube $J \in \mathscr{W}$ with $\ell(J) = 2\ell(L)$ and nonempty intersection with L. It then follows that, for $\rho := r + 2\sqrt{m}\,\ell(L) = (1 + 2\sqrt{m}\,c_s)r$, $B_\rho(0, \pi_0)$ intersects J. Again upon assuming N_0 sufficiently large, such ρ is necessarily smaller than 1. On the other hand, since $2\sqrt{m}c_s < 1$ we then have $c_s\rho < 2\,c_s\,r \leq 2\,\ell(L) = \ell(J)$.

Next observe that $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}\rho}) \leq C\boldsymbol{m}_0\rho^{2-2\delta_2}$ for some constant C and for every $\rho \geq r$. Indeed, if ρ is smaller than a threshold r_0 but larger than r, then $\mathbf{B}_{6\sqrt{m}\rho}$ is contained in the ball $\mathbf{B}_J$ for some ancestor J of L with $\ell(J) \leq \bar{C}\rho$, where the constant $\bar{C}$ and the threshold r_0 depend upon the various parameters. Then, $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r}) \leq C\mathbf{E}(T, \mathbf{B}_J) \leq C\,\boldsymbol{m}_0\,\rho^{2-2\delta_2}$. If instead $\rho \geq r_0$, we then use simply $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}\rho}) \leq C(r_0)\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}}) \leq C(r_0)\,\boldsymbol{m}_0$. This estimate also has the consequence that, if $\pi(\rho)$ is an optimal m-plane in $\mathbf{B}_{6\sqrt{m}\rho}$, then $|\pi_L - \pi(\rho)| \leq C\boldsymbol{m}_0^{1/2}\rho^{1-\delta_2}$.

We next consider the notation introduced in Section 7.3, the corresponding cubes $L \subset H \subset J$ and the π_H-approximation f introduced there. If $L \in \mathscr{W}^e$, then by (7.23) we get

$$\int_{B_{\ell/8}(x,\pi_H)} \mathcal{G}(f, Q\,[\![\boldsymbol{\eta}\circ f]\!])^2 \geq \bar{c}\,\boldsymbol{m}_0\,\ell^{m+4-2\delta_2} \geq \tilde{c}\,\boldsymbol{m}_0\,r^{m+4-2\delta_2}\,, \tag{9.1}$$

where $x = \mathbf{p}_{\pi_H}(x_H)$. On the other hand, if $L \in \mathscr{W}^h$, we can argue as in the proof of Proposition 3.1 and use Theorem A.1 to conclude the existence of at least two stripes $\mathbf{S}_1$ and $\mathbf{S}_2$, at distance $\tilde{c}\,\boldsymbol{m}_0^{1/2m}\ell^{1+\beta_2}$ with the property that any slice $\langle T, \mathbf{p}_{\pi_H}, z\rangle$ with $z \in B_{\ell/8}(x,\pi_H)$ must intersect both of them. Since for $x \in K$ such slice coincides with $f(x)$, we then have

$$\begin{aligned}\int_{B_{\ell/8}(x,\pi_H)} \mathcal{G}(f, Q\,[\![\boldsymbol{\eta}\circ f]\!])^2 &\geq \tilde{c}\,\boldsymbol{m}_0^{1/m}\ell^{m+2+2\beta_2} - C\boldsymbol{m}_0^{1/m}\ell^{2+2\beta_2}|K| \\ &\geq \tilde{c}\,\boldsymbol{m}_0^{1/m}\ell^{m+2+2\beta_2} - C\boldsymbol{m}_0^{1+\gamma_1+1/m}\ell^{(m+2-2\delta_2)(1+\gamma_1)+2\beta_2} \\ &\geq \tilde{c}\,\boldsymbol{m}_0 r^{m+4-2\delta_2}\,.\end{aligned} \tag{9.2}$$

Rescale next through $\iota_{0,r}$ and consider $T' := (\iota_{0,r})_\sharp T$. We also rescale the graph of the corresponding π_H-approximation f to the graph of a map g, which then has the following properties. If $B \subset \pi_H$ is the rescaling of the ball $B_{\ell/8}(x,\pi_H)$, then $B \subset B_{3/2}$ and the radius of B is given by $c_s/8$. On B we have the estimate

$$\int_B \mathcal{G}(g, Q\,[\![\boldsymbol{\eta}\circ g]\!])^2 = r^{-m-2}\int_{B_{\ell/8}(x,\pi_H)} \mathcal{G}(f, Q\,[\![\boldsymbol{\eta}\circ f]\!])^2 \geq \tilde{c}\,\boldsymbol{m}_0 r^{2-2\delta_2}\,. \tag{9.3}$$

The Lipschitz constant of g is the same of that of f and hence controlled by $C\boldsymbol{m}_0^{\gamma_1} r^{\gamma_1}$. On the other hand, we have

$$\hat{\boldsymbol{m}}_0 := \max\left\{\mathbf{E}(T', \mathbf{B}_{\sqrt{m}6}), \mathbf{c}(\Sigma')^2\right\} \leq \max\{C\boldsymbol{m}_0 r^{2-2\delta_2}, \mathbf{c}(\Sigma)^2 r^2\} \leq C\boldsymbol{m}_0 r^{2-2\delta_2}\,. \tag{9.4}$$

