# BV regularity and differentiability properties of a class of upper semicontinuous functions

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## 1 Introduction

We study a class of upper semicontinuous functions $f : \mathbb{R}^d \to \mathbb{R}$ whose hypograph $\mathrm{hypo}\, f$ (see Definition 1) satisfies a geometric regularity property, namely: there exist $c > 0$, $\theta \in ]0, 1]$ such that for each $P$ on the boundary of $\mathrm{hypo}\, f$ there exists a unitary Fréchet (outer) normal $v \in N^F_{\mathrm{hypo}\, f}(P) \cap \mathbb{S}^d$ to $\mathrm{hypo}\, f$ with

$$\langle v, P - Q\rangle \le c\|P - Q\|^{1+\theta} \qquad \text{for every } Q \in \mathrm{hypo}\, f. \tag{1}$$

Geometrically speaking, this inequality expresses the fact that, in a neighborhood of each point $P$ on the boundary of $\mathrm{hypo}\, f$, there exists a "subquadratic" smooth hypersurface $\Gamma(P)$ whose intersection with $\mathrm{hypo}\, f$ reduces to $P$. One could also say that $\Gamma(P)$ is *supertangent* to $\mathrm{hypo}\, f$ in a generalized sense. When $\theta = 1$ condition (1) means that the open sphere of center $P - \frac{v}{2c}$ and radius $\frac{1}{2c}$ lies outside $\mathrm{hypo}\, f$ and touches the boundary of $\mathrm{hypo} f$ at $P$. This property is also called *exterior sphere condition* and was studied by several authors, mainly in connection with regularity problems arising in the control theory.

If we strenghten the exterior sphere condition by requiring (1) to hold for *every* $v \in N^F_{\mathrm{hypo}\, f}(P) \cap \mathbb{S}^d$ (while in its formulation this is required just for *at least one* normal) with $\theta = 1$, we are in the class of functions whose hypograph has *positive reach* in the sense of Federer. In finite dimension, sets of positive reach were introduced by Federer in [13] as a generalization of convex sets and sets with $C^2$-boundary. If moreover we are also allowed to take $c = 0$, then the set is convex.

Upper semicontinuous functions whose hypograph has positive reach share several regularity properties with concave functions: it was proved in [6] that around a.e. points of their domain they are actually Lipschitz continuous, and twice differentiable a.e. In [8], [9] and [10] some regularity results were proved for the minimum time function of control problems; under suitable weak controllability assumptions, the latter is proved to have epigraph or hypograph with locally positive reach, thus generalizing the results of [4] and [5].

However, it is easy to give examples where the hypograph of the minimum time function does not satisfy an exterior sphere property, so that the results of [9, 15] can not be applied. Let us consider the constant control system

$$\begin{cases} x'(t) = 0, \\ y'(t) = u(t) \in [0, 1], \\ (x(0), y(0)) = (x_0, y_0) \in \mathbb{R}^2, \end{cases} \tag{2}$$

together with the target $\mathcal{T} = \{(x, \beta) \mid \beta \geq f(x)\}$, where $f(x) = 1$ if $x \leq 0$ and $f(x) = -x^{\frac{2}{3}}$ if $x > 0$.

The minimum time to reach the target $\mathcal{T}$ subject to the above control system is denoted by $T$. It can be proved (see the Appendix) that hypo $T$ does not satisfy an exterior sphere condition, but still enjoys the weaker uniformity regularity property (1) with $\theta = 1/2$.

The previous considerations motivate us to study the class $\mathscr{F}(\Omega)$ of real functions defined on $\Omega \subset \mathbb{R}^d$ satisfying condition (1) in order to provide a new regularity class which, hopefully, will cover the regularity properties for the minimum time function of certain classes of nonlinear control systems and differential inclusions (see [3]) that do not satisfy an exterior sphere condition. We will refer to this property as $N$-*regularity* (see Definition 2). We state our first general result, whose main ideas were presented in our recent paper [14], for closed set $K \subset \mathbb{R}^{d+1}$ concerning the structure and dimension of the set $K^{(j)}$ of points on $\partial K$ where the Fréchet normal cone to $\partial K$ has dimension larger than or equal to $j$. This result generalizes a similar result proved by Federer for sets with positive reach. Indeed, it shows that $K^{(j)}$ can be covered by countably many Lipschitz graphs of $d - j + 1$ variables.

**Theorem 1.** *Let $K \subseteq \mathbb{R}^{d+1}$ be closed; then $K^{(j)}$ is countably $\mathscr{H}^{d-j+1}$-rectifiable. In particular, also $K_{\pm}^{(j)}$ are countably $\mathscr{H}^{d-j+1}$-rectifiable.*

The sets $K_{\pm}^{(j)}$ are here defined in the same way of $K^{(j)}$ by taking the normal cone to, respectively, $K$ and $\overline{\mathbb{R}^{d+1} \setminus K}$; see Definition 5. Concerning the differentiability properties of functions, we denote by $\mathscr{S}_f$ the set of non-differentiability points of $f$ and prove the following result:

**Theorem 2.** *Let $\Omega \subseteq \mathbb{R}^d$ be a nonempty open set and $f : \Omega \to \mathbb{R}$ be an upper semicontinuous function with $f \in L^\infty_{\mathrm{loc}}(\Omega)$. Assume that the closed set $K :=$ hypo $f$ is $N$-regular in $\Omega \times \mathbb{R}$. Then $f \in BV_{\mathrm{loc}}(\Omega)$ and $\mathscr{L}^d(\mathscr{S}_f) = 0$. In particular, $f$ is differentiable a.e.*

## 2 Notation

Let $K$ be a closed subset of $\mathbb{R}^d$, $S \subseteq \mathbb{R}^d$, $x = (x_1, \ldots, x_d) \in K$, $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$, $r > 0$. We denote by $\langle \cdot, \cdot \rangle$, the usual *scalar product* in $\mathbb{R}^d$; $\partial S$, $\mathrm{int}(S)$, $\overline{S}$, the *topological boundary, interior* and *closure* of $S$, respectively; $\mathcal{P}(S) := \{B \subseteq$

$\mathbb{R}^d : B \subseteq S\}$, the *power set* of $S$; $\mathbb{B}^d := \{w \in \mathbb{R}^d : \|w\| < 1\}$, the *unit open ball* (centered at the origin); $\mathbb{S}^{d-1} := \{w \in \mathbb{R}^d : \|w\| = 1\} = \partial\mathbb{B}^d$, the *unit sphere* (centered at the origin); $B(y,r) := \{z \in \mathbb{R}^d : \|z-y\| < r\} = y + r\mathbb{B}^d$, the *open ball* of center $y$ and radius $r$; $d_K(y) := \mathrm{dist}(y,K) = \min\{\|z-y\| : z \in K\}$, the *distance* of $y$ from $K$; $\pi_K(y) := \{z \in K : \|z-y\| = d_K(y)\}$, the *set of projections* of $y$ onto $K$: if $\pi_K(y)$ contains an unique element $\xi$, we will write $\pi_K(y) = \xi$. $\mathscr{H}^p(S)$ and $\dim_{\mathscr{H}}(S)$, the *p-dimensional Hausdorff measure* and the *Hausdorff dimension* of $S$. The *characteristic function* $\chi_S : \mathbb{R}^d \to \{0,1\}$ of $S$ is defined as $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ if $x \notin S$. If $V, W \subseteq \mathbb{R}^d$ are two subset of $\mathbb{R}^d$, we will write $V \subset\subset W$ if $V$ is bounded and $\overline{V} \subseteq W$. Given a set $X$, $\mathrm{card}(X)$ denotes the number of its elements. The *Fréchet normal cone* and the *Bouligand tangent cone* to $K$ at $x$ are defined respectively by

$$N_K^F(x) := \left\{ v \in \mathbb{R}^d : \limsup_{\substack{y \to x \\ y \in K \setminus \{x\}}} \left\langle v, \frac{y-x}{\|y-x\|} \right\rangle \leq 0 \right\};$$

$$T_K^F(x) := \left\{ \lambda\xi \in \mathbb{R}^d : \lambda \geq 0, \exists \{y_n\}_n \subseteq K \setminus \{x\}, y_n \to x \text{ s.t. } \xi = \lim_{n\to\infty} \frac{y_n - x}{\|y_n - x\|} \right\}.$$

**Definition 1.** *Let $\Omega \subseteq \mathbb{R}^d$ and $f : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ be a function. For $x \in \Omega$ fixed we denote by $\overline{f}(x) := \limsup_{\substack{y\to x \\ y \neq x}} f(y)$; $\widetilde{f}(x) := \limsup_{y \to x} f(y) = \max\{f(x), \overline{f}(x)\}$; $\underline{f}(x) := \liminf_{\substack{y\to x \\ y \neq x}} f(y)$; $\underset{\sim}{f}(x) := \liminf_{y\to x} f(y) = \min\{f(x), \underline{f}(x)\}$; $\mathrm{dom}(f) := \{z \in \Omega : f(z) \in \mathbb{R}\}$, the* domain *of $f$; $\mathrm{hypo}\, f := \{(z,\beta) \in \Omega \times \mathbb{R} : \beta \leq f(z)\}$, the* hypograph *of $f$; $\mathrm{epi}\, f := \{(z,\alpha) \in \Omega \times \mathbb{R} : \alpha \geq f(z)\}$, the* epigraph *of $f$; $\partial^F f(x) := \{v \in \mathbb{R}^d : (-v,1) \in N^F_{\mathrm{hypo}\, f}(x, f(x))\}$; $\partial_F f(x) := \{v \in \mathbb{R}^d : (v,-1) \in N^F_{\mathrm{epi}\, f}(x, f(x))\}$. We say that $f$ is* upper *(respectively,* lower*)* semicontinuous *if $f(x) \geq \overline{f}(x)$ (resp., if $f(x) \leq \underline{f}(x)$) for any $x \in \Omega$. The sets $\partial^F f(x)$ and $\partial_F f(x)$ are called respectively the* Fréchet superdifferential *and the* Fréchet subdifferential *of $f$ at $x$.*

If $\Omega \subseteq \mathbb{R}^d$ is open , we denote by $BV(\Omega)$ the set of *function of bounded variation in $\Omega$*, and if $u \in BV(\Omega)$, we denote by $\|Du\|$ the total variation of the vector-valued measure $Du$. The *perimeter of $E$ in $\Omega$* is $P(E,\Omega) = \|D\chi_E\|(\Omega)$.

Let $A \subseteq \mathbb{R}^d$ and $0 \leq p \leq d$. Let $k \in \mathbb{N}$, we say that $A \subseteq \mathbb{R}^d$ is *countably $\mathscr{H}^k$-rectifiable* if $A \subseteq \mathcal{N} \cup \bigcup_{i=1}^{\infty} S_i$, where $S_i$ are suitable $k$-dimensional Lipschitz surfaces and $\mathscr{H}^k(\mathcal{N}) = 0$.

