STRICT INTERIOR APPROXIMATION OF SETS OF FINITE PERIMETER AND FUNCTIONS OF BOUNDED VARIATION

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Abstract. It is well known that sets of finite perimeter can be strictly approximated by smooth sets, while, in general, one cannot hope to approximate an *open* set Ω of finite perimeter in $\mathbb{R}^n$ strictly *from within*. In this note we show that, nevertheless, the latter type of approximation is possible under the mild hypothesis that the $(n-1)$-dimensional Hausdorff measure of the topological boundary $\partial\Omega$ equals the perimeter of Ω. We also discuss an optimality property of this hypothesis, and we establish a corresponding result on strict approximation of BV-functions from a prescribed Dirichlet class.

1. Introduction and statement of the results

For arbitrary $n \in \mathbb{N}$, the perimeter $\mathrm{P}(\Omega)$ of a measurable set Ω in $\mathbb{R}^n$ is defined as

$$\mathrm{P}(\Omega) := \sup\left\{ \int_\Omega \operatorname{div}\varphi \,\mathrm{d}\mathscr{L}^n \,:\, \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n,\mathbb{R}^n),\, \sup_{\mathbb{R}^n}|\varphi| \le 1 \right\} \in [0,\infty]$$

(with the n-dimensional Lebesgue measure $\mathscr{L}^n$). As a consequence of the divergence theorem, this definition generalizes the more classical notions of perimeter, and in particular, for sets Ω with smooth boundary, one has the identity $\mathrm{P}(\Omega) = \mathscr{H}^{n-1}(\partial\Omega)$ with the $(n-1)$-dimensional Hausdorff measure $\mathscr{H}^{n-1}$. Even for non-smooth sets Ω, the above notion of perimeter is very reasonable and useful, but in general one only has the inequality (see [1, Proposition 3.62])

$$\mathrm{P}(\Omega) \le \mathscr{H}^{n-1}(\partial\Omega)\,. \tag{1.1}$$

Though a set Ω with finite perimeter $\mathrm{P}(\Omega) < \infty$ in $\mathbb{R}^n$ is in several regards well-behaved, one often needs to approximate Ω with smooth sets in such a way that also the perimeters converge, and indeed it is very well known (see [1, Theorem 3.42]) that this is always possible: one can find open sets Ω_ε with smooth boundaries in $\mathbb{R}^n$ such that Ω_ε converges to Ω in measure and

$$\mathscr{H}^{n-1}(\partial\Omega_\varepsilon) = \mathrm{P}(\Omega_\varepsilon) \text{ converges to } \mathrm{P}(\Omega) \text{ for } \varepsilon \searrow 0. \tag{1.2}$$

In this paper we are concerned with strict interior approximation: that means, we investigate whether, for an open Ω, one can additionally take

$$\Omega_\varepsilon \Subset \Omega\,. \tag{1.3}$$

For bounded Lipschitz domains Ω, this requirement can be achieved by the constructions described in [18, 17, 9]. For more general Ω, in contrast, some related problems have been considered in [20, 23, 24, 19, 21, 22], but it seems that approximations with (1.2) and (1.3) have not been found. The main obstacle in this regard is that the analogous approximation is impossible for *arbitrary* bounded open sets with finite perimeter in $\mathbb{R}^n$: indeed, when one

Date: November 5, 2014.

2010 *Mathematics Subject Classification.* Primary 28A75, 26B30, 41A63, 41A30; Secondary 28A78, 26B15.

The research leading to these results has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement GeMeThnES n° 246923.

considers $\Omega = (0,1)^{n-1} \times \big[(0,1) \setminus \{\frac{1}{2}\}\big]$, then it follows (using the lower semicontinuity [1, Proposition 3.38(b)] of the perimeter on both halves of Ω) that for all approximations Ω_ε one necessarily has $\liminf_{\varepsilon \searrow 0} \mathrm{P}(\Omega_\varepsilon) \geq 2n{+}2$, but $\mathscr{H}^{n-1}(\partial\Omega) = 2n{+}1$ and $\mathrm{P}(\Omega) = 2n$ are strictly smaller.

Here, we will show that these and similar examples are ruled out and strict interior approximation is again possible if one only imposes the mild extra assumption (1.4) below. This is actually the first main result of the present paper, which will be proved in Section 3:

Theorem 1.1 (strict interior approximation of the perimeter)**.** *Suppose that Ω is a bounded open set in $\mathbb{R}^n$ whose topological boundary is well-behaved in the sense of*

$$\mathscr{H}^{n-1}(\partial\Omega) = \mathrm{P}(\Omega)\,. \tag{1.4}$$

Then, for every $\varepsilon > 0$, there exists an open set Ω_ε with smooth boundary in $\mathbb{R}^n$ and with

$$\Omega_\varepsilon \Subset \Omega\,, \qquad \Omega \setminus \Omega_\varepsilon \subset \mathcal{N}_\varepsilon(\partial\Omega) \cap \mathcal{N}_\varepsilon(\partial\Omega_\varepsilon)\,, \qquad \mathrm{P}(\Omega_\varepsilon) \leq \mathrm{P}(\Omega) + \varepsilon\,, \tag{1.5}$$

where we have used the notation $\mathcal{N}_\varepsilon(\,\cdot\,)$ for ε-neighborhoods of sets in $\mathbb{R}^n$.

Several comments on this approximation theorem follow.

First of all, we remark that boundedness of Ω is not essential, but is just imposed in order to simplify our statements and proofs. In fact, the extension of Theorem 1.1 (and of Theorem 1.2 stated below) to unbounded Ω is discussed in [5, Section 3.3].

Furthermore, by De Giorgi's structure theorem [1, Theorem 3.59] (combined with [1, Proposition 3.6]), whenever the perimeter $\mathrm{P}(\Omega)$ is finite, then one has equality

$$\mathrm{P}(\Omega) = \mathscr{H}^{n-1}(\mathcal{F}\Omega) \tag{1.6}$$

with the measure of the reduced boundary $\mathcal{F}\Omega$ of Ω. In this case, we can thus rephrase the hypothesis (1.4) — also taking into account that $\mathcal{F}\Omega \subset \partial\Omega$ holds by definition — as the requirement

$$\mathscr{H}^{n-1}(\partial\Omega \setminus \mathcal{F}\Omega) = 0\,. \tag{1.7}$$

In this light, (1.4) yields a measure-theoretic control on the topological boundary $\partial\Omega$, and it seems plausible that such a condition may be relevant in order to incorporate the topological condition (1.3) into the measure-theoretic theory of the perimeter. A more detailed discussion of (1.4), including the proof of a restricted optimality property, is postponed to Section 5.

Turning from the hypothesis of Theorem 1.1 to its conclusion, we observe that (1.5) implies $\bigcup_{\varepsilon>0} \Omega_\varepsilon = \Omega$ and $\lim_{\varepsilon\searrow 0} \|\mathbb{1}_{\Omega_\varepsilon} - \mathbb{1}_\Omega\|_{\mathrm{L}^1(\mathbb{R}^n)} = \lim_{\varepsilon\searrow 0} \mathscr{L}^n(\Omega \setminus \Omega_\varepsilon) = 0$. Via the lower semicontinuity of the perimeter (and the smoothness of $\partial\Omega_\varepsilon$), we thus infer that (1.2) is valid for the approximations of the theorem. Furthermore, whenever $\mathrm{P}(\Omega)$ is finite, it equals the total variation $|\mathrm{D}\mathbb{1}_\Omega|(\mathbb{R}^n)$ of the gradient measure of $\mathbb{1}_\Omega$, so that the approximating sequence $(\mathbb{1}_{\Omega_{1/k}})_{k\in\mathbb{N}}$ converges strictly in $\mathrm{BV}(\mathbb{R}^n)$ to $\mathbb{1}_\Omega$ (in the sense of [1, Definition 3.14]). In addition, we also record that the first two assertions in (1.5) imply the bounds

$$\mathrm{d}_{\mathcal{H}}(\Omega_\varepsilon, \Omega) \leq \varepsilon \quad \text{and} \quad \mathrm{d}_{\mathcal{H}}(\partial\Omega_\varepsilon, \partial\Omega) \leq \varepsilon \tag{1.8}$$

in the Hausdorff distance[1] $\mathrm{d}_{\mathcal{H}}$.

Last but not least, let us emphasize that the precise form of the last condition in (1.5) is significant; indeed, if one requires only the weaker bound

$$\mathrm{P}(\Omega_\varepsilon) \leq C\mathscr{H}^{n-1}(\partial\Omega) \tag{1.9}$$

with a certain dimensional constant $C \in (1,\infty)$, then the existence of interior approximations Ω_ε with (1.9) can be concluded from a standard argument, which is based on a covering of $\partial\Omega$

[1]For non-empty, bounded subsets A and B of $\mathbb{R}^n$, the Hausdorff distance is defined by $\mathrm{d}_{\mathcal{H}}(A,B) := \max\big\{\sup_{a\in A} \operatorname{dist}(a,B),\, \sup_{b\in B} \operatorname{dist}(A,b)\big\} \in [0,\infty)\,.$

with suitable balls and obtains the Ω_ε by removing these balls from Ω (compare with [1, Proof of Proposition 3.62]). The same argument has been used in connection with [7, Proposition 8.1], where a statement similar to (1.5) has been claimed, but — as an inspection of the proof reveals — only a statement of the type (1.9) has been established. Indeed, it seems that (1.5) cannot be obtained via a covering of $\partial\Omega$ with balls, but that rather coverings with suitably flat objects are needed. In the proof of Theorem 1.1 we will see that such refined coverings can be constructed, when one involves ideas from the proof of De Giorgi's structure theorem in order to first decompose $\partial\Omega$ into a negligible set and countably many locally flat-looking pieces.

