REGULARITY OF AREA MINIMIZING CURRENTS III: BLOW-UP

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Abstract. This is the last of a series of three papers in which we give a new, shorter proof of a slightly improved version of Almgren's partial regularity of area minimizing currents in Riemannian manifolds. Here we perform a blow-up analysis deducing the regularity of area minimizing currents from that of Dir-minimizing multiple valued functions.

0. Introduction

In this paper we complete the proof of a slightly improved version of the celebrated Almgren's partial regularity result for area minimizing currents in a Riemannian manifold (see [1]), namely Theorem 0.3 below.

Assumption 0.1. Let $\varepsilon_0 > 0$, $m, \bar{n} \in \mathbb{N} \setminus \{0\}$ and $l \in \mathbb{N}$. We denote by

(M) $\Sigma \subset \mathbb{R}^{m+n} = \mathbb{R}^{m+\bar{n}+l}$ an embedded $m+\bar{n}$-dimensional submanifold of class C^{3,ε_0};

(C) T an integral current of dimension m with $\mathrm{spt}(T) \subset \Sigma$, area minimizing in Σ.

Definition 0.2. For T and Σ as in Assumption 0.1 we define

$$\mathrm{Reg}(T) := \big\{x \in \mathrm{spt}(T) :\ \mathrm{spt}(T) \cap \mathbf{B}_r(x) \text{ is a } C^{3,\varepsilon_0} \text{ submanifold for some } r > 0\big\}, \quad (0.1)$$

$$\mathrm{Sing}(T) := \mathrm{spt}(T) \setminus \big(\mathrm{spt}(\partial T) \cup \mathrm{Reg}(T)\big). \quad (0.2)$$

The partial regularity result proven first by Almgren [1] under the more restrictive hypothesis $\Sigma \in C^5$ gives an estimate on the Hausdorff dimension $\dim_H(\mathrm{Sing}(T))$ of $\mathrm{Sing}(T)$.

Theorem 0.3. *$\dim_H(\mathrm{Sing}(T)) \leq m-2$ for any $m, \bar{n}, l, T$ and Σ as in Assumption 0.1.*

In this note we complete the proof of Theorem 0.3, based on our previous works [3, 4, 5], thus providing a new, and much shorter, account of the most fundamental regularity result in geometric measure theory; we refer to [4] for a more extended general introduction. The proof is carried by contradiction: in the sequel we will always assume the following.

Assumption 0.4 (Contradiction)**.** There exist $m \geq 2, \bar{n}, l$, Σ and T as in Assumption 0.1 such that $\mathcal{H}^{m-2+\alpha}(\mathrm{Sing}(T)) > 0$ for some $\alpha > 0$.

The hypothesis $m \geq 2$ in Assumption 0.4 is justified by the well-known fact that $\mathrm{Sing}(T) = \emptyset$ when $m = 1$ (in this case $\mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$ is the union of finitely many non-intersecting geodesic segments). Starting from Assumption 0.4, we make a careful blow-up analysis, split in the following steps.

0.1. **Flat tangent planes.** We first reduce to flat blow-ups around a given point, which in the sequel is assumed to be the origin. These blow-ups will also be chosen so that the size of the singular set satisfies a uniform estimate from below (cp. Section 1).

0.2. **Intervals of flattening.** For appropriate rescalings of the current around the origin we take advantage of the center manifold constructed in [5], which gives a good approximation of the average of the sheets of the current at some given scale. However, since it might fail to do so at different scales, in Section 2 we introduce a *stopping condition* for the center manifolds and define appropriate *intervals of flattening* $I_j = [s_j, t_j]$. For each j we construct a different center manifold $\mathcal{M}_j$ and approximate the (rescaled) current with a suitable multi-valued map on the normal bundle of $\mathcal{M}_j$.

0.3. **Finite order of contact.** A major difficulty in the analysis is to prove that the minimizing current has finite order of contact with the center manifold. To this aim, in analogy with the case of harmonic multiple valued functions (cp. [2, Section 3.4]), we introduce a variant of the *frequency function* and prove its almost monotonicity and boundedness. This analysis, carried in Sections 3, 4 and 5, relies on the variational formulas for images of multiple valued maps as computed in [3] and on the careful estimates of [5]. Our frequency function differs from that of Almgren and allows for simpler estimates.

0.4. **Convergence to Dir-minimizer and contradiction.** Based on the previous steps, we can blow-up the Lipschitz approximations from the center manifold $\mathcal{M}_j$ in order to get a limiting Dir-minimizing function on a flat m-dimensional domain. We then show that the singularities of the rescaled currents converge to singularities of that limiting Dir-minimizer, contradicting the partial regularity of [2, Section 3.6] and, hence, proving Theorem 0.3.

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1. Flat tangent cones

Definition 1.1 (Q-points)**.** For $Q \in \mathbb{N}$, we denote by $\mathrm{D}_Q(T)$ the points of density Q of the current T, and set

$$\mathrm{Reg}_Q(T) := \mathrm{Reg}(T) \cap \mathrm{D}_Q(T) \quad \text{and} \quad \mathrm{Sing}_Q(T) := \mathrm{Sing}(T) \cap \mathrm{D}_Q(T).$$

Definition 1.2 (Tangent cones)**.** For any $r > 0$ and $x \in \mathbb{R}^{m+n}$, $\iota_{x,r}$ is the map $y \mapsto \frac{y-x}{r}$ and $T_{x,r} := (\iota_{x,r})_\sharp T$. The classical monotonicity formula (see [9] and [4, Lemma A.1]) implies that, for every $r_k \downarrow 0$ and $x \in \mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$, there is a subsequence (not relabeled) for which T_{x,r_k} converges to an integral cycle S which is a cone (i.e., $S_{0,r} = S$ for all $r > 0$ and $\partial S = 0$) and is (locally) area-minimizing in $\mathbb{R}^{m+n}$. Such a cone will be called, as usual, *a tangent cone to T at x*.

Fix $\alpha > 0$. By Almgren's stratification theorem (see [9, Theorem 35.3]), for $\mathcal{H}^{m-2+\alpha}$-a.e. $x \in \operatorname{spt}(T) \setminus \operatorname{spt}(\partial T)$, there exists a subsequence of radii $r_k \downarrow 0$ such that T_{x,r_k} converge to an integer multiplicity flat plane. Similarly, for measure-theoretic reasons, if T is as in Assumption 0.4, then for $\mathcal{H}^{m-2+\alpha}$-a.e. $x \in \operatorname{spt}(T) \setminus \operatorname{spt}(\partial T)$ there is a subsequence $s_k \downarrow 0$ such that $\liminf_k \mathcal{H}^{m-2+\alpha}_\infty(\mathrm{D}_Q(T_{x,s_k}) \cap \mathbf{B}_1) > 0$ (see again [9]). Obviously there would then be $Q \in \mathbb{N}$ and $x \in \operatorname{Sing}_Q(T)$ where both subsequences exist. The two subsequences might, however, differ: in the next proposition we show the existence of one point and a single subsequence along which *both* conclusions hold. Concerning the relevant notation (for instance regarding the excess $\mathbf{E}$) we refer to [4, 5].

Proposition 1.3 (Contradiction sequence)**.** *Under Assumption 0.4, there are $m, n, Q \geq 2$, Σ and T as in Assumption 0.1, reals $\alpha, \eta > 0$, and a sequence $r_k \downarrow 0$ such that $0 \in \mathrm{D}_Q(T)$ and the following holds:*

$$\lim_{k\to+\infty} \mathbf{E}(T_{0,r_k}, \mathbf{B}_{6\sqrt{m}}) = 0, \tag{1.1}$$

$$\lim_{k\to+\infty} \mathcal{H}^{m-2+\alpha}_\infty(\mathrm{D}_Q(T_{0,r_k}) \cap \mathbf{B}_1) > \eta, \tag{1.2}$$

$$\mathcal{H}^m\big((\mathbf{B}_1 \cap \operatorname{spt}(T_{0,r_k})) \setminus \mathrm{D}_Q(T_{0,r_k})\big) > 0 \quad \forall\, k \in \mathbb{N}. \tag{1.3}$$

The proof is based on the following lemma.

Lemma 1.4. *Let S be an m-dimensional area minimizing integral cone in $\mathbb{R}^{m+n}$ such that $\partial S = 0$, $Q = \Theta(S,0) \in \mathbb{N}$, $\mathcal{H}^m\big(D_Q(S)\big) > 0$ and $\mathcal{H}^{m-1}(\operatorname{Sing}_Q(S)) = 0$. Then, S is an m-dimensional plane with multiplicity Q.*

Proof. For each $x \in \operatorname{Reg}_Q(S)$, let r_x be such that $S \llcorner \mathbf{B}_{2r_x}(x) = Q\,[\![\Gamma]\!]$ for some regular submanifold Γ and set

$$U := \bigcup_{x\in\operatorname{Reg}_Q(S)} \mathbf{B}_{r_x}(x).$$

Obviously, $\operatorname{Reg}_Q(S) \subset U$; hence, by assumption, it is not empty. Fix $x \in \operatorname{spt}(S) \cap \partial U$. Let next $(x_k)_{k\in\mathbb{N}} \subset \operatorname{Reg}_Q(S)$ be such that $\operatorname{dist}(x, \mathbf{B}_{r_{x_k}}(x_k)) \to 0$. We necessarily have that $r_{x_k} \to 0$: otherwise we would have $x \in \mathbf{B}_{2r_{x_k}}(x_k)$ for some k, which would imply $x \in \operatorname{Reg}_Q(S) \subset U$, i.e. a contradiction. Therefore, $x_k \to x$ and, by [9, Theorem 35.1],

$$Q = \limsup_{k\to+\infty} \Theta(S, x_k) \leq \Theta(S,x) = \lim_{\lambda\downarrow 0} \Theta(S, \lambda x) \leq \Theta(S,0) = Q.$$

This implies $x \in \mathrm{D}_Q(S)$. Since $x \in \partial U$, we must then have $x \in \operatorname{Sing}_Q(S)$. Thus, we conclude that $\mathcal{H}^{m-1}(\operatorname{spt}(S) \cap \partial U) = 0$. It follows from the standard theory of rectifiable currents (cp. Lemma A.2) that $S' := S \llcorner U$ has 0 boundary in $\mathbb{R}^{m+n}$. Moreover, since S is an area minimizing cone, the same clearly holds for S'. By definition of U we have $\Theta(S', x) = Q$ for $\|S'\|$-a.e. x and, by semicontinuity,

$$Q \leq \Theta(S', 0) \leq \Theta(S, 0) = Q.$$

We apply Allard's theorem and deduce that S' is regular, i.e. S' is an m-plane with multiplicity Q. Finally, from $\Theta(S',0) = \Theta(S,0)$, we infer $\mathbf{M}(S \llcorner \mathbf{B}_1) = \mathbf{M}(S' \llcorner \mathbf{B}_1)$ and then $S' = S$. □

Proof of Proposition 1.3. Let $m > 1$ be the smallest integer for which Theorem 0.3 fails. By Almgren's stratification theorem (cp. Theorem A.3), there must be an integer rectifiable area minimizing current R of dimension m and a positive integer Q such that the Hausdorff dimension of $\mathrm{Sing}_Q(R)$ is larger than $m-2$. We fix the smallest Q for which such a current R exists. Recall that, by the upper semicontinuity of the density and a straightforward application of Allard's regularity theorem (see Theorem A.1), $\mathrm{Sing}_1(R) = \emptyset$, i.e. $Q > 1$.

Let $\alpha \in]0,1]$ be such that $\mathcal{H}^{m-2+\alpha}(\mathrm{Sing}_Q(R)) > 0$. By [9, Theorem 3.6] there exists a point $x \in \mathrm{Sing}_Q(R)$ such that $\mathrm{Sing}_Q(R)$ has positive $\mathcal{H}^{m-2+\alpha}_\infty$-upper density: i.e., assuming without loss of generality $x = 0$ and $\partial R \llcorner \mathbf{B}_1 = 0$, there exists $r_k \downarrow 0$ such that

$$\lim_{k\to+\infty} \mathcal{H}^{m-2+\alpha}_\infty\big(\mathrm{Sing}_Q(R_{0,r_k}) \cap \mathbf{B}_1\big) = \lim_{k\to+\infty} \frac{\mathcal{H}^{m-2+\alpha}_\infty\big(\mathrm{Sing}_Q(R) \cap \mathbf{B}_{r_k}\big)}{r_k^{m-2+\alpha}} > 0\,.$$

Up to a subsequence (not relabelled) we can assume that $R_{0,r_k} \to S$, with S a tangent cone. If S is a multiplicity Q flat plane, then we set $T := R$ and we are done: indeed, (1.3) is satisfied by Theorem A.1, because $0 \in \mathrm{Sing}(R)$ and $\|R\| \geq \mathcal{H}^m \llcorner \mathrm{spt}(R)$.

Assume therefore that S is *not* an m-dimensional plane with multiplicity Q. Taking into account the convergence of the total variations for minimizing currents [9, Theorem 34.5] and the upper semicontinuity of $\mathcal{H}^{m-2+\alpha}_\infty$ under the Hausdorff convergence of compact sets, we get

$$\mathcal{H}^{m-2+\alpha}_\infty\big(\mathrm{D}_Q(S) \cap \bar{\mathbf{B}}_1\big) \geq \liminf_{k\to+\infty} \mathcal{H}^{m-2+\alpha}_\infty\big(\mathrm{D}_Q(R_{0,r_k}) \cap \bar{\mathbf{B}}_1\big) > 0. \tag{1.4}$$

We claim that (1.4) implies

$$\mathcal{H}^{m-2+\alpha}_\infty(\mathrm{Sing}_Q(S)) > 0. \tag{1.5}$$

Indeed, if all points of $\mathrm{D}_Q(S)$ are singular, then this follows from (1.4) directly. Otherwise, $\mathrm{Reg}_Q(S)$ is not empty and, hence, $\mathcal{H}^m(D_Q(S) \cap \mathbf{B}_1) > 0$. In this case we can apply Lemma 1.4 and infer that, since S is not regular, then $\mathcal{H}^{m-1}(\mathrm{Sing}_Q(S)) > 0$ and (1.5) holds.

We can, hence, find $x \in \mathrm{Sing}_Q(S) \setminus \{0\}$ and $r_k \downarrow 0$ such that

$$\lim_{k\to+\infty} \mathcal{H}^{m-2+\alpha}_\infty\big(\mathrm{Sing}_Q(S_{x,r_k}) \cap \mathbf{B}_1\big) = \lim_{k\to+\infty} \frac{\mathcal{H}^{m-2+\alpha}_\infty\big(\mathrm{Sing}_Q(S) \cap \mathbf{B}_{r_k}(x)\big)}{r_k^{m-2+\alpha}} > 0.$$

Up to a subsequence (not relabelled), we can assume that S_{x,r_k} converges to S_1. Since S_1 is a tangent cone to the cone S at $x \neq 0$, S_1 splits off a line, i.e. $S_1 = S_2 \times\!\!\!\!\times [\![\mathbb{R}v]\!]$, for some area minimizing cone S_2 in $\mathbb{R}^{m-1+n}$ and some $v \in \mathbb{R}^{m+n}$ (cp. the arguments in [9, Lemma 35.5]). Since m is, by assumption, the smallest integer for which Theorem 0.3 fails, $\mathcal{H}^{m-3+\alpha}(\mathrm{Sing}(S_2)) = 0$ and, hence, $\mathcal{H}^{m-2+\alpha}(\mathrm{Sing}_Q(S_1)) = 0$. On the other hand, arguing as for (1.4), we have

$$\mathcal{H}^{m-2+\alpha}_\infty(\mathrm{D}_Q(S_1) \cap \bar{\mathbf{B}}_1) \geq \limsup_{k\to+\infty} \mathcal{H}^{m-2+\alpha}_\infty(\mathrm{D}_Q(S_{x,r_k}) \cap \bar{\mathbf{B}}_1) > 0.$$

Thus $\mathrm{Reg}_Q(S_1) \neq \emptyset$ and, hence, $\mathcal{H}^m(D_Q(S_1)) > 0$. We can apply Lemma 1.4 again and conclude that S_1 is an m-dimensional plane with multiplicity Q. Therefore, the proposition follows taking $T := \tau_\sharp S$, with τ the translation map $y \mapsto y - x$, and Σ the tangent plane at 0 to the original Riemannian manifold. □

2. Intervals of flattening

For the sequel we fix the constant $c_s := \frac{1}{16\sqrt{m}}$ and notice that $2^{-N_0-1} < c_s$, where N_0 is the parameter introduced in [5, Assumption 1.5]. It is always understood that the parameters $\beta_2, \delta_2, \gamma_2, \varepsilon_2, \kappa, C_2, C_h, M_0, N_0$ in [5] are chosen in such a way that all the theorems and propositions therein are applicable, cf. [5, Remark 1.6]. We recall also the notation introduced in [5, Assumption 1.2]: $\mathbf{c}(\Sigma) := \sup_{p\in\Sigma} \|D^2\Psi_p\|_{C^{1,\varepsilon_0}}$, where $\Psi_p : T_p\Sigma \to (T_p\Sigma)^\perp$ is the function parametrizing Σ and $\boldsymbol{m}_0 = \max\left\{\mathbf{c}(\Sigma)^2, \mathbf{E}\left(T, \mathbf{B}_{6\sqrt{m}}\right)\right\}$.