Moreover, denoting by $\hat{\mathbf{C}}$ the cylinder $\mathbf{C}_4(0,\pi_H)$, we have that

$$\|\mathbf{G}_g - T'\|(\hat{\mathbf{C}}) \leq C\boldsymbol{m}_0^{1+\gamma_1} r^{2+\gamma_1/2}\,. \tag{9.5}$$

Finally, since $|\pi - \pi_H| \leq C\boldsymbol{m}_0^{1/2} r^{1-\delta_2}$ and because $\mathcal{M}'$ is the graph of a function $\boldsymbol{\varphi}'$ with $\|D\boldsymbol{\varphi}'\|_{C^{2,\kappa}} \leq C\hat{\boldsymbol{m}}_0^{1/2}$ and $\|\boldsymbol{\varphi}'\|_{C^0} \leq C\hat{\boldsymbol{m}}_0^{1/2m}$, by (9.4) we can actually conclude that $\mathcal{M}'$ is the graph over π_H of a map $\hat{\boldsymbol{\varphi}} : \pi_H \to \pi_H^\perp$ with $\|D\hat{\boldsymbol{\varphi}}\|_{C^{2,\alpha}} \leq C\boldsymbol{m}_0^{1/2} r^{1-\delta_2}$. Similarly, the $\mathcal{M}'$-approximating map $x \mapsto F'(x) := \sum_i [\![x + N_i'(x)]\!]$ coincides with T' over a subset $\mathcal{K}' \subset \mathcal{M}'$ with $|\mathcal{M}' \setminus \mathcal{K}'| \leq \hat{\boldsymbol{m}}_0^{1+\gamma_2} \leq C\boldsymbol{m}_0^{1+\gamma_2} r^{(2-2\delta_2)(1+\gamma_2)}$.

Consider now the projection $\mathbf{p}'$ over $\mathcal{M}'$ and hence define the set $\mathcal{J} := \mathbf{p}'(\mathrm{spt}(T') \cap \mathrm{Gr}(g) \cap \mathrm{Im}(F'))$. It turns out that $|\mathcal{B}_2 \setminus \mathcal{J}| \leq \boldsymbol{m}_0^{1+\gamma_2} r^{(2-2\delta_2)(1+\gamma_2)}$. On the other hand, if $G : \mathcal{B}_2 \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ is the map with $\mathbf{T}_G = \mathbf{G}_g \llcorner \mathbf{p}'^{-1}(\mathcal{B}_2)$ given by [5, Theorem 5.1], we then have that $G \equiv F$ on $\mathcal{J}$. Consider therefore a point $p \in \mathcal{J}$ with $p = (x, \hat{\boldsymbol{\varphi}}(x))$ and consider that for this point we have by [5, Theorem 5.1 (5.3)]

$$\mathcal{G}(g(x), Q\,[\![\boldsymbol{\eta}\circ g(x)]\!]) \leq \mathcal{G}(g(x), Q\,[\![\hat{\boldsymbol{\varphi}}(x)]\!]) \leq C\mathcal{G}(G(p), Q\,[\![p]\!])\,.$$

Therefore, using (9.4) we can easily estimate

$$\int_{\mathbf{B}_2\cap\mathcal{M}'}|N'|^2 \geq \int_{\mathbf{B}_2}(G(p), Q\,[\![p]\!])^2 - C\hat{\boldsymbol{m}}_0^{1/m}|\mathcal{B}_2\setminus\mathcal{J}|$$
$$\geq \bar{c}_1\,\boldsymbol{m}_0 r^{2-2\delta_2} - C\boldsymbol{m}_0^{1/m+1+\gamma_2}r^{(2-2\delta_2)(1-\gamma_2)} \geq \bar{c}_2\,\boldsymbol{m}_0 r^{2-2\delta_2}\,, \qquad (9.6)$$

where all the constants are independent of ε_2 and the latter is supposed to be sufficiently small. Thus finally, by (9.4) we conclude

$$\int_{\mathbf{B}_2\cap\mathcal{M}'}|N'|^2 \geq \bar{c}_s\hat{\boldsymbol{m}}_0 = \bar{c}_s\max\{\mathbf{E}(T',\mathbf{B}_{6\sqrt{m}}),\mathbf{c}(\Sigma')^2\}\,.$$

□

Appendix A. Height bound revisited

In this section we prove a strengthened version of the so-called "height bound" (see [8, Lemma 5.3.4]), which appeared first in [1]. Our proof follows closely that of [11].

Theorem A.1. *Let Q, m, $\bar{n}$ and n be positive integers. Then there are $\varepsilon > 0$ and C with the following property. Assume:*

- (h1) *$\Sigma\subset\mathbb{R}^{m+n}$ is an $(m+\bar{n})$-dimensional C^2 submanifold with $\mathbf{A} := \|A_\Sigma\|_0 \leq \varepsilon$;*
- (h2) *R is an integer rectifiable m-current with* $\mathrm{spt}(R)\subset\Sigma$ *and area-minimizing in Σ;*
- (h3) *$\partial R\llcorner\mathbf{C}_r(x_0) = 0$, $(\mathbf{p}_{\pi_0})_\sharp R\llcorner\mathbf{C}_r(x_0) = Q\,[\![B_r(\mathbf{p}_{\pi_0}(x_0),\pi_0)]\!]$ and $E := \mathbf{E}(R,\mathbf{C}_r(x_0)) < \varepsilon$.*

Then there are $k\in\mathbb{N}$, points $\{y_1,\ldots,y_k\}\subset\mathbb{R}^n$ and positive integers $Q_1,\ldots,Q_k$ such that:

- (i) *having set $\sigma := CE^{1/2m}$, the open sets $\mathbf{S}_i := \mathbb{R}^m\times(y_i+]-r\sigma, r\sigma[^n)$ are pairwise disjoint and* $\mathrm{spt}(R)\cap\mathbf{C}_{r(1-\sigma|\log E|)}(x_0)\subset\cup_i\mathbf{S}_i$;
- (ii) $(\mathbf{p}_{\pi_0})_\sharp[R\llcorner(\mathbf{C}_{r(1-\sigma|\log E|)}(x_0)\cap\mathbf{S}_i)] = Q_i\,[\![B_{r(1-\sigma|\log E|)}(\mathbf{p}_{\pi_0}(x_0),\pi_0)]\!]$ $\forall i\in\{1,\ldots,k\}$.
- (iii) *for every $p\in\mathrm{spt}(R)\cap\mathbf{C}_{r(1-\sigma|\log E|)}(x_0)$ we have $\Theta(R,p) < \max\{k_i\}+\frac{1}{2}$.*

Remark A.2. Obviously, $\sum_i Q_i = Q$ and hence $1\leq k\leq Q$. Most likely the bound on the radius of the inner cylinder could be improved to $1-\sigma$. However this would not give us any advantage in the rest of the paper and hence we do not pursue the issue here.

Proof. We first observe that, without loss of generality, we can assume $x_0 = 0$ and $r = 1$. Moreover, (iii) follows from (i) and (ii) through the monotonicity formula. Indeed, let $p\in\mathrm{spt}(R)$ be such that $\mathbf{B}_\rho(p) := \mathbf{B}_{E^{1/2m}}(p)\subset\mathbf{C}_{1-\sigma|\log E|}(x_0) =: \mathbf{C}'$. p must be contained in one of the $\mathbf{S}_i$, say $\mathbf{S}_1$. Consider the current $R_1 = R\llcorner(\mathbf{S}_1\cap\mathbf{C}')$. Observe that R_1 must be area-minimizing in Σ, $\Theta(R_1,p) = \Theta(R,p)$ and that $\mathbf{E}(R_1,\mathbf{C}')\leq E$. On the other hand, if $\|A_\Sigma\|$ is smaller than a geometric constant, the monotonicity formula implies

$$\mathbf{M}(R_1\llcorner\mathbf{C}_\rho(p)) \geq \mathbf{M}(R_1\llcorner\mathbf{B}_\rho(p)) \geq \omega_m(\Theta(R,p)-\tfrac{1}{4})\rho^m = \omega_m(\Theta(R,p)-\tfrac{1}{4})E^{1/2}\,.$$

On the other hand, $\mathbf{M}(R_1\llcorner\mathbf{C}_\rho(p))\leq\omega_m k_1\rho^m + E$. Thus, if E is smaller than a geometric constant, we ensure $\Theta(R,p)\leq k_1+\frac{1}{2}$. This means that, having proved (i) and (ii) for $\sigma = CE^{1/2m}$, (iii) would hold if we redefine σ as $(C+1)E^{1/2m}$.

The proof of (i) and (ii) is by induction on Q. The starting step $Q = 1$ is Federer's classical statement (cp. with [8, Lemma 5.3.4] and [11, Lemma 2]) and though its proof can be easily concluded from what we describe next, our only concern will be to prove the inductive step. Hence, from now on we assume that the theorem holds for all multiplicities up to $Q-1 \geq 1$ and we prove it for Q. Indeed, we will show a slightly weaker assertion, i.e. the existence of numbers $a_1, \ldots, a_k \in \mathbb{R}$ such that the conclusions (i) and (ii) apply when we replace $\mathbf{S}_i$ with $\mathbf{\Sigma}_i = \mathbb{R}^{m+n-1} \times]a_i - \sigma, a_i + \sigma[$. The general statement is obviously a simple corollary. To simplify the notation we use $\bar{\mathbf{p}}$ in place of $\mathbf{p}_{\pi_0}$.

Step 1. Let $r \geq \frac{1}{2}$ and $a - b > 2\eta = 2C_0 E^{1/2m}$, where C_0 is a constant depending only on m and n, which will be determined later. We denote by $W_r(a,b)$ the open set $B_r \times \mathbb{R}^{n-1} \times]a,b[$. In this step we show

$$\|R\|(W_r(a,b)) \leq \tfrac{2Q-1}{2Q}\omega_m r^m \quad \Longrightarrow \quad \mathrm{spt}(R) \cap W_{r-\eta}(a+\eta, b-\eta) = \emptyset \qquad . \tag{A.1}$$

Without loss of generality, we assume $a = 0$. For each $\tau \in]0, \frac{b}{2}[$ consider the currents $R_\tau := R \llcorner W_r(\tau, b-\tau)$ and $S_\tau := \bar{\mathbf{p}}_\sharp R_\tau$. It follows from the slicing theory that S_τ is a locally integral current for a.e. τ. There are then functions $f_\tau \in BV_{loc}(B_r)$ which take integer values and such that $S_\tau = f_\tau [\![B_r]\!]$. Since $\|f_\tau\|_1 = \mathbf{M}(S_\tau) \leq \|R\|(W_r(0,b)) \leq \frac{2Q-1}{2Q}\omega_m r^m$, f_τ must vanish on a set of measure at least $\frac{\omega_m}{2Q} r^m$. By the relative Poincaré inequality,

$$\mathbf{M}(S_\tau)^{1-1/m} = \|f_\tau\|_{L^1}^{1-1/m} \leq C\|Df_\tau\|(B_r) = C\|\partial(\bar{\mathbf{p}}_\sharp R_\tau)\|(B_r) \leq C\|\partial R_\tau\|(\mathbf{C}_r)\,.$$

We introduce the slice $\langle R, \tau\rangle$ relative to the map $x_{m+n} : \mathbb{R}^{m+n} \to \mathbb{R}$ which is the projection on the last coordinate factor. Then the usual slicing theory gives that

$$(\mathbf{M}(S_\tau))^{1-1/m} \leq C\|\partial R_\tau\|(\mathbf{C}_r) = C\mathbf{M}\big(\langle R,\tau\rangle - \langle R, b-\tau\rangle\big) \quad \text{for a.e. } \tau\,. \tag{A.2}$$

Let now $\bar{\tau}$ be the supremum of τ's such that $\mathbf{M}(S_t) \geq \sqrt{E}\ \forall t < \tau$. If $\mathbf{M}(S_0) < \sqrt{E}$, we then set $\bar{\tau} := 0$. If $\bar{\tau} > 0$, observe that, for a.e. $\tau \in [0, \bar{\tau}[$ we have

$$E^{\frac{m-1}{2m}} \leq (\mathbf{M}(S_\tau))^{1-1/m} \leq C\left(\mathbf{M}\big(\langle R,\tau\rangle - \langle R, b-\tau\rangle\big)\right)\,. \tag{A.3}$$

Integrate (A.3) between 0 and $\bar{\tau}$ to conclude

$$\bar{\tau}E^{\frac{m-1}{2m}} \leq C\int_0^{\bar{\tau}} \mathbf{M}\big(\langle R,\tau\rangle - \langle R, \ell-\tau\rangle\big)\,d\tau = \int_{W_r(0,\bar{\tau})\cup W_r(b-\bar{\tau},b)} |\vec{R}\llcorner dx_{m+n}|\,d\|R\|\,. \tag{A.4}$$

We then achieve $\bar{\tau}E^{\frac{m-1}{2m}} \leq C\sqrt{E}$, i.e. $\bar{\tau} \leq \bar{C}E^{1/2m}$, applying Cauchy-Schwartz and recalling

$$\int_{\mathbf{C}_1} |\vec{R}\llcorner dx_{m+n}|^2\,d\|R\| \leq \mathbf{E}(R, \mathbf{C}_1) = E\,.$$

($\bar{C}$ depends only on m and n). Set $C_0 := \bar{C} + 2$ and recall that $\eta = C_0 E^{1/2m}$. Observe also that there must be a sequence of $\tau_k \downarrow \bar{\tau}$ with $\mathbf{M}(S_{\tau_k}) < \sqrt{E}$. Therefore,

$$\|R\|(W_r(\bar{\tau}, b-\bar{\tau})) \leq \liminf_{k\to\infty} \|R\|(W_e(\tau_k, b-\tau_k)) \leq \liminf_{k\to\infty} \mathbf{M}(S_{\tau_k}) + E \leq 2\sqrt{E}\,. \tag{A.5}$$

Assume now the existence of $p \in \mathrm{spt}(T) \cap W_{r-\eta}(\eta, b-\eta)$. By the properties of area-minimizing currents, $\Theta(T,p) \geq 1$. Set $\rho := 2E^{1/2m}$ and $B' := \mathbf{B}_\rho(p) \subset W_r(\bar{\tau}, \ell - \bar{\tau})$. By the monotonicity formula, $\|R\|(B') \geq c\,2^m \omega_m \sqrt{E}$, where c depends only on $\mathbf{A}$ (recall that

$\rho \leq 1$) and approaches 1 as $\mathbf{A}$ approaches 0. Thus, for ε_2 sufficiently small, this would contradict (A.5). We have therefore proved (A.1).

Step 2. We are now ready to conclude the proof of (i) and (ii). Assume

$$\max\left\{\|R\|(W_1(0,\infty)), \|R\|(W_1(-\infty,0))\right\} \leq \tfrac{1}{2}\mathbf{M}(R)\,. \tag{A.6}$$

Divide the interval $[0,1[$ into $Q+1$ intervals $[a_i, a_{i+1}[$ and let $W^i := W_1(a_i, a_{i+1})$. For each i consider $S^i := \bar{\mathbf{p}}_\sharp(T \llcorner W^i)$. Observe that there must be one i for which $\mathbf{M}(S^i) \leq (1-\frac{1}{2Q})\omega_m$. Otherwise we would have

$$\omega_m Q + E \geq \mathbf{M}(R) \geq \sum \mathbf{M}(S^i) \geq \omega_m\left(Q + \tfrac{Q-1}{2Q}\right),$$

which is obviously a contradiction if E is sufficiently small.

It follows from Step 1 that there must be an i so that $\mathrm{spt}(T)$ does not intersect $W_{1-\eta}(a_i+\eta, a_{i+1}-\eta)$. Consider $R_1 := R\llcorner W_{1-\eta}(-\infty, a_i+\eta)$ and $R_2 := R\llcorner W_{1-\eta}\times(a_{i+1}-\eta,\infty)$. By the constancy theorem $\bar{\mathbf{p}}_\sharp R_i = k_i\,[\![B_{1-\eta}]\!]$, where both k_i's are integers. Indeed, having assumed that E is sufficiently small, each k_i must be nonnegative and their sum is Q. There are now two possibilities.

(a) Both k_i's are positive. In this case R_1 and R_2 satisfy again the assumptions of the Theorem with $1-\eta$ in place of 1. After a suitable rescaling we can apply the inductive hypothesis to both currents and hence get the desired conclusion.

(b) One k_i is zero. In this case $\mathbf{M}(R_i) \leq E$ and it cannot be R_1, since $\mathbf{M}(R_1) \geq \frac{1}{2}\mathbf{M}(R)$ by (A.6). Thus it is R_2 and, arguing as at the end of Step 1, we conclude $R\llcorner W_{1-2\eta}(]a_{i+1}+2\eta,\infty[) = 0$.

In case (b) we repeat the argument splitting $]-1,0]$ into $Q+1$ intervals. Once again, either we can "separate" the current into two pieces and apply the inductive hypothesis, or we conclude that $\mathrm{spt}(R\llcorner \mathbf{C}_{1-4\eta}) \subset W_{1-4\eta}(-1-\eta, 1+\eta) =: W_{1-4\eta}(a_0,b_0)$. If this is the case, we apply once again the argument above and either we "separate" $R^1 := R\llcorner \mathbf{C}_{1-6\eta}\times\mathbb{R}^n$ into two pieces, or we conclude that $\mathrm{spt}(R^1) \subset W_{1-6\eta}(a_1,b_1)$, where

$$b_1 - a_1 \leq (b_0-a_0)\left(1-\tfrac{1}{Q+1}+\eta\right) \leq 2\left(1-\tfrac{1}{Q+2}\right)$$

(provided ε_2 is smaller than a geometric constant). We now iterate this argument at most $c_0|\log E|$ times, stopping if at any step we "separate" the current and can apply the inductive hypothesis, or if the resulting current is contained in $W_{1-(4+2k)\eta}(a_k,b_k)$ for some a_k, b_k with $b_k - a_k \leq c_1 E^{1/2m}$. The constant c_1 is chosen larger than 1 and in such a way that, if $\ell > c_1 E^{1/2m}$, then $\ell\frac{Q}{Q+1}+\eta \leq \frac{Q+1}{Q+2}\ell$. Observe that, since $\eta = C_0 E^{1/2m}$, c_1 depends only upon Q, m and n

If the procedure does not stop until we reach $c_0|\log E|$ iterations, then we conclude that the current $R\llcorner \mathbf{C}_{1-(4+2c_0|\log E|)\eta}$ is supported in $W_{1-(4+2c_0|\log E|)\eta}(a,b)$ where

$$b-a \leq (2+\eta)\left(\tfrac{Q+1}{Q+2}\right)^{c_0|\log E|}.$$

c_0 is chosen so that this last number is smaller than $c_1 E^{1/2m}$, i.e. so that

$$c_0 |\log E| \log \tfrac{Q+1}{Q+2} \leq -\log(2+\eta) - \log c_1 + \tfrac{1}{2m}\log E\,. \tag{A.7}$$

Observe that both $\log E$ and $\log \frac{Q+1}{Q+2}$ are negative. Moreover, since c_1 is a geometric constant, if ε_2 is chosen small enough, $|\log E| \geq 2m(\log(2+\eta) + \log c_1)$. Thus (A.7) is surely fulfilled when $c_0 \geq \frac{1}{m}\left|\log \frac{Q+2}{Q+1}\right|^{-1}$. □

Appendix B. Changing coordinates for classical functions

Lemma B.1. *There are constants $c_0, C > 0$ with the following properties. Assume that*

(i) $|\varkappa - \varkappa_0| \leq c_0$, $r \leq 1$;

(ii) $p = (q,u) \in \varkappa \times \varkappa^\perp$ *and* $f, g : B^m_{7r}(q,\varkappa) \to \varkappa^\perp$ *are Lipschitz functions such that*

$$\mathrm{Lip}(f), \mathrm{Lip}(g) \leq c_0 \quad \textit{and} \quad |f(q) - u| + |g(q) - u| \leq c_0\, r.$$

Then there are two maps $f', g' : B_{5r}(p,\varkappa_0) \to \varkappa_0^\perp$ such that

(a) $\mathbf{G}_{f'} = \mathbf{G}_f \llcorner \mathbf{C}_{5r}(p,\varkappa_0)$ *and* $\mathbf{G}_{g'} = \mathbf{G}_g \llcorner \mathbf{C}_{5r}(p,\varkappa_0)$;

(b) $\|f' - g'\|_{L^1(B_{5r}(p,\varkappa_0))} \leq C\, \|f - g\|_{L^1(B_{7r}(p,\varkappa))}$;

(c) *if* $f \in C^4(B_{7r}(p,\varkappa))$ *(resp.* $C^{3,\kappa}(B_{7r}(p,\varkappa))$*) then* $f' \in C^4(B_{5r}(p,\varkappa_0))$ *(resp.* $C^{3,\kappa}(B_{5r}(p,\varkappa_0))$*) with the estimates*

$$\|f' - u'\|_{C^3} \leq \Phi\left(|\varkappa - \varkappa_0|, \|f - u\|_{C^3}\right), \tag{B.1}$$

$$\|D^4 f'\|_{C^0} \leq \Lambda\left(|\varkappa - \varkappa_0|, \|f - u\|_{C^3}\right)\left(1 + \|D^4 f\|_{C^0}\right), \tag{B.2}$$

$$\Big(\textit{resp.} \qquad [D^3 f']_\kappa \leq \Lambda_\kappa\left(|\varkappa - \varkappa_0|, \|f - u\|_{C^3}\right)\left(1 + [D^3 f]_\kappa\right)\Big), \tag{B.3}$$

where Φ, Λ and Λ_κ are smooth functions with $\Phi(\cdot, 0) \equiv 0$;

(d) $\|f' - g'\|_{W^{1,2}(B_{5r}(p,\varkappa_0))} \leq C(1 + \|D^2 f\|_{C^0})\|f - g\|_{W^{1,2}(B_{7r}(p,\varkappa))}$.

All the conclusions of the Lemma still hold if we replace the exterior radius $7r$ and interior radius $5r$ with ρ and s: the corresponding constants (and functions Φ and λ) will then depend also on the ratio $\frac{\rho}{s}$.

Proof. The case of two general radii s and ρ follows easily from that of $\rho = 7r$ and $s = 5r$ and a simple covering argument. In what follows, given a pair of points $x \in \varkappa, y \in \varkappa^\perp$ we use the notation (x,y) for the vector $x+y$. By translation we can assume that $(q,u) = (0,0)$. Consider then the maps $F, G : B_{7r}(0,\varkappa) \to \varkappa_0^\perp$ and $I, J : B_{7r}(0,\varkappa) \to \varkappa_0$ given by

$$F(x) = \mathbf{p}_{\varkappa_0^\perp}((x, f(x))) \quad \text{and} \quad G(x) = \mathbf{p}_{\varkappa_0^\perp}((x, g(x))),$$

$$I(x) = \mathbf{p}_{\varkappa_0}((x, f(x))) \quad \text{and} \quad J(x) = \mathbf{p}_{\varkappa_0}((x, g(x))).$$

Obviously, if c_0 is sufficiently small, I and J are injective Lipschitz maps. Hence, $\mathrm{Gr}_{\varkappa_0}(f)$ and $\mathrm{Gr}_{\varkappa_0}(g)$ coincide, in the new coordinates, with the graphs of the functions f' and g' defined respectively in $D := I(B_{7r}(0,\varkappa))$ and $\tilde{D} := J(B_{7r}(0,\varkappa))$ by $f' = F \circ I^{-1}$ and $g' = G \circ J^{-1}$. If c_0 is chosen sufficiently small, then

$$\mathrm{Lip}(I),\ \mathrm{Lip}(J),\ \mathrm{Lip}(I^{-1}),\ \mathrm{Lip}(J^{-1}) \leq 1 + C\, c_0, \tag{B.4}$$

and

$$|I(q) - q'|, |J(q) - q'| \leq C\, c_0\, r, \tag{B.5}$$

where the constant C is only geometric. Clearly, (B.4) and (B.5) easily imply that $B_{5r}(0, \varkappa_0) \subset D \cap \tilde{D}$ when c_0 is smaller than a geometric constant, thereby implying (a). Conclusion (c) is a simple consequence of the inverse function theorem. Finally we claim that, for small c_0,

$$|f'(x') - g'(x')| \leq 2\, |f(I^{-1}(x')) - g(I^{-1}(x'))| \quad \forall\, x' \in B_r(q'), \tag{B.6}$$

from which, using the change of variables formula for biLipschitz homeomorphisms and (B.4), (b) follows.