## 3 Standing hypothesis and first consequences

**Definition 2.** *Let $U \subseteq \mathbb{R}^{d+1}$ be open and $K \subseteq \mathbb{R}^{d+1}$ be nonempty and relatively closed in $U$. We say that $K$ is* $N$-regular in $U$ *if there exists an upper semicontinuous multifunction $N : \partial K \cap U \rightrightarrows \mathbb{S}^d$ such that for every $x \in \partial K \cap U$ the following two properties hold:*

*(N1)* $\emptyset \neq N(x) \subseteq N_K^F(x) \cap \mathbb{S}^d$;

*(N2)* *there exist* $\delta_x \in ]0, dist\,(x, \partial U)[$ *and a continuous function* $\omega_x : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\lim_{r\to 0^+} \omega_x(r)/r = 0$ *and satisfying the following uniformity property: for every* $y_1 \in \left(x + \delta_x \mathbb{B}^{d+1}\right) \cap \partial K$ *there exists* $\nu(y_1) \in N(y_1)$ *such that*

$$\langle \nu(y_1), y_2 - y_1 \rangle \leq \omega_x(\|y_2 - y_1\|) \text{ for all } y_2 \in \left(x + \delta_x \mathbb{B}^{d+1}\right) \cap K.$$

*We will say that* $K \subseteq \mathbb{R}^{d+1}$ *is* $N$-regular *if* $K$ *is* $N$*-regular in* $\mathbb{R}^{d+1}$*. Possibly replacing the set-valued map* $N$ *with* $x \mapsto \overline{N(x)}$*, when* $K$ *is* $N$*-regular in* $U$ *we can always assume that* $N$ *has closed graph.*

*Example 1.* Every set $K$ that is the closure of an open $C^1$ domain is $N$-regular, moreover a closed convex set $C$ is $N$-regular with $N(x) = N_C^F(x) \cap \mathbb{S}^d$

**Definition 3.** *Let* $U \subseteq \mathbb{R}^{d+1}$ *be open and* $K \subseteq \mathbb{R}^{d+1}$ *be nonempty and relatively closed in* $U$*; let also* $z \in \partial K \cap U$, $\theta \in ]0, 1]$ *and* $C \geq 0$*. We define*

$$\mathscr{N}_K^{C,\theta,U}(z) := \left\{\zeta \in \mathbb{R}^{d+1} : \langle \zeta, z' - z\rangle \leq C \cdot \|\zeta\| \cdot \|z' - z\|^{1+\theta} \right. \\ \left. \text{for all } z' \in K \cap U\right\}. \tag{3}$$

*If* $K$ *is closed,* $U = \mathbb{R}^{d+1}$ *and* $z \in \partial K$ *we will simply write* $\mathscr{N}_K^{C,\theta}(z)$ *instead of* $\mathscr{N}_K^{C,\theta,\mathbb{R}^{d+1}}(z)$*. We notice that if* $\zeta \in \mathscr{N}_K^{C,\theta,U}(x)$*, then* $\mu\zeta \in \mathscr{N}_K^{C,\theta,U}(x)$ *for all* $\mu \geq 0$ *and the multifunction* $\mathscr{N}_K^{C,\theta,U} : \partial K \cap U \rightrightarrows \mathbb{R}^{d+1}$ *has closed graph.*

*Let now* $\Omega \subseteq \mathbb{R}^d$ *be nonempty and open and* $f : \Omega \to \mathbb{R}$ *be upper semicontinuous. By adapting the previous definition, for* $(x, \beta_x) \in \partial\mathrm{hypo}\, f \cap (\Omega \times \mathbb{R})$ *we define* $\hat{\mathscr{N}}_{\mathrm{hypo}\, f}^{C,\theta}(x, \beta_x)$ *as the set of those* $(v, \lambda) \in \mathbb{R}^d \times \mathbb{R}$ *such that* $\forall (y, \beta) \in \mathrm{hypo}\, f$.

$$\langle (v, \lambda), (y - x, \beta - \beta_x)\rangle \leq C\|(v, \lambda)\| \left(\|y - x\|^{1+\theta} + |\beta - \beta_x|^{1+\theta}\right) \tag{4}$$

*We notice that there exist constants* $c_1, c_2 > 0$ *depending only on* $d$ *and* $\theta$ *such that*

$$\mathscr{N}_{\mathrm{hypo}\, f}^{c_1 C,\theta,\Omega\times\mathbb{R}}(x, \beta_x) \subseteq \hat{\mathscr{N}}_{\mathrm{hypo}\, f}^{C,\theta}(x, \beta_x) \subseteq \mathscr{N}_{\mathrm{hypo}\, f}^{c_2 C,\theta,\Omega\times\mathbb{R}}(x, \beta_x).$$

*It is clear from the definition that also* $\hat{\mathscr{N}}_{\mathrm{hypo}\, f}^{c,\theta} : \partial\, \mathrm{hypo}\, f \cap (\Omega \times \mathbb{R}) \rightrightarrows \mathbb{R}^{d+1}$ *has closed graph.*

We are ready now to introduce the classes of sets and functions subject of our investigation.

**Definition 4.** *Let* $U \subseteq \mathbb{R}^{d+1}$ *and* $\Omega \subseteq \mathbb{R}^d$ *be open. We define:*

$$\begin{aligned}
\mathscr{F}^U :=& \{K \subseteq U : K \text{ is relatively closed in } U \text{ and } \exists\, C \geq 0,\, 0 < \theta \leq 1 \text{ s.t.} \\
& \qquad \mathscr{N}_K^{C,\theta,U}(z) \neq \{0\} \text{ for all } z \in \partial K \cap U\} \\
\mathscr{F} :=& \mathscr{F}^{\mathbb{R}^{d+1}} \\
\mathscr{F}(\Omega) :=& \{f : \Omega \to \mathbb{R} : f \text{ u.s.c., } \mathrm{hypo}\, f \in \mathscr{F}^{\Omega\times\mathbb{R}}\} \\
=& \{f : \Omega \to \mathbb{R} : f \text{ u.s.c., } \exists C \geq 0,\ 0 < \theta \leq 1 \text{ such that} \\
& \qquad \hat{\mathscr{N}}_{\mathrm{hypo}\, f}^{C,\theta}(x, \beta_x) \neq \{0\}\ \forall (x, \beta_x) \in \partial\, \mathrm{hypo}\, f \cap (\Omega \times \mathbb{R})\}.
\end{aligned}$$

If $K \in \mathscr{F}^U$, then there exist $C > 0$, $0 < \theta \leq 1$ such that $K$ is $N$-regular in $U$ with

$$N(x) := \mathscr{N}_K^{C,\theta,U}(x) \cap \mathbb{S}^d \subseteq N_K^F(x), \quad \omega_x(r) := r^{1+\theta} \qquad \forall x \in \partial K \cap U.$$

The upper semicontinuity of $N$ follows from the fact that $\mathscr{N}_K^{C,\theta,U}(x)$ has closed graph.

We refer the reader to [13, 11] for a survey of the properties satisfied by sets with positive reach, on which the class $\mathscr{F}$ is modeled.

## 4 Regularity results for sets

In this section we will prove regularity results for the boundary of a closed set $K \subseteq \mathbb{R}^{d+1}$ in a quite general setting. They will be used later to prove fine regularity properties for functions in the class $\mathscr{F}(\Omega)$.

The first result extends an analogous result for the class of sets with positive reach proved by Federer in Remark 4.15 of [13]. Roughly speaking it states that points with *large* normal cone are relatively *few*.

**Definition 5.** *Let $K \subseteq \mathbb{R}^{d+1}$ be closed; for $j = 1, ..., d+1$ we define*

$$K^{(j)} := \left\{x \in \partial K : \dim\left(N_{\partial K}^F(x)\right) \geq j\right\}, \tag{5}$$

$$K_+^{(j)} := \left\{x \in \partial K : \dim\left(N_K^F(x)\right) \geq j\right\}, \tag{6}$$

$$K_-^{(j)} := \left\{x \in \partial K : \dim\left(N_{\overline{\mathbb{R}^{d+1} \setminus K}}^F(x)\right) \geq j\right\}. \tag{7}$$

*We notice that $K^{(j_1)} \supseteq K^{(j_2)}$, $K_+^{(j_1)} \supseteq K_+^{(j_2)}$, $K_-^{(j_1)} \supseteq K_-^{(j_2)}$ if $1 \leq j_1 \leq j_2 \leq d+1$, and that $K_\pm^{(j)} \subseteq K^{(j)}$. Clearly, $K^{(1)} = \{x \in \partial K : N_{\partial K}^F(x) \neq \{0\}\}$.*

In order to use local arguments, we will need the following estimate which gives some *uniformity* with respect to the elements of the normal cone, which can be proved exploiting compactness of $N_{\partial K}^F(x) \cap \mathbb{S}^d$: for every $x \in K^{(1)}$ and $0 < \varepsilon \leq 1$ we have

$$\begin{aligned} \delta(x,\varepsilon) := \frac{1}{2} \sup \Big\{ \ & \delta \in \mathbb{R} : \langle v, y - x \rangle \leq \varepsilon \|y - x\|, \\ & \text{for all } y \in \partial K \cap \left(x + \delta \mathbb{B}^{d+1}\right), \, v \in N_{\partial K}^F(x) \cap \mathbb{S}^d \Big\} > 0 \end{aligned} \tag{8}$$

We are now ready to prove the first main result of the paper.

*Proof (Proof of Theorem 1.).* We begin by constructing a countable covering $\{K_{n,m,h,l}^{(j)}\}_{n,m,h,l \in \mathbb{N}}$ of $K^{(j)}$; we will prove later that each element of the covering is rectifiable and this will establish our result.

Define the function $w : (\mathbb{R}^{d+1})^j \to [0,1]$

$$w(v_1, \ldots, v_j) := \min \left\{ \left\| \sum_{i=1}^{j} \alpha_i v_i \right\| : \alpha_i \in \mathbb{R}, \, \sum_{i=1}^{j} |\alpha_i| = 1 \right\}.$$

We notice that $w$ is continuous and invariant under permutations of its arguments, so if $V = \{v_1, \dots, v_j\}$ we will write $w(V)$ instead of $w(v_1, ..., v_j)$ . We have that $w(V) = 0$ iff the elements of $V$ are linearly dependent.

Let $\{a_l\}_{l\in\mathbb{N}}$ be a countable dense set in $\mathbb{R}^{d+1}$. For every $x \in K^{(j)}$ choose $V_x \subseteq N^F_{\partial K}(x) \cap \mathbb{S}^d$ with $\operatorname{card} V_x = j$ and $w(V_x) > 0$, $V_x = \{v_x^{(i)}\}_{i=1,\dots,j}$. Consider the countable set

$$\mathcal{A}^{(j)} := \left\{V' \subseteq \mathbb{Q}^{d+1} : \operatorname{card}(V') = j,\, w(V') > 0\right\},$$

Being $\mathcal{A}^{(j)}$ countable, we can order its elements and write $\mathcal{A}^{(j)} = \{V'_n\}_{n\in\mathbb{N}}$. We set $V_n^{(j)} = \operatorname{Span}(V'_n)$ and consider the countable set of $j$-dimensional planes $\mathcal{V}^{(j)} := \{V_n^{(j)}\}_{n\in\mathbb{N}}$. Define also $W_n^{(j)} := (V_n^{(j)})^\perp$, $n \in \mathbb{N}$, and $W^{(j)} := \{W_n^{(j)}\}_{n\in\mathbb{N}}$. Given $n, m, h, l \in \mathbb{N}$, let $v_1, \dots, v_j \in \mathbb{Q}^{d+1}$ be such that $V'_n = \{v_1, \dots, v_j\}$ and set

$$K^{(j)}_{n,m,h,l} := \left\{ x \in K^{(j)} \cap \left(a_l + \tfrac{1}{2(h+1)}\mathbb{B}^{d+1}\right) : \begin{array}{l} w(V_x) \ge \frac{1}{m+3},\ \delta\big(x, \frac{1}{2(m+3)^2}\big) \ge \frac{1}{h+1}, \\ \big\|v_x^{(i)} - v_i\big\| \le \frac{1}{2(m+3)^2} \text{ for } i = 1, \dots, j \end{array} \right\},$$

where $\delta(x, \frac{1}{2(m+3)^2})$ is as in (8) with $\varepsilon = (2(m+3)^2)^{-1}$.