By Proposition 1.3 and simple rescaling arguments, we assume in the sequel the following.

Assumption 2.1. Under Assumption 0.4, there exist $m, n, Q \geq 2$, $\alpha, \eta > 0$, T, Σ and a sequence of radii $r_k \downarrow 0$ as in Proposition 1.3 which satisfy

$$T_0\Sigma = \mathbb{R}^{m+\bar{n}} \times \{0\}, \quad \mathrm{spt}(\partial T) \cap \mathbf{B}_{6\sqrt{m}} = \emptyset, \quad 0 \in D_Q(T), \tag{2.1}$$

$$\|T\|(\mathbf{B}_{6\sqrt{m}r}) \leq r^m \left(Q\,\omega_m(6\sqrt{m})^m + \varepsilon_3^2\right) \quad \text{for all } r \in (0,1), \tag{2.2}$$

$$\mathbf{c}(\Sigma) \leq \varepsilon_3, \tag{2.3}$$

where ε_3 is a positive constant to be specified later, smaller than the ε_2 of [5].

2.1. **Defining procedure.** We set

$$\mathcal{R} := \left\{r \in]0,1] : \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r}) \leq \varepsilon_3^2\right\}. \tag{2.4}$$

Observe that, if $\{s_k\} \subset \mathcal{R}$ and $s_k \uparrow s$, then $s \in \mathcal{R}$. We cover $\mathcal{R}$ with a collection $\mathcal{F} = \{I_j\}_j$ of intervals $I_j =]s_j, t_j]$ defined according as follows. $t_0 := \max \mathcal{R}$ and, for given t_j, consider

- $T_j := (\iota_{0,t_j})_\sharp T$, $\Sigma_j := \iota_{0,t_j}(\Sigma)$;
- $\mathcal{M}_j$ the corresponding center manifold constructed in [5, Theorem 1.12] with respect to an m-plane π_j such that $\mathbf{E}(T_j, \mathbf{B}_{6\sqrt{m}}, \pi_j) = \mathbf{E}(T_j, \mathbf{B}_{6\sqrt{m}})$ (since $\mathbf{c}(\iota_{0,r}(\Sigma)) \leq r\mathbf{c}(\Sigma)$, [5, Theorem 1.12] is indeed applicable); the manifold $\mathcal{M}_j$ is then the graph of a map $\boldsymbol{\varphi}_j : \pi_j \supset [-4,4]^m \to \pi_j^\perp$, and we set $\boldsymbol{\Phi}_j(x) := (x, \boldsymbol{\varphi}_j(x)) \in \pi_j \times \pi_j^\perp$.

Then, one of the following possibilities occurs:

(Stop) either there is $r \in]0,3]$ and a cube L of the Whitney decomposition $\mathscr{W}$ of [5, Proposition 1.7] (applied to the current T_j) such that

$$\ell(L) \geq c_s\, r \qquad \text{and} \qquad L \cap B_r(0) \neq \emptyset\,; \tag{2.5}$$

(Go) or there exists no radius as in (Stop).

In case (Go) holds, we set $s_j := 0$, i.e. $I_j :=]0, t_j]$, and end the procedure. Otherwise, we let $s_j := \bar{r}\, t_j$ where $\bar{r}$ is the maximum radius satisfying (Stop). We choose then t_{j+1} as the largest element in $\mathcal{R} \cap]0, s_j]$ and proceed iteratively.

2.2. **First consequences.** The following is a list of easy consequences of the definition.

Proposition 2.2. *Assuming ε_3 sufficiently small, then the following holds:*

(i) $s_j < \frac{t_j}{2}$ *and the family $\mathcal{F}$ is either countable and $t_j \downarrow 0$, or finite and $I_j =]0, t_j]$ for the largest j;*

(ii) *the union of the intervals of $\mathcal{F}$ cover $\mathcal{R}$, and for k large enough the radii r_k in Assumption 2.1 belong to $\mathcal{R}$;*

(iii) *if $r \in]\frac{s_j}{t_j}, 3[$ and $J \in \mathscr{W}_n$ intersects $B := \mathbf{p}_{\pi_j}(\mathcal{B}_r(p_j))$, with $p_j := \mathbf{\Phi}_j(0)$, then J is in the domain of influence (see* [5, Corollary 3.2]*) of a cube $H \in \mathscr{W}_e$ with*

$$\ell(H) \leq 8\, c_s\, r \quad \textit{and} \quad \max\{\mathrm{dist}\,(H,B), \mathrm{dist}\,(H,J)\} \leq 2\sqrt{m}\,\ell(H) \leq r;$$

(iv) $\mathbf{E}(T_j, \mathbf{B}_r) \leq C\varepsilon_3^2\, r^{2-2\delta_2}$ *for every $r \in]\frac{s_j}{t_j}, 3[$.*

(v) $\sup\{\mathrm{dist}(x, \mathcal{M}_j) : x \in \mathrm{spt}(T_j) \cap \mathbf{p}_j^{-1}(\mathcal{B}_r(p_j))\} \leq C\,(\boldsymbol{m}_0^j)^{\frac{1}{2m}} r^{1+\beta_2}$ *for every $r \in]\frac{s_j}{t_j}, 3[$, where $\boldsymbol{m}_0^j := \max\{\mathbf{c}(\Sigma_j), \mathbf{E}(T_j, \mathbf{B}_{6\sqrt{m}})\}$.*

Proof. We start noticing that $s_j \leq \frac{t_j}{2}$ follows from the choice of c_s because all cubes in the Whitney decomposition have side-length at most 2^{-N_0}. In particular, this implies that the iterative procedure either never stops, leading to $t_j \downarrow 0$, or it stops because $s_j = 0$ and $]0, t_j] \subset \mathcal{R}$, thus proving (i). The first part of (ii) follows straightforwardly from the choice of t_{j+1}, and the last assertion holds from $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m} r_k}) \to 0$.

Regarding (iii), consider $H \in \mathscr{W}_e$ as in [5, Corollary 3.2] and choose $k \in \mathbb{N} \setminus \{0\}$ such that $\ell(H) = 2^k\,\ell(J)$. Observe that $\|\boldsymbol{\varphi}_j\|_{C^{3,\kappa}} \leq C\varepsilon_3$ by [5, Theorem 1.12], where the constant C is independent of ε_3. If ε_3 is sufficiently small, we can assume

$$B_{r/2}(\pi_j) \subset B \subset B_r(\pi_j)\,. \tag{2.6}$$

Let $x \in J$ and $y \in H$ be two points maximizing $|x-y|$. Then, by [5, Corollary 3.2],

$$\mathrm{dist}\,(B,H) \leq |x-y| \leq 2^{k+1}\sqrt{m}\,\ell(J) = 2\sqrt{m}\,\ell(H).$$

Therefore, the conclusion is trivial if $r \geq \frac{1}{16}$ because $\ell(H) \leq 2^{-N_0}$. Assume then $r \leq \frac{1}{16}$ and note that H intersects $B_{2r+2\sqrt{m}\,\ell(H)}$. Let $\rho := 2r + 2\sqrt{m}\ell(H)$. Observe that $2r < \rho < 1$. Since for such ρ (Stop) cannot hold, we have that

$$\ell(H) < c_s\,\left(2r + 2\sqrt{m}\,\ell(H)\right) \leq 4\,c_s\,r + \frac{\ell(H)}{2}.$$

Therefore, we conclude that $\ell(H) \leq 8\,c_s r$ and $\mathrm{dist}(H,B) \leq 16\sqrt{m}\,c_s r < r$.

We now turn to (iv). If $r \geq 2^{-N_0}$, then obviously

$$\mathbf{E}(T_j, \mathbf{B}_r) \leq (4\sqrt{m}\,2^{N_0})^{m+2-2\delta_2} r^{2-2\delta_2}\mathbf{E}(T_j, \mathbf{B}_{4\sqrt{m}}) \leq (4\sqrt{m}\,2^{N_0})^{m+2-2\delta_2} r^{2-2\delta_2}\varepsilon_3^2\,.$$

Otherwise, let $k \geq N_0$ be the smallest natural number such that $2^{-k} < r$ and let $L \in \mathscr{W}^k \cup \mathscr{S}^k$ be a cube of the Whitney decomposition [5, Section 1.1] which contains the origin. From condition (Go), it follows that $L \notin \mathscr{W}$. Therefore, $\mathbf{B}_r \subset \mathbf{B}_L$, where $\mathbf{B}_L$ is the ball of [5, Section 1.1]. Thus, by [5, Proposition 1.7], we get

$$\mathbf{E}(T_j, \mathbf{B}_r) \leq C\mathbf{E}(T_j, \mathbf{B}_L) \leq C\varepsilon_3^2\, r^{2-2\delta_2}.$$

Finally, (v) follows from [5, Corollary 2.2 (ii)], because by (Go), for every $r \in]\frac{s_j}{t_j}, 3[$, every cube $L \in \mathscr{W}$ which intersects $B_r(0)$ satisfies $\ell(L) < c_s r$. □

3. Frequency function and first variations

Consider the following Lipschitz (piecewise linear) function $\phi : [0+\infty[\to [0,1]$ given by

$$\phi(r) := \begin{cases} 1 & \text{for } r \in [0, \frac{1}{2}], \\ 2-2r & \text{for } r \in]\frac{1}{2}, 1], \\ 0 & \text{for } r \in]1, +\infty[. \end{cases}$$

For every interval of flattening $I_j =]s_j, t_j]$, let N_j be the normal approximation of T_j on $\mathcal{M}_j$ in [5, Theorem 2.4]. To simplify the notation, in this section we drop the subscript $_j$ and set $d(p) := d_{\mathcal{M}}(p, \mathbf{\Phi}(0))$ where $d_{\mathcal{M}}$ is the geodesic distance on $\mathcal{M}$. Moreover we omit the measure $\mathcal{H}^m$ in the integrals over regions of $\mathcal{M}$.

Definition 3.1 (Frequency function)**.** For every $r \in]0, 3[$ we define:

$$\mathbf{D}_N(r) := \int_{\mathcal{M}} \phi\left(\frac{d(p)}{r}\right) |DN|^2(p)\, dp \quad \text{and} \quad \mathbf{H}_N(r) := -\int_{\mathcal{M}} \phi'\left(\frac{d(p)}{r}\right) \frac{|N|^2(p)}{d(p)}\, dp\,.$$

If $\mathbf{H}_N(r) > 0$, we define the *frequency function* as $\mathbf{I}_N(r) := \frac{r\,\mathbf{D}_N(r)}{\mathbf{H}_N(r)}$.

To simplify further the notation, N might be sometimes omitted as subscript. The following is the main analytical estimate of the paper, which allows us to exclude infinite order of contact among the different sheets of a minimizing current.

Theorem 3.2 (Main frequency estimate)**.** *If ε_3 is sufficiently small, then there exists a constant $C > 0$ such that, for every $[a, b] \subset [\frac{s}{t}, 3]$ with $\mathbf{H}|_{[a,b]} > 0$, it holds*

$$\mathbf{I}(a) \leq C(1 + \mathbf{I}(b)). \tag{3.1}$$

The proof exploits four identities collected in Proposition 3.5, which will be proved in the next sections.

Definition 3.3. We let $\partial_{\hat{r}}$ denote the derivative along geodesics starting at $\mathbf{\Phi}(0)$. We set

$$\mathbf{E}(r) := -\int_{\mathcal{M}} \phi'\left(\tfrac{d(p)}{r}\right) \sum_{i=1}^{Q} \langle N_i(p), \partial_{\hat{r}} N_i(p)\rangle\, dp\,, \tag{3.2}$$

$$\mathbf{G}(r) := -\int_{\mathcal{M}} \phi'\left(\tfrac{d(p)}{r}\right) d(p)\, |\partial_{\hat{r}} N(p)|^2\, dp \quad \text{and} \quad \mathbf{\Sigma}(r) := \int_{\mathcal{M}} \phi\left(\tfrac{d(p)}{r}\right) |N|^2(p)\, dp\,. \tag{3.3}$$

Remark 3.4. Observe that all these functions of r are absolutely continuous and, therefore, classically differentiable at almost every r. Moreover, the following rough estimate easily follows from [5, Theorem 2.4] and the condition (Stop):

$$\mathbf{D}(r) \leq C\, \boldsymbol{m}_0\, r^{m+2-2\delta_2} \quad \text{for every} \quad r \in \left]\tfrac{s}{t}, 3\right[\,. \tag{3.4}$$

Proposition 3.5 (First variations estimates)**.** *There exist dimensional constants $C, \gamma_3 > 0$ such that, if the hypotheses of Theorem 3.2 hold and $\mathbf{I} \geq 1$, then*

$$\left|\mathbf{H}'(r) - \tfrac{m-1}{r}\,\mathbf{H}(r) - \tfrac{2}{r}\,\mathbf{E}(r)\right| \leq C\mathbf{H}(r), \tag{3.5}$$

$$\left|\mathbf{D}(r) - r^{-1}\mathbf{E}(r)\right| \leq C\mathbf{D}(r)^{1+\gamma_3} + C\varepsilon_3^2\,\mathbf{\Sigma}(r), \tag{3.6}$$

$$\left|\mathbf{D}'(r) - \tfrac{m-2}{r}\,\mathbf{D}(r) - \tfrac{2}{r^2}\,\mathbf{G}(r)\right| \leq C\mathbf{D}(r) + C\mathbf{D}(r)^{\gamma_3}\mathbf{D}'(r) + r^{-1}\mathbf{D}(r)^{1+\gamma_3}, \tag{3.7}$$

$$\mathbf{\Sigma}(r) + r\,\mathbf{\Sigma}'(r) \leq C\,r^2\,\mathbf{D}(r)\ \leq Cr^{2+m}\varepsilon_3^2. \tag{3.8}$$

We assume for the moment the proposition and prove the theorem.

Proof of Theorem 3.2. Set $\mathbf{\Omega}(r) := \log\big(\max\{\mathbf{I}(r), 1\}\big)$. To prove (3.1) it is enough to show $\mathbf{\Omega}(a) \leq C + \mathbf{\Omega}(b)$. If $\mathbf{\Omega}(a) = 0$, then there is nothing to prove. If $\mathbf{\Omega}(a) > 0$, let $b' \in]a, b]$ be the supremum of t such that $\mathbf{\Omega} > 0$ on $]a, t[$. If $b' < b$, then $\mathbf{\Omega}(b') = 0 \leq \mathbf{\Omega}(b)$. Therefore, by possibly substituting $]a, b[$ with $]a, b'[$, we can assume that $\mathbf{\Omega} > 0$, i.e. $\mathbf{I} > 1$, on $]a, b[$.