In order to prove (B.6), consider any $x' \in B_r(q')$, set $x := I^{-1}(x')$ and

$$p_1 := (x, f(x)) \in \varkappa \times \varkappa^\perp, \quad p_2 := (x, g(x)) \in \varkappa \times \varkappa^\perp \quad \text{and} \quad p_3 := (x', g'(x')) \in \varkappa_0 \times \varkappa_0^\perp.$$

Obviously $|f'(x') - g'(x')| = |p_1 - p_3|$ and $|f(x) - g(x)| = |p_1 - p_2|$. Note that, $g(x) = f(x)$ if and only if $g'(x') = f'(x')$, and in this case (B.6) follows trivially. If this is not the case, the triangle with vertices p_1, p_2 and p_3 is non-degenerate. Let θ_i be the angle at p_i. Note that, $\mathrm{Lip}(g) \leq c_0$ implies $|\frac{\pi}{2} - \theta_2| \leq C c_0$ and $|\varkappa - \varkappa_0| \leq c_0$ implies $|\theta_1| \leq C c_0$, for some dimensional constant C. Since $\theta_3 = \pi - \theta_1 - \theta_2$, we conclude as well $|\frac{\pi}{2} - \theta_3| \leq C c_0$. Therefore, if c_0 is small enough, we have $1 \leq 2 \sin\theta_3$, so that, by the Sinus Theorem,

$$|f'(x') - g'(x')| = |p_1 - p_3| = \frac{\sin\theta_2}{\sin\theta_3}\, |p_1 - p_2| \leq 2\, |p_1 - p_2| = 2\, |f(x) - g(x)|,$$

thus concluding the claim.

We finally come to (d). The estimate $\|f' - g'\|_{L^2} \leq C \|f - g\|_{L^2}$ is an obvious consequence of (B.6). Given next a point p in the graph of f, resp. in the graph of g, we denote by $\sigma(p)$, resp. $\tau(p)$, the oriented tangent plane to the corresponding graphs. Observe that the points are described by the pairs $(x', f(x'))$ and $(x', g(x'))$, in the coordinates $\kappa \times \kappa^\perp$, and by $(I^{-1}(x'), f(I^{-1}(x)'))$ and $(J^{-1}(x'), g(J^{-1}(x')))$, in the coordinates $\varkappa_0 \times \varkappa_0^\perp$. Thus

$$\begin{aligned}
&|\nabla f'(x') - \nabla g'(x')| \leq C|\sigma(p) - \tau(q)| \leq C|\nabla f(I^{-1}(x')) - \nabla g(J^{-1}(x')| \\
\leq& C|\nabla f(I^{-1}(x')) - \nabla f(J^{-1}(x'))| + C|\nabla f(J^{-1}(x')) - \nabla g(J^{-1}(x'))| \\
\leq& C\|D^2 f\|_{C^0} |I^{-1}(x') - J^{-1}(x')| + C|\nabla f(J^{-1}(x')) - \nabla g(J^{-1}(x'))| \\
\leq& C\|D^2 f\|_{C^0} |f'(x') - g'(x')| + C|\nabla f(J^{-1}(x')) - \nabla g(J^{-1}(x'))|\,.
\end{aligned} \tag{B.7}$$

Integrating this last inequality in x' and changing variables we then conclude

$$\|\nabla f' - \nabla g'\|^2 \leq C \|\nabla f - \nabla g\|_{L^2} + C \|D^2 f\|_{C^0} \|f' - g'\|_{L^2}\,,$$

which, together with the L^2 estimate, gives (d). □

Appendix C. Two interpolation inequalities

Lemma C.1. *Let $A > 0$ and $\psi \in C^2(B_\rho, \mathbb{R}^n)$ satisfy $\|\psi\|_{L^1} \leq A\,\rho^{m+1}$ and $\|\Delta\psi\|_{L^\infty} \leq \rho^{-1}\,A$. Then, for every $r < \rho$ there is a constant $C > 0$ (depending only on m and $\frac{\rho}{r}$) such that*

$$\rho^{-1}\,\|\psi\|_{L^\infty(B_r)} + \|D\psi\|_{L^\infty(B_r)} \leq C\,A. \tag{C.1}$$

Proof. By a simple covering argument we can, w.l.o.g., assume $\rho = 3r$. Moreover, if we apply the scaling $\psi_r(x) := r^{-1}\psi(rx)$ we see that $\|\psi_r\|_{L^1(B_3)} = (\rho/3)^{-m-1}\|\psi\|_{L^1(B_\rho)}$, $\|\psi_r\|_\infty = (\rho/3)^{-1}\|\psi\|_\infty$, $\|D\psi_\rho\|_\infty = \|D\psi\|_\infty$ and $\|\Delta\psi_\rho\|_\infty = (\rho/3)\|\Delta\psi\|_\infty$. We can therefore assume $r = 1$. Consider the harmonic function $\zeta : B_2 \to \mathbb{R}$ with boundary data $\psi|_{\partial B_2}$,

$$\begin{cases} \Delta\zeta = 0 & \text{in } B_2,\\ \zeta = \psi & \text{on } \partial B_2. \end{cases}$$

Set $u := \psi - \zeta$ and note that $u = 0$ on ∂B_2, $\|\Delta u\|_{C^0(B_2)} \leq A$. Hence, using the Poincaré inequality, we can estimate the L^1-norm of u in the following way:

$$\|u\|_{L^1} \leq \|u\|_{L^2} \leq C\,\|Du\|_{L^2} \leq C\,\left(\int_{B_2} |\Delta u\, u|\right)^{1/2} \leq C\,\|\Delta u\|_{C^0}^{1/2}\,\|u\|_{L^1}^{1/2} \leq CA\,.$$