Next, for arbitrary $n, N \in \mathbb{N}$, we turn to approximation results for $\mathbb{R}^N$-valued functions of bounded variation on an open subset Ω of $\mathbb{R}^n$. We fix $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n, \mathbb{R}^N)$, for every $u \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ we set

$$\overline{u}(x) := \begin{cases} u(x) & \text{for } x \in \Omega \\ u_0(x) & \text{for } x \in \mathbb{R}^n \setminus \Omega \end{cases},$$

and we define

$$\mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big) := \{u \in \mathrm{BV}(\Omega, \mathbb{R}^N) \,:\, \overline{u} \in \mathrm{BV}(\mathbb{R}^n, \mathbb{R}^N)\}\,.$$

We observe that $\mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$ contains the Dirichlet class $\mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N) := u_0\big|_\Omega + \mathrm{W}^{1,1}_0(\Omega, \mathbb{R}^N)$ and is closed in the weak-$*$ topology of $\mathrm{BV}(\Omega, \mathbb{R}^N)$. Hence, $\mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$ may be seen as a natural replacement for $\mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N)$ in many regards, for instance when dealing with minimization problems for variational integrals in Dirichlet classes (compare [5, Section 2.2]). If Ω has sharp external cusps, then $\mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$ is strictly smaller than $\mathrm{BV}(\Omega, \mathbb{R}^N)$, but whenever Ω has a bounded Lipschitz boundary, then [1, Corollary 3.89] yields $\mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big) = \mathrm{BV}(\Omega, \mathbb{R}^N)$; thus, in general, functions in $\mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$ do not attain the boundary values of u_0 in any reasonable sense, but rather their derivative comprises some information about the deviation from these boundary values.

We now focus on the strict approximation of functions $u \in \mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$ from $\mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N)$. Whenever this approximation is possible for all functions u in the class, then $\mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$ is in fact the sequential closure of $\mathrm{W}^{1,1}_{u_0}(\Omega, \mathbb{R}^N)$ in both the weak-$*$ and the strict topology of $\mathrm{BV}(\Omega, \mathbb{R}^N)$ — and thus often the smallest possible replacement. Before stating our corresponding result, we mention a variety of closely related approximation theorems in the literature: first, the simpler case of functions u which do attain[2] the boundary values u_0 on $\partial\Omega$ is treated in [12, Remark 2.12], [6, Lemma B.1], [14, Lemma 1] and does not depend on the regularity of $\partial\Omega$; generalized boundary conditions are considered in [3, Lemma 5.1] and [13, Lemma A.2] under a C^2-assumption on $\partial\Omega$, and in [16, Theorem 3.1] for $u_0 \equiv 0$ and Lipschitz boundaries; finally, [2, Fact 3.3] deals with arbitrary generalized boundary data on bounded Lipschitz domains, but it seems that a complete proof in this generality has only been published in [6, Lemma B.2]. Our second main result, which will be established in Section 4, extends all these statements to cases with possibly irregular boundaries as follows.

Theorem 1.2 (strict approximation of BV-functions from a given Dirichlet class)**.** *Suppose that Ω is a bounded open set in $\mathbb{R}^n$ with $\mathscr{H}^{n-1}(\partial\Omega) = \mathrm{P}(\Omega) < \infty$. Then, for $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n, \mathbb{R}^N)$, $u \in \mathrm{BV}_{u_0}\big(\overline{\Omega}, \mathbb{R}^N\big)$, and every $\varepsilon > 0$ there exists a*

$$u_\varepsilon \in u_0\big|_\Omega + \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)$$

with

$$\int_\Omega |u_\varepsilon - u| \,\mathrm{d}\mathscr{L}^n \le \varepsilon \quad \textit{and} \quad |(\mathscr{L}^n, \mathrm{D}u_\varepsilon)|(\Omega) \le |(\mathscr{L}^n, \mathrm{D}\overline{u})|\big(\overline{\Omega}\big) + \varepsilon\,. \tag{1.10}$$

[2]In our terminology, attainment of the boundary values means $|\mathrm{D}\overline{u}|(\partial\Omega) = 0$; compare also [15, Theorem 1] for the construction of certain piecewise affine approximations in this situation.

Here, $(\mathscr{L}^n, \mathrm{D}\overline{u})$ denotes the $\mathbb{R}^{Nn+1}$-valued measure on $\mathbb{R}^n$ whose first component is $\mathscr{L}^n$ and whose other components are given by the $\mathbb{R}^{Nn}$-valued gradient measure $\mathrm{D}\overline{u}$, and $(\mathscr{L}^n, \mathrm{D}u_\varepsilon)$ is understood analogously. Writing $\mathrm{D}\overline{u} = (\nabla\overline{u})\mathscr{L}^n + \mathrm{D}^{\mathrm{s}}\overline{u}$ for the Lebesgue decomposition of $\mathrm{D}\overline{u}$ with respect to $\mathscr{L}^n$, the second inequality in (1.10) can thus be restated as

$$\int_\Omega \sqrt{1+|\nabla u_\varepsilon|^2}\,\mathrm{d}\mathscr{L}^n \le \int_\Omega \sqrt{1+|\nabla \overline{u}|^2}\,\mathrm{d}\mathscr{L}^n + |\mathrm{D}^{\mathrm{s}}\overline{u}|\big(\overline{\Omega}\big) + \varepsilon\,.$$

Finally, we stress that — via the Reshetnyak continuity theorem [1, Theorem 2.39] — we can also achieve variants of (1.10), as for instance $|\mathrm{D}u_\varepsilon|(\Omega) \le |\mathrm{D}\overline{u}|\big(\overline{\Omega}\big) + \varepsilon$.

Acknowledgement. The author would like to thank Lisa Beck for several comments on a preliminary version of the manuscript.

2. Some notation and terminology

We have tried to keep the notation of this paper as close as possible to the standard, and most of our terminology follows the one in the monographs [1, 12]. Nonetheless, we comment on some specific conventions:

Elementary geometry. First of all, $\mathrm{B}_\varrho(x_0)$ is the open ball with center x and radius ϱ in $\mathbb{R}^n$, and we set $\omega_n := \mathscr{L}^n(\mathrm{B}_1(0))$ (and $\omega_0 := 1$). For a vector subspace T of $\mathbb{R}^n$ and $y \in \mathbb{R}^n$, we write $T(y)$ for the orthogonal projection of y on T, and we set $T^\perp(y) := y - T(y)$. With this notation, the cone with axis T and opening parameter $\varepsilon > 0$ is defined as

$$C_\varepsilon(T) := \{y \in \mathbb{R}^n \,:\, |T^\perp(y)| \le \varepsilon |T(y)|\}\,. \tag{2.1}$$

Reduced boundaries. Following [1, Definition 3.54], we write $\mathcal{F}\Omega$ for the reduced boundary of any $\mathscr{L}^n$-measurable subset Ω of $\mathbb{R}^n$, taken in the largest open set in which Ω has locally finite perimeter. By $\nu_\Omega(x)$ we denote the (generalized) *exterior* unit normal to Ω at $x \in \mathcal{F}\Omega$, and we set $\mathrm{B}^\pm_\varrho(x) := \{y \in \mathrm{B}_\varrho(x) \,:\, \pm y \cdot \nu_\Omega(x) \ge 0\}$ (where the relevant Ω is indicated by the context). Finally, $\mathrm{Tan}^{n-1}(\mathcal{F}\Omega, x)$ is the approximate tangent space at $x \in \mathcal{F}\Omega$, that is an $(n-1)$-dimensional subspace T of $\mathbb{R}^n$ such that, for $\varrho \searrow 0$, the measures $\mathscr{H}^{n-1} \llcorner \frac{\mathcal{F}\Omega - x}{\varrho}$ locally weak-$*$ converge to $\mathscr{H}^{n-1} \llcorner T$ on $\mathbb{R}^n$; compare [1, Definition 2.79]. By [1, Theorem 3.59], $\mathrm{Tan}^{n-1}(\mathcal{F}\Omega, x)$ exists and equals the orthogonal complement of $\nu_\Omega(x)$ for $x \in \mathcal{F}\Omega$.

3. Strict interior approximation of the perimeter

We start this section with a lemma on the decomposition of $\mathcal{F}\Omega$ into flat pieces.