Set for simplicity $\mathbf{F}(r) := \mathbf{D}(r)^{-1} - r\mathbf{E}(r)^{-1}$, and compute

$$-\mathbf{\Omega}'(r) = \frac{\mathbf{H}'(r)}{\mathbf{H}(r)} - \frac{\mathbf{D}'(r)}{\mathbf{D}(r)} - \frac{1}{r} \stackrel{(3.6)}{=} \frac{\mathbf{H}'(r)}{\mathbf{H}(r)} - \frac{r\mathbf{D}'(r)}{\mathbf{E}(r)} - \mathbf{D}'(r)\mathbf{F}(r) - \frac{1}{r}.$$

By Proposition 3.5 (applicable because $\mathbf{I} > 1$), if ε_3 is sufficiently small, then

$$\frac{\mathbf{D}(r)}{2} \stackrel{(3.6)\,\&\,(3.8)}{\leq} \frac{\mathbf{E}(r)}{r} \stackrel{(3.6)\,\&\,(3.8)}{\leq} 2\,\mathbf{D}(r), \tag{3.9}$$

$$\frac{\mathbf{H}'(r)}{\mathbf{H}(r)} \stackrel{(3.5)}{\leq} \frac{m-1}{r} + C + \frac{2}{r}\,\frac{\mathbf{E}(r)}{\mathbf{H}(r)}, \tag{3.10}$$

$$|\mathbf{F}(r)| \stackrel{(3.6)}{\leq} C\,\frac{r(\mathbf{D}(r)^{1+\gamma_3} + \mathbf{\Sigma}(r))}{\mathbf{D}(r)\,\mathbf{E}(r)} \stackrel{(3.9)}{\leq} C\,\mathbf{D}(r)^{\gamma_3-1} + C\,\frac{\mathbf{\Sigma}(r)}{\mathbf{D}(r)^2}, \tag{3.11}$$

$$\begin{aligned}
-\frac{r\mathbf{D}'(r)}{\mathbf{E}(r)} &\stackrel{(3.7)}{\leq} \left(C - \frac{m-2}{r}\right)\frac{r\mathbf{D}(r)}{\mathbf{E}(r)} - \frac{2}{r}\,\frac{\mathbf{G}(r)}{\mathbf{E}(r)} + C\frac{r\mathbf{D}(r)^{\gamma_3}\mathbf{D}'(r) + \mathbf{D}(r)^{1+\gamma_3}}{\mathbf{E}(r)}\\
&\leq C - \frac{m-2}{r} + \frac{C}{r}\mathbf{D}(r)|\mathbf{F}(r)| - \frac{2}{r}\,\frac{\mathbf{G}(r)}{\mathbf{E}(r)} + Cr\mathbf{D}(r)^{\gamma_3-1}\mathbf{D}'(r) + C\frac{\mathbf{D}(r)^{\gamma_3}}{r}\\
&\stackrel{(3.8),\,(3.11)\,\&\,(3.4)}{\leq} C - \frac{m-2}{r} - \frac{2}{r}\,\frac{\mathbf{G}(r)}{\mathbf{E}(r)} + Cr\mathbf{D}(r)^{\gamma_3-1}\mathbf{D}'(r) + C\,r^{\gamma_3\,m-1}.
\end{aligned} \tag{3.12}$$

By Cauchy-Schwartz, we have

$$\frac{\mathbf{E}(r)}{r\mathbf{H}(r)} \leq \frac{\mathbf{G}(r)}{r\mathbf{E}(r)}. \tag{3.13}$$

Thus, by (3.4), (3.10), (3.12) and (3.13), we conclude

$$\begin{aligned}
-\mathbf{\Omega}'(r) &\leq C + C\,r^{\gamma_3\,m-1} + Cr\mathbf{D}(r)^{\gamma_3-1}\mathbf{D}'(r) - \mathbf{D}'(r)\mathbf{F}(r)\\
&\stackrel{(3.11)}{\leq} C\,r^{\gamma_3\,m-1} + C\mathbf{D}(r)^{\gamma_3-1}\mathbf{D}'(r) + C\frac{\mathbf{\Sigma}(r)\mathbf{D}'(r)}{\mathbf{D}(r)^2}.
\end{aligned} \tag{3.14}$$

Integrating (3.14) we conclude:

$$\mathbf{\Omega}(a) - \mathbf{\Omega}(b) \leq C + C\left(\mathbf{D}(b)^{\gamma_3} - \mathbf{D}(a)^{\gamma_3}\right) + C\left[\frac{\mathbf{\Sigma}(a)}{\mathbf{D}(a)} - \frac{\mathbf{\Sigma}(b)}{\mathbf{D}(b)} + \int_a^b \frac{\mathbf{\Sigma}'(r)}{\mathbf{D}(r)}\,dr\right] \stackrel{(3.8)}{\leq} C. \qquad \square$$

The rest of the section is devoted to the proof of Proposition 3.5.

3.1. **Estimates on $\mathbf{H}'$: proof of** (3.5)**.** Set $q := \mathbf{\Phi}(0)$. Let $\exp : B_3 \subset T_q\mathcal{M} \to \mathcal{M}$ be the exponential map and $\mathbf{J}\exp$ its Jacobian. Note that $d_{\mathcal{M}}(\exp(y), q) = |y|$ for every $y \in B_3$. By the area formula, setting $y = rz$, we can write $\mathbf{H}$ in the following way:

$$\mathbf{H}(r) = -r^{m-1}\int_{T_q\mathcal{M}} \frac{\phi'\,(|z|)}{|z|}\,|N|^2(\exp(rz))\,\mathbf{J}\exp(rz)\,dx$$

where the integration is always intended with respect to $\mathcal{H}^m$. Therefore, differentiating under the integral sign, we easily get (3.5):

$$\begin{aligned}
\mathbf{H}'(r) = &-(m-1)\,r^{m-2}\int_{T_q\mathcal{M}} \frac{\phi'(|z|)}{|z|}\,|N|^2(\exp(rz))\,\mathbf{J}\exp(rz)\,dz\\
&- 2\,r^{m-1}\int_{T_q\mathcal{M}} \phi'(|z|)\sum_i \langle N_i(\exp(rz)), \partial_{\hat r} N_i(\exp(rz))\rangle\;\mathbf{J}\exp(rz)\,dz\\
&- r^{m-1}\int_{T_q\mathcal{M}} \frac{\phi'(|z|)}{|z|}\,|N|^2(\exp(rz))\,\frac{d}{dr}\mathbf{J}\exp(rz)\,dz\\
= &\;\frac{m-1}{r}\,\mathbf{H}(r) + \frac{2}{r}\,\mathbf{E}(r) + O(1)\,\mathbf{H}(r),
\end{aligned}$$

where we used that $\frac{d}{dr}\mathbf{J}\exp(r\,z) = O(1)$, because $\mathcal{M}$ is a $C^{3,\kappa}$ submanifold and hence exp is a $C^{2,\kappa}$ map (see Proposition A.4). $\square$

3.2. **$\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$: proof of** (3.8)**.** We show the following more precise estimates.

Lemma 3.6. *There exists a dimensional constant $C > 0$ such that*

$$\mathbf{\Sigma}(r) \leq C\,r^2\,\mathbf{D}(r) + Cr\mathbf{H}(r) \quad \textit{and} \quad \mathbf{\Sigma}'(r) \leq C\mathbf{H}(r), \tag{3.15}$$

$$\int_{\mathcal{B}_r(q)} |N|^2 \leq C\,\mathbf{\Sigma}(r) + C\,r\,\mathbf{H}(r)\,, \tag{3.16}$$

$$\int_{\mathcal{B}_r(q)} |DN|^2 \leq C\,\mathbf{D}(r) + C\,r\mathbf{D}'(r). \tag{3.17}$$

In particular, if $\mathbf{I} \geq 1$, then (3.8) *holds and*

$$\int_{\mathcal{B}_r(q)} |N|^2 \leq C\,r^2\mathbf{D}(r). \tag{3.18}$$

Proof. Observe that $\psi(p) := \phi\big(\frac{d(p)}{r}\big)|N|^2(p)$ is a Lipschitz function with compact support in $\mathbf{B}_r(q)$. We therefore use the Poincaré inequality: $\mathbf{\Sigma}(r) = \int_{\mathcal{M}} \psi \leq Cr \int_{\mathcal{M}} |D\psi|$. The constant C depends on the smoothness of $\mathcal{M}$ and, therefore, not on the interval of flattening. We compute

$$\begin{aligned}\mathbf{\Sigma}(r) &\leq -C \int_{\mathcal{M}} \phi'(r^{-1}d(p))|N|^2(p) + C\,r \int_{\mathcal{M}} \phi(r^{-1}d(p))|N||DN| \\ &\leq Cr\mathbf{H}(r) + C\mathbf{\Sigma}(r)^{1/2}\left(r^2\mathbf{D}(r)\right)^{1/2} \leq Cr\mathbf{H}(r) + \tfrac{1}{2}\mathbf{\Sigma}(r) + C\,r^2\mathbf{D}(r),\end{aligned}$$

which gives the first part of (3.15). The remaining inequality is straightforward:

$$\mathbf{\Sigma}'(r) = -\int_{\mathcal{M}} \frac{d(p)}{r^2}\,\phi'\left(\frac{d(p)}{r}\right)|N|^2(p) \leq C\mathbf{H}(r)\,.$$

Since $\phi' = 0$ on $]0, \frac{1}{2}[$ and $\phi' = -2$ on $]\frac{1}{2}, 1[$, we easily deduce

$$\int_{\mathcal{B}_r(q)\setminus\mathcal{B}_{r/2}(q)} |N|^2 \leq r\,\mathbf{H}(r),$$

$$r\mathbf{D}'(r) = -\int \frac{d(p)}{r}\phi'\left(\frac{d(p,q)}{r}\right)|DN|^2 \geq \int_{\mathcal{B}_r(q)\setminus\mathcal{B}_{r/2}(q)} |DN|^2\,.$$

On the other hand, since $\phi = 1$ on $[0, \frac{1}{2}]$, (3.16) and (3.17) readily follow. Therefore, in the hypothesis $\mathbf{I} \geq 1$, i.e. $\mathbf{H} \leq r\mathbf{D}$, we conclude (3.8) from (3.15). □

3.3. **First variations.** To prove the remaining estimates in Proposition 3.5 we exploit the first variation of T along some vector fields X. The variations are denoted by $\delta T(X)$. We fix a neighborhood $\mathbf{U}$ of $\mathcal{M}$ and the normal projection $\mathbf{p} : \mathbf{U} \to \mathcal{M}$ as in [5, Assumption 2.1]. By [5, Theorem 2.4] $\mathbf{p} \in C^{2,\kappa}$ and [3, Assumption 3.1] holds. We will consider two types of variations:

- the *outer variations*, where $X(p) = X_o(p) := \phi\left(\frac{d(\mathbf{p}(p))}{r}\right)(p - \mathbf{p}(p))$.
- the *inner variations*, where $X(p) = X_i(p) := Y(\mathbf{p}(p))$ with

$$Y(p) := \frac{d(p)}{r}\,\phi\left(\frac{d(p)}{r}\right)\frac{\partial}{\partial \hat{r}} \quad \forall\, p \in \mathcal{M}.$$

Note that X_i is the infinitesimal generator of a one parameter family of diffeomorphisms Φ_ε defined as $\Phi_\varepsilon(p) := \Psi_\varepsilon(\mathbf{p}(p)) + p - \mathbf{p}(p)$, where Ψ_ε is the one-parameter family of biLipschitz homeomorphisms of $\mathcal{M}$ generated by Y.

Consider now the map $F(p) := \sum_i [\![p + N_i(p)]\!]$ and the current $\mathbf{T}_F$ associated to its image (cf. [3] for the notation). Observe that X_i and X_o are supported in $\mathbf{p}^{-1}(\mathcal{B}_r(q))$ but none of them is *compactly* supported. However, recalling Proposition 2.2 (v) and the minimizing property of T in Σ, we deduce that $\delta T(X) = \delta T(X^T) + \delta T(X^\perp) = \delta T(X^\perp)$, where $X = X^T + X^\perp$ is the decomposition of X in the tangent and normal components to

$T\Sigma$. Then, we have

$$
\begin{aligned}
&|\delta \mathbf{T}_F(X)| \leq |\delta \mathbf{T}_F(X) - \delta T(X)| + |\delta T(X^\perp)| \\
&\leq \underbrace{\int_{\mathrm{spt}(T)\setminus \mathrm{Im}(F)} |\mathrm{div}_{\vec{T}} X| \; d\|T\| + \int_{\mathrm{Im}(F)\setminus \mathrm{spt}(T)} \left|\mathrm{div}_{\vec{\mathbf{T}}_F} X\right| \; d\|\mathbf{T}_F\|}_{\mathrm{Err}_4} + \underbrace{\left|\int \mathrm{div}_{\vec{T}} X^\perp \, d\|T\|\right|}_{\mathrm{Err}_5}. \qquad (3.19)
\end{aligned}
$$

Set now for simplicity $\varphi_r(p) := \phi\big(\frac{d(p)}{r}\big)$. We wish to apply [3, Theorem 4.2] to conclude

$$
\delta \mathbf{T}_F(X_o) = \int_{\mathcal{M}} \Big(\varphi_r \, |DN|^2 + \sum_{i=1}^{Q} N_i \otimes \nabla \varphi_r : DN_i\Big) + \sum_{j=1}^{3} \mathrm{Err}_j^o, \qquad (3.20)
$$

where the errors Err_j^o correspond to the error terms $\mathrm{Err_j}$ of [3, Theorem 4.2]. This would imply

$$
\mathrm{Err}_1^o = -Q \int_{\mathcal{M}} \varphi_r \langle H_{\mathcal{M}}, \boldsymbol{\eta} \circ N \rangle, \qquad (3.21)
$$

$$
|\mathrm{Err}_2^o| \leq C \int_{\mathcal{M}} |\varphi_r| |A|^2 |N|^2, \qquad (3.22)
$$

$$
|\mathrm{Err}_3^o| \leq C \int_{\mathcal{M}} \Big(|\varphi_r| \big(|DN|^2 |N| |A| + |DN|^4\big) + |D\varphi_r| \big(|DN|^3 |N| + |DN| |N|^2 |A|\big)\Big), \qquad (3.23)
$$

where $H_{\mathcal{M}}$ is the mean curvature vector of $\mathcal{M}$. Note that [3, Theorem 4.2] requires the C^1 regularity of φ_r. We overcome this technical obstruction applying [3, Theorem 4.2] to a standard smoothing of ϕ and then passing into the limit (the obvious details are let to the reader). Plugging (3.20) into (3.19), we then conclude

$$
\big|D(r) - r^{-1} E(r)\big| \leq \sum_{j=1}^{5} \big|\mathrm{Err}_j^o\big| \,, \qquad (3.24)
$$

where Err_4^o and Err_5^o correspond respectively to Err_4 and Err_5 of (3.19) when $X = X_o$. With the same argument, but applying this time [3, Theorem 4.3] to $X = X_i$, we get

$$
\delta \mathbf{T}_F(X_i) = \frac{1}{2} \int_{\mathcal{M}} \Big(|DN|^2 \mathrm{div}_{\mathcal{M}} Y - 2 \sum_{i=1}^{Q} \langle DN_i : (DN_i \cdot D_{\mathcal{M}} Y) \rangle\Big) + \sum_{j=1}^{3} \mathrm{Err}_j^i \,, \qquad (3.25)
$$

where this time the errors Err_j^i correspond to the error terms Err_j of [3, Theorem 4.3], i.e.

$$
\mathrm{Err}_1^i = -Q \int_{\mathcal{M}} \big(\langle H_{\mathcal{M}}, \boldsymbol{\eta} \circ N \rangle \, \mathrm{div}_{\mathcal{M}} Y + \langle D_Y H_{\mathcal{M}}, \boldsymbol{\eta} \circ N \rangle\big) \,, \qquad (3.26)
$$

$$
|\mathrm{Err}_2^i| \leq C \int_{\mathcal{M}} |A|^2 \left(|DY| |N|^2 + |Y| |N| \, |DN|\right) , \qquad (3.27)
$$

$$
|\mathrm{Err}_3^i| \leq C \int_{\mathcal{M}} \Big(|Y| |A| |DN|^2 \big(|N| + |DN|\big) + |DY| \big(|A| \, |N|^2 |DN| + |DN|^4\big)\Big) . \qquad (3.28)
$$

Straightforward computations (again appealing to Proposition A.4) lead to

$$D_{\mathcal{M}} Y(p) = \phi'\left(\frac{d(p)}{r}\right) \frac{d(p)}{r^2} \frac{\partial}{\partial \hat{r}} \otimes \frac{\partial}{\partial \hat{r}} + \phi\left(\frac{d(p)}{r}\right)\left(\frac{\mathrm{Id}}{r} + O(1)\right), \tag{3.29}$$

$$\mathrm{div}_{\mathcal{M}}\, Y(p) = \phi'\left(\frac{d(p)}{r}\right) \frac{d(p)}{r^2} + \phi\left(\frac{d(p)}{r}\right)\left(\frac{m-1}{r} + O(1)\right). \tag{3.30}$$

Plugging (3.29) and (3.30) into (3.25) and using (3.19) we then conclude

$$\left|\mathbf{D}'(r) - (m-2) r^{-1} \mathbf{D}(r) - 2 r^{-2} \mathbf{G}(r)\right| \leq C \mathbf{D}(r) + \sum_{j=1}^{5} \left|\mathrm{Err}_j^i\right|. \tag{3.31}$$

Proposition 3.5 is then proved by the estimates of the errors terms done in the next section.

4. Error estimates

We start with some preliminary considerations.

4.1. **Family of subregions.** Set $q := \mathbf{\Phi}(0)$. We select a family of subregions of $\mathcal{B}_r(p) \subset \mathcal{M}$. Denote by B and ∂B respectively $\mathbf{p}_\pi(\mathcal{B}_r(q))$ and $\mathbf{p}_\pi(\partial \mathcal{B}_r(q))$, where π is the reference m-dimensional plane of the construction of the center manifold $\mathcal{M}$. Since $\|\boldsymbol{\varphi}\|_{C^{3,\kappa}} \leq C \varepsilon_3^{1/m}$ (cf. [5, Theorem 1.12]), by Proposition A.4 we can assume that B is a C^2 convex set and that the maximal curvature of ∂B is everywhere smaller than $\frac{2}{r}$. Thus:

$$\forall z \in B \quad \text{there is a ball } B_{r/2}(y, \pi) \subset B \text{ whose closure touches } \partial B \text{ at } z. \tag{4.1}$$

Definition 4.1 (Family of cubes)**.** We first define a family $\mathcal{T}$ of cubes in the Whitney decomposition $\mathscr{W}$ as follows:

(i) $\mathcal{T}$ includes all $L \in \mathscr{W}_h \cup \mathscr{W}_e$ which intersect B;
(ii) if $L' \in \mathscr{W}_n$ intersects B and belongs to the domain of influence $\mathscr{W}_n(L)$ of the cube $L \in \mathscr{W}_e$ as in [5, Corollary 3.2], then $L \in \mathcal{T}$.