Choose now $a \in]0,1[$ and $s \in]1,\infty[$ such that $\frac{1}{m} + a\left(\frac{1}{s} - \frac{2}{m}\right) + 1 - a < 0$ (which exist because for $s \to \infty$ and $a \to 1$ the expression converges to $-\frac{1}{m}$). By a classical interpolation inequality, (see [9])

$$\|Du\|_{L^\infty} \leq C\,\|D^2u\|_{L^s}^a \|u\|_{L^1}^{1-a} + C\|u\|_{L^1}\,.$$

Using the L^s-estimate for the Laplacian, we deduce

$$\|Du\|_{L^\infty} \leq C\,\|\Delta u\|_{L^s}^a \|u\|_{L^1}^{1-a} + C\|u\|_{L^1} \leq C\,\|\Delta u\|_\infty^a \|u\|_{L^1}^{1-a} + \|u\|_{L^1} \leq C\,A\,. \tag{C.2}$$

From (C.2) and $u|_{\partial B_2} = 0$ it follows trivially $\|u\|_{L^\infty} \leq A$. To infer (C.1), we observe that, by $\|\zeta\|_{L^1(B_2)} \leq \|u\|_{L^1(B_2)} + \|\psi\|_{L^1(B_2)} \leq C\,A$ and the harmonicity of ζ,

$$\|\zeta\|_{L^\infty(B_1)} + \|D\zeta\|_{L^\infty(B_1)} \leq C\|\zeta\|_{L^1(B_2)} \leq C\,A\,.$$

□

Lemma C.2. *For every m, $r < s$ and κ there is a positive constant C (depending on m, κ and $\frac{s}{r}$) with the following property. Let f be a $C^{3,\kappa}$ function in the ball $B_s \subset \mathbb{R}^m$. Then*

$$\|D^j f\|_{C^0(B_r)} \leq Cr^{-m-j}\|f\|_{L^1(B_s)} + Cr^{3+\kappa-j}[D^3 f]_{\kappa,B_s} \qquad \forall j \in \{0,1,2,3\}\,. \tag{C.3}$$

Proof. A simple covering argument reduces the lemma to the case $s = 2r$. Moreover, define $f_r(x) := f(rx)$ to see that we can assume $r = 1$. So our goal is to show

$$\sum_{j=0}^{3} |D^j f(y)| \leq C\|f - g\|_{L^1} + C[D^3 f]_\kappa \qquad \forall y \in B_1, \forall f \in C^{3,\kappa}(B_2)\,. \tag{C.4}$$

By translating it suffices then to prove the estimate

$$\sum_{j=0}^{3} |D^j f(0)| \leq C\|f\|_{L^1(B_1)} + C[D^3 f]_{\kappa,B_1} \qquad \forall f \in C^{3,\kappa}(B_1)\,. \tag{C.5}$$

Consider now the space of polynomials R in m variables of degree at most 3, which we write as $R = \sum_{j=0}^3 A_j x^j$. This is a finite dimensional vector space, on which we can define the norms $|R| := \sum_{j=0}^3 |A_j|$ and $\|R\| := \int_{B_1} |R(x)|\,dx$. These two norms must then be equivalent, so there is a constant C (depending only on m), such that $|R| \leq C\|R\|$ for any such polynomial. In particular, if P is the Taylor polynomial of third order for f at the point 0, we conclude

$$\sum_{j=0}^{3} |D^j f(0)| = |P| \leq C\|P\| = C \int_{B_1} |P(x)|\,dx \leq C\|f\|_{L^1(B_1)} + C\|f-P\|_{L^1(B_1)}$$
$$\leq C\|f\|_{L^1} + C[D^3 f]_\kappa\,. \qquad \square$$

References

[1] F. J. Almgren, Jr. Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure. *Ann. of Math. (2)*, 87:321–391, 1968.

[2] Frederick J. Almgren, Jr. *Almgren's big regularity paper*, volume 1 of *World Scientific Monograph Series in Mathematics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2000.

[3] Camillo De Lellis and Emanuele Spadaro. Center manifold: a case study. *Discrete Contin. Dyn. Syst.*, 31(4):1249–1272, 2011.

[4] Camillo De Lellis and Emanuele Spadaro. Q-valued functions revisited. *Mem. Amer. Math. Soc.*, 211(991):vi+79, 2011.

[5] Camillo De Lellis and Emanuele Spadaro. Multiple valued functions and integral currents. *Preprint*, 2013.

[6] Camillo De Lellis and Emanuele Spadaro. Regularity of area-minimizing currents I: gradient L^p estimates. *Preprint*, 2013.

[7] Camillo De Lellis and Emanuele Spadaro. Regularity of area-minimizing currents III: blow-up. *Preprint*, 2013.

[8] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[9] L. Nirenberg. An extended interpolation inequality. *Ann. Scuola Norm. Sup. Pisa (3)*, 20:733–737, 1966.

[10] Tristan Rivière. A lower-epiperimetric inequality for area-minimizing surfaces. *Comm. Pure Appl. Math.*, 57(12):1673–1685, 2004.

[11] Richard Schoen and Leon Simon. A new proof of the regularity theorem for rectifiable currents which minimize parametric elliptic functionals. *Indiana Univ. Math. J.*, 31(3):415–434, 1982.

[12] Leon Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University*. Australian National University Centre for Mathematical Analysis, Canberra, 1983.