POINCARÉ-TYPE INEQUALITY FOR INTRINSIC LIPSCHITZ CONTINUOUS VECTOR FIELDS IN THE HEISENBERG GROUP

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Abstract. Scope of this paper is to prove a Poincaré type inequality for a family of non linear vector fields, whose coefficients are only Lipschitz continuous with respect to the distance induced by the vector fields themselves.

1. Introduction

The Poincaré inequality is one of the main tools in the proof of regularity of solutions of divergence form PDE's equations. In particular, it is fundamental in the so called Moser iteration technique used to obtain Harnack inequalities and Hölder continuity for solutions. Indeed, the proof of the Harnack inequality by means of the Moser technique can be reduced to verifying a suitable Poincaré inequality (see [30] for the details). Conversely, a parabolic Harnack inequality implies a version of the Poincaré inequality as shown by Saloff-Coste in [48]. It is well known (see [34] or [22, 25, 38]) that this type of inequality is satisfied for smooth vector fields satisfying an Hörmander type rank condition.

The Poincaré inequality for non smooth vector fields was first attached in [23]. Here the authors considered vector fields in diagonal form

$$X_i = \lambda_i(x)\partial_i \quad i = 1, \ldots, n$$

and they only require that λ_i's satisfy a reverse Hölder type inequality (see also [24]). Later on in [36], the authors developed a general approach to prove Poincaré inequalities for (possibly nonsmooth) vector fields. In the recent paper [41], the Poincaré inequality is proved by developing the method described in [36] for Euclidean Lipschitz vector fields with commutators which satisfy some additional structural conditions. We also quote the paper [39] in which the author generalized the approach developed in [41] to families of Lipschitz continuous vector fields satisfying the Hörmander condition of step two with low regularity assumptions on the commutators. In [8] authors prove the Poincaré inequality for a family of $C^{r-1,1}$ vector fields satisfying the Hörmander rank condition of step $r \geq 2$. We point out that all these proofs are based on refinements of the so called Nagel-Stein-Weinger's Lemma proved in [44] and the doubling condition for the balls of the ambient metric space.

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In [26] authors study the relationship between the validity of the Poincaré inequality and the existence of representation formulas for functions as (fractional) integral transforms of first-order vector fields. They show that Poincaré inequality leads (and in fact is often equivalent to) to a suitable representation formula. This approach was later developed in [11] in which another proof of the representation formula relying on Jerison-Poincaré inequality has been given. Finally, in [14] a general representation formula is proved in terms of the fundamental solution of an Hörmander type sublaplacian.

It is well known how to attack the regularity problem for solutions of non linear differential equations of the form

$$\sum_{i,j=1}^{m} X_i(a_{ij}(u)\, X_j u) = f \tag{1}$$

where X_i are smooth vector fields, a_{ij} are regular functions and f is a given source term. The situation is tremendously different if the non linearity shows up in the vector fields, rather than in the coefficients. Equations involving non linear vector fields naturally arise while studying curvature equations ([15]), Monge-Ampére equation ([50]), mathematical finance equation ([18]) or some fine properties of surfaces in the sub-Riemannian setting ([2, 4, 5, 16, 17, 27]).

In [15], while studying properties of graphs of functions $u : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$ with prescribed Levi curvature, authors were lead to study the following fully nonlinear and totally degenerate equation

$$X^2u + Y^2u = k(\xi, u)\frac{(1 + a^2 + b^2)^{3/2}}{(1 + u_t^2)^{1/2}} \tag{2}$$

where $X(p) := \partial_x + a(p)\partial_t$, $Y(p) := \partial_y + b(p)\partial_t$ and $a = a(\nabla u)$, $b = b(\nabla u)$ are suitable bounded functions depending on the gradient of u. Since X, Y are not self-adjoint and their coefficients are only bounded, then Poincaré and Sobolev inequalities for viscosity solutions of (2) a priori do not hold. To overcame these difficulties and in order to prove regularity results for equation (2) authors implemented a suitable approximation procedure which can be considered as an extension in this setting of the classical Schauder approach. In particular an appropriate notion of Lipschitz continuity for the coefficients a and b was introduced and approximate Poincaré and Sobolev inequalities were proved in term of this notion.

Vector fields of the same type arise also while studying sub-Riemannian mean curvature equation in the Heisenberg group $\mathbb{H}^n$ for $n \geq 2$. We recall that $\mathbb{H}^n$ can be identified with $(\mathbb{R}^{2n+1}, \cdot)$ where $\cdot$ is a suitable non commutative group operation. Moreover, the associated Lie algebra $\mathfrak{h}_n$ admits the stratification $\mathfrak{h}_n = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ where $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are subspaces of $\mathfrak{h}_n$ of dimension $2n$ and 1 respectively and $\mathfrak{h}_2 = [\mathfrak{h}_1, \mathfrak{h}_1]$ and all the other commutators vanish. From now on we will denote by $\nabla_{\mathbb{H}} := (\nabla_1^{\mathbb{H}}, \ldots, \nabla_{2n}^{\mathbb{H}})$ a basis of $\mathfrak{h}_1$.
In this setting an intrinsic notion of regular hypersurface has been introduced, since the classical Euclidean notion doesn't work (see [27, 35]). More precisely an intrinsic regular hypersurface M can be locally given as zero level set of $C^1_{\mathbb{H}}$ function f, with non vanishing gradient $(\nabla_1^{\mathbb{H}}, \ldots, \nabla_{2n}^{\mathbb{H}})$, where by $C^1_{\mathbb{H}}$ we will denote the set of functions f admitting continuous horizontal distributional derivatives $\nabla_i^{\mathbb{H}} f$ with $i = 1, \ldots, 2n$.