Definition 4.2 (Associated balls B^L)**.** By Proposition 2.2 (iii), $\ell(L) \leq 8 c_s r \leq r$ and $\mathrm{dist}(L, B) \leq 2\sqrt{m}\, \ell(L)$ for each $L \in \mathcal{T}$. Let x_L be the center of L and:

(a) if $x_L \in \overline{B}$, we then set $s(L) := \ell(L)$ and $B^L := B_{s(L)}(x_L, \pi)$;
(b) otherwise we consider the ball $B_{r(L)}(x_L, \pi) \subset \pi$ whose closure touches $\overline{B}$ at exactly one point $p(L)$, we set $s(L) := r(L) + \ell(L)$ and define $B^L := B_{s(L)}(x_L, \pi)$.

We proceed to select a countable family $\mathscr{T}$ of pairwise disjoint balls $\{B^L\}$. We let $S := \sup_{L \in \mathcal{T}} s(L)$ and start selecting a maximal subcollection $\mathscr{T}_1$ of pairwise disjoint balls with radii larger than $S/2$. Clearly, $\mathscr{T}_1$ is finite. In general, at the stage k, we select a maximal subcollection $\mathscr{T}_k$ of pairwise disjoint balls which do not intersect any of the previously selected balls in $\mathscr{T}_1 \cup \ldots \cup \mathscr{T}_{k-1}$ and which have radii $r \in]2^{-k} S, 2^{1-k} S]$. Finally, we set $\mathscr{T} := \bigcup_k \mathscr{T}_k$.

Definition 4.3 (Family of pairs cube-balls $(L, B(L)) \in \mathscr{Z}$)**.** Recalling (4.1) and $\ell(L) \le r$, it easy to see that there exist balls $B_{\ell(L)/4}(q_L, \pi) \subset B^L \cap B$ which lie at distance at least $\ell(L)/4$ from ∂B. We denote by $B(L)$ one of such balls and by $\mathscr{Z}$ the collection of pairs $(L, B(L))$ with $B^L \in \mathscr{T}$.

Next, we partition the cubes of $\mathscr{W}$ which intersect B into disjoint families $\mathscr{W}(L)$ labelled by $(L, B(L)) \in \mathscr{Z}$ in the following way. Let $H \in \mathscr{W}$ have nonempty intersection with B. Then, either H is in $\mathcal{T}$, or is in the domain of influence of some $J \in \mathcal{T}$. By Proposition 2.2, the distance between J and H is at most $2\sqrt{m}\ell(J)$ and, hence, $H \subset B_{3\sqrt{m}\ell(J)}(x_J)$. By construction there is a $B^L \in \mathscr{T}$ with $B^J \cap B^L \neq \emptyset$ and radius $s(L) \ge \frac{s(J)}{2}$. We then prescribe $H \in \mathscr{W}(L)$. Observe that $s(L) \le 4\sqrt{m}\ell(L)$ and $s(J) \ge \ell(J)$. Therefore, $\ell(J) \le 8\sqrt{m}\ell(L)$ and $|x_J - x_L| \le 5s(L) \le 20\sqrt{m}\ell(L)$. This implies that

$$H \subset B_{3\sqrt{m}\ell(J)}(x_J) \subset B_{3\sqrt{m}\ell(J)+20\sqrt{m}\ell(L)}(x_L) \subset B_{30\sqrt{m}\ell(L)}(x_L)\,.$$

For later reference, we collect the main properties of the above constructions in the following lemma.

Lemma 4.4. *The following holds.*

(i) *If $(L, B(L)) \in \mathscr{Z}$, then $L \in \mathscr{W}_e \cup \mathscr{W}_h$, the radius of $B(L)$ is $\frac{\ell(L)}{4}$, $B(L) \subset B^L \cap B$ and* $\mathrm{dist}\,(B(L), \partial B) \ge \frac{\ell(L)}{4}$.

(ii) *If the pairs $(L, B(L)), (L', B(L')) \in \mathscr{Z}$ are distinct, then L and L' are distinct and $B(L) \cap B(L') = \emptyset$.*

(iii) *The cubes $\mathscr{W}$ which intersect B are partitioned into disjoint families $\mathscr{W}(L)$ labeled by $(L, B(L)) \in \mathscr{Z}$ such that, if $H \in \mathscr{W}(L)$, then $H \subset B_{30\sqrt{m}\ell(L)}(x_L)$.*

4.2. **Basic estimates in the subregions.** For notational convenience, we order the family $\mathscr{Z} = \{(J_i, B(J_i))\}_{i\in\mathbb{N}}$, and set

$$\mathcal{B}^i := \mathbf{\Phi}(B(J_i)) \qquad \mathcal{U}_i = \cup_{H\in\mathscr{W}(J_i)}\mathbf{\Phi}(H) \cap \mathcal{B}_r(q)\,.$$

Observe that the distance between $\mathcal{B}^i$ and $\partial\mathcal{B}_r(q)$ is larger than that between $B(L_i)$ and $\partial B = \mathbf{p}_\pi(\partial\mathcal{B}_r(q))$. Thus, by Lemma 4.4 (i), $\varphi_r(p) = \phi\big(\frac{d(p)}{r}\big)$ satisfies

$$\inf_{p\in\mathcal{B}^i} \varphi_r(p) \ge (4r)^{-1}\ell_i\,, \tag{4.2}$$

where $\ell_i := \ell(J_i)$. From this and Lemma 4.4 (iii), we also obtain

$$\sup_{p\in\mathcal{U}_i}\varphi_r(p) - \inf_{p\in\mathcal{U}_i}\varphi_r(p) \le C\mathrm{Lip}(\varphi_r)\ell_i \le \frac{C}{r}\,\ell_i \stackrel{(4.2)}{\le} C\inf_{p\in\mathcal{B}_i}\varphi_r(p)\,,$$

which translates into

$$\sup_{p\in\mathcal{U}_i}\varphi_r(p) \le C\inf_{p\in\mathcal{B}^i}\varphi_r(p)\,. \tag{4.3}$$

Moreover, set $\mathcal{V}_i := \mathcal{U}_i \setminus \mathcal{K}$, where $\mathcal{K}$ is the coincidence set of [5, Theorem 2.4]. From [5, Theorem 2.4], we derive the following estimates:

$$\int_{\mathcal{U}_i} |\boldsymbol{\eta} \circ N| \leq C \boldsymbol{m}_0\, \ell_i^{2+m+\gamma_2} + C \int_{\mathcal{U}^i} |N|^{2+\gamma_2}, \tag{4.4}$$

$$\int_{\mathcal{U}_i} |DN|^2 \leq C \boldsymbol{m}_0\, \ell_i^{m+2-2\delta_2}, \tag{4.5}$$

$$\|N\|_{C^0(\mathcal{U}_i)} + \sup_{p \in \mathrm{spt}(T) \cap \mathbf{p}^{-1}(\mathcal{U}_i)} |p - \mathbf{p}(p)| \leq C \boldsymbol{m}_0^{1/2m} \ell_i^{1+\beta_2}, \tag{4.6}$$

$$\mathrm{Lip}(N|_{\mathcal{U}_i}) \leq C \boldsymbol{m}_0^{\gamma_2} \ell_i^{\gamma_2}, \tag{4.7}$$

$$\mathbf{M}(T \llcorner \mathbf{p}^{-1}(\mathcal{V}_i)) + \mathbf{M}(\mathbf{T}_F \llcorner \mathbf{p}^{-1}(\mathcal{V}_i)) \leq C \boldsymbol{m}_0^{1+\gamma_2} \ell_i^{m+2+\gamma_2}. \tag{4.8}$$

To prove these estimates, observe first that $\sum_{H \in \mathscr{W}(J_i)} \ell(H)^m \leq C \ell_i^m$, because all $H \in \mathscr{W}(J_i)$ are disjoint and contained in a ball of radius comparable to ℓ_i. This in turn implies that $\sum_{H \in \mathscr{W}(J_i)} \ell(H)^{m+\varepsilon} \leq C \ell_i^{m+\varepsilon}$, because $\ell(H) \leq \ell_i$ for any $H \in \mathscr{W}(L)$. Thus:

- (4.4) follows summing the estimate of [5, Theorem 2.4 (2.4)] applied with $a = 1$ to $\mathbf{\Phi}(H)$ with $H \in \mathscr{W}(J_i)$;
- (4.5) follows from summing the estimate of [5, Theorem 2.4 (2.3)] applied to $\mathbf{\Phi}(H)$ with $H \in \mathscr{W}(J_i)$;
- (4.6) follows from [5, Theorem 2.4 (2.1)] and [5, Corollary 2.2 (ii)];
- (4.7) follows from [5, Theorem 2.4 (2.1)];
- (4.8) follows summing [5, Theorem 2.4 (2.2)] applied to $\mathbf{\Phi}(H)$ with $H \in \mathscr{W}(J_i)$.

The last ingredient for the completion of the proof of Proposition 3.5 are the following three key estimates which are derived from the analysis of the construction of the center manifold in [5].

Lemma 4.5. *Under the assumptions of Proposition 3.5, it holds*

$$\sum_i \big(\inf_{\mathcal{B}^i} \varphi_r\big)\, \boldsymbol{m}_0\, \ell_i^{m+2+\gamma_2/4} \leq C \mathbf{D}(r), \tag{4.9}$$

$$\sum_i \boldsymbol{m}_0\, \ell_i^{m+2+\gamma_2/4} \leq C \left(\mathbf{D}(r) + r \mathbf{D}'(r)\right). \tag{4.10}$$

Moreover, for every $t > 0$ there exists $C(t) > 0$ and $\gamma(t) > 0$ such that

$$\sum_i \boldsymbol{m}_0^t \Big[\ell_i{}^t + \Big(\inf_{\mathcal{B}^i} \varphi_r\Big)^{t/2} \ell_i^{t/2}\Big] \leq C(t) \mathbf{D}(r)^{\gamma(t)}. \tag{4.11}$$

Proof. Recall that, from [5, Propositions 3.1 and 3.3], we have

$$\int_{\mathcal{B}^i} |N|^2 \geq c\, \boldsymbol{m}_0^{1/m} \ell_i^{m+2+2\beta_2} \quad \text{if } L_i \in \mathscr{W}_h, \tag{4.12}$$

$$\int_{\mathcal{B}^i} |DN|^2 \geq c\, \boldsymbol{m}_0\, \ell_i^{m+2-2\delta_2} \quad \text{if } L_i \in \mathscr{W}_e\,. \tag{4.13}$$

Therefore, by Lemma 3.6, (4.2), (4.12) and (4.13), it follows easily that, for suitably chosen $\gamma(t), C(t) > 0$,

$$\sum_i \boldsymbol{m}_0^t \Big[\ell_i^t + \Big(\inf_{\mathcal{B}^i} \varphi_r\Big)^{t/2} \ell_i^{t/2}\Big] \leq C(t) \sum_i \Big(\int_{\mathcal{B}^i} \varphi_r(|DN|^2 + |N|^2)\Big)^{\gamma(t)} \overset{(3.18)\,\&\,\mathbf{I}\geq 1}{\leq} C(t)\mathbf{D}(r)^{\gamma(t)}.$$

Next, (4.9) follows similarly because the $\mathcal{B}^i$ are disjoint and $8\,\beta_2 < \gamma_2$:

$$\sum_i \big(\inf_{\mathcal{B}^i} \varphi_r\big)\boldsymbol{m}_0\, \ell_i^{m+2+\gamma_2/4} \leq C \sum_i \int_{\mathcal{B}^i} \varphi_r(|DN|^2 + |N|^2) \overset{(3.18)\,\&\,\mathbf{I}\geq 1}{\leq} C\mathbf{D}(r).$$

Finally, arguing as above we conclude that

$$\sum_i \boldsymbol{m}_0\, \ell_i^{m+2+\gamma_2/4} \leq C \int_{\mathcal{B}_r(q)} \big(|DN|^2 + |N|^2\big) \overset{(3.17)\,\&\,(3.18)}{\leq} C\big(\mathbf{D}(r) + r\mathbf{D}'(r)\big). \qquad \square$$

4.3. **Proof of Proposition 3.5:** (3.6) **and** (3.7)**.** We can now pass to estimate the errors terms in (3.6) and (3.7) in order to conclude the proof of Proposition 3.5.

Errors of type 1. By [5, Theorem 1.12], the map $\boldsymbol{\varphi}$ defining the center manifold satisfies $\|D\boldsymbol{\varphi}\|_{C^{2,\kappa}} \leq C\,\boldsymbol{m}_0^{1/2}$, which in turn implies $\|H_{\mathcal{M}}\|_{L^\infty} + \|DH_{\mathcal{M}}\|_{L^\infty} \leq C\,\boldsymbol{m}_0^{1/2}$ (recall that $H_{\mathcal{M}}$ denotes the mean curvature of $\mathcal{M}$). Therefore, by (4.3), (4.4), (4.9) and (4.11), we get

$$\begin{aligned}
|\mathrm{Err}_1^o| &\leq C \int_{\mathcal{M}} \varphi_r\, |H_{\mathcal{M}}|\, |\boldsymbol{\eta} \circ N| \\
&\leq C\,\boldsymbol{m}_0^{1/2} \sum_j \Big(\big(\sup_{\mathcal{U}_i} \varphi_r\big)\, \boldsymbol{m}_0\, \ell_j^{2+m+\gamma_2} + C \int_{\mathcal{U}_j} \varphi_r |N|^{2+\gamma_2}\Big) \\
&\leq C\mathbf{D}(r)^{1+\gamma} + C \sum_j \boldsymbol{m}_0^{1/2}\, \ell_j^{\gamma_2(1+\beta_2)} \int_{\mathcal{U}_j} \varphi_r |N|^2 \leq C\mathbf{D}(r)^{1+\gamma_3},
\end{aligned}$$

and analogously

$$\begin{aligned}
\big|\mathrm{Err}_1^i\big| &\leq C\, r^{-1} \int_{\mathcal{M}} \big(|H_{\mathcal{M}}| + |D_Y H\mathcal{M}|\big) |\boldsymbol{\eta} \circ N| \\
&\leq C\, r^{-1} \boldsymbol{m}_0^{1/2} \sum_j \Big(\boldsymbol{m}_0\, \ell_j^{2+m+\gamma_2} + C \int_{\mathcal{U}_j} |N|^{2+\gamma_2}\Big) \leq C\, r^{-1} \mathbf{D}(r)^{\gamma} \big(\mathbf{D}(r) + r\,\mathbf{D}'(r)\big).
\end{aligned}$$

Errors of type 2. From $\|A\|_{C^0} \leq C\|D\boldsymbol{\varphi}\|_{C^2} \leq C\boldsymbol{m}_0^{1/2} \leq C\varepsilon_3$, it follows that $\mathrm{Err}_2^o \leq C\varepsilon_3^2 \boldsymbol{\Sigma}(r)$. Moreover, since $|DX_i| \leq Cr^{-1}$, Lemma 3.6 gives

$$\big|\mathrm{Err}_2^i\big| \leq Cr^{-1} \int_{\mathcal{B}_r(p_0)} |N|^2 + C \int \varphi_r |N||DN| \leq C\mathbf{D}(r).$$

Errors of type 3. Clearly, we have

$$|\mathrm{Err}_3^o| \leq \underbrace{\int \varphi_r\left(|DN|^2|N| + |DN|^4\right)}_{I_1} + \underbrace{C\, r^{-1}\int_{\mathcal{B}_r(q)} |DN|^3|N|}_{I_2} + \underbrace{C\, r^{-1}\int_{\mathcal{B}_r(q)} |DN||N|^2}_{I_3}\,.$$

We estimate separately the three terms (recall that $\gamma_2 > 4\delta_2$):

$$\begin{aligned}
I_1 &\leq \int_{\mathcal{B}_r(p_0)} \varphi_r(|N|^2|DN| + |DN|^3) \leq I_3 + C\sum_j \sup_{\mathcal{U}_j} \varphi_r \boldsymbol{m}_0^{1+\gamma_2}\ell_j^{m+2+\gamma_2/2}\\
&\stackrel{(4.9)\,\&\,(4.11)}{\leq} I_3 + C\mathbf{D}(r)^{1+\gamma_3},\\
I_2 &\leq Cr^{-1}\sum_j \boldsymbol{m}_0^{1+1/2m+\gamma_2}\ell_j^{m+3+\beta_2+\gamma_2/2} \stackrel{(4.3)}{\leq} C\sum_j \boldsymbol{m}_0^{1+1/2m+\gamma_2}\ell_j^{m+2+\beta_2+\gamma_2/2}\inf_{\mathcal{B}^j}\varphi_r\\
&\stackrel{(4.9)\,\&\,(4.11)}{\leq} C\mathbf{D}(r)^{1+\gamma_3},\\
I_3 &\leq Cr^{-1}\sum_j \boldsymbol{m}_0^{\gamma_2}\ell_j^{\gamma_2}\int_{\mathcal{U}_j}|N|^2 \stackrel{(4.11)}{\leq} Cr^{-1}\mathbf{D}(r)^{\gamma_3}\int_{\mathcal{B}_r(q)}|N|^2 \stackrel{(3.18)}{\leq} C\mathbf{D}(r)^{1+\gamma_3}.
\end{aligned}$$

For what concerns the inner variations, we have

$$|\mathrm{Err}_3^i| \leq C\int_{\mathcal{B}_r(q)}\left(r^{-1}|DN|^3 + r^{-1}|DN|^2|N| + r^{-1}|DN||N|^2\right).$$

The last integrand corresponds to I_3, while the remaining part can be estimated as follows:

$$\begin{aligned}
\int_{\mathcal{B}_r(q)} r^{-1}(|DN|^3 + |DN|^2|N|) &\leq C\sum_j r^{-1}(\boldsymbol{m}_0^{\gamma_2}\ell_j^{\gamma_2} + \boldsymbol{m}_0^{1/2m}\ell_j^{1+\beta_2})\int_{\mathcal{U}_j}|DN|^2\\
&\stackrel{(4.11)}{\leq} C\, r^{-1}\mathbf{D}(r)^{\gamma}\int_{\mathcal{B}_r(q)}|DN|^2\\
&\leq C\mathbf{D}(r)^{\gamma_3}\left(\mathbf{D}'(r) + r^{-1}\mathbf{D}(r)\right).
\end{aligned}$$

Errors of type 4. We compute explicitly

$$|DX_o(p)| \leq 2\,|p-\mathbf{p}(p)|\,\frac{|Dd(\mathbf{p}(p),q)|}{r} + \varphi_r(p)\,|D(p-\mathbf{p}(p))| \leq C\left(\frac{|p-\mathbf{p}(p)|}{r} + \varphi_r(p)\right).$$

It follows readily from (3.19), (4.6) and (4.8) that

$$\begin{aligned}
|\mathrm{Err}_4^o| &\leq \sum_i C\Big(r^{-1}\boldsymbol{m}_0^{1/2m}\ell_i^{1+\beta_2} + \sup_{\mathcal{U}_i}\varphi_r\Big)\boldsymbol{m}_0^{1+\gamma_2}\ell_i^{m+2+\gamma_2}\\
&\stackrel{(4.2)\,\&\,(4.3)}{\leq} C\sum_i\left[\boldsymbol{m}_0^{\gamma_2}\ell_i^{\gamma_2/4}\right]\inf_{\mathcal{B}_i}\varphi_r\,\boldsymbol{m}_0\,\ell_i^{m+2+\gamma_2/4} \stackrel{(4.9)\,\&(4.11)}{\leq} C\mathbf{D}(r)^{1+\gamma_3}.
\end{aligned} \tag{4.14}$$

Similarly, since $|DX_i| \leq Cr^{-1}$, we get

$$\mathrm{Err}_4^i \leq Cr^{-1} \sum_j \Big(\boldsymbol{m}_0^{\gamma_2} \ell_j^{\gamma_2/2}\Big) \boldsymbol{m}_0\, \ell_j^{m+2+\gamma_2/2} \overset{(4.10)\,\&\,(4.11)}{\leq} C\mathbf{D}(r)^\gamma \left(\mathbf{D}'(r) + r^{-1}\mathbf{D}(r)\right) .$$

Errors of type 5. Integrating by part Err_5, we get

$$\mathrm{Err}_5 = \left|\int \langle X^\perp, h(\vec{T}(p))\rangle d\|T\|\right| \leq \underbrace{\left|\int \langle X^\perp, h(\vec{\mathbf{T}}_F(p))\rangle d\|\mathbf{T}_F\|\right|}_{I_2}$$
$$+ \underbrace{\int_{\mathrm{spt}(T)\setminus \mathrm{Im}(F)} |X^\perp||h(\vec{T}(p))|d\|T\| + \int_{\mathrm{Im}(F)\setminus \mathrm{spt}(T)} |X^\perp||h(\vec{\mathbf{T}}_F(p)|d\|\mathbf{T}_F\|,}_{I_1}$$

where $h(\vec{\lambda})$ is the trace of A_Σ on the m-vector $\vec{\lambda}$, i.e. $h(\vec{\lambda}) := \sum_{k=1}^m A_\Sigma(v_k, v_k)$ with $v_1, \ldots, v_m$ is orthonormal vectors such that $v_1 \wedge \ldots \wedge v_m = \vec{\lambda}$.

Since $|X| \leq C$, I_1 can be easily estimated as Err_4:

$$I_2 \leq C \sum_j (\sup_{\mathcal{U}_i} \varphi_r) \boldsymbol{m}_0^{1+\gamma_2}\, \ell_j^{m+2+\gamma_2} \leq C\mathbf{D}^{1+\gamma}(r).$$

For what concerns I_2, we argue differently for the outer and the inner variations. For Err_5^o, observe that $|X^{o\perp}(p)| = \varphi_r(\mathbf{p}(p))|\mathbf{p}_{T_p\Sigma^\perp}(p - \mathbf{p}(p))|$. On the other hand, we also have

$$|\mathbf{p}_{T_p\Sigma^\perp}(p - \mathbf{p}(p))| \leq C\mathbf{c}(\Sigma)|p - \mathbf{p}(p)|^2 \leq C\boldsymbol{m}_0^{1/2}|p - \mathbf{p}(p)|^2 \quad \forall\, p \in \Sigma.$$

Therefore, we can estimate

$$I_2^o \leq C\, \boldsymbol{m}_0 \int \varphi_r |N|^2 \leq C\, \varepsilon_3^2 \mathbf{\Sigma}(r).$$

For the inner variations, denote by $\nu_1, \ldots, \nu_l$ an orthonormal frame for $T_p\Sigma^\perp$ of class C^{2,ε_0} (cf. [3, Appendix A]) and set $h_p^j(\vec{\lambda}) := -\sum_{k=1}^m \langle D_{v_k}\nu_j(p), v_k\rangle$ whenever $v_1 \wedge \ldots \wedge v_m = \vec{\lambda}$ is an m-vector of $T_p\Sigma$ (with $v_1, \ldots, v_m$ orthonormal). For the sake of simplicity, we write

$$h_p^j := h_p^j(\vec{\mathbf{T}}_F(p)) \quad \text{and} \quad h_p = \sum_{j=1}^l h_p^j \nu_j(p),$$

$$h_{\mathbf{p}(p)}^j := h_{\mathbf{p}(p)}^j(\vec{\mathcal{M}}) \quad \text{and} \quad h_{\mathbf{p}(p)} = \sum_{j=1}^l h_{\mathbf{p}(p)}^j \nu_j(\mathbf{p}(p)).$$

By the C^2 regularity of ν_j, we deduce that

$$h_p - h_{\mathbf{p}(p)} = \sum_j \nu_j(p)(h_p^j - h_{\mathbf{p}(p)}^j) + \sum_j \big(\nu_j(p) - \nu_j(\mathbf{p}(p))\big) h_{\mathbf{p}(p)}^j$$
$$= \sum_j \nu_j(p)(h_p^j - h_{\mathbf{p}(p)}^j) + \sum_j D\nu_j(\mathbf{p}(p)) \cdot (p - \mathbf{p}(p)) h_{\mathbf{p}(p)}^j + O(|p - \mathbf{p}(p)|^2). \tag{4.15}$$

On the other hand, $X^i(\mathbf{p}(p)) = Y(\mathbf{p}(p))$ is tangent to $\mathcal{M}$ in $\mathbf{p}(p)$ and hence orthogonal to $h_{\mathbf{p}(p)}$. Thus

$$\langle X^i(p), h_p\rangle = \langle X^i(p), (h_p - h_{\mathbf{p}(p)})\rangle = \sum_j \langle X^i(\mathbf{p}(p)), D\nu_j(\mathbf{p}(p))\cdot(p-\mathbf{p}(p))\rangle h^j_{\mathbf{p}(p)}$$
$$+ \sum_j \langle \nu_j(p), X^i(p)\rangle \big(h^j_p - h^j_{\mathbf{p}(p)}\big) + O\left(|p-\mathbf{p}(p)|^2\right)$$
$$= \sum_j \langle X^i(\mathbf{p}(p)), D\nu_j(\mathbf{p}(p))\cdot(p-\mathbf{p}(p))\rangle h^j_{\mathbf{p}(p)} + O\left(|\vec{T}(p) - \vec{\mathcal{M}}||p-\mathbf{p}(p)| + |p-\mathbf{p}(p)|^2\right), \tag{4.16}$$

where we used elementary calculus to infer that $|\langle X^i(p), \nu_j(p)\rangle| \leq C|p-\mathbf{p}(p)|$ and

$$|h^j_p - h^j_{\mathbf{p}(p)}| \leq C\left(|\vec{T}(p) - \vec{\mathcal{M}}| + |p-\mathbf{p}(p)|\right).$$

Using (4.16) and the expansion of the area functional in [3, Theorem 3.2], we can now conclude the estimate on I^i_2:

$$I^i_2 = \left|\int \langle X^i, h_p\rangle d\|\mathbf{T}_F\|\right| = \left|\sum_{i=1}^Q \int_{\mathcal{M}} \langle Y, h_{F_i}\rangle \mathbf{J}F_i\right|$$
$$\stackrel{(4.16)}{\leq} Q\left|\int_{\mathcal{M}} \sum_{j=1}^l \langle Y, D\nu_j\cdot(\boldsymbol{\eta}\circ N)\rangle h^j_{\mathbf{p}(\cdot)}\right| + C\int_{\mathcal{M}} \varphi_r(|N|^2 + |DN|^2)$$
$$\leq C\,\boldsymbol{m}_0 \int_{\mathcal{M}} \varphi_r\,|\boldsymbol{\eta}\circ N| + C\int_{\mathcal{M}} \varphi_r(|N|^2+|DN|^2) =: J_1 + J_2,$$

where we used that $|\langle Y, D\nu_j\rangle| \leq C\varphi_r \|A_\Sigma\|_{C^0} \leq C\varphi_r\,\boldsymbol{m}_0^{1/2}$ and $|h^j_{\mathbf{p}(p)}| \leq C\|A_\Sigma\|_{C^0} \leq \boldsymbol{m}_0^{1/2}$. Clearly J_1 can be estimated as Err^i_1 and J_2 as Err^i_2, thus concluding the proof.

5. Boundedness of the frequency

In this section we prove that the frequency function $\mathbf{I}_{N_j}$ remains bounded along the different center manifolds corresponding to the intervals of flattening. To simplify the notation, we set $p_j := \mathbf{\Phi}_j(0)$ and write simply $\mathcal{B}_\rho$ in place of $\mathcal{B}_\rho(p_j)$.

Theorem 5.1 (Boundedness of the frequency functions)**.** *Let T be as in Assumption 2.1. If the intervals of flattening are $j_0 < \infty$, then there is $\rho > 0$ such that*

$$\mathbf{H}_{N_{j_0}} > 0 \text{ on }]0,\rho[\quad \text{and} \quad \limsup_{r\to 0} \mathbf{I}_{N_{j_0}}(r) < \infty\,. \tag{5.1}$$

If the intervals of flattening are infinitely many, then there is a number $j_0 \in \mathbb{N}$ such that

$$\mathbf{H}_{N_j} > 0 \text{ on }]\tfrac{s_j}{t_j}, 3[\text{ for all } j \geq j_0 \quad \text{and} \quad \sup_{j\geq j_0}\ \sup_{r\in]\frac{s_j}{t_j},3[} \mathbf{I}_{N_j}(r) < \infty\,. \tag{5.2}$$

Proof. Consider the first alternative. Note that N_{j_0} cannot vanish identically on a ball $\mathcal{B}_r$ for any $r > 0$, i.e. there must be a radius $0 < \rho < r$ such that $\mathbf{H}(\rho) = \mathbf{H}_{N_{j_0}}(\rho) > 0$. Indeed, by [5, Propositions 3.1 and 3.3] and the condition (Stop), N_{j_0} can vanish identically in $\mathcal{B}_r$ only if no cube of the Whitney decomposition $\mathscr{W}$ intersects the projection of $\mathcal{B}_r$ onto the plane π_{j_0}. But then T_{j_0} would coincide with $Q[\![\mathcal{M}]\!]$ in $\mathbf{B}_{3r/4}$, that is 0 would be a regular point of T_{j_0}, and hence of T.

Next we claim that $\mathbf{H}(r) > 0$ for every $r \leq \rho$. If not, let r_0 be the largest zero of $\mathbf{H}$ which is smaller than ρ. By Theorem 3.2, there is a constant C such that $\mathbf{I}(r) \leq C(1 + \mathbf{I}(\rho))$ for every $r \in]r_0, \rho[$. By letting $r \downarrow r_0$, we then conclude

$$r_0 \mathbf{D}(r_0) \leq C(1 + \mathbf{I}(\rho))\mathbf{H}(r_0) = 0,$$

that is, $N_j|_{\mathcal{B}_{r_0}} \equiv 0$ which we have already excluded. Therefore, since $\mathbf{H} > 0$ on $]0, \rho[$, we can now apply Theorem 3.2 to conclude (5.1).

In the second case, we partition the extrema t_j of the intervals of flattening into two different classes:

(A) $t_j = s_{j-1}$,
(B) $t_j < s_{j-1}$.

If t_j belongs to (A), set $r := \frac{s_{j-1}}{t_{j-1}}$. Let $L \in \mathscr{W}_{j-1}$ be a cube of the Whitney decomposition such that $c_s r \leq \ell(L)$ and $L \cap B_r(0, \pi_{j-1}) \neq \emptyset$. We are in the position to apply [5, Proposition 3.6] for the comparison of two center manifolds: there exists a constant $\bar{c}_s > 0$ such that

$$\int_{\mathbf{B}_2 \cap \mathcal{M}_j} |N_j|^2 \geq \bar{c}_s \boldsymbol{m}_0^j := \bar{c}_s \max\left\{\mathbf{E}(T_j, \mathbf{B}_{6\sqrt{m}}), \mathbf{c}(\Sigma_j)^2\right\},$$

which obviously gives $\mathbf{H}_{N_j}(3) \geq c\boldsymbol{m}_0^j$. By (3.4), we conclude that $\mathbf{I}_{N_j}(3)$ is smaller than a given constant, independent of j. Arguing as above, we also conclude that, for j large enough, $\mathbf{H}_{N_j}$ is positive on the entire interval $]\frac{s_j}{t_j}, 3[$. Hence, we can apply Theorem 3.2 to conclude (5.2).

In the case t_j belongs to the class (B), then, by construction $\mathbf{E}(T_j, \mathbf{B}_{6\sqrt{m}}) \geq \frac{\varepsilon_3^2}{2}$. In fact, assuming the bound would not hold, we could assume, by the monotonicity formula, that T_j converges to a cone T, from which we would actually conclude that $\mathbf{E}(T_j, \mathbf{B}_{12\sqrt{m}}) \geq \frac{3}{4}\varepsilon_3^2$ for sufficiently large j, violating the very definition of t_j. With the same argument we conclude that

$$\frac{\varepsilon_3^2}{2} \leq \liminf_{j\to\infty, j\in(B)} \mathbf{E}(T_j, \mathbf{B}_3)\,.$$

Thus, by Lemma 5.2 below, we conclude $\liminf_{j\to\infty, j\in(B)} \mathbf{H}_{N_j}(3) > 0$. Since $\mathbf{D}_{N_j}(3) \leq C\boldsymbol{m}_0^j \leq C\varepsilon_3^2$, we conclude that $\limsup_{j\to\infty, j\in(B)} \mathbf{I}_{N_j}(3) > 0$, and conclude as before. □

Lemma 5.2. *Assume the intervals of flattening are infinitely many and* $r_j \in]\frac{s_j}{t_j}, 3[$ *is a subsequences (not relabelled) with* $\lim_j \|N_j\|_{L^2(\mathcal{B}_{r_j} \setminus \mathcal{B}_{r_j/2})} = 0$. *Then,* $\mathbf{E}(T_j, \mathbf{B}_{r_j}) \to 0$.

Proof. Note that, if $r_j \to 0$, then necessarily $\mathbf{E}(T_j, \mathbf{B}_{r_j}) \to 0$ by Proposition2.2 (iv). Therefore, up to a subsequence, we can assume the existence of $c > 0$ such that

$$r_j \geq c \quad \text{and} \quad \mathbf{E}(T_j, \mathbf{B}_{8\sqrt{m}}) \geq c. \tag{5.3}$$

After the extraction of a further subsequence, we can assume the existence of r such that

$$\int_{\mathcal{B}_r \setminus \mathcal{B}_{\frac{3r}{4}}} |N_j|^2 \to 0, \tag{5.4}$$

and the existence of a cone $S \in \mathscr{C}(T, 0)$ such that $T_j \to S$. Note that, by (5.3), S is not a multiplicity Q flat m-plane. Consider the orthogonal projection $\mathbf{q}_j : \mathbb{R}^{m+n} \to \pi_j$, where π_j is the m-dimensional plane of the construction of the center manifold $\mathcal{M}_j$. Assuming ε_3 is sufficiently small, we have $U_j := B_{\frac{15}{16}r}(\pi_j) \setminus B_{\frac{13}{16}r}(\pi_j) \subset \mathbf{q}_j(\mathcal{B}_r \setminus \mathcal{B}_{\frac{3}{4}r})$. Consider the Whitney decomposition $\mathscr{W}_j$ leading to the construction of $\mathcal{M}_j$: if the decomposition is empty, then T_j coincides with $\mathcal{M}_j$. Otherwise, set

$$d_j := \max \left\{ \ell(J) : J \in \mathscr{W}_j \quad \text{and} \quad J \cap U_j \neq \emptyset \right\}.$$

Let $J_j \in \mathscr{W}_j$ be such that $U_j \cap J_j \neq \emptyset$ and $d_j = \ell(J_j)$. If the stopping condition for J_j is either (HT) or (EX), recalling that $\ell(J_j) \leq c_s r$, we choose ball $B^j \subset U_j$ of radius $\frac{d_j}{2}$ and at distance at most $\sqrt{m} d_j$ from J_j. If the stopping condition for J_j is (NN), J_j is in the domain of influence of $K_j \in \mathscr{W}_e$. By Proposition 2.2 we can then choose a ball $B^j \subset U_j$ of radius $\frac{\ell(K_j)}{8}$ at distance at most $3\sqrt{m}\ell(K_j)$ from K_j. If the stopping condition is (HT), we then have by [5, Proposition 3.1]

$$\int_{\mathcal{B}_r \setminus \mathcal{B}_{\frac{3r}{4}}} |N_j|^2 \geq \int_{\mathbf{\Phi}_j(B^j)} |N_j|^2 \geq c \left(\boldsymbol{m}_0^j\right)^{\frac{1}{m}} d_j^{m+2+2\beta_2}.$$

If the stopping condition is either (NN) or (EX), by [5, Proposition 3.3] we have

$$\int_{\mathcal{B}_r \setminus \mathcal{B}_{\frac{3r}{4}}} |N_j|^2 \geq \int_{\mathbf{\Phi}_j(B^j)} |N_j|^2 \geq c d_j^2 \int_{\mathbf{\Phi}_j(B^j)} |DN_j|^2 \geq c \boldsymbol{m}_0^j d_j^{m+4-2\delta_2}. \tag{5.5}$$

In both cases we conclude that $d_j \to 0$.

By [5, Corollary 2.2], $\mathrm{spt}(T_j) \cap \mathbf{\Phi}_j(U_j)$ is contained in a d_j-tubular neighborhood of $\mathcal{M}_j$, which we denote by $\hat{\mathbf{U}}_j$. Moreover, again assuming that ε_3 is sufficiently small, we can assume $\mathbf{B}_t \setminus \mathbf{B}_s \cap \mathcal{M}_j \subset \mathbf{\Phi}_j(U_j)$ for some appropriate choice of $s < t$, independent of j. Finally, by [5, Theorem 1.12] we can assume that (up to subsequences) $\mathcal{M}_j$ converges to $\mathcal{M}$ in C^3. We thus conclude that $S \llcorner \mathbf{B}_t \setminus \mathbf{B}_s$ is supported in $\mathcal{M} \cap \mathbf{B}_t \setminus \mathbf{B}_s$ and, hence, by the constancy theorem, $S \llcorner \mathbf{B}_t \setminus \mathbf{B}_s = Q_0 \llbracket \mathcal{M} \cap \mathbf{B}_t \setminus \mathbf{B}_s \rrbracket$ for some integer Q_0. Observe also that, if $\mathbf{p}_j : \hat{\mathbf{U}}_j \to \mathcal{M}_j$ is the orthogonal projection onto $\mathcal{M}_j$, by [5, Theorem 2.4] we also have $(\mathbf{p}_j)_\sharp(T_j \llcorner \mathbf{B}_t \setminus \mathbf{B}_s) = Q \llbracket \mathcal{M}_j \cap \mathbf{B}_t \setminus \mathbf{B}_s \rrbracket$. We therefore conclude that $Q_0 = Q$. Since S is a cone without boundary, $\partial(S \llcorner \mathbf{B}_t) = Q \llbracket \mathcal{M} \cap \partial \mathbf{B}_t \rrbracket$, i.e. $S = Q \llbracket 0 \rrbracket \times\!\!\!\!\times \llbracket \mathcal{M} \cap \partial \mathbf{B}_t \rrbracket$. By Allard's regularity theorem, S is regular in a neighborhood of 0 and, therefore, it is an m-plane with multiplicity Q, which gives the desired contradiction. □

A corollary of Theorem 5.1 is the following.

Corollary 5.3 (Reverse Sobolev)**.** *Let T be as in Assumption 2.1. Then, there exists a constant $C>0$ such that, for every interval of flattening I_j and for every $r \in I_j$, there is $s \in]\frac{3}{2}r, 3r[$ such that*

$$\int_{\mathcal{B}_s(\mathbf{\Phi}_j(0))} |DN_j|^2 \leq \frac{C}{r^2} \int_{\mathcal{B}_s(\mathbf{\Phi}_j(0))} |N_j|^2 \,. \tag{5.6}$$

Proof. By Theorem 5.1, there exists $C>0$ depending on the current T such that

$$\int_{\frac{3}{2}r}^{3r} dt \int_{\mathcal{B}_t(\mathbf{\Phi}_j(0))} |DN_j|^2 = \frac{3}{2} r\, \mathbf{D}_{N_j}(3r) \leq C\, \mathbf{H}_{N_j}(3r) = \frac{C}{r}\int_{\frac{3}{2}r}^{3r} dt \int_{\partial B_t(\mathbf{\Phi}_j(0))} |N_j|^2 \,.$$

Therefore, there must be $s \in [\frac{3}{2}r, 3r]$ such that

$$\int_{\mathcal{B}_s(\mathbf{\Phi}_j(0))} |DN_j|^2 \leq \frac{C}{r} \int_{\partial\mathcal{B}_s(\mathbf{\Phi}_j(0))} |N_j|^2 \,. \tag{5.7}$$

On the other hand, using the fundamental theorem of calculus, for every $\delta > 0$ sufficiently small, we get

$$\begin{aligned} s \int_{\partial\mathcal{B}_s(\mathbf{\Phi}_j(0))} |N_j|^2 &\leq C \int_{\mathcal{B}_s(\mathbf{\Phi}_j(0))} \left(|N_j|^2 + s\,|N_j||DN_j|\right) \\ &\leq \delta\, s^2 \int_{\mathcal{B}_s(\mathbf{\Phi}_j(0))} |DN_j|^2 + \frac{C}{\delta}\int_{\mathcal{B}_s(\mathbf{\Phi}_j(0))} |N_j|^2 \,. \end{aligned} \tag{5.8}$$

Suitably choosing δ in (5.8), we easily conclude (5.6) from (5.7). □

6. Final blow-up sequence and capacitary argument

Let T be a current as in the Assumption 2.1. By Proposition 2.2 we can assume that for each radius r_k there is an interval of flattening $I_{j(k)}$ containing r_k. In order to simplify the notation, with a slight abuse, we denote $I_{j(k)}$ as $I_k =]s_k, t_k]$ (so that, in particular, two distinct radii r_k and $r_{k'}$ might belong to the same interval of flattening).

We define the sequence of blow-up maps which will lead to the proof of Almgren's partial regularity result Theorem 0.3. To this aim, for every $r_k \in I_k$, we consider the corresponding radius $\bar{s}_k \in \left]\frac{3}{2}r_k, 3r_k\right[$ provided in Corollary 5.3 and set $\bar{r}_k := \frac{2\bar{s}_k}{3t_k}$. We then rescale and translate the currents and maps accordingly:

- $\bar{T}_k = (\iota_{0,\bar{r}_k t_k})_\sharp T = (\iota_{0,\bar{r}_k})_\sharp T_k$, $\bar{\Sigma}_k = \iota_{0,\bar{r}_k}(\Sigma_k)$ and $\bar{\mathcal{M}}_k := \iota_{0,\bar{r}_k}(\mathcal{M}_k)$;
- $\bar{N}_k : \bar{\mathcal{M}}_k \to \mathbb{R}^{m+n}$ is the rescaled $\bar{\mathcal{M}}_k$-normal approximations given by

$$\bar{N}_k(p) = \frac{1}{\bar{r}_k} N_k(\bar{r}_k p).$$

Since by assumption $T_0\bar{\Sigma}_k = \mathbb{R}^{m+\bar{n}} \times \{0\}$, the ambient manifolds $\bar{\Sigma}_k$ converge to $\mathbb{R}^{m+\bar{n}} \times \{0\}$ locally in C^{3,ε_0}. Moreover, since $\frac{1}{2} < \frac{r_k}{\bar{r}_k t_k} < 1$, it follows from Proposition 1.3 that

$$\mathbf{E}(\bar{T}_k, \mathbf{B}_{\frac{1}{2}}) \leq C\mathbf{E}(T, \mathbf{B}_{r_k}) \to 0.$$

By the standard regularity theory of area minimizing currents and Assumption 2.1, this implies that $\bar{T}_k$ locally converge (and in the Hausdorff sense for what concerns the supports) to a minimizing tangent cone which is an m-plane with multiplicity Q contained in $\mathbb{R}^{m+\bar{n}} \times \{0\}$. Without loss of generality, we can assume that T_k locally converge to $Q\,[\![\pi_0]\!]$. Moreover, from Proposition 1.3 it follows that

$$\mathcal{H}^{m-2+\alpha}_\infty(\mathrm{D}_Q(\bar{T}_k) \cap \mathbf{B}_1) \geq \mathcal{H}^{m-2+\alpha}_\infty(\mathrm{D}_Q(T) \cap \mathbf{B}_{r_k}) \geq \eta > 0. \tag{6.1}$$

We can, therefore, apply [5, Proposition 3.4] and infer that

$$\bar{\ell}_k := \bar{r}_k^{-1} \sup\left\{\ell(L) : L \in \mathscr{W}_e^k \text{ and } L \cap B_{3\bar{r}_k}(0, \pi_k) \neq \emptyset\right\} = o(1). \tag{6.2}$$

In the next lemma, we show that the rescaled center manifolds $\bar{\mathcal{M}}_k$ converge locally to the flat m-plane π_0, thus leading to the following natural definition for the blow-up maps $N_k^b : B_3 \subset \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ given by

$$N_k^b(x) := \mathbf{h}_k^{-1} \bar{N}_k(\mathbf{e}_k(x))\,, \tag{6.3}$$

where $\mathbf{h}_k := \|\bar{N}_k\|_{L^2(\mathcal{B}_{\frac{3}{2}})}$ and $\mathbf{e}_k : B_3 \subset \mathbb{R}^m \simeq T_0\bar{\mathcal{M}}_k \to \bar{\mathcal{M}}_k$ denotes the exponential map.

Lemma 6.1 (Vanishing lemma). *Under the Assumption 2.1, the following hold:*

(i) $\bar{r}_k \boldsymbol{m}_0^k \to 0$;

(ii) *the rescaled center manifolds* $\bar{\mathcal{M}}_k$ *converge (up to subsequences) to* $\mathbb{R}^m \times \{0\}$ *in* $C^{3,\kappa/2}(\mathbf{B}_4)$ *and the maps* $\mathbf{e}_k$ *converge in* $C^{2,\kappa/2}$ *to the identity map* $\mathrm{id} : B_3 \to B_3$;

(iii) *there exists a constant* $C > 0$ *such that, for every* k,

$$\int_{B_{\frac{3}{2}}} |DN_k^b|^2 \leq C. \tag{6.4}$$

Proof. To show (i), note that, if $\liminf_k \bar{r}_k > 0$, then $\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}t_k}) \leq C\mathbf{E}(\bar{T}_k, \mathbf{B}_{6\sqrt{m}\bar{r}_k^{-1}}) \to 0$ because $\bar{T}_k$ converges to $Q\,[\![\pi_0]\!]$. Therefore, from the obvious fact $\mathbf{c}(\bar{\Sigma}_k) \to 0$, we easily deduce that, under the assumption $\liminf_k \bar{r}_k > 0$, then necessarily $\boldsymbol{m}_0^k \to 0$. Next, using $\bar{r}_k \boldsymbol{m}_0^k \to 0$ and the estimate of [5, Theorem 1.12], it follows easily that $\bar{\mathcal{M}}_k - \bar{p}_k$, with $\bar{p}_k := r_k^{-1}\mathbf{\Phi}_k(0)$, converge (up to subsequences) to a plane in $C^{3,\kappa/2}(\mathbf{B}_4)$. By Proposition 2.2 (v) we deduce that $\mathcal{M}_k$ converges to π_0 as well. Therefore, by Proposition A.4 the maps $\mathbf{e}_k$ converge to the identity in $C^{2,\kappa/2}$ (indeed, by standard arguments they must converge to the exponential map on the – totally geodesic! – submanifold $\mathbb{R}^m \times \{0\}$) . Finally, (iii) is a simple consequence of Corollary 5.3 □

The main result about the blow-up maps N_k^b is the following.

Theorem 6.2 (Final blow-up). *Up to subsequences, the maps* N_k^b *converge strongly in* $L^2(B_{\frac{3}{2}})$ *to a function* $N_\infty^b : B_{\frac{3}{2}} \to \mathcal{A}_Q(\{0\} \times \mathbb{R}^{\bar{n}} \times \{0\})$ *which is Dir-minimizing in* $B_{\frac{5}{4}}$ *and satisfies* $\|N_\infty^b\|_{L^2(B_{\frac{3}{2}})} = 1$ *and* $\boldsymbol{\eta} \circ N_\infty^b \equiv 0$.

We postpone the proof of Theorem 6.2 to the next section and show next the conclusion of Theorem 0.3.

6.1. **Proof of Theorem 0.3: capacitary argument.** Let N^b_∞ be as in Theorem 6.2 and

$$\Upsilon := \left\{x \in \bar B_{\frac{5}{4}} : N^b_\infty(x) = Q\,[\![0]\!]\right\}.$$

Since $\boldsymbol{\eta} \circ N^b_\infty \equiv 0$ and $\|N^b_\infty\|_{L^2(B_{3/2})} = 1$, from the regularity of Dir-minimizing Q-valued functions (cf. [2, Theorem 0.11]), we know that $\mathcal{H}^{m-2+\alpha}_\infty(\Upsilon) = 0$. We show in the three steps that this contradicts Assumption 0.4.

Step 1. We cover Υ by balls $\{\mathbf{B}_{\sigma_i}(x_i)\}$ in such a way that $\sum_i \omega_{m-2+\alpha}(3\sigma_i)^{m-2+\alpha} \leq \frac{\eta}{2}$, where η is the constant in (6.1). By the compactness of Υ, such a covering can be chosen finite. Let $\sigma > 0$ be a radius whose specific choice will be given only at the very end, and such that $0 < 40\sigma \leq \min \sigma_i$. Denote by Λ_k the set of Q points of $\bar T_k$ far away from the singular set Υ:

$$\Lambda_k := \{p \in D_Q(\bar T_k) \cap \mathbf{B}_1 : \operatorname{dist}(p, \Upsilon) > 4 \min \sigma_i\}.$$

Clearly, $\mathcal{H}^{m-2+\alpha}_\infty(\Lambda_k) \geq \frac{\eta}{2}$. By the Hölder continuity of Dir-minimizing functions (cf. [2, Theorem 0.9]), there is a positive constant $\vartheta = \vartheta(\min \sigma_i) > 0$ (independent of σ) such that

$$\fint_{B_{2\sigma}(x)} |N^b_\infty|^2 \geq 2\,\vartheta \qquad \forall\, x \in B_{\frac{5}{4}} \text{ with } \operatorname{dist}(x, \Upsilon) \geq 3 \min \sigma_i.$$

Therefore, by Theorem 6.2, one can deduce that

$$\fint_{\mathcal{B}_{2\sigma}(x)} \mathcal{G}(\bar N_k, Q\,[\![\boldsymbol{\eta} \circ \bar N_k]\!])^2 \geq \vartheta \mathbf{h}_k^2 \qquad \forall\, x \in \Gamma_k := \mathbf{p}_{\bar{\mathcal{M}}_k}(\Lambda_k). \tag{6.5}$$

Step 2. Next we claim that, for every $p \in \Lambda_k$, there is a radius $\varrho_p \leq 2\sigma$ such that, for k large enough and a suitably chosen geometric constant $c_0 > 0$ (in particular, independent of σ), it holds

$$\frac{c_0\,\vartheta}{\sigma^\alpha}\mathbf{h}_k^2 \leq \frac{1}{\varrho_p^{m-2+\alpha}} \int_{\mathcal{B}_{\varrho_p}(\mathbf{p}_{\bar{\mathcal{M}}_k}(p))} |D\bar N_k|^2, \tag{6.6}$$

$$\mathcal{B}_{\varrho_p}(\mathbf{p}_{\bar{\mathcal{M}}_k}(p)) \subset \mathbf{B}_{4\varrho_p}(p)\,. \tag{6.7}$$