It turns out that $K^{(j)} \subseteq \bigcup_{n,m,h,l\in\mathbb{N}} K^{(j)}_{n,m,h,l}$: given $x \in K^{(j)}$, we choose in this sequence the indexes: $m$, $n$, $h$, $l$, to fulfill the properties yielding $x \in K^{(j)}_{n,m,h,l}$.

We prove now that for any $x_1, x_2 \in K^{(j)}_{n,m,h,l}$ the orthogonal projection $\pi_{W_n^{(j)}} : K^{(j)}_{n,m,h,l} \to W_n^{(j)}$ satisfies

$$\|\pi_{W_n^{(j)}}(x_2 - x_1)\|^2 \ge \frac{m+1}{m+3}\,\|x_2 - x_1\|^2. \tag{9}$$

Indeed, we notice that if $V'_n = \{v_1, \dots, v_j\}$, then each $v_i$ is near to a normal vector both at $x_1$, and at $x_2$. By exploiting the definition of $K^{(j)}_{n,m,h,l}$, this fact yields:

$$|\langle v_i, x_2 - x_1\rangle| \le \frac{1}{(m+3)^2}\|x_2 - x_1\| \quad \text{for every } i = 1, \dots, j.$$

Given $v \in V_n^{(j)}$, $v \neq 0$, we can find (in a unique way) $\alpha_i \in \mathbb{R}$, $i = 1, \dots, j$ such that $v = \sum_{i=1}^j \alpha_i v_i$; therefore

$$\left|\left\langle \frac{v}{\|v\|}, x_2 - x_1\right\rangle\right| \le \frac{\sum_{i=1}^j |\alpha_i| \cdot |\langle v_i, x_2 - x_1\rangle|}{\|\sum_{i=1}^j \alpha_i v_i\|} \le \frac{\|x_2 - x_1\|}{(m+3)^2}\frac{\sum_{i=1}^j |\alpha_i|}{\|\sum_{i=1}^j \alpha_i v_i\|}.$$

Set $\beta_i := \alpha_i / \sum_{s=1}^j |\alpha_s|$; we have $\sum_{i=1}^j |\beta_i| = 1$ and thus

$$\begin{aligned}\left|\left\langle \frac{v}{\|v\|}, x_2 - x_1\right\rangle\right| &\le \frac{\|x_2 - x_1\|}{(m+3)^2}\frac{1}{\|\sum_{i=1}^j \beta_i v_i\|} \le \frac{\|x_2 - x_1\|}{(m+3)^2}\frac{1}{w(v_1, \dots, v_j)} \\ &\le \frac{2}{m+3}\|x_2 - x_1\|\end{aligned}$$

because $w(v_1, \dots, v_j) \geq (2(m+3))^{-1}$. Therefore,

$$\|\pi_{W_n^{(j)}}(x_2 - x_1)\|^2 = \|x_2 - x_1\|^2 - \langle \pi_{V_n^{(j)}}(x_2 - x_1), x_2 - x_1 \rangle \geq \frac{m+1}{m+3} \|x_2 - x_1\|^2.$$

By (9), for each $n, m, h, l$ the inverse map $\pi^{-1}_{W_n^{(j)}} : \pi_{W_n^{(j)}}(K^{(j)}_{n,m,h,l}) \to K^{(j)}_{n,m,h,l}$. is Lipschitz continuous and, by Kirszbraun's Theorem, it can be extended to a Lipschitz function defined on the whole $W_n^{(j)}$. This ends the proof.

## 5 Application to functions: *BV* regularity and structure of singular set

In this section we will apply the results obtained in the previous one to closed sets that can be written as hypographs of upper semicontinuous functions possessing at least one normal direction at a.e. point of the boundary of their hypograph; our goal is to obtain regularity results for such functions.

**Definition 6.** *Let $\Omega$ be a nonempty open subset of $\mathbb{R}^d$ and $f : \Omega \to \mathbb{R}$ be a function. For each $x \in \Omega$, we define*

$$\begin{aligned} J_f &:= \{x \in \Omega : \widetilde{f}(x) \neq \underset{\sim}{f}(x)\} = \{x \in \Omega : f \text{ is not continuous at } x\}, \\ S_f &:= \{x \in \Omega \setminus J_f : \big(\mathbb{S}^{d-1} \times \{0\}\big) \cap N^F_{\operatorname{hypo} f}(x, f(x)) \neq \emptyset\}, \\ \mathscr{S}_f &:= J_f \cup S_f. \end{aligned}$$

We begin with a trivial corollary of Theorem 1, dealing with the singularities corresponding to large dimension of the normal cone.

**Corollary 1.** *Let $\Omega$ be a nonempty open subset of $\mathbb{R}^d$ and $f : \Omega \to \mathbb{R}$ be an upper semicontinuous function. Set $K = \operatorname{hypo} f$ and assume that $N^F_K(x, \beta) \neq \{0\}$ for $\mathscr{H}^d$-a.e. $(x, \beta) \in \partial K \cap (\Omega \times \mathbb{R})$. Then for $\mathscr{L}^d$-almost every $x \in \Omega$ there exists $\zeta_x \in \mathbb{S}^d$ such that $N^F_K(x, \beta) \subseteq \mathbb{R}\zeta_x$ for all $\beta$ with $(x, \beta) \in \partial K \cap (\Omega \times \mathbb{R})$.*

*Proof.* By Theorem 1, $K^{(2)}$ is $\mathscr{H}^{d-1}$-rectifiable and hence $\mathscr{H}^d$-negligible. If $\pi : \Omega \times \mathbb{R} \to \Omega$ denotes the canonical projection on $\Omega$, then $\Omega \cap \big(\pi(\partial K \setminus K^{(1)}) \cup \pi(K^{(2)})\big)$ is $\mathscr{L}^d$-negligible, hence $E := \Omega \setminus \big(\pi(\partial K \setminus K^{(1)}) \cup \pi(K^{(2)})\big)$ has the same measure of $\Omega$. The results follows.

We recall the following result, proved in Theorem 1.2 of [14]:

**Theorem 3.** *Let $\Omega$ be a nonempty open subset of $\mathbb{R}^d$ and let $f \in BV_{\mathrm{loc}}(\Omega)$ be an upper semicontinuous function; set $K := \operatorname{hypo} f$. Assume that for $\mathscr{H}^d$-a.e. $(x, \beta_x) \in \partial K \cap (\Omega \times \mathbb{R})$ it holds $N^F_K(x, \beta_x) \neq \{0\}$. Then $\mathscr{L}^d(\mathscr{S}_f) = 0$.*

We are going to study the regularity properties of upper semicontinuous functions $f$ such that hypo$f$ is $N$-regular. One of our primary goals is to estimate the size of the singular set $\mathscr{S}_f$; to this aim it will be important to assume that $f$ is of class $BV$.

We can now prove the second main result of the paper:

*Proof (of Theorem 2).* Let us prove that $f \in BV(U)$ for any open set $U$ such that $U \subset\subset \Omega$. According to Theorem 1, we have that $\partial K$ is rectifiable, whence $P(K, U \times \mathbb{R}) = P(K, U \times ]-2\|f\|_{L^\infty}, 2\|f\|_{L^\infty}[) < \infty$. According to Theorem 4 in [2], we have that $f \in BV_{\mathrm{loc}}(\Omega)$, so we can apply Theorem 3.

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