It has been proved in [16, 28] that, up a change of variables, regular hypersurfaces can be locally represented as graphs of the form:

$$M = \{(\phi(x), x) \in \mathbb{R}^{2n+1} : x \in \omega \subset \mathbb{R}^{2n}\}, \tag{3}$$

where the function ϕ is regular with respect to the projection on $\mathbb{R}^{2n}$ of the family $\nabla_{\mathbb{H}}$. A possible choice of these projected vectors fields (see also [2, 16]) is the following one:

$$\nabla_i^\phi = \partial_{x_i} - x_{i+n}\partial_{x_{2n}},\ \nabla_n^\phi = \partial_{x_n} + 2\phi(x)\partial_{x_{2n}}, \nabla_{i+n}^\phi = \partial_{x_{i+n}} + x_i\partial_{x_{2n}},\ i = 1, \dots, n-1. \tag{4}$$

In the same papers a quasi-distance d_ϕ associated to the function ϕ is defined. The relation between this notion, the one introduced in [15], and the exponential distance associated to the vector fields (4) has been studied in [17]. In addition, in [16] a new notion of Lipschitz continuous function with respect to the distance d_ϕ has been introduced. Precisely, in their definition a function $\phi : \omega \subset \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ is called intrinsic Lipschitz if $(\phi, \mathrm{d}_\phi) \longrightarrow \mathbb{R}$ is Lipschitz in the classical sense for the so called graph distance d_ϕ on ω defined in (12). In [27] the authors proposed an equivalent notion of Lipschitz continuous function, which applies for surfaces of arbitrary codimension and in the particular setting of this paper it is equivalent to the one given in [16]. We also refer to [17] for comparison of this distance and the cc-distance. The class of Lipschitz continuous functions has been successfully applied in the problem of rectifiability in $\mathbb{H}^n$ (see [27]) and a lot of interesting properties of this class have been recently studied (see [42, 51]). See also [21] where the notions of intrinsic graphs and of intrinsic Lipschitz graphs within general Carnot groups are studied.

Vector fields in (4) have been recently applied for studying intrinsic minimal graphs in $\mathbb{H}^n$ (see for istance [3, 13, 9, 10, 20, 49] and the references therein). In particular, the following mean curvature equation has been introduced for intrinsic minimal graphs

$$\sum_{i=1}^{2n-1} \nabla_i^\phi \Big(\frac{\nabla_i^\phi \phi}{\sqrt{1 + |\nabla^\phi \phi|^2}} \Big) = 0 \quad \text{in } \omega \tag{5}$$

where $\phi : \omega \subset \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ is a regular function and $\nabla^\phi \phi := (\nabla_1^\phi \phi, \dots, \nabla_{2n-1}^\phi \phi)$. We note that equation (5) is formally equivalent to the classical minimal surface equation. Existence result of variational solutions are proven in [29] and [49]; approximation of a minimal boundary by means of intrinsic Lipschitz functions has been recently made in [43]. Nevertheless, as far as we known, regularity results for intrinsic minimal graphs are known only under the additional assumption that $|\nabla^\phi \phi| + |\partial_{x_{2n}} \phi|$ is bounded, see [9, 10].

The Poincaré inequality for intrinsic Lipschitz functions is the natural analogous in this setting of the instrument used in classical Euclidean setting to fill this gap.

In the present paper we will prove a Poincaré type inequality for intrinsic Lipschitz functions. More precisely, in view of some possible applications to the regularity of solutions to (5), we will prove the inequality for functions which belong to an intrinsic Sobolev space, modeled on ϕ in a viscosity sense (see for example [9, 10]).

Definition 1.1. *Let $\phi : \omega \subset \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ be an intrinsic Lipschitz continuous function.*

We say that a function $\psi : \omega \subset \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ belongs to the space $W_\phi(\omega)$ if there exist sequences $\{\psi_k\}_{k\in\mathbb{N}}$ and $\{\phi_k\}_{k\in\mathbb{N}}$ in $C^\infty(\omega)$ such that

(i) *$\psi_k \to \psi$ in $L^1_{loc}(\omega)$ as $k \to +\infty$;*
(ii) *$\phi_k \to \phi$ uniformly in ω as $k \to +\infty$;*

(iii) $|\nabla^{\phi_k}\psi_k(x)| \leq M \ \forall x \in \omega$ *and* k;
(iv) $\nabla^{\phi_k}\psi_k \rightharpoonup^* \nabla^{\phi}\psi$ *as* $k \to +\infty$,

for some positive constant M.