In order to show this, consider $\bar\ell_k$ in (6.2). By [5, Proposition 3.1], for every $p \in \Lambda_k \cap \mathbf{B}_{\frac{5}{2}\bar r_k}$, $x = \mathbf{p}_{\bar{\mathcal{M}}_k}(p)$ cannot belong to $\mathbf{\Phi}(L)$ for some $L \in \mathscr{W}_h$. Thus either it belongs to $\mathbf{\Phi}(L)$ for some $L \in \mathscr{W}_e \cup \mathscr{W}_n$ or it belongs to the contact set $\mathbf{\Phi}(\mathbf{\Gamma})$ and hence $\bar N_k = Q\,[\![0]\!]$. In the case $L \in \mathscr{W}_n$, by Proposition 2.2 (iii), there exists a cube $H \in \mathscr{W}_e$ such that L belongs to the domain of influence of H and $\operatorname{dist}(H, L) \leq 2\sqrt{m}\ell(L)$. For k large enough, we can then apply [5, Proposition 3.5] with $\eta_2 = \frac{\vartheta}{4}$ for every $p \in \mathrm{D}_Q(T_k) \cap \mathbf{B}_{\frac{3}{2}\bar r_k}$. Therefore, there exists a constant $\bar s < 1$ such that, for $x = \mathbf{p}_{\bar{\mathcal{M}}_k}(p) \in \mathbf{\Phi}(L)$,

$$\fint_{\mathcal{B}_{\bar s \ell(L)}(x)} \mathcal{G}(N_k, Q\,[\![\boldsymbol{\eta} \circ N_k]\!])^2 \leq \frac{\vartheta}{4\omega_m \ell(L)^{m-2}} \int_{\mathcal{B}_{\ell(L)}(x)} |DN_k|^2\,,$$

that is, rescaling to $\bar{\mathcal{M}}_k$, there exists $t_x \leq \bar\ell_k$ such that

$$\fint_{\mathcal{B}_{\bar s t_x}(x)} \mathcal{G}(\bar N_k, Q\,[\![\boldsymbol{\eta} \circ \bar N_k]\!])^2 \leq \frac{\vartheta}{4\omega_m t_x^{m-2}} \int_{\mathcal{B}_{t_x}(x)} |D\bar N_k|^2\,. \tag{6.8}$$

By (6.2) we can assume that, provided k is large enough, then $t_x = \frac{\ell(L)}{\bar r_k} \leq \bar\ell_k \leq \sigma$ for every $x \in \Gamma_k$. Moreover, from Proposition 2.2 (v) and Lemma 6.1, for k large enough, we get

$$|p - x| \leq C(\boldsymbol{m}_0^k)^{\frac{1}{2m}} \bar r_k^{\beta_2} t_x < \bar s\, t_x. \tag{6.9}$$

In case x belongs to the contact set $\mathbf{\Phi}(\mathbf{\Gamma})$, then $p = x$ and $N_k(x) = Q\,[\![0]\!]$. Therefore

$$\lim_{t\downarrow 0} \fint_{\mathcal{B}_t(x)} \mathcal{G}(\bar N_k, Q\,[\![\boldsymbol{\eta}\circ \bar N_k]\!])^2 = 0$$

and we choose $t_x < \sigma$ such that

$$\fint_{\mathcal{B}_{\bar s t_x}(x)} \mathcal{G}(\bar N_k, Q\,[\![\boldsymbol{\eta}\circ \bar N_k]\!])^2 \leq \frac{\vartheta}{4}\mathbf{h}_k^2\,. \tag{6.10}$$

We show that we can choose $\varrho_p \in]\bar s\, t_x, 2\sigma[$ such that (6.6) holds. To this aim, for each $x \in \Gamma_k$, we can distinguish two cases. Either

$$\frac{1}{\omega_m t_x^{m-2}} \int_{\mathcal{B}_{t_x}(x)} |DN_k|^2 \geq \mathbf{h}_k^2\,, \tag{6.11}$$

and (6.6) follows with $\varrho_p = t_{\mathbf{p}_{\bar{\mathcal{M}}_k}(p)}$. Or (6.11) does not hold, and we argue as follows. We use first (6.8) (in case $x \notin \mathbf{\Phi}(\mathbf{\Gamma})$) or (6.10) (when $x \in \mathbf{\Phi}(\mathbf{\Gamma})$) to get

$$\fint_{\mathcal{B}_{\bar s r_x}(x)} \mathcal{G}(\bar N_k, Q\,[\![\boldsymbol{\eta}\circ \bar N_k]\!])^2 \leq \frac{\vartheta}{4}\mathbf{h}_k^2\,. \tag{6.12}$$

Then, we show by contradiction that there exists a radius $\varrho_y \in [\bar s t_x, 2\sigma]$ such that (6.6) holds. Indeed, if this were not the case, setting for simplicity $f := \mathcal{G}(\bar N_k, Q\,[\![\boldsymbol{\eta}\circ \bar N_k]\!])$ and letting j be the smallest integer such that $2^{-j}\sigma \leq \bar s t_x$, we can estimate as follows

$$\begin{aligned}
\fint_{\mathcal{B}_{2\sigma}(x)} f^2 &\leq 2\fint_{\mathcal{B}_{\bar s t_x}(x)} f^2 + \sum_{i=0}^{j}\left(\fint_{\mathcal{B}_{2^{1-i}\sigma}(x)} f^2 - \fint_{\mathcal{B}_{2^{-i}\sigma}(x)} f^2\right)\\
&\stackrel{(6.12)}{\leq} \frac{\vartheta}{2}\mathbf{h}_k^2 + C\sum_{i=1}^{j}\frac{1}{(2^{-j}\sigma)^{m-2}}\int_{\mathcal{B}_{2^{1-i}\sigma}(x)} |D\bar N_k|^2\\
&\leq \frac{\vartheta}{2}\mathbf{h}_k^2 + Cc_0\frac{\vartheta}{\sigma^\alpha}\mathbf{h}_k^2\sum_{i=1}^{j}(2^{-j}\sigma)^\alpha \leq \mathbf{h}_k^2\left(\frac{\vartheta}{2} + C(\alpha)c_0\vartheta\right)\,.
\end{aligned}$$

In the second line we have used the simple Morrey inequality

$$\left|\fint_{\mathcal{B}_{2t}(x)} f^2 - \fint_{\mathcal{B}_t(x)} f^2\right| \leq \frac{C}{t^{m-2}}\int_{\mathcal{B}_{2t}(x)} |Df|^2 \leq \frac{C}{t^{m-2}}\int_{\mathcal{B}_{2t}(x)} |D\bar N_k|^2\,.$$

The constant C depends only upon the regularity of the underlying manifold $\bar{\mathcal{M}}_k$, and, hence, can assumed independent of k.

Since $C(\alpha)$ depends only on α, m and Q, for c_0 chosen sufficiently small the latter inequality would contradict (6.5). Note that (6.7) follows by a simple triangular inequality.

Step 3. Finally, we show that (6.6) and (6.7) lead to a contradiction. Consider a covering of Λ_k with balls $\mathbf{B}^i := \mathbf{B}_{20\varrho_{p_i}}(p_i)$ with the property that the corresponding balls $\mathbf{B}_{4\varrho_{p_i}}(p_i)$ are disjoint. We then can estimate

$$\begin{aligned}\frac{\eta}{2} &\le C(m)\sum_i \varrho_{p_i}^{m-2+\alpha} \stackrel{(6.6)}{\le} \frac{C(m)}{c_0}\frac{\sigma^\alpha}{\vartheta \mathbf{h}_k^2}\sum_i \int_{\mathcal{B}_{\varrho_{p_i}}(\mathbf{p}_{\bar{\mathcal{M}}_k}(p_i))} |D\bar{N}_k|^2 \\ &\le \frac{C(m)}{c_0}\frac{\sigma^\alpha}{\vartheta \mathbf{h}_k^2}\int_{\mathcal{B}_{\frac{3}{2}}} |D\bar{N}_k|^2 \stackrel{(6.4)}{\le} C\frac{\sigma^\alpha}{\vartheta},\end{aligned}$$

where $C(m) > 0$ is a dimensional constant. In the last line we have used that, thanks to (6.7), the balls $\mathcal{B}_{\varrho_{p_i}}(p_{\bar{\mathcal{M}}_k}(p_i))$ are pairwise disjoint and that, provided σ is smaller than $\frac{1}{32}$ and k large enough, they are all contained in $\mathcal{B}_{\frac{3}{2}}$. Since ϑ and c_0 are independent of σ, the above inequality reaches the desired contradiction as soon as σ is fixed sufficiently small.

7. Harmonicity of the limit

In this section we prove Theorem 6.2 and conclude our argument.

7.1. **First estimates.** Let $\bar{F}_k : \mathcal{B}_{\frac{3}{2}} \subset \bar{\mathcal{M}}_k \to \mathcal{A}_Q(\mathbb{R}^{m+n})$ be the multiple valued map given by $\bar{F}_k(x) := \sum_i [\![x + (\bar{N}_k)_i(x)]\!]$ and, to simplify the notation, set $\mathbf{p}_k := \mathbf{p}_{\bar{\mathcal{M}}_k}$. The following estimates are easy consequences of the rescaling and the previous analysis: there exists a suitable exponent $\gamma > 0$ such that

$$\mathrm{Lip}(\bar{N}_k) \le C\mathbf{h}_k^\gamma \quad \text{and} \quad |\bar{N}_k| \le C(\boldsymbol{m}_0^k \bar{r}_k)^\gamma, \tag{7.1}$$

$$\mathbf{M}((\mathbf{T}_{\bar{F}_k} - \bar{T}_k)\llcorner(\mathbf{p}_k^{-1}(\mathcal{B}_{\frac{3}{2}})) \le C\mathbf{h}_k^{2+2\gamma}, \tag{7.2}$$

$$\int_{\mathcal{B}_{\frac{3}{2}}} |\boldsymbol{\eta} \circ \bar{N}_k| \le C\mathbf{h}_k^2\,. \tag{7.3}$$

Indeed, using the domain decomposition of Section 4.1 (note that $\frac{3}{2}\bar{r}_k \in]\frac{s_k}{t_k}, 3[$) and arguing in an analogous way we infer that

$$\|N_k\|_{C^0(\mathcal{B}_{\frac{3}{2}\bar{r}_k}(p_k))} \le C(\boldsymbol{m}_0^k)^{\frac{1}{2m}}\bar{r}_k^{1+\beta_2} \quad \text{and} \quad \mathrm{Lip}(N_k|_{\mathcal{B}_{\frac{3}{2}\bar{r}_k}(p_k)}) \le C(\boldsymbol{m}_0^k)^{\gamma_2}\max_i \ell_i^{\gamma_2}$$

$$\mathbf{M}\big((\mathbf{T}_{F_k} - T_k)\llcorner \mathbf{p}_k^{-1}(\mathcal{B}_{\frac{3}{2}\bar{r}_k}(p_k))\big) \le \sum_i (\boldsymbol{m}_0^k)^{1+\gamma_2}\ell_i^{m+2+\gamma_2},$$

$$\int_{\mathcal{B}_{\frac{3}{2}\bar{r}_k}(p_k)} |\boldsymbol{\eta} \circ N_k| \le C\boldsymbol{m}_0\bar{r}_k\sum_i \ell_i^{2+m+\gamma_2} + \frac{C}{\bar{r}_k}\int_{\mathcal{B}_{\frac{3}{2}\bar{r}_k}(\boldsymbol{\Phi}_k(0))} |N_k|^2\,,$$

where this time, for the latter inequality we have used [5, Theorem 2.4 (2.4)] with $a = \bar{r}_k$. On the other hand, by (3.17) and Corollary 5.3, we see that

$$\sum_i \boldsymbol{m}_0^k \ell_i^{m+2+\frac{\gamma_2}{4}} \le C\int_{\mathcal{B}_{\frac{3}{2}\bar{r}_k}(p_k)} (|DN_k|^2 + |N_k|^2) \le C\int_{\mathcal{B}_{\bar{s}_k}(p_k)} |N_k|^2, \tag{7.4}$$

from which (7.1)-(7.3) follow by a simple rescaling.

It is then clear that the strong L^2 convergence of N^b_k is a consequence of these bounds and of the Sobolev embedding (cf. [2, Proposition 2.11]); whereas, by (7.3),

$$\int_{\mathcal{B}_{\frac{3}{2}}} |\boldsymbol{\eta}\circ N^b_\infty| = \lim_{k\to+\infty}\int_{\mathcal{B}_{\frac{3}{2}}} |\boldsymbol{\eta}\circ N^b_k| \leq C \lim_{k\to+\infty} \mathbf{h}_k^2 = 0\,.$$

Finally, note that N^b_∞ must take its values in $\{0\}\times\mathbb{R}^{\bar n}\times\{0\}$. Indeed, considering the tangential part of $\bar N_k$ given by $\bar N^T_k(x) := \sum_i \left[\!\left[\mathbf{p}_{T_x\bar\Sigma_k}(\bar N_k(x))_i\right]\!\right]$, it is simple to verify that $\mathcal{G}(\bar N_k, \bar N_k^T)\leq C|\bar N_k|^2$, which leads to

$$\int \mathcal{G}(N^b_k, \mathbf{h}_k^{-1}\bar N^T_k\circ\mathbf{e}_k)^2 \stackrel{(7.1)}{\leq} C(\boldsymbol{m}_0^k\bar r_k)^{2\gamma}\to 0 \quad \text{as } k\to+\infty,$$

and, by the convergence of $\bar\Sigma_k$ to $\mathbb{R}^{m+\bar n}\times\{0\}$, gives the claim.

7.2. **A suitable trivialization of the normal bundle.** By Lemma 6.1, we can consider for every $\bar{\mathcal{M}}_k$ an orthonormal frame of $(T\bar{\mathcal{M}}_k)^\perp$, $\nu^k_1,\ldots,\nu^k_{\bar n},\varpi^k_1,\ldots\varpi^k_l$ with the property that $\nu^k_j(x)\in T_x\bar\Sigma_k$, $\varpi^k_j(x)\perp T_x\bar\Sigma_k$ and (cf. [3, Lemma A.1])

$$\nu^k_j\to e_{m+j}\quad\text{and}\quad \varpi^k_j\to e_{m+\bar n+j}\quad \text{in } C^{2,\kappa/2}(\bar{\mathcal{M}}_k) \text{ as } k\uparrow\infty$$

(for every j). It is simple to check that there exists a map $\psi_k:\bar{\mathcal{M}}_k\times\mathbb{R}^{\bar n}\to\mathbb{R}^l$ converging to 0 in $C^{2,\kappa/2}$ (uniformly bounded in $C^{2,\kappa}$) and such that, for every $v\in T_p\bar{\mathcal{M}}_k$ with $|v|\leq\delta$ for some sufficiently small δ (independent of k),

$$p+v\in\bar\Sigma_k \iff v^\perp = \psi_k(p, v^T),$$

with $v^T = (\langle v,\nu^k_1\rangle,\ldots,\langle v,\nu^k_{\bar n}\rangle)\in\mathbb{R}^{\bar n}$ and $v^\perp = (\langle v,\varpi^k_1\rangle,\ldots,\langle v,\varpi^k_l\rangle)\in\mathbb{R}^l$. To see this, consider the map

$$\Phi_k:\bar{\mathcal{M}}_k\times\mathbb{R}^{\bar n}\times\mathbb{R}^l\ni(p,z,w)\mapsto p+z^j\nu^k_j+w^j\varpi^k_j\in\mathbb{R}^{m+n},$$

where we use the Einstein convention of summation over repeated indices. It is simple to show that $D\Phi_k(p,0)=\mathrm{Id}$ for every $p\in\bar{\mathcal{M}}_k$ and, hence, $\Phi_k^{-1}(\bar\Sigma_k)$ can be written locally as a graph of a function ψ_k satisfying the claim above.

Note that, by construction we also have that $\psi_k(p,0) = |D_w\psi_k(p,0)| = 0$ for every $p\in\bar{\mathcal{M}}_k$, which in turn implies

$$|D_x\psi_k(x,w)|\leq C|w|^{1+\kappa},\quad |D_w\psi_k(x,w)|\leq C|w|\quad\text{and}\quad |\psi_k(x,w)|\leq C|w|^2. \tag{7.5}$$

Given now any Q-valued map $u=\sum_i [\![u_i]\!]:\bar{\mathcal{M}}_k\to\mathcal{A}_Q(\{0\}\times\mathbb{R}^{\bar n}\times\{0\})$ with $\|u\|_{L^\infty}\leq\delta$, we can consider the map $\mathbf{u}_k := \psi_k(x,u)$ defined by

$$x\mapsto\sum_i \left[\!\left[(u_i)^j\nu^k_j(x)+\psi^j_k(x,u_i(x))\varpi^k_j(x)\right]\!\right],$$

where we set $(u_i)^j := \langle u_i(x), e_{m+j}\rangle$, $\psi^j_k(x,u_i(x)) := \langle\psi_k(x,u_i(x)), e_{m+\bar n+j}\rangle$ (again we use Einstein's summation convention). Then, the differential map $D\mathbf{u}_k := \sum_i [\![D(\mathbf{u}_k)_i]\!]$ is given by

$$D(\mathbf{u}_k)_i = D(u_i)^j \nu_j^k + \left[D_x \psi_k^j(x,u_i) + D_w(\psi_k^j, u_i) Du_i\right] \varpi_j^k \\ + (u_i)^j D\nu_j^k + \psi_k^j(x,u_i) D\varpi_j^k\,.$$

Taking into account that $\|D\nu_i^k\|_{C^0} + \|D\varpi_j^k\|_{C^0} \to 0$ as $k\to+\infty$, by (7.5) we deduce that

$$\left|\int \left(|D\mathbf{u}_k|^2 - |Du|^2\right)\right| \leq C\int \left(|Du|^2|u| + |Du||u|^{1+\kappa} + |u|^{2+2\kappa}\right) + o(1)\int \left(|u|^2 + |Du|^2\right)\,. \tag{7.6}$$

Now we clearly have $\bar N_k(x) = \psi_k(x,\bar u_k)$ for some Lipschitz $\bar u_k : \bar{\mathcal{M}}_k \to \mathcal{A}_Q(\mathbb{R}^{\bar n})$ with $\|\bar u_k\|_{L^\infty} = o(1)$ by (7.1). Setting $u_k^b := \bar u_k \circ \mathbf{e}_k$, we conclude from (5.6), (7.1) and (7.6) that

$$\lim_{k\to+\infty} \int_{B_{\frac32}} \left(|DN_k^b|^2 - \mathbf{h}_k^{-2}|Du_k^b|^2\right) = 0, \tag{7.7}$$

and N_∞^b is the limit of $\mathbf{h}_k^{-1}u_k^b$.

7.3. **Competitor function.** We now show the Dir-minimizing property of N_∞^b. Clearly, there is nothing to prove if its Dirichlet energy vanishes. We can therefore assume that there exists $c_0>0$ such that

$$c_0\mathbf{h}_k^2 \leq \int_{\mathcal{B}_{\frac32}} |D\bar N_k|^2\,. \tag{7.8}$$

Assume there is a radius $t\in \left]\frac54,\frac32\right[$ and a function $f: B_{\frac32}\to\mathcal{A}_Q(\mathbb{R}^{\bar n})$ such that

$$f|_{B_{\frac32}\setminus B_t} = N_\infty^b|_{B_{\frac32}\setminus B_t} \quad\text{and}\quad \mathrm{Dir}(f,B_t) \leq \mathrm{Dir}(N_\infty^b, B_t) - 2\,\delta,$$

for some $\delta>0$. We can apply [4, Proposition 3.5] to the functions $\mathbf{h}_k^{-1}u_k^b$ and find $r\in]t,2[$ and competitors v_k^b such that, for k large enough,

$$v_k^b|_{\partial B_r} = u_k^b|_{\partial B_r}, \quad \mathrm{Lip}(v_k^b)\leq C\mathbf{h}_k^\gamma, \quad |v_k^b| \leq C(\boldsymbol{m}_0^k \bar r_k)^\gamma, \\ \int_{B_{\frac32}} |\boldsymbol{\eta}\circ v_k^b| \leq C\mathbf{h}_k^2 \quad\text{and}\quad \int_{B_{\frac32}} |Dv_k^b|^2 \leq \int |Du_k^b|^2 - \delta\,\mathbf{h}_k^2\,,$$

where $C>0$ is a constant independent of k. Clearly, by Lemma 6.1 and (7.5), the maps $\tilde N_k = \psi_k(x, v_k^b\circ \mathbf{e}_k^{-1})$ satisfy

$$\tilde N_k \equiv \bar N_k \quad\text{in } \mathcal{B}_{\frac32}\setminus\mathcal{B}_t, \quad \mathrm{Lip}(\tilde N_k)\leq C\mathbf{h}_k^\gamma, \quad |\tilde N_k| \leq C(\boldsymbol{m}_0^k\bar r_k)^\gamma, \\ \int_{B_{\frac32}} |\boldsymbol{\eta}\circ\tilde N_k| \leq C\mathbf{h}_k^2 \quad\text{and}\quad \int_{B_{\frac32}} |D\tilde N_k|^2 \leq \int_{B_{\frac32}} |D\bar N_k|^2 - \delta\mathbf{h}_k^2.$$

7.4. **Competitor current.** Consider finally the map $\tilde{F}_k(x) = \sum_i [\![x+\tilde{N}_i(x)]\!]$. The current $\mathbf{T}_{\tilde{F}_k}$ coincides with $\mathbf{T}_{\bar{F}_k}$ on $\mathbf{p}_k^{-1}(\mathcal{B}_{\frac{3}{2}} \setminus \mathcal{B}_t)$. Define the function $\varphi_k(p) = \mathrm{dist}_{\bar{\mathcal{M}}_k}(0, \mathbf{p}_k(p))$ and consider for each $s \in \left]t, \frac{3}{2}\right[$ the slices $\langle \mathbf{T}_{\tilde{F}_k} - \bar{T}_k, \varphi_k, s\rangle$. By (7.2) we have

$$\int_t^{\frac{3}{2}} \mathbf{M}(\langle \mathbf{T}_{\tilde{F}_k} - \bar{T}_k, \varphi_k, s\rangle) \leq C\mathbf{h}_k^{2+\gamma}\,.$$

Thus we can find for each k a radius $\sigma_k \in \left]t, \frac{3}{2}\right[$ on which $\mathbf{M}(\langle \mathbf{T}_{\tilde{F}_k} - \bar{T}_k, \varphi_k, \sigma_k\rangle) \leq C\mathbf{h}_k^{2+\gamma}$. By the isoperimetric inequality (see [4, Remark 4.3]) there is a current S_k such that

$$\partial S_k = \langle \mathbf{T}_{\tilde{F}_k} - \bar{T}_k, \varphi_k, \sigma_k\rangle, \quad \mathbf{M}(S_k) \leq C\mathbf{h}_k^{(2+\gamma)m/(m-1)} \quad \text{and} \quad \mathrm{spt}(S_k) \subset \bar{\Sigma}_k.$$

Our competitor current is, then, given by

$$Z_k := \bar{T}_k \llcorner (\mathbf{p}_k^{-1}(\bar{\mathcal{M}}_k \setminus \mathcal{B}_{\sigma_k})) + S_k + \mathbf{T}_{\tilde{F}_k} \llcorner (\mathbf{p}_k^{-1}(\mathcal{B}_{\sigma_k})).$$

Note that Z_k is supported in $\bar{\Sigma}_k$ and has the same boundary as $\bar{T}_k$. On the other hand, by (7.2) and the bound on $\mathbf{M}(S_k)$, we have

$$\mathbf{M}(\tilde{T}_k) - \mathbf{M}(\bar{T}_k) \leq \mathbf{M}(\mathbf{T}_{\bar{F}_k}) - \mathbf{M}(\mathbf{T}_{\tilde{F}_k}) + C\mathbf{h}_k^{2+2\gamma}\,. \tag{7.9}$$

Denote by A_k and by H_k respectively the second fundamental forms and mean curvatures of the manifolds $\bar{\mathcal{M}}_k$. Using the Taylor expansion of [3, Theorem 3.2], we achieve

$$\begin{aligned}\mathbf{M}(\tilde{T}_k) - \mathbf{M}(\bar{T}_k) \leq \frac{1}{2}\int_{\mathcal{B}_\rho} \left(|D\tilde{N}_k|^2 - |D\bar{N}_k|^2\right) + C\|H_k\|_{C^0} \int \left(|\boldsymbol{\eta}\circ \bar{N}_k| + |\boldsymbol{\eta}\circ\tilde{N}_k|\right) \\ + \|A_k\|^2_{C^0} \int \left(|\bar{N}_k|^2 + |\tilde{N}_k|^2\right) + o(\mathbf{h}_k^2) \leq -\frac{\delta}{2}\mathbf{h}_k^2 + o(\mathbf{h}_k^2)\,,\end{aligned} \tag{7.10}$$

where in the last inequality we have taken into account Lemma 6.1. Clearly, (7.10) and (7.9) contradict the minimizing property of $\bar{T}_k$ for k large enough and concludes the proof.

Appendix A. Some technical Lemmas

The following is a special case of Allard's ε-regularity theory (see [9, Chapter 5]).

Theorem A.1. *Assume T is area minimizing, $x \in \mathrm{D}_Q(T)$ and $\|T\|((\mathrm{spt}(T)\cap U)\setminus D_Q) = 0$ in some neighborhood U of x. Then, $x \in \mathrm{Reg}(T)$. In particular, $\mathrm{D}_1(T) \subset \mathrm{Reg}(T)$.*

Proof. By simple considerations on the density, the tangent cones at x must necessarily be all m-dimensional planes with multiplicity Q. This allows to apply Allard's theorem and conclude that, in neighborhood of x, $\mathrm{spt}(T)$ is necessarily the graph of a C^{1,κ_0} function for some $\kappa_0 > 0$. Let $u : \mathbb{R}^m \to \mathbb{R}^{\bar{n}+l}$ be the corresponding function and $\Psi : \mathbb{R}^{m+\bar{n}} \to \mathbb{R}^l$ a C^{3,ε_0} function whose graph describes Σ. Let $\bar{u}$ consist of the first $\bar{n}$ coordinates functions of u. We then have that $\bar{u}$ minimizes an elliptic functional of the form $\int \Phi(x, \bar{u}(x), D\bar{u}(x))\,dx$ where $(x, v, p) \mapsto \Phi(x, v, p)$ and $(x, v, p) \to D_p\Phi(x, v, p)$ are of class C^{2,ε_0}. We can then apply the classical regularity theory to conclude that $\bar{u} \in C^{3,\varepsilon_0}$ (see, for instance, [8, Theorem 9.2]), thereby concluding that x belongs to $\mathrm{Reg}(T)$ according to Definition 0.2. Fix next *any* $x \in D_1(T)$. By the upper semicontinuity of the density Θ (cp. [9]), $\Theta \leq \frac{3}{2}$ in a neighborhood U of x, which implies $\|T\|((\mathrm{spt}(T) \cap U) \setminus D_1) = 0$. □

Next, we prove the following technical lemma.

Lemma A.2. *Let T be an area minimizing integer rectifiable current of dimension m in $\mathbb{R}^{m+n}$ and U an open set such that $\mathcal{H}^{m-1}(\partial U \cap \mathrm{spt}(T)) = 0$ and $(\partial T)\llcorner U = 0$. Then $\partial(T\llcorner U) = 0$.*

Proof. Observe that the current $T\llcorner U$ has locally finite mass. Consider therefore $V \subset\subset \mathbb{R}^{m+n}$. By the slicing Theorem [7, 4.2.1] applied to $\mathrm{dist}(\cdot, \partial U)$ we conclude that $S_r := T\llcorner(V \cap U \cap \{\mathrm{dist}\,(x, \partial U) > r\})$ is a normal current in $\mathbf{N}_m(V)$ for a.e. r. Since $\mathbf{M}(T\llcorner(V\cap U) - S_r) \to 0$ as $r \downarrow 0$, we conclude that $T\llcorner(U \cap V)$ is in the $\mathbf{M}$ closure of $\mathbf{N}_m(V)$. Thus, by [7, 4.1.17], $T\llcorner U$ is a flat chain in $\mathbb{R}^{m+n}$. By [7, 4.1.12], $\partial(T\llcorner U)$ is also a flat chain. It is easy to check that $\mathrm{spt}(\partial(T\llcorner U)) \subset \partial U \cap \mathrm{spt}(T)$. Thus we can apply [7, Theorem 4.1.20] to conclude that $\partial(T\llcorner U) = 0$. □

Recall the following theorem (for the proof see [9, Theorem 35.3]).

Theorem A.3. *If T is an integer rectifiable area minimizing current in Σ, then*

$$\mathcal{H}^{m-3+\alpha}_\infty\Big(\mathrm{spt}(T) \setminus \Big(\mathrm{spt}(\partial T) \cup \bigcup_{Q\in\mathbb{N}} \mathrm{D}_Q(T)\Big)\Big) = 0 \quad \forall\, \alpha > 0.$$

We finally prove the following result (first proved by Allard in an unpublished note and hence reported in [1]).

Proposition A.4. *Set $\pi := \mathbb{R}^m \times \{0\} \subset \mathbb{R}^{m+n}$ and let $\mathcal{M}$ be the graph of $C^{3,\kappa}$ function $\boldsymbol{\varphi} : \pi \supset B_3(0) \to \mathbb{R}^m$, with $\boldsymbol{\varphi}(0) = 0$. Then the exponential map $\exp : B_3(0) \to \mathcal{M}$ belongs to the class $C^{2,\kappa}$. Moreover, if $\|\boldsymbol{\varphi}\|_{C^{3,\kappa}}$ is sufficiently small, then the set $\mathbf{p}_\pi(\exp(B_r(0))) \subset \pi$ is (for all $r < 3$) a convex set and the maximal curvature of its boundary is less than $\frac{2}{r}$.*

Proof. We first observe that the metric on $\mathcal{M}$ and the Christoffel symbols Γ are, respectively, $C^{2,\kappa}$ and $C^{1,\kappa}$, in any $C^{3,\kappa}$ chart. The Riemann curvature tensor is also $C^{1,\kappa}$, because one can use the Gauss equation and compute the sectional curvatures from the second fundamental form, which is $C^{1,\kappa}$.

Let next $\Phi(v,t) := \exp(vt)$. Fix a $C^{3,\kappa}$ coordinate patch on Σ where 0 is the origin and observe that $t \mapsto \gamma(t) = \Phi(v,t)$ satisfies the system of ordinary differential equation

$$\gamma_j''(t) = \sum_{ik} \Gamma_j^{ik}(\gamma(t))\gamma_i'(t)\gamma_k'(t),$$

with the initial conditions $\gamma(0) = 0$ and $\gamma'(0) = v$ (here Γ_j^{ik} denote the Christoffel symbols in the system of coordinates). It follows thus that the maps Φ and $\partial_t\Phi$ are $C^{1,\kappa}$.

Fix now a tangent vector e at 0, a point $p = \exp(v) \in \mathcal{M}$ and make a radial parallel transport of e along the geodesic segment $[0,1] \ni t \mapsto \exp(tv)$ to define $e(p)$. We claim that the corresponding vector field is $C^{1,\kappa}$. Indeed, fix any orthonormal tangent frame $f_1, \ldots f_m$ which is $C^{2,\kappa}$. Set $e(\exp(tv)) = \sum_i \alpha_{v,i}(t) f_i(\Phi(t,v))$ and observe observe that the parallel transport is equivalent to solving the ODE

$$\alpha_{v,i}'(t) = -\sum_j \alpha_{v,j}(t) F_{ij}(\Phi(t,v), \partial_t\Phi(t,v))$$

where, for $q \in \Sigma$ and $w \in T_q\Sigma$, $\sum_j F_{ij}(q,w) f_j = \nabla_w f_i(q)$. It turns out that $(t,v) \mapsto F_{ij}(\Phi(t,v), \partial_t \Phi(t,v))$ is then a $C^{1,\kappa}$ map and hence the claimed regularity of $(v,t) \mapsto \alpha_{v,i}(t)$ follows from the standard theory of ODEs.

We conclude that there exists a (not necessarily orthonormal) frame $e_1, \ldots, e_m$ of class $C^{1,\kappa}$ which is parallel along geodesic rays emanating from the origin. Next, consider the map $(w,v,t) \mapsto \partial_w \Phi(t,v)$ where w varies in $\mathbb{R}^m$. Fix w and v and consider again the curve $\gamma(t)$ above and the vector $\eta_{v,w}(t) = \partial_w \Phi(t,v)$. Observe that the vector η satisfies the Jacobi equation along the geodesic γ, with initial data $\eta_{v,w}(0) = 0$ and $\eta'_{v,w}(0) = w$. If we write the vector field in the frame e_i as $\eta(t) = \sum_i \eta_i(t) e_i(\gamma(t))$, the Jacobi equation is

$$\eta''_{v,w,i}(t) = \sum_j R_{\gamma(t)}(e_j(\gamma(t)), \gamma'(t), \gamma'(t), e_i(\gamma(t)))\eta_{v,w,j}(t),$$

where R depends on the Riemann tensor and the matrix $\langle e_i, e_j\rangle$. Taking into account that $\gamma(t) = \Phi(v,t)$ and $\gamma'(t) = \partial_t \Phi(v,t)$ we conclude that $\eta_{v,w,i}$ satisfies an ODE of the type $\eta''_{v,w,i}(t) = \Lambda(v,t,\eta_{v,w,i}(t))$ where the function Λ is $C^{1,\kappa}$ in all its entries. We thus conclude that the map $(v,w,t) \mapsto \eta_{v,w}(t) = \partial_w \Phi(v,t)$ is a $C^{1,\kappa}$ map. Since $d\exp(v)(w) = \partial_w \Phi(v,1)$, this implies that the exponential map is $C^{2,\kappa}$.

As for the last conclusion, for $\|\boldsymbol{\varphi}\|_{C^{3,\kappa}}$ vanishing we conclude from the discussion above that $\mathbf{p}_\pi \circ \exp$ is C^2 close to the identity, which implies the desired statement. □

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