Lemma 3.1. *Consider an $\mathscr{L}^n$-measurable set Ω in $\mathbb{R}^n$. For every $\varepsilon > 0$, the reduced boundary $\mathcal{F}\Omega$ can be decomposed into countably many disjoint Borel subsets $R_1, R_2, R_3, \ldots$ such that*

$$x \in R_i \implies \begin{cases} \mathscr{L}^n(\Omega \cap \mathrm{B}^+_\varrho(x)) \le \varepsilon^2 \varrho^n \text{ for all } \varrho \in \big(0, \tfrac{1}{i}\big]\,, \\ R_i \cap \mathrm{B}_{1/i}(x) \subset x + \mathrm{C}_\varepsilon(\mathrm{Tan}^{n-1}(\mathcal{F}\Omega, x))\,. \end{cases}$$

Proof. For $j \in \mathbb{N}$, we set inductively

$$A_j := \left\{ x \in \mathcal{F}\Omega \,:\, \begin{matrix} \mathscr{H}^{n-1}(\mathcal{F}\Omega \cap \mathrm{B}_\varrho(x)) \ge \frac{1}{2}\omega_{n-1}\varrho^{n-1} \\ \mathscr{L}^n(\Omega \cap \mathrm{B}^+_\varrho(x)) \le \varepsilon^2 \varrho^n \end{matrix} \text{ for all } \varrho \le \tfrac{1}{j} \right\} \setminus \bigcup_{m=1}^{j-1} A_m\,.$$

Evidently the A_j are disjoint, by Fubini's theorem they are Borel sets, and by De Giorgi's structure theorem [1, Theorem 3.59] their union is all of $\mathcal{F}\Omega$. By a well-known argument, taken from the proof of the structure theorem, one can moreover show that for every $x \in A_j$ there exists some $r_x > 0$ (depending also on ε) with

$$A_j \cap \mathrm{B}_{r_x}(x) \subset x + \mathrm{C}_\varepsilon(\mathrm{Tan}^{n-1}(\mathcal{F}\Omega, x))\,. \tag{3.1}$$

The required argument for (3.1) is described in the proof of [1, Theorem 2.83(ii)], but for the sake of clarity we sketch it in the following. Indeed, if (3.1) failed for arbitrarily small radii, then, abbreviating $T := \mathrm{Tan}^{n-1}(\mathcal{F}\Omega, x)$, we could find a sequence $(x_\ell)_{\ell\in\mathbb{N}}$ in $A_j \setminus [x+\mathrm{C}_\varepsilon(T)]$ such that $\varrho_\ell := |x_\ell - x|$ converges to 0 for $\ell \to \infty$. By a straightforward computation, we would get $\mathrm{B}_{\chi\varrho_\ell}(x_\ell) \cap [x+\mathrm{C}_{\varepsilon/2}(T)] = \emptyset$ for $\chi := \varepsilon/(2+2\varepsilon)$, and in combination with the choice of A_j, this would in turn give $(\mathscr{H}^{n-1}\llcorner\mathcal{F}\Omega)(\mathrm{B}_{(1+\chi)\varrho_\ell}(x) \setminus [x+\mathrm{C}_{\varepsilon/2}(T)]) \geq \mathscr{H}^{n-1}(\mathcal{F}\Omega \cap \mathrm{B}_{\chi\varrho_\ell}(x_\ell)) \geq \frac{1}{2}\omega_{n-1}(\chi\varrho_\ell)^{n-1}$ for $\ell \gg 1$. Hence, for the rescalings $\Omega_\ell := (\Omega - x)/\varrho_\ell$, we would have $(\mathscr{H}^{n-1}\llcorner\mathcal{F}\Omega_\ell)(\mathrm{B}_{1+\chi}(0) \setminus \mathrm{C}_{\varepsilon/2}(T)) \geq \frac{1}{2}\omega_{n-1}\chi^{n-1}$ for $\ell \gg 1$, and every local weak-$*$ limit of the measures $\mathscr{H}^{n-1}\llcorner\mathcal{F}\Omega_\ell$ would have positive mass in $\overline{\mathrm{B}_{1+\chi}(0) \setminus \mathrm{C}_{\varepsilon/2}(T)}$, but, by [1, Theorem 3.59(b)], the limit measure is $\mathscr{H}^{n-1}\llcorner T$, which does not have this property. This contradiction completes the derivation of (3.1).

Now the r_x can moreover be taken[3] as Borel functions of x, and when we choose an enumeration $(j(1),k(1)), (j(2),k(2)), (j(3),k(3)), \ldots$ of $\mathbb{N} \times \mathbb{N}$ with $j(i) \leq i$ and $k(i) \leq i$ for all $i \in \mathbb{N}$, then it is easy to see that the sets

$$R_i := \{x \in A_{j(i)} \,:\, r_x \geq k(i)^{-1}\} \setminus (R_1 \cup R_2 \cup \ldots \cup R_{i-1})$$

have the required properties. □

The following proof of Theorem 1.1 is essentially a special case of the argument used for Proposition 4.1 below. Nevertheless, we will first carry out the details in the more geometric and slightly less technical situation of the theorem.

Proof of Theorem 1.1. We first observe that it suffices to construct arbitrary measurable Ω_ε with (1.5), not necessarily with smooth boundaries. Indeed, as soon as we achieve this, we can mollify the characteristic functions of these sets and employ the coarea formula in order to select smooth Ω_ε as superlevel sets of the mollifications. This reasoning is detailed in the proof of the classical approximation property [1, Theorem 3.42], and we omit further details.

Furthermore, we claim that it suffices to establish the claim only with the weaker requirement

$$\Omega \setminus \Omega_\varepsilon \subset \mathcal{N}_\varepsilon(\partial\Omega) \tag{3.2}$$

instead of the second condition in (1.5). To see this, we assume that the weaker form of (1.5) holds for sets $\widetilde{\Omega}_{1/k}$ with $k \in \mathbb{N}$ (where $1/k$ replaces ε), and we show by an elementary contradiction argument that, for every given $\varepsilon > 0$, we can choose the desired Ω_ε as one of the $\widetilde{\Omega}_{1/k}$ with $k \in \mathbb{N}$. Indeed, if this were not possible, we would necessarily have $\Omega \setminus \widetilde{\Omega}_{1/k} \not\subset \mathcal{N}_\varepsilon(\partial\widetilde{\Omega}_{1/k})$ for all $k \geq \varepsilon^{-1}$. Then, we could choose $x_k \in (\Omega \setminus \widetilde{\Omega}_{1/k}) \setminus \mathcal{N}_\varepsilon(\partial\widetilde{\Omega}_{1/k})$, and a subsequence of the x_k would converge to some $x \in \overline{\Omega}$ with $\mathrm{dist}(x, \widetilde{\Omega}_{1/k}) > \mathrm{dist}(x_k, \widetilde{\Omega}_{1/k}) - \varepsilon/2 = \mathrm{dist}(x_k, \partial\widetilde{\Omega}_{1/k}) - \varepsilon/2 \geq \varepsilon/2$ for $k \gg 1$ (where the last two steps exploit $x_k \notin \widetilde{\Omega}_{1/k}$ and $x_k \notin \mathcal{N}_\varepsilon(\partial\widetilde{\Omega}_{1/k})$, respectively). Consequently, we could also find a $y \in \Omega$ with $\mathrm{dist}(y, \widetilde{\Omega}_{1/k}) \geq \varepsilon/2$ for $k \gg 1$, but this would result in a contradiction, as we started with $\Omega \setminus \widetilde{\Omega}_{1/k} \subset \mathcal{N}_{1/k}(\partial\Omega)$ and would thus have $y \in \widetilde{\Omega}_{1/k}$ for $k > \mathrm{dist}(y, \partial\Omega)^{-1}$.

In view of the preceding reduction steps, the case $\mathrm{P}(\Omega) = \infty$ can be concluded by the simple choice $\Omega_\varepsilon := \{x \in \Omega \,:\, \mathrm{dist}(x, \partial\Omega) > \varepsilon/2\}$, and in the following we assume $\mathrm{P}(\Omega) < \infty$. We further work with a given $\varepsilon \leq \frac{1}{2}$ and the corresponding sets R_i of Lemma 3.1. By definition of

[3]For instance, fixing a dense subset $\{\varrho_m \,:\, m \in \mathbb{N}\}$ of $(0,\infty)$, one can choose $r_x = \sup_{m\in\mathbb{N}} r_x^m$, where r_x^m equals ϱ_m in the case $A_j \cap \mathrm{B}_{\varrho_m}(x) \subset x + \mathrm{C}_\varepsilon(\mathrm{Tan}^{n-1}(\mathcal{F}\Omega, x))$ and equals 0 otherwise.

the Hausdorff measure[4], there exist $x_{i,j} \in \mathbb{R}^n$ and $r_{i,j} \in (0,\infty)$ with $3r_{i,j} \leq \min\left\{\frac{1}{i}, \varepsilon\right\}$,

$$R_i \subset \bigcup_{j=1}^{\infty} \mathrm{B}_{r_{i,j}}(x_{i,j}), \quad \text{and} \quad \sum_{j=1}^{\infty} \omega_{n-1} r_{i,j}^{n-1} \leq \mathscr{H}^{n-1}(R_i) + 2^{-i}\varepsilon .$$

Evidently, we can assume $R_i \cap \mathrm{B}_{r_{i,j}}(x_{i,j}) \neq \emptyset$ so that this intersection contains at least one point $\widetilde{x}_{i,j}$, for all $i,j \in \mathbb{N}$. Abbreviating $T_{i,j} := \mathrm{Tan}^{n-1}(\mathcal{F}\Omega, \widetilde{x}_{i,j})$ and taking $3r_{i,j} \leq \frac{1}{i}$ into account, the assertions of Lemma 3.1 imply

$$\begin{gathered} \mathscr{L}^n(\Omega \cap \mathrm{B}^+_{3r_{i,j}}(\widetilde{x}_{i,j})) \leq 3^n \varepsilon^2 r_{i,j}^n , \\ R_i \cap \mathrm{B}_{r_{i,j}}(x_{i,j}) \subset R_i \cap \mathrm{B}_{1/i}(\widetilde{x}_{i,j}) \subset \widetilde{x}_{i,j} + \mathrm{C}_\varepsilon(T_{i,j}) . \end{gathered} \tag{3.3}$$

Recalling the notations introduced around (2.1), we infer that $R_i \cap \mathrm{B}_{r_{i,j}}(x_{i,j})$ is also contained in the (quite flat) cylinder

$$\{y \in \mathbb{R}^n \,:\, |T_{i,j}(y - x_{i,j})| < r_{i,j} \text{ and } |T_{i,j}^\perp(y - \widetilde{x}_{i,j})| < 2\varepsilon r_{i,j}\} .$$

We will slightly enlarge this cylinder in order to get control on a part of its boundary. To this end we take some $h_{i,j} \in [2\varepsilon r_{i,j}, 3\varepsilon r_{i,j}]$ with

$$\mathscr{H}^{n-1}\big(\{y \in \Omega \,:\, |T_{i,j}(y - x_{i,j})| < r_{i,j} \text{ and } T_{i,j}^\perp(y - \widetilde{x}_{i,j}) = h_{i,j}\nu_\Omega(\widetilde{x}_{i,j})\}\big) \leq 3^n \varepsilon r_{i,j}^{n-1} .$$

Such a choice is possible by (3.3) and a Fubini type argument, as in view of $\varepsilon \leq \frac{1}{2}$ all the relevant sets are indeed contained in $\Omega \cap \mathrm{B}^+_{3r_{i,j}}(\widetilde{x}_{i,j})$. We now introduce

$$C_{i,j} := \{y \in \mathbb{R}^n \,:\, |T_{i,j}(y - x_{i,j})| < r_{i,j} \text{ and } |T_{i,j}^\perp(y - \widetilde{x}_{i,j})| < h_{i,j}\} \Subset \mathrm{B}_\varepsilon(\widetilde{x}_{i,j}) ,$$

and record that top and bottom of $C_{i,j}$ have each $\mathscr{H}^{n-1}$-measure $\omega_{n-1} r_{i,j}^{n-1}$, while the side has measure $2h_{i,j}(n-1)\omega_{n-1} r_{i,j}^{n-2}$. Involving also the preceding choice of $h_{i,j}$, we get

$$\mathscr{H}^{n-1}(\Omega \cap \partial C_{i,j}) \leq \big[\omega_{n-1} + 3^n\varepsilon + 6\varepsilon(n-1)\omega_{n-1}\big] r_{i,j}^{n-1} . \tag{3.4}$$

Now we exploit the reformulation (1.7) of (1.4), and we cover also $\partial\Omega \setminus \mathcal{F}\Omega$ by countably many balls $\mathrm{B}_{\varrho_1}(y_1), \mathrm{B}_{\varrho_2}(y_2), \mathrm{B}_{\varrho_3}(y_3), \ldots$ with radii $\varrho_k \leq \varepsilon/2$ such that $\mathrm{B}_{\varrho_k}(y_k) \cap \partial\Omega \neq \emptyset$ and $\sum_{k=1}^\infty \varrho_k^{n-1} \leq \varepsilon$. Since the $C_{i,j}$ cover $\mathcal{F}\Omega$, the $C_{i,j}$ and $\mathrm{B}_{\varrho_k}(y_k)$ together form an open cover of $\partial\Omega$. Exploiting boundedness of Ω, we choose a finite sub-cover

$$S := C_{i_1,j_1} \cup C_{i_2,j_2} \cup \ldots \cup C_{i_M,j_M} \cup \mathrm{B}_{\varrho_{k_1}}(y_{k_1}) \cup \mathrm{B}_{\varrho_{k_2}}(y_{k_2}) \cup \ldots \cup \mathrm{B}_{\varrho_{k_N}}(y_{k_N})$$

of $\partial\Omega$, and we take

$$\Omega_\varepsilon := \Omega \setminus \overline{S} .$$

Then, Ω_ε is an open set with $\Omega_\varepsilon \Subset \Omega$ and $\Omega \setminus \Omega_\varepsilon \subset \overline{S} \subset \mathcal{N}_\varepsilon(\partial\Omega)$. Thus, the first condition in (1.5) and the weaker form (3.2) of the second condition are valid. In order to verify the third condition in (1.5), we exploit in turn (1.1), the preceding definition of Ω_ε, (3.4), the choices of

[4]At this point, we work with the spherical Hausdorff measure, but we also use results from [1], formulated with the standard Hausdorff measure. Nevertheless, our argument is consistent, as we only evaluate these measures on $\mathscr{H}^{n-1}$-rectifiable sets, where they coincide by [11, Theorem 3.2.26]. Notice in addition that the theory of the perimeter can also be developed using only the spherical measure from the very beginning.

the $C_{i,j}$ and $\mathrm{B}_{\varrho_k}(y_k)$, the fact that the R_i are disjoint in $\mathcal{F}\Omega$, and (1.6). In this way we deduce

$$\begin{aligned}
\mathrm{P}(\Omega_\varepsilon) \le \mathscr{H}^{n-1}(\partial\Omega_\varepsilon) &\le \sum_{i,j=1}^{\infty} \mathscr{H}^{n-1}(\Omega \cap \partial C_{i,j}) + \sum_{k=1}^{\infty} \mathscr{H}^{n-1}(\partial \mathrm{B}_{\varrho_k}(y_k)) \\
&\le (1{+}A_n\varepsilon) \sum_{i,j=1}^{\infty} \omega_{n-1} r_{i,j}^{n-1} + A_n \sum_{k=1}^{\infty} \varrho_k^{n-1} \\
&\le (1{+}A_n\varepsilon) \sum_{i=1}^{\infty} \left[\mathscr{H}^{n-1}(R_i) + 2^{-i}\varepsilon\right] + A_n\varepsilon \\
&= (1{+}A_n\varepsilon)[\mathscr{H}^{n-1}(\mathcal{F}\Omega) + \varepsilon] + A_n\varepsilon \\
&\le \mathrm{P}(\Omega) + (1{+}2A_n{+}A_n\mathrm{P}(\Omega))\varepsilon
\end{aligned}$$

with the dimensional constant $A_n := \max\{3^n/\omega_{n-1}{+}6(n{-}1), n\omega_n\}$. The same reasoning applies with the smaller quantity $\varepsilon/(1{+}2A_n{+}A_n\mathrm{P}(\Omega))$ in place of ε. Thus, we obtain the last assertion of (1.5), and the proof is complete. $\square$

4. Strict approximation of BV-functions from a given Dirichlet class

For a BV-function w, defined near $\mathcal{F}\Omega$, we write $w^+_{\mathcal{F}\Omega}$ for the exterior trace and $w^-_{\mathcal{F}\Omega}$ for the interior trace on $\mathcal{F}\Omega$ (with orientation given by the exterior normal ν_Ω); see [1, Theorem 3.77]. With this notation, Theorem 1.1 corresponds essentially to the special case of a constant w in the next proposition.

Proposition 4.1. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$ with $\mathscr{H}^{n-1}(\partial\Omega) = \mathrm{P}(\Omega) < \infty$. Then, for $w \in \mathrm{BV}(\mathbb{R}^n,\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^n,\mathbb{R}^N)$ and every $\varepsilon > 0$, there exists an open set Ω_ε with finite perimeter in $\mathbb{R}^n$ such that*

$$\Omega_\varepsilon \Subset \Omega\,, \quad \mathscr{L}^n(\Omega \setminus \Omega_\varepsilon) \le \varepsilon\,, \quad \int_{\mathcal{F}\Omega_\varepsilon} |w^-_{\mathcal{F}\Omega_\varepsilon}|\,\mathrm{d}\mathscr{H}^{n-1} \le \int_{\mathcal{F}\Omega} |w^-_{\mathcal{F}\Omega}|\,\mathrm{d}\mathscr{H}^{n-1} + \varepsilon\,.$$

Proof. We assume $\varepsilon \le \frac{1}{2}$ and take $L := \max\{1, \|w\|_{\mathrm{L}^\infty(\mathbb{R}^n,\mathbb{R}^N)}\}$. Working with the sets R_i from Lemma 3.1, we pass on to the sets R_i^* of all $x \in R_i$ such that, for some $r_x > 0$ and all $\varrho \in (0, r_x]$, we have

$$\mathscr{H}^{n-1}(R_i \cap \mathrm{B}_\varrho(x)) \ge \tfrac{1}{1+\varepsilon}\omega_{n-1}\varrho^{n-1}\,, \tag{4.1}$$

$$\int_{\mathrm{B}^-_\varrho(x)} |w - w^-_{\mathcal{F}\Omega}(x)|\,\mathrm{d}\mathscr{L}^n \le \varepsilon^2\varrho^n\,, \tag{4.2}$$

$$\int_{R_i \cap \mathrm{B}_\varrho(x)} |w^-_{\mathcal{F}\Omega} - w^-_{\mathcal{F}\Omega}(x)|\,\mathrm{d}\mathscr{H}^{n-1} \le \varepsilon\mathscr{H}^{n-1}(R_i \cap \mathrm{B}_\varrho(x))\,. \tag{4.3}$$

With this terminology we have $\mathscr{H}^{n-1}(R_i \setminus R_i^*) = 0$ for the following reasons: the almost-everywhere validity of the first property is guaranteed by (one implication of) the Besicovitch-Marstrand-Mattila theorem [1, Theorem 2.63, Theorem 2.83(i)]; similarly, for the second property we rely on the defining property of traces [1, Theorem 3.77], and in connection with the third one we exploit the fact that $\mathscr{H}^{n-1}$-a. e. point of R_i is a Lebesgue point of the L^∞-function $w^-_{\mathcal{F}\Omega}$ with respect to the finite Radon measure $\mathscr{H}^{n-1} \llcorner R_i$. The family

$$\left\{\overline{\mathrm{B}_\varrho(x)} \,:\, x \in R_i^*,\, 0 < 2\varrho \le \min\{r_x, \tfrac{1}{i}\}\,,\, \mathscr{H}^{n-1}(R_i \cap \partial\mathrm{B}_\varrho(x)) = 0\right\}$$

is a fine cover of R_i^*, and by the Vitali covering theorem (see [1, Theorem 2.19]) for the Radon measure $\mathscr{H}^{n-1} \llcorner R_i$ we can find a countable subfamily $\left\{\overline{\mathrm{B}_{r_{i,j}}(x_{i,j})} \,:\, j \in \mathbb{N}\right\}$, consisting of disjoint balls with $x_{i,j} \in R_i^*$ and $2r_{i,j} \le \min\{r_{x_{i,j}}, \frac{1}{i}\}$, such that

$$\mathscr{H}^{n-1}\biggl(R_i \setminus \bigcup_{j=1}^{\infty} \mathrm{B}_{r_{i,j}}(x_{i,j})\biggr) = 0 \tag{4.4}$$

holds. As a consequence of Lemma 3.1, we have

$$\mathscr{L}^n(\Omega \cap \mathrm{B}^+_{2r_{i,j}}(x_{i,j})) \le 2^n\varepsilon^2 r_{i,j}^n\,, \tag{4.5}$$

$$R_i \cap \mathrm{B}_{r_{i,j}}(x_{i,j}) \subset x_{i,j} + \mathrm{C}_\varepsilon(T_{i,j}) \tag{4.6}$$

with the abbreviation $T_{i,j} := \mathrm{Tan}^{n-1}(\mathcal{F}\Omega, x_{i,j})$. Next we introduce the cylinders

$$C_{i,j} := \{y \in \mathbb{R}^n \,:\, |T_{i,j}(y{-}x_{i,j})| < r_{i,j} \text{ and } |T_{i,j}^\perp(y{-}x_{i,j})| < h_{i,j}\}\,,$$

where $h_{i,j} \in (\varepsilon r_{i,j}, 2\varepsilon r_{i,j}]$ will now be chosen such that we get good estimates for parts of $\partial C_{i,j}$. Indeed, via (4.5), (4.2), the observation that $C_{i,j} \Subset \mathrm{B}_{2r_{i,j}}(x_{i,j})$ holds (whenever $h_{i,j} \le 2\varepsilon r_{i,j}$), and Fubini's theorem we can fix $h_{i,j} \in (\varepsilon r_{i,j}, 2\varepsilon r_{i,j}]$ such that, for

$$\partial_+ C_{i,j} := \{y \in \mathbb{R}^n \,:\, |T_{i,j}(y{-}x_{i,j})| < r_{i,j} \text{ and } T_{i,j}^\perp(y{-}x_{i,j}) = h_{i,j}\nu_\Omega(x_{i,j})\}\,,$$

we have

$$\mathscr{H}^{n-1}(\Omega \cap \partial_+ C_{i,j}) \le 2^{n+1}\varepsilon r_{i,j}^{n-1}\,, \tag{4.7}$$

such that $\mathscr{H}^{n-1}$-a. e. point in

$$\partial_- C_{i,j} := \{y \in \mathbb{R}^n \,:\, |T_{i,j}(y{-}x_{i,j})| < r_{i,j} \text{ and } T_{i,j}^\perp(y{-}x_{i,j}) = -h_{i,j}\nu_\Omega(x_{i,j})\}$$

is a Lebesgue point of w, and such that we have

$$\int_{\partial_- C_{i,j}} |w - w^-_{\mathcal{F}\Omega}(x_{i,j})|\,\mathrm{d}\mathscr{H}^{n-1} \le 2^{n+1}\varepsilon r_{i,j}^{n-1}\,. \tag{4.8}$$

From (4.4), (4.6), and $h_{i,j} > \varepsilon r_{i,j}$ we infer

$$\mathscr{H}^{n-1}\biggl(R_i \setminus \bigcup_{j=1}^{\infty} C_{i,j}\biggr) = 0\,, \tag{4.9}$$

and via the boundedness of w, via the fact that the side $\mathcal{F}C_{i,j} \setminus (\partial_+ C_{i,j} \cup \partial_- C_{i,j})$ of $C_{i,j}$ has $\mathscr{H}^{n-1}$-measure $2h_{i,j}(n{-}1)\omega_{n-1}r_{i,j}^{n-2}$, and via (4.8), (4.7), (4.1), (4.3) we arrive at

$$\begin{aligned}
&\int_{\Omega\cap\mathcal{F}C_{i,j}} |w^+_{\mathcal{F}C_{i,j}}|\,\mathrm{d}\mathscr{H}^{n-1} \\
&\quad\le \int_{\partial_- C_{i,j}} |w|\,\mathrm{d}\mathscr{H}^{n-1} + L\mathscr{H}^{n-1}(\Omega \cap \partial_+ C_{i,j}) + 2h_{i,j}(n{-}1)L\omega_{n-1}r_{i,j}^{n-2} \\
&\quad\le |w^-_{\mathcal{F}\Omega}(x_{i,j})|\mathscr{H}^{n-1}(\partial_- C_{i,j}) + \bigl[2^{n+1} + 2^{n+1}L + 4(n{-}1)L\omega_{n-1}\bigr]\varepsilon r_{i,j}^{n-1} \\
&\quad\le |w^-_{\mathcal{F}\Omega}(x_{i,j})|\omega_{n-1}r_{i,j}^{n-1} + \tfrac{1}{2}\bigl(\widehat{L}{-}2\bigr)\varepsilon\omega_{n-1}r_{i,j}^{n-1} \\
&\quad\le |w^-_{\mathcal{F}\Omega}(x_{i,j})|(1{+}\varepsilon)\mathscr{H}^{n-1}(R_i \cap \mathrm{B}_{r_{i,j}}(x_{i,j})) + \bigl(\widehat{L}{-}2\bigr)\varepsilon\mathscr{H}^{n-1}(R_i \cap \mathrm{B}_{r_{i,j}}(x_{i,j})) \\
&\quad\le (1{+}\varepsilon)\int_{R_i\cap\mathrm{B}_{r_{i,j}}(x_{i,j})} |w^-_{\mathcal{F}\Omega}|\,\mathrm{d}\mathscr{H}^{n-1} + \widehat{L}\varepsilon\mathscr{H}^{n-1}(R_i \cap \mathrm{B}_{r_{i,j}}(x_{i,j}))\,,
\end{aligned} \tag{4.10}$$

where we abbreviated $\widehat{L} := (2^{n+3}/\omega_{n-1}{+}8n)L$. Via (1.7), Lemma 3.1, and (4.9), we get

$$\mathscr{H}^{n-1}\biggl(\partial\Omega \setminus \bigcup_{i,j=1}^{\infty} C_{i,j}\biggr) = 0\,.$$

Then, as in the proof of Theorem 1.1, we cover $\partial\Omega \setminus \bigcup_{i,j=1}^{\infty} C_{i,j}$ by countably many balls $\mathrm{B}_{\varrho_1}(y_1), \mathrm{B}_{\varrho_2}(y_2), \mathrm{B}_{\varrho_3}(y_3), \ldots$ with radii $\varrho_k \le 1$ such that $\sum_{k=1}^{\infty} \varrho_k^{n-1} \le \varepsilon$. The $C_{i,j}$ and $\mathrm{B}_{\varrho_k}(y_k)$ form an open cover of the compactum $\partial\Omega$, and we can choose a finite sub-cover

$$S := C_{i_1,j_1} \cup C_{i_2,j_2} \cup \ldots \cup C_{i_M,j_M} \cup \mathrm{B}_{\varrho_{k_1}}(y_{k_1}) \cup \mathrm{B}_{\varrho_{k_2}}(y_{k_2}) \cup \ldots \cup \mathrm{B}_{\varrho_{k_N}}(y_{k_N})\,.$$

Then

$$\Omega_\varepsilon := \Omega \setminus \overline{S} \Subset \Omega$$

is an open set with finite perimeter in $\mathbb{R}^n$ (this follows, for instance, from the fact that the multiplication of $\mathrm{BV} \cap \mathrm{L}^\infty$-functions is still in $\mathrm{BV} \cap \mathrm{L}^\infty$). Next we employ (4.10), where we control the right-hand side of this estimate via the fact that $(\mathrm{B}_{r_{i,j}}(x_{i,j}))_{j\in\mathbb{N}}$ is a disjoint family of balls (for fixed $i \in \mathbb{N}$). Exploiting that also the R_i are disjoint, we deduce

$$\begin{aligned}
\int_{\mathcal{F}\Omega_\varepsilon} |w^-_{\mathcal{F}\Omega_\varepsilon}|\,\mathrm{d}\mathscr{H}^{n-1} &\le \sum_{i,j=1}^{\infty} \int_{\Omega\cap\mathcal{F}C_{i,j}} |w^+_{\mathcal{F}C_{i,j}}|\,\mathrm{d}\mathscr{H}^{n-1} + Ln\omega_n \sum_{k=1}^{\infty} \varrho_k^{n-1} \\
&\le \sum_{i=1}^{\infty} \left[(1{+}\varepsilon)\int_{R_i} |w^-_{\mathcal{F}\Omega}|\,\mathrm{d}\mathscr{H}^{n-1} + \widehat{L}\varepsilon\mathscr{H}^{n-1}(R_i)\right] + Ln\omega_n\varepsilon \\
&\le (1{+}\varepsilon)\int_{\mathcal{F}\Omega} |w^-_{\mathcal{F}\Omega}|\,\mathrm{d}\mathscr{H}^{n-1} + \widehat{L}\varepsilon\mathscr{H}^{n-1}(\mathcal{F}\Omega) + Ln\omega_n\varepsilon\,.
\end{aligned}$$

Finally, relying on $r_{i,j} \le 1$ and $\varrho_k \le 1$, on (4.1), and on the fact that all $R_i \cap \mathrm{B}_{r_{i,j}}(x_{i,j})$ are disjoint, we have

$$\begin{aligned}
\mathscr{L}^n(\Omega\setminus\Omega_\varepsilon) &\le \sum_{i,j=1}^{\infty} \mathscr{L}^n(C_{i,j}) + \sum_{k=1}^{\infty} \mathscr{L}^n(\mathrm{B}_{\varrho_k}(y_k)) \\
&\le 4\varepsilon \sum_{i,j=1}^{\infty} \omega_{n-1} r_{i,j}^n + \omega_n \sum_{k=1}^{\infty} \varrho_k^n \\
&\le 4\varepsilon(1{+}\varepsilon) \sum_{i,j=1}^{\infty} \mathscr{H}^{n-1}(R_i \cap \mathrm{B}_{r_{i,j}}(x_{i,j})) + \omega_n \sum_{k=1}^{\infty} \varrho_k^{n-1} \\
&\le \Big[6\mathscr{H}^{n-1}(\mathcal{F}\Omega) + \omega_n\Big]\varepsilon\,.
\end{aligned}$$

Possibly decreasing ε suitably, the last two estimates yield the claims. □

Next we apply Proposition 4.1 with suitable truncations of $\overline{u}{-}u_0$ in place of w. Involving also a mollification procedure, this leads to a

Proof of Theorem 1.2. We fix $\varepsilon > 0$, for $L \ge 0$ we set

$$\overline{u}^L := \begin{cases} \overline{u} & \text{if } |\overline{u}| \le L \\ L\frac{\overline{u}}{|\overline{u}|} & \text{if } |\overline{u}| > L \end{cases} \quad\text{and}\quad u_0^L := \begin{cases} u_0 & \text{if } |u_0| \le L \\ L\frac{u_0}{|u_0|} & \text{if } |u_0| > L \end{cases},$$

and we start with the observation that $\overline{u}^L$ equals u_0^L outside Ω and satisfies

$$|(\mathscr{L}^n, \mathrm{D}\overline{u}^L)|\big(\overline{\Omega}\big) \le |(\mathscr{L}^n, \mathrm{D}\overline{u})\big(\overline{\Omega}\big)|\,. \tag{4.11}$$

Now we fix also L (depending on ε) so large that we have

$$\int_\Omega |\overline{u}^L{-}u|\,\mathrm{d}\mathscr{L}^n \le \varepsilon \quad\text{and}\quad \int_\Omega \big(|u_0^L{-}u_0| + |\nabla u_0^L{-}\nabla u_0|\big)\,\mathrm{d}\mathscr{L}^n \le \varepsilon\,.$$

Then we apply Proposition 4.1 to the bounded function $\overline{u}^L{-}u_0^L$, finding open sets $\Omega_\varepsilon \Subset \Omega$ with $\mathrm{P}(\Omega_\varepsilon) < \infty$,

$$\mathscr{L}^n(\Omega\setminus\Omega_\varepsilon) \le \varepsilon\,,$$

and

$$\int_{\mathcal{F}\Omega_\varepsilon} |(\overline{u}^L)^-_{\mathcal{F}\Omega_\varepsilon} - u_0^L| \,\mathrm{d}\mathscr{H}^{n-1} \le \int_{\mathcal{F}\Omega} |(\overline{u}^L)^-_{\mathcal{F}\Omega} - u_0^L| \,\mathrm{d}\mathscr{H}^{n-1} + \varepsilon = |\mathrm{D}\overline{u}^L|(\mathcal{F}\Omega) + \varepsilon \tag{4.12}$$

(where we have just written u_0^L, as all one- or both-sided sided traces of the $\mathrm{W}^{1,1}$-function u_0^L coincide). Now we use that, by the absolute continuity of the Lebesgue integral, there exists an increasing $\eta\colon [0,\infty) \to [0,\infty)$ with $\lim_{\delta\searrow 0} \eta(\delta) = 0$ and

$$\int_A \big(|u| + |u_0| + |\nabla u_0|\big) \,\mathrm{d}\mathscr{L}^n \le \eta(\mathscr{L}^n(A)) \quad \text{for all measurable subsets } A \text{ of } \Omega\,.$$

We also introduce the notation M_ε for a smoothing operator, based on mollification with a radius $r_\varepsilon < \operatorname{dist}(\Omega_\varepsilon, \mathbb{R}^n \setminus \Omega)$ which is taken small enough that

$$\int_\Omega |\mathrm{M}_\varepsilon(\mathbb{1}_{\Omega_\varepsilon}\overline{u}^L) - \mathbb{1}_{\Omega_\varepsilon}\overline{u}^L| \,\mathrm{d}\mathscr{L}^n \le \varepsilon\,,$$

$$\int_\Omega \big(|\mathrm{M}_\varepsilon(\mathbb{1}_{\Omega_\varepsilon} u_0^L) - \mathbb{1}_{\Omega_\varepsilon} u_0^L| + |\mathrm{M}_\varepsilon(\mathbb{1}_{\Omega_\varepsilon}\nabla u_0^L) - \mathbb{1}_{\Omega_\varepsilon}\nabla u_0^L|\big) \,\mathrm{d}\mathscr{L}^n \le \varepsilon$$

hold. Consequently, we can estimate

$$\begin{aligned} \int_\Omega |\mathrm{M}_\varepsilon(\mathbb{1}_{\Omega_\varepsilon}\overline{u}^L) - u| \,\mathrm{d}\mathscr{L}^n &\le \int_\Omega \big(|\mathrm{M}_\varepsilon(\mathbb{1}_{\Omega_\varepsilon}\overline{u}^L) - \mathbb{1}_{\Omega_\varepsilon}\overline{u}^L| + |\overline{u}^L - u|\big) \,\mathrm{d}\mathscr{L}^n + \int_{\Omega\setminus\Omega_\varepsilon} |u| \,\mathrm{d}\mathscr{L}^n \\ &\le 2\varepsilon + \eta(\varepsilon) \end{aligned} \tag{4.13}$$

and, very similarly,

$$\int_\Omega |\mathrm{M}_\varepsilon(\mathbb{1}_{\Omega_\varepsilon} u_0^L) - u_0| \,\mathrm{d}\mathscr{L}^n + \int_\Omega |\mathrm{M}_\varepsilon(\mathbb{1}_{\Omega_\varepsilon}\nabla u_0^L) - \nabla u_0| \,\mathrm{d}\mathscr{L}^n \le 2\varepsilon + \eta(\varepsilon)\,. \tag{4.14}$$

At this stage, we define u_ε as the restriction of $u_0 + \mathrm{M}_\varepsilon(\mathbb{1}_{\Omega_\varepsilon}(\overline{u}^L - u_0^L))$ to Ω. Then, by the choice of r_ε, we have $u_\varepsilon \in u_0\big|_\Omega + \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^N)$, and (4.13) and (4.14) imply

$$\int_\Omega |u_\varepsilon - u| \,\mathrm{d}\mathscr{L}^n \le \int_\Omega |\mathrm{M}_\varepsilon(\mathbb{1}_{\Omega_\varepsilon}\overline{u}^L) - u| \,\mathrm{d}\mathscr{L}^n + \int_\Omega |u_0 - \mathrm{M}_\varepsilon(\mathbb{1}_{\Omega_\varepsilon} u_0^L)| \,\mathrm{d}\mathscr{L}^n \le 4\varepsilon + 2\eta(\varepsilon)\,.$$

With the help of [1, Theorem 3.84], we compute, $\mathscr{L}^n$-a. e. on Ω,

$$\begin{aligned} \nabla u_\varepsilon &= \nabla u_0 + \mathrm{M}_\varepsilon \mathrm{D}(\mathbb{1}_{\Omega_\varepsilon}(\overline{u}^L - u_0^L)) \\ &= \nabla u_0 - \mathrm{M}_\varepsilon(\mathbb{1}_{\Omega_\varepsilon}\nabla u_0^L) + \mathrm{M}_\varepsilon(\mathrm{D}\overline{u}^L \llcorner \Omega_\varepsilon) - \mathrm{M}_\varepsilon\big([(\overline{u}^L)^-_{\mathcal{F}\Omega_\varepsilon} - u_0^L] \otimes \nu_{\Omega_\varepsilon} \mathscr{H}^{n-1} \llcorner \mathcal{F}\Omega_\varepsilon\big) \end{aligned}$$

(where we understand, as usual, the mollification of a measure as a function). Starting from this formula, we now use (4.14) and the inequality $\|\mathrm{M}_\varepsilon(\mu)\|_{\mathrm{L}^1(\mathbb{R}^n,\mathbb{R}^m)} \le |\mu|(\mathbb{R}^n)$ (which holds for every $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^n$) to deduce

$$\begin{aligned} |(\mathrm{M}_\varepsilon(\mathbb{1}_\Omega)\mathscr{L}^n, \mathrm{D}u_\varepsilon)|(\Omega) &\le \int_\Omega |\nabla u_0 - \mathrm{M}_\varepsilon(\mathbb{1}_{\Omega_\varepsilon}\nabla u_0^L)| \,\mathrm{d}\mathscr{L}^n + \big\|\mathrm{M}_\varepsilon(\mathscr{L}^n \llcorner \Omega, \mathrm{D}\overline{u}^L \llcorner \Omega_\varepsilon)\big\|_{\mathrm{L}^1(\Omega,\mathbb{R}^{Nn+1})} \\ &\qquad + \|\mathrm{M}_\varepsilon([(\overline{u}^L)^-_{\mathcal{F}\Omega_\varepsilon} - u_0^L] \otimes \nu_{\Omega_\varepsilon} \mathscr{H}^{n-1} \llcorner \mathcal{F}\Omega_\varepsilon)\|_{\mathrm{L}^1(\Omega,\mathbb{R}^{Nn})} \\ &\le 2\varepsilon + \eta(\varepsilon) + |(\mathscr{L}^n, \mathrm{D}\overline{u}^L)|(\Omega) + \int_{\mathcal{F}\Omega_\varepsilon} |(\overline{u}^L)^-_{\mathcal{F}\Omega_\varepsilon} - u_0^L| \,\mathrm{d}\mathscr{H}^{n-1} \\ &\le 3\varepsilon + \eta(\varepsilon) + |(\mathscr{L}^n, \mathrm{D}\overline{u}^L)|(\Omega \cup \mathcal{F}\Omega) \le 3\varepsilon + \eta(\varepsilon) + |(\mathscr{L}^n, \mathrm{D}\overline{u})|\big(\overline{\Omega}\big)\,, \end{aligned}$$

where we also involved (4.12) and (4.11) in the last steps. Combining the preceding estimate with

$$|\mathscr{L}^n - \mathrm{M}_\varepsilon(\mathbb{1}_\Omega)\mathscr{L}^n|(\Omega) = \int_\Omega |1 - \mathrm{M}_\varepsilon(\mathbb{1}_\Omega)| \,\mathrm{d}\mathscr{L}^n \le \mathscr{L}^n(\Omega \setminus \Omega_\varepsilon) \le \varepsilon\,,$$

we conclude

$$|(\mathscr{L}^n, \mathrm{D}u_\varepsilon)|(\Omega) \le |(\mathscr{L}^n, \mathrm{D}\overline{u})|(\overline{\Omega}) + 4\varepsilon + \eta(\varepsilon)\,.$$

Decreasing ε suitably, we have established (1.10), and the proof is complete. $\square$

5. On the hypothesis $\mathscr{H}^{n-1}(\partial\Omega) = \mathrm{P}(\Omega)$

In this section, we are concerned with the basic assumption $\mathscr{H}^{n-1}(\partial\Omega) = \mathrm{P}(\Omega)$, which we have imposed in Theorems 1.1 and 1.2. We emphasize that this hypothesis is actually not completely new, but has already occurred in a related context, namely in the trace theorems [4, Teoremi 5, 7, 10]. In the following, we prove some optimality of the assumption in the 2-dimensional case, and we discuss a refined example for its necessity.

Theorem 5.1 (optimality property in $\mathbb{R}^2$)**.** *Consider a bounded open set Ω in $\mathbb{R}^2$ such that Ω has finitely many connected components, while $\partial\Omega$ has at most countably many path-connected components. If, for every $\varepsilon > 0$, there exists an open set Ω_ε with smooth boundary in $\mathbb{R}^2$ and with* (1.5)*, then we already have $\mathscr{H}^1(\partial\Omega) = \mathrm{P}(\Omega)$.*

Proof. For $\Omega \neq \emptyset$, we denote the number of connected components of Ω by $M \in \mathbb{N}$, and we write $\Omega^1, \Omega^2, \ldots, \Omega^M$ for these components. We record that the open and connected sets Ω^i are automatically path-connected, and we fix one point p^i in each Ω^i. For sufficiently large k (and only those we will consider), (1.5) implies that all p^i are contained in $\Omega_{1/k}$, and we can define Ω^i_k as the path-connected component of $\Omega_{1/k}$ which contains p^i. For the open set $\Omega^\sharp_k := \bigcup_{i=1}^M \Omega^i_k \Subset \Omega$ we will now show

$$\lim_{k\to\infty} \mathrm{d}_{\mathcal{H}}(\partial\Omega^\sharp_k, \partial\Omega) = 0\,. \tag{5.1}$$

Indeed, from $\partial\Omega^\sharp_k \subset \partial\Omega_{1/k}$ and the observations (1.8) we deduce

$$\sup_{x\in\partial\Omega^\sharp_k} \mathrm{dist}(x,\partial\Omega) \le \sup_{x\in\partial\Omega_{1/k}} \mathrm{dist}(x,\partial\Omega) \le \mathrm{d}_{\mathcal{H}}(\partial\Omega_{1/k},\partial\Omega) \overset{k\to\infty}{\longrightarrow} 0\,.$$

Hence, the claim (5.1) could only fail if we had

$$\gamma := \limsup_{k\to\infty} \sup_{x\in\partial\Omega} \mathrm{dist}(x,\partial\Omega^\sharp_k) > 0$$

and thus $\lim_{l\to\infty} \mathrm{dist}(x_l,\partial\Omega^\sharp_{k_l}) = \gamma$ for some subsequence $(k_l)_{l\in\mathbb{N}}$ and some sequence $(x_l)_{l\in\mathbb{N}}$ in $\partial\Omega$ which converges to $x_\infty \in \partial\Omega$. Now, on one hand, we would have $\mathrm{dist}(x_\infty,\partial\Omega^\sharp_{k_l}) \ge \gamma/2$ for $l \gg 1$, while on the other hand we could find an $x \in \Omega$ with $|x - x_\infty| < \gamma/2$ and a path from x to an p^i in Ω. For sufficiently large l, the whole path would be contained in Ω_{1/k_l}, therefore we would have $x \in \Omega^i_{k_l} \subset \Omega^\sharp_{k_l}$ and $\mathrm{dist}(x_\infty,\partial\Omega^\sharp_{k_l}) \le |x - x_\infty| < \gamma/2$. The latter estimate contradicts the previously observed lower bound for $\mathrm{dist}(x_\infty,\partial\Omega^\sharp_{k_l})$, and thus (5.1) is proved.

Next we write $(R^j)_{j\in\mathbb{N}}$ for the family of path-connected components of $\partial\Omega$ (where, in case of finitely many components, we understand $R^j = \emptyset$ for large j). For the moment we will work with $R^1, R^2, \ldots, R^N$, where $N \in \mathbb{N}$ is arbitrary, but fixed. We define E^j_k as the path-connected component of $\mathbb{R}^n \setminus \overline{\Omega^\sharp_k}$ which contains R^j, we set

$$R^{i,j}_k := \partial\Omega^i_k \cap \partial E^j_k\,,$$

and we claim:

$$R^{i,j}_k \text{ is path-connected,} \tag{5.2}$$

$$\liminf_{k\to\infty} \mathrm{d}_{\mathcal{H}}\left(\bigcup_{i=1}^M \bigcup_{j=1}^N R^{i,j}_k, X_N\right) = 0 \text{ for some closed } X_N \text{ with } \bigcup_{j=1}^N R^j \subset X_N\,. \tag{5.3}$$

In order to prove (5.2) we take arbitrary $y, \widetilde{y} \in R_k^{i,j}$. As $\partial\Omega_{1/k}$ is smooth and bounded, we can write the path-connected component of $R_k^{i,j}$ which contains y as the image of a smooth *closed* curve ℓ. Moreover, by the path-connectedness of Ω_k^i and E_k^j we can connect y and $\widetilde{y}$ by curves $c_{\text{int}}\colon [0,1] \to \mathbb{R}^2$ and $c_{\text{ext}}\colon [0,1] \to \mathbb{R}^2$ with $c_{\text{int}}(0) = c_{\text{ext}}(0) = y$ and $c_{\text{int}}(1) = c_{\text{ext}}(1) = \widetilde{y}$ such that c_{int} remains inside $\Omega_k^\sharp$ and c_{ext} remains outside $\overline{\Omega_k^\sharp}$ (apart from the endpoints). Now, as $\operatorname{Im}\ell$ locally separates $\Omega_k^\sharp$ and $\mathbb{R}^n \setminus \overline{\Omega_k^\sharp}$, this means that $c_{\text{int}}(t)$ and $c_{\text{ext}}(t)$ lie on different sides of the loop $\operatorname{Im}\ell$ for $0 < t \ll 1$. As neither c_{int} nor c_{ext} can cross $\operatorname{Im}\ell \subset \partial\Omega_k^\sharp$, we infer that the common endpoint $\widetilde{y}$ is contained in $\operatorname{Im}\ell$. But then a part of ℓ connects y and $\widetilde{y}$ in $R_k^{i,j}$, and we have shown that $R_k^{i,j}$ is path-connected.

Turning to (5.3), we first observe that the validity of the convergence for *some* closed set X_N is immediate by the Blaschke selection theorem (see [1, Theorem 6.1] or [10, Theorem 3.16]), so it only remains to justify the claimed inclusion of $\bigcup_{j=1}^N R^j$ in X_N. To this end, we fix $j_0 \in \{1, 2, \ldots, N\}$ and consider an arbitrary $x \in R^{j_0} \subset \partial\Omega$. By the compactness of $\partial\Omega_k^\sharp$, we can find $x_k \in \partial\Omega_k^\sharp$ with $|x_k - x| = \operatorname{dist}(x, \partial\Omega_k^\sharp)$; then the line segment from x_k to x does not intersect $\Omega_k^\sharp$, hence this segment is contained in $E_k^{j_0}$, and we have $x_k \in \partial E_k^{j_0}$. In view of $\partial\Omega_k^\sharp = \bigcup_{i=1}^M \partial\Omega_k^i$ we can also fix an $i_0 \in \{1, 2, \ldots, M\}$ such that $x_k \in \partial\Omega_k^{i_0}$ and therefore $x_k \in R_k^{i_0,j_0}$ hold for infinitely many $k \in \mathbb{N}$. Taking into account $|x_k - x| = \operatorname{dist}(x, \partial\Omega_k^\sharp) \le \mathrm{d}_{\mathcal{H}}(\partial\Omega, \partial\Omega_k^\sharp)$ and (5.1), the x_k converge to x, and hence x is contained in the Hausdorff limit X_N of $\bigcup_{i=1}^M \bigcup_{j=1}^N R_k^{i,j}$. We have thus shown $R^{j_0} \subset X_N$, and the derivation of (5.3) is complete.

In view of (5.2), the number of connected components of $\bigcup_{i=1}^M \bigcup_{j=1}^N R_k^{i,j}$ is bounded by MN, and then [8, Corollary 35.15] (that is essentially Golab's theorem; compare also [10, Theorem 3.18]) guarantees lower semicontinuity of $\mathscr{H}^1$ along the convergence in (5.3). We can thus deduce

$$\mathscr{H}^1(X_N) \le \limsup_{k\to\infty} \mathscr{H}^1\left(\bigcup_{i=1}^M \bigcup_{j=1}^N R_k^{i,j}\right).$$

Using the inclusion $\bigcup_{j=1}^N R^j \subset X_N$ from (5.3) on the left-hand side and $R_k^{i,j} \subset \partial\Omega_{1/k}$ on the right-hand side, we infer

$$\mathscr{H}^1\left(\bigcup_{j=1}^N R^j\right) \le \limsup_{k\to\infty} \mathscr{H}^1(\partial\Omega_{1/k}).$$

Finally, we take into account that $\mathscr{H}^1(\partial\Omega_{1/k}) = \mathrm{P}(\Omega_{1/k}) \le \mathrm{P}(\Omega) + 1/k$ holds by the smoothness of $\partial\Omega_{1/k}$ and (1.5) so that we end up with

$$\mathscr{H}^1\left(\bigcup_{j=1}^N R^j\right) \le \mathrm{P}(\Omega).$$

As N is arbitrary and $\bigcup_{j=1}^\infty R^j$ equals $\partial\Omega$, this shows $\mathscr{H}^1(\partial\Omega) \le \mathrm{P}(\Omega)$. As observed in (1.1), the opposite inequality is always valid, and thus the proof is complete. □

In the introduction we have given a concrete example for the failure of the boundary hypothesis $\mathscr{H}^{n-1}(\partial\Omega) = \mathrm{P}(\Omega)$, but in that case the problematic set is — in some sense — quite negligible. Indeed, in the mentioned example and in many similar situations, one may pass to an open set $\widetilde{\Omega}$ which satisfies $\mathscr{H}^{n-1}(\partial\widetilde{\Omega}) = \mathrm{P}(\widetilde{\Omega})$, allows strict interior approximation, and is equivalent in the sense of $\mathscr{L}^n(\widetilde{\Omega}\Delta\Omega) = 0$. To some extent, this concept of equivalence seems reasonable, as it leaves the perimeter and the related measure-theoretic notions invariant, but — evidently — it changes the topological notions of interior approximation and boundary.

Anyhow, every equivalence class of open sets contains a maximal (with respect to set inclusion) representative $\Omega^* := \{x \in \mathbb{R}^n : \mathscr{L}^n(\mathrm{B}_\varrho(x) \setminus \Omega) = 0 \text{ for some } \varrho > 0\}$, which moreover has minimal topological boundary in the class. Whenever the boundary hypothesis holds or strict interior approximation is possible for an open set Ω, then this is a fortiori the case for Ω^*. Nonetheless, in the following we show that these assertions may fail even for Ω^*.

Example 5.2. *For $n \geq 2$, we take*

$$\Omega := (0,1)^n \setminus \overline{\bigcup_{i=1}^{\infty} \mathrm{B}_{\varrho_i}\big(y_i, \tfrac{1}{2}\big)} \subset \mathbb{R}^n \,,$$

where the $\varrho_i > 0$ and $y_i \in \mathbb{R}^{n-1}$ are chosen such that the $(n-1)$-dimensional balls $\mathrm{B}_{\varrho_i}(y_i)$ are disjoint with dense union in $[0,1]^{n-1}$ and such that we have $\sum_{i=1}^\infty \omega_{n-1}\varrho_i^{n-1} < 1$. Then, the Cantor type set

$$A := [0,1]^{n-1} \setminus \bigcup_{i=1}^{\infty} \mathrm{B}_{\varrho_i}(y_i) \subset \mathbb{R}^{n-1}$$

has positive $\mathscr{L}^{n-1}$-measure, Ω^* equals Ω, and we have

$$\mathrm{P}(\Omega) \leq \mathscr{H}^{n-1}(\partial\Omega) = 2n + n\omega_n \sum_{i=1}^{\infty} \varrho_i^{n-1} + \mathscr{L}^{n-1}(A) \,.$$

In addition, the lower semicontinuity of the perimeter, applied on $\Omega_\pm := \{x \in \Omega : \pm(x_n - \frac{1}{2}) > 0\}$, shows that interior approximations Ω_ε necessarily fulfill

$$\liminf_{\varepsilon \searrow 0} \mathrm{P}(\Omega_\varepsilon) \geq 2n + n\omega_n \sum_{i=1}^{\infty} \varrho_i^{n-1} + 2\mathscr{L}^{n-1}(A) \,.$$

Hence, strict interior approximation of Ω, in the sense of (1.5), is impossible.

To close this section, we remark that the situation may change for non-open sets. Looking at the representative $\Omega^1 := \{x \in \mathbb{R}^n : \lim_{\varrho \searrow 0} \varrho^{-n}\mathscr{L}^n(\mathrm{B}_\varrho(x) \setminus \Omega) = 0\}$ (which is even larger than Ω^*, has even smaller boundary, but is not necessarily open), for instance, one may still hope to find strict approximations $\Omega_\varepsilon \Subset \Omega^1$. However, we stress that such Ω_ε may potentially touch $\partial\Omega$ and even $\partial\Omega^1$; thus, this type of approximation seems vaguely related to ideas of [23, 24, 19], but quite different from the considerations of the present paper.

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