Then, our main result is the following

Theorem 1.2. *Let* ω *be a bounded and open subset of* $\mathbb{R}^{2n}$ *with* $n \geq 2$, *and let* $\phi : \omega \to \mathbb{R}$ *be an intrinsic Lipschitz function and* $\psi \in W_\phi(\omega)$. *Then there exist positive constants* C_1, C_2 *with* $C_2 > 1$ *(depending continuosly on the Lipschitz constant* L_ϕ *of* ϕ*) such that*

$$\int_{U_\phi(\bar{x},r)} |\psi(y) - \psi_{U_\phi(\bar{x},r)}| \, \mathrm{d}\mathcal{L}^{2n}(y) \leq C_1\, r \int_{U_\phi(\bar{x},C_2\, r)} |\nabla^\phi \psi(y)| \, \mathrm{d}\mathcal{L}^{2n}(y), \tag{6}$$

for every $U_\phi(\bar{x}, C_2\, r) \subset \omega$, *where*

$$U_\phi(x,r) := \{y \in \omega : \mathrm{d}_\phi(x,y) < r\}\,. \tag{7}$$

Here $\psi_{U_\phi(\bar{x},r)}$ *denotes the mean of* ψ *on the ball* $U_\phi(\bar{x}, r)$ *with respect to the Lebesgue measure, i.e.*

$$\psi_{U_\phi(\bar{x},r)} := \frac{1}{\mathcal{L}^{2n}(U_\phi(\bar{x},r))} \int_{U_\phi(\bar{x},r)} \psi(y) \, \mathrm{d}\mathcal{L}^{2n}(y)\,. \tag{8}$$

Corollary 1.3. *If* $\phi : \omega \to \mathbb{R}$ *is an intrinsic Lipschitz function then there exist positive constants* C_1, C_2 *with* $C_2 > 1$ *(depending continuosly on the Lipschitz constant* L_ϕ *of* ϕ*) such that*

$$\int_{U_\phi(\bar{x},r)} |\phi(y) - \phi_{U_\phi(\bar{x},r)}| \, \mathrm{d}\mathcal{L}^{2n}(y) \leq C_1\, r \int_{U_\phi(\bar{x},C_2\, r)} |\nabla^\phi \phi(y)| \, \mathrm{d}\mathcal{L}^{2n}(y), \tag{9}$$

for every $U_\phi(\bar{x}, C_2\, r) \subset \omega$.

In order to clarify the statement in what follow we briefly describe our approach. As proved in [2] an intrinsic Lipschitz function can have a low Euclidean regularity (at most 1/2-Hölder continuous) and this lack of regularity prevent us to apply in our setting the classical techniques for proving Poincaré inequality with respect to a family of vector fields. Nevertheless the explicit expression of the vector fields $\nabla_1^\phi, \dots, \nabla_{2n-1}^\phi$ ensures the validity of the Hörmander condition. Indeed we have

$$[\nabla_1^\phi, \nabla_{n+1}^\phi] = \partial_{x_{2n}}.$$

Hence, even though the vector field ∇_n^ϕ has only intrinsic Lipschitz coefficients, the family ∇^ϕ spans the whole tangent space at every point. Moreover, always exploiting the explicit structure of the family ∇^ϕ we can approximate it by a suitable family of smooth vector fields. If the vector fields had C^1 coefficients this reduction can be made in very general setting, with the so called freezing method of Rothschild and Stein introduced in [46] and slightly simplified in [19]. In our case the lack of regularity of the coefficients does not allow to apply directly the freezing method and hence an ad hoc method has to be introduced. Precisely, if $\omega \subset \mathbb{R}^{2n}$, $n \geq 2$ and $\phi : \omega \longrightarrow \mathbb{R}$ is an intrinsic Lipschitz function then for every $x_0 \in \omega$ we define

$$\nabla_i^{\phi(x_0)} := \nabla_i^\phi \text{ for } i = 1, \dots, 2n-1, i \neq n, \quad \nabla_n^{\phi(x_0)} := \partial_{x_n} + 2\phi(x_0)\partial_{x_{2n}}. \tag{10}$$

The main advantage of working with the family of vector fields defined in (10) is that they have C^∞ coefficients, they satisfy an Hörmander type condition analogous to the one satisfied by the family ∇^ϕ and they can be considered as a zero order approximation family of ∇^ϕ. In particular, every $\psi \in C^\infty(\omega)$ can be represented by means of a suitable representation formula (proved in [7, 14]) in terms of the vector fields $\nabla_i^{\phi(x_0)}$, the fundamental solution Γ_{x_0} of the Laplacian operator

$$\mathcal{L}_{\phi(x_0)} := \sum_{i=1}^{2n-1} (\nabla_i^{\phi(x_0)})^2$$

and the superlevel sets $\Omega_{\phi(x_0)}(x_0, r)$ of Γ_{x_0}, which are equivalent to the balls $U_\phi(x_0, r)$.

In order to prove Theorem 1.2 in Section 3 we will first modify the aforementioned representation formula to obtain another representation formula in terms of the family ∇^ϕ. Successively, using an approximation result for intrinsic Lipschitz functions contained in [42] (see also [17] for a refinement) we prove that the representation formula proved in Section 3 still holds for intrinsic Lipschitz functions. Finally, in Section 4 we will provide the proof of Theorem 1.2 using an approach similar to the one proposed in [30].

2. Preliminaries

2.1. **Lipschitz continuous functions with respect to nonlinear vector fields.** Let $\omega \subset \mathbb{R}^{2n}$, with $n \geq 2$ be open and let $\phi : \omega \longrightarrow \mathbb{R}$ be a continuous function. Let us introduce the following family of vector fields

$$\begin{aligned} \nabla_i^\phi(x) &= \partial_{x_i} - x_{i+n}\partial_{x_{2n}}, \quad i = 1, \dots, n-1, \\ \nabla_n^\phi(x) &= \partial_{x_n} + 2\phi(x)\partial_{x_{2n}}, \\ \nabla_i^\phi(x) &= \partial_{x_i} + x_{i-n}\partial_{x_{2n}}, \quad i = n+1 \dots, 2n-1. \end{aligned} \tag{11}$$

These vector fields have been introduced in [2] and in [16] in the context of intrinsic graphs in the Heisenberg group and successively studied in [4, 5]. Similar vector fields show up in many other contexts both geometric [15], [50] and of mathematical finance [18]. The Lie algebra generated by the family $\nabla^\phi := (\nabla_1^\phi, \dots, \nabla_{2n-1}^\phi)$ has maximum rank at every point, hence it is possible to connect each couple of points in ω with an integral curve, and the Carnot-Carathéodory distance d_{cc} associated to ∇^ϕ is well defined, see [17]. It has been proved in [17] that if ϕ is Lipschitz continuous with respect to the d_{cc} distance then d_{cc} is locally equivalent to the following function, introduced in [2] and [16]:

$$\mathrm{d}_\phi(x, y) = \frac{1}{2}\max\left\{|\hat{x} - \hat{y}|_{\mathbb{R}^{2n-1}}, \sigma_\phi(x, y)\right\} + \frac{1}{2}\max\left\{|\hat{x} - \hat{y}|_{\mathbb{R}^{2n-1}}, \sigma_\phi(y, x)\right\}, \tag{12}$$

where for every $x = (x_1, \dots, x_{2n}) \in \mathbb{R}^{2n}$ we denote by $\hat{x} := (x_1, \dots, x_{2n-1}) \in \mathbb{R}^{2n-1}$,

$$\sigma_\phi(x, y) := |y_{2n} - x_{2n} - 2\phi(x)(y_n - x_n) + \sigma(x, y)|^{1/2} \quad x,\, y \in \omega \tag{13}$$

and

$$\sigma(x, y) := \sum_{i=1}^{n-1}(y_{i+n}x_i - x_{i+n}y_i). \tag{14}$$

Since the distance d_ϕ is explicit, we will always prefer it instead of the d_{cc} one.

Definition 2.1. *We say that $\phi : \omega \subset \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ is an intrinsic Lipschitz continuous function in ω and we write $\phi \in Lip(\omega)$, if there is a constant $L > 0$ such that:*

$$|\phi(x) - \phi(y)| \le L\, \mathrm{d}_\phi(x,y) \qquad \forall x, y \in \omega. \tag{15}$$

The Lipschitz constant of ϕ in ω is the infimum of the numbers L such that (15) *holds and we write $L_{\phi,\omega}$ (or simply L_ϕ) to denote it. We also say that ϕ is a locally intrinsic Lipschitz function, and we write $\phi \in Lip_{loc}(\omega)$ if $\phi \in Lip(\omega')$ for every $\omega' \Subset \omega$.*

Remark 2.2. *It immediately follows from the explicit expression of d_ϕ (see also* [15]*) that, if $\phi \in Lip(\omega)$ then d_ϕ is a quasi-distance on ω. Precisely,*

$$\mathrm{d}_\phi(x,y) = 0 \Longleftrightarrow x = y;$$
$$\mathrm{d}_\phi(x,y) = \mathrm{d}_\phi(y,x);$$

and for each $x, y, z \in \omega$:

$$\mathrm{d}_\phi(x,y) \le \tag{16}$$

$$\le \mathrm{d}_\phi(x,z) + \mathrm{d}_\phi(y,z) + |\phi(x) - \phi(z)|^{1/2}|x_n - z_n|^{1/2} + |\phi(y) - \phi(z)|^{1/2}|y_n - z_n|^{1/2}$$

so that

$$\mathrm{d}_\phi(x,y) \le (1 + L_\phi)^{1/2}(\mathrm{d}_\phi(x,z) + \mathrm{d}_\phi(y,z))\,.$$

Remark 2.3. *It is easy to see that, if $\phi \in Lip(\omega)$, then*

$$\sigma_\phi(y,x) \le \sigma_\phi(x,y) + |\phi(x) - \phi(y)|^{1/2}|x_n - y_n|^{1/2} \quad \forall x, y \in \omega$$

whence, by (12)*,*

$$\mathrm{d}_\phi(x,y) \le |\hat{x} - \hat{y}|_{\mathbb{R}^{2n-1}} + \sigma_\phi(x,y) + |\phi(x) - \phi(y)|^{1/2}|x_n - y_n|^{1/2} \quad \forall x, y \in \omega. \tag{17}$$

A detailed analysis of $Lip(\omega)$ can be found in [17, 27], here we recall only the properties that we will need for the proof of Theorem 1.2.

Note that $Lip(\omega)$ does not turn to be a vector space (see [49, Remark 4.2]). Nevertheless, the intrinsic Lipschitz functions amount to a thick class of functions. Indeed, it holds that ([27, Propositions 4.8 and 4.11])

$$Lip_E(\omega) \subsetneq Lip_{loc}(\omega) \subsetneq C^{1/2}_{loc}(\omega)\,, \tag{18}$$

where, $Lip_E(\omega)$ and $C^{1/2}_{loc}(\omega)$ denote the classes of real-valued Euclidean Lipschitz and locally 1/2-Euclidean-Hölder continuous functions on ω respectively.

Theorem 2.4. *(*[28]*) If $\phi \in Lip(\omega)$ then ϕ is ∇^ϕ-differentiable for $\mathcal{L}^{2n}$-a.e $x \in \omega$, in the sense defined in* [2]*. Besides, for $\mathcal{L}^{2n}$-a.e $x \in \omega$ there is a unique vector $\nabla^\phi\phi(x) \in \mathbb{R}^{2n-1}$ called $\nabla^\phi-$gradient of ϕ such that*

$$\phi(x) = \phi(y) + \left\langle \nabla^\phi\phi(x), \tilde{\pi}(y) \right\rangle + o(\mathrm{d}_\phi(x,y)) \quad \text{as } y \to x$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^{2n-1}$ and $\tilde{\pi}(x_1, \ldots, x_{2n-1}, x_{2n}) := (x_1, \ldots, x_{2n-1})$, $\forall x \in \mathbb{R}^{2n-1}$.

In [17] the following estimates for L_ϕ are proved. Precisely, for each $\bar{x} \in \omega$ and each $r > 0$ sufficiently small there is $c_1 > 0$ depending only on $\|\nabla^\phi \phi\|_{L^\infty(\omega)}$ such that

$$L_{\phi, U_\phi(\bar{x}, r)} \leq c_1 \|\nabla^\phi \phi\|_{L^\infty(\omega)},$$

and there is $c_2 = c_2(n) > 0$ such that

$$\|\nabla^\phi \phi\|_{L^\infty(\omega)} \leq c_2 L_\phi (L_\phi + 1)$$

where $U_\phi(x, r)$ is defined in (7).

It has been recently proved in [42] the following approximation result for intrinsic Lipschitz functions:

Theorem 2.5. *Let $\omega \subset \mathbb{R}^{2n}$ be a bounded open set and let $\phi \in Lip(\omega)$. Then there exists a sequence $\{\phi_k\}$ with $\phi_k \in C^\infty(\omega)$ such that*

(i) *$\phi_k \to \phi$ uniformly in ω as $k \to \infty$,*
(ii) *$|\nabla^{\phi_k} \phi_k(x)| \leq \|\nabla^\phi \phi\|_{L^\infty(\omega)}$ $\forall x \in \omega$.*

We also quote the paper [17] where we proved that every $\phi \in Lip(\omega)$ can be approximated by a sequence $\{\phi_k\}_{k \in \mathbb{N}}$ of smooth functions satisfying (i), (ii) and also

$$\nabla^{\phi_k} \phi_k(x) \to \nabla^\phi \phi(x) \quad \mathcal{L}^{2n} - \text{a.e in } \omega.$$

A detailed analysis and further properties of $Lip(\omega)$ can be found in [27].

2.2. **Local approximation of the vector fields.** By (18) if $\phi : \omega \subset \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ is an intrinsic Lipschitz function then the family ∇^ϕ has coefficients which are, from an Euclidean point of view, only Hölder continuous. To overcome this lack of regularity on the coefficients we use the approach introduced in [44] and subsequently refined in [19]. The idea is to associate to the family ∇^ϕ a new family of Hörmander vector fields with regular coefficients. Precisely, for each $x_0 \in \omega$ we consider the family of vector fields $\nabla^{\phi(x_0)} = (\nabla_1^{\phi(x_0)}, \dots, \nabla_{2n-1}^{\phi(x_0)})$ introduced in (10). We will also call

$$\hat{\nabla}_i^{\phi(x_0)} = \nabla_i^{\phi(x_0)}, i = 1, \cdots, 2n-1, \quad \hat{\nabla}_{2n}^{\phi(x_0)} = \partial_{x_{2n}},$$

and we will denote by $\hat{\nabla}^{\phi(x_0)}$ the family $\left(\hat{\nabla}_1^{\phi(x_0)}, \dots, \hat{\nabla}_{2n}^{\phi(x_0)}\right)$.

Since the point $x_0 \in \omega$ is fixed, then the vector fields $\nabla^{\phi(x_0)}$ are C^∞ and nilpotent whose generated Lie algebra is isomorphic to $\mathcal{G} = \mathbb{H}^{n-1} \times \mathbb{R}$. We denote by $\mathcal{Q}$ the homogeneous dimension of $\mathcal{G}$ and by

$$\tilde{\delta}_\lambda : \mathcal{G} \longrightarrow \mathcal{G} \text{ the dilation family canonically associated to } \mathcal{G}. \tag{19}$$

We can repeat for the family $\hat{\nabla}^{\phi(x_0)}$ the general procedure known for nilpotent vector fields. Namely, for each $x \in \mathbb{R}^{2n}$ we use the exponential mapping:

$$Exp_{\phi(x_0), x} : \mathcal{G} \longrightarrow \mathbb{R}^{2n}, \quad Exp_{\phi(x_0), x}(\tilde{y}) := \exp\Big(\sum_{i=1}^{2n} \tilde{y}_i \hat{\nabla}_i^{\phi(x_0)}\Big)(x),$$

where we have identified the element $\tilde{y} \in \mathcal{G}$ with its coordinates on the basis $\hat{\nabla}^{\phi(x_0)}$. In coordinates we get:

$$Exp_{\phi(x_0),x}(\tilde{y}) = \Big(x_1 + \tilde{y}_1, \ldots, x_{2n-1} + \tilde{y}_{2n-1}, x_{2n} + \tilde{y}_{2n} + 2\tilde{y}_n\phi(x_0) - \sigma(\tilde{y}, x)\Big) \tag{20}$$

where $\sigma(\cdot,\cdot)$ is as in (14). The inverse mapping of $Exp_{\phi(x_0),x}$ will be denoted by

$$Log_{\phi(x_0),x} : \mathbb{R}^{2n} \longrightarrow \mathcal{G}$$

and an easy computations provides

$$Log_{\phi(x_0),x}(y) = \Big(y_1 - x_1, \ldots, y_{2n-1} - x_{2n-1}, y_{2n} - x_{2n} - 2\phi(x_0)(y_n - x_n) - \sigma(x, y)\Big), \tag{21}$$

where as before we have identified the element $Log_{\phi(x_0),x}(y) \in \mathcal{G}$ with its coordinates on the basis $\hat{\nabla}^{\phi(x_0)}$. We will define for every $x, y \in \mathbb{R}^{2n}$

$$\mathrm{d}_{\phi(x_0)}(x, y) := \|Log_{\phi(x_0),x}(y)\|,$$

where $\|(\tilde{x}_1, \ldots, \tilde{x}_{2n})\| := \max\{|(\tilde{x}_1, \ldots, \tilde{x}_{2n-1})|_{\mathbb{R}^{2n-1}}, |\tilde{x}_{2n}|^{\frac{1}{2}}\}$ and $|\cdot|_{\mathbb{R}^{2n-1}}$ denotes the Euclidean norm in $\mathbb{R}^{2n-1}$.

Remark 2.6. *Let $\omega \subset \mathbb{R}^{2n}$ be open and bounded and $\phi \in Lip(\omega)$ then $\mathrm{d}_\phi(x, y)$ can be expressed as follows:*

$$\mathrm{d}_\phi(x, y) = \frac{1}{2}\Big(\mathrm{d}_{\phi(x)}(x, y) + \mathrm{d}_{\phi(y)}(y, x)\Big), \tag{22}$$

indeed, by (21)

$$\mathrm{d}_{\phi(x)}(x, y) = \max\Big\{|\hat{x} - \hat{y}|_{\mathbb{R}^{2n-1}}, |y_{2n} - x_{2n} - 2\phi(x)(y_n - x_n) + \sigma(y, x)|^{\frac{1}{2}}\Big\}. \tag{23}$$

Moreover, by a simple calculation we obtain that the functions $\mathrm{d}_{\phi(x)}$ and d_ϕ are equivalent. Precisely, there exist $C_1, C_2 > 1$ depending only on L_ϕ such that for each $x, y \in \omega$

$$C_2\mathrm{d}_{\phi(y)}(y, x) \le \mathrm{d}_\phi(x, y) \le C_1\mathrm{d}_{\phi(y)}(y, x), \tag{24}$$

$$C_2\mathrm{d}_{\phi(x)}(x, y) \le \mathrm{d}_\phi(x, y) \le C_1\mathrm{d}_{\phi(x)}(x, y). \tag{25}$$

Besides, there exists a positive constant $C = C(L_\phi)$ such that for each $x, y, z \in \omega$

$$\mathrm{d}_{\phi(x)}(x, y) \le C\Big(\mathrm{d}_{\phi(x)}(x, z) + \mathrm{d}_{\phi(z)}(z, y)\Big). \tag{26}$$

In order to study the dependence of the vector fields $\hat{\nabla}^{\phi(x_0)}$ on the variable x_0 we recognize that the map

$$Log_{\phi(x_0),x_0} : \mathbb{R}^{2n} \longrightarrow \mathcal{G}$$

changes the families $\nabla^{\phi(x_0)}$ and $\hat{\nabla}^{\phi(x_0)}$ into the family ∇ and $\hat{\nabla}$ respectively, where:

$$\begin{aligned} &\nabla_i := \nabla_i^{\phi(x_0)}, \quad \hat{\nabla}_j := \nabla_j^{\phi(x_0)} \quad \text{for } i \in \{1, \ldots, 2n-1\}, j \in \{1, \ldots, 2n\}, i, j \ne n, \\ &\nabla_n := \partial_{x_n}, \quad \hat{\nabla}_n = \partial_{x_n}. \end{aligned} \tag{27}$$

Precisely, for each $\psi \in C^\infty(\mathbb{R}^n)$, if we define

$$\tilde{\psi}(\tilde{x}) := \psi(Log^{-1}_{\phi(x_0),x_0}(\tilde{x})), \tag{28}$$

then

$$\hat{\nabla}_i^{\phi(x_0)}\psi(x) = \hat{\nabla}_i\tilde{\psi}(Log_{\phi(x_0),x_0}(x)), \qquad \forall i \in \{1,\dots,2n\}.$$

Obviously, the exponential distance $\tilde{\mathrm{d}}$ associated to the vector fields ∇ is smooth, independent of x_0 and such that

$$\begin{aligned} &\tilde{\mathrm{d}}(0,\tilde{x}) = \|\tilde{x}\|, \quad \forall \tilde{x} \in \mathbb{R}^{2n},\\ &\mathrm{d}_{\phi(x_0)}(x,y) = \tilde{\mathrm{d}}(Log_{\phi(x_0),x_0}(x), Log_{\phi(x_0),x_0}(y)) \qquad \forall x,y,x_0 \in \omega. \end{aligned} \tag{29}$$

2.3. **Sub-Laplacian and fundamental solution.** Let us call sub-Laplacian the second order differential operator defined as

$$\mathcal{L}_{\phi(x_0)} := \sum_{i=1}^{2n-1} (\nabla_i^{\phi(x_0)})^2. \tag{30}$$

It is well known that $\mathcal{L}_{\phi(x_0)}$ admits a fundamental solution which we will denote by $\Gamma_{\phi(x_0)}$ (see for [7] for the details). This operator is changed by the map $Log_{\phi(x_0),x_0}$ into the sub-Laplacian operator

$$\mathcal{L} := \sum_{i=1}^{2n-1} (\nabla_i)^2.$$

That is, for each $\psi \in C^\infty(\mathbb{R}^{2n})$

$$(\mathcal{L}_{\phi(x_0)}\psi)(x) = (\mathcal{L}\tilde{\psi})(Log_{\phi(x_0),x_0}(x)) \quad \forall\, x \in \mathbb{R}^{2n},$$

where $\tilde{\psi}$ is defined in (28).

Clearly the operator $\mathcal{L}$ has a fundamental solution Γ of class C^∞ far from the pole $\tilde{x} = \tilde{y}$, which is homogeneous of degree $2 - \mathcal{Q}$ with respect to the dilation family $\tilde{\delta}_\lambda$, defined in (19) (see [7] and the references therein). This means that there exist positive constants C_1, C_2 such that for every $\tilde{x}$ and $\tilde{y}$ in $\mathbb{R}^{2n}$, $\tilde{x} \neq \tilde{y}$

$$\begin{aligned} \frac{C_1}{\tilde{\mathrm{d}}(\tilde{x},\tilde{y})^{\mathcal{Q}-2}} \le \Gamma(\tilde{x},\tilde{y}) &\le \frac{C_2}{\tilde{\mathrm{d}}(\tilde{x},\tilde{y})^{\mathcal{Q}-2}};\\ |\nabla_i\Gamma(\tilde{x},\tilde{y})| &\le \frac{C_2}{\tilde{\mathrm{d}}(\tilde{x},\tilde{y})^{\mathcal{Q}-1}};\\ |\nabla_j\nabla_i\Gamma(\tilde{x},\tilde{y})| &\le \frac{C_2}{\tilde{\mathrm{d}}(\tilde{x},\tilde{y})^{\mathcal{Q}}}, \end{aligned} \tag{31}$$

for every $i, j = 1, \dots, 2n-1$ (see [7, 47]). Besides, the fundamental solution $\Gamma_{\phi(x_0)}$ of $\mathcal{L}_{\phi(x_0)}$ can be explicitly written in terms of Γ as

$$\Gamma_{\phi(x_0)}(x,y) = \Gamma(Log_{\phi(x_0),x_0}(x), Log_{\phi(x_0),x_0}(y)), \tag{32}$$

and

$$\nabla_i^{\phi(x_0)}\Gamma_{\phi(x_0)}(x,y) = \nabla_i\Gamma(Log_{\phi(x_0),x_0}(x), Log_{\phi(x_0),x_0}(y)),$$

for $i = 1, \dots, 2n-1$. It follows that the inequalities in (31) are satisfied also for $\Gamma_{\phi(x_0)}(x,y)$ and $\mathrm{d}_{\phi(x_0)}(x,y)$ with the same constants. In particular it is clear that these constants are independent of x_0. Using the estimates on $\Gamma_{\phi(x_0)}$ it follows that the sphere of the metric $\mathrm{d}_{\phi(x_0)}$ are equivalent to the superlevels of the fundamental solution $\Gamma_{\phi(x_0)}$:

$$\Omega_{\phi(x_0)}(x,r) = \left\{ y \in \mathbb{R}^{2n} \mid \Gamma_{\phi(x_0)}(x,y) > r^{2-Q} \right\}, \ r > 0, \tag{33}$$

and that for every fixed $x_0 \in \omega$ the set $\Omega_{\phi(x_0)}(x_0,r)$ has regular boundary (see [14]). In particular, from (25), (29) and (31), there exists $r_0, \alpha > 0$ with $\alpha = \alpha(L_\phi)$ such that for any $x_0 \in \omega$ and $r \le r_0$

$$\Omega_{\phi(x_0)}(x_0, r/\alpha) \subset U_\phi(x_0,r) \subset \Omega_{\phi(x_0)}(x_0, \alpha\, r), \tag{34}$$

where

$$U_\phi(x_0,r) := \{ y \in \omega \mid \mathrm{d}_\phi(x_0,y) < r \}. \tag{35}$$

By (32) we have that

$$\Omega_{\phi(x_0)}(x,r) = \left\{ y \in \mathbb{R}^{2n} \mid \Gamma(Log_{\phi(x_0),x_0}(x), Log_{\phi(x_0),x_0}(y)) > r^{2-Q} \right\}, \tag{36}$$

in particular the sets $\Omega_{\phi(x_0)}(x_0,r)$ can be expressed in terms of the superlevels of the fundamental solution Γ as follows:

$$\begin{aligned} \Omega_{\phi(x_0)}(x_0,r) &= \left\{ y \in \mathbb{R}^{2n} \mid \Gamma(0, Log_{\phi(x_0),x_0}(y)) > r^{2-Q} \right\} \\ &= Exp_{\phi(x_0),x_0}(\tilde{\Omega}(0,r)), \end{aligned} \tag{37}$$

where

$$\tilde{\Omega}(0,r) := \left\{ \tilde{y} \in \mathbb{R}^{2n} \mid \Gamma(0,\tilde{y}) > r^{2-Q} \right\}. \tag{38}$$

We will also denote

$$K(\tilde{y}) := \Gamma^{-\frac{1}{(Q-2)}}(0,\tilde{y}), \ \tilde{y} \in \mathbb{R}^{2n}, \tag{39}$$

so that, we can rewrite $\tilde{\Omega}(0,r)$ as:

$$\tilde{\Omega}(0,r) = \left\{ \tilde{y} \in \mathbb{R}^{2n} \mid K(\tilde{y}) < r \right\}. \tag{40}$$

3. A representation formula in terms of the intrinsic gradient

Let us fix $\omega \subset \mathbb{R}^{2n}$ open and bounded, $n \ge 2$ and $\phi, \psi \in C^\infty(\omega)$. The aim of this section is to prove a representation formula for ψ in terms of its intrinsic gradient $\nabla^\phi \psi$ on the superlevels $\Omega_{\phi(x_0)}(x_0,r)$ of $\Gamma_{\phi(x_0)}$. To obtain this result we use an already known representation formula for general operators in Lie groups which can be found in [7, 14]. In our case the aforementioned result can be stated as follow:

Proposition 3.1. *For every $x_0 \in \omega$ and $R > 0$ such that $\Omega_{\phi(x_0)}(x_0, R) \subset \omega$ we have*

(41)
$$\psi(x_0) = \frac{\mathcal{Q}}{(\mathcal{Q}-2)(1-\frac{1}{2^{\mathcal{Q}}})R^{\mathcal{Q}}} \int_{\Omega_{\phi(x_0)}(x_0,R)\setminus\Omega_{\phi(x_0)}(x_0,\frac{R}{2})} \frac{|\nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)|^2}{\Gamma_{\phi(x_0)}^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}(x_0,y)} \psi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)$$
$$+ \frac{\mathcal{Q}}{(1-\frac{1}{2^{\mathcal{Q}}})R^{\mathcal{Q}}} \int_{\frac{R}{2}}^{R} r^{\mathcal{Q}-1} \int_{\Omega_{\phi(x_0)}(x_0,r)} \left\langle \nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y), \nabla^{\phi(x_0)}\psi(y) \right\rangle \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r.$$

Here $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean scalar product in $\mathbb{R}^{2n-1}$.

Remark 3.2. *We explicitly note that, if we choose $\psi \equiv 1$, then from (41) we get:*

$$1 = \frac{C(\mathcal{Q})}{R^{\mathcal{Q}}} \int_{\Omega_{\phi(x_0)}(x_0,R)\setminus\Omega_{\phi(x_0)}(x_0,\frac{R}{2})} \frac{|\nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)|^2}{\Gamma_{\phi(x_0)}^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}(x_0,y)}\,\mathrm{d}\mathcal{L}^{2n}(y) \tag{42}$$

where $C(\mathcal{Q}) := \frac{\mathcal{Q}}{(\mathcal{Q}-2)(1-\frac{1}{2^{\mathcal{Q}}})}$.

This remark allows to say that (41) represents a function ψ as the sum of its mean on a suitable set and the gradient $\nabla^{\phi(x_0)}\psi$. Hence, it seems natural to give the following definition

Definition 3.3. *For every $x_0 \in \omega$ and $R > 0$ such that $\Omega_{\phi(x_0)}(x_0, R) \subset \omega$ we define the following mean of ψ, on the set $\Omega_{\phi(x_0)}(x_0, R) \setminus \Omega_{\phi(x_0)}(x_0, \frac{R}{2})$, in terms of the fundamental solution $\Gamma_{\phi(x_0)}$:*

$$\bar{m}(\psi,\phi,R)(x_0) := \frac{C(\mathcal{Q})}{R^{\mathcal{Q}}} \int_{\Omega_{\phi(x_0)}(x_0,R)\setminus\Omega_{\phi(x_0)}(x_0,\frac{R}{2})} \frac{|\nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)|^2}{\Gamma_{\phi(x_0)}^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}(x_0,y)} \psi(y)\,\mathrm{d}\mathcal{L}^{2n}(y).$$

In the sequel we will need another mean of ψ on the same set $\Omega_{\phi(x_0)}(x_0, R) \setminus \Omega_{\phi(x_0)}(x_0, \frac{R}{2})$, precisely we denote:

$$m(\psi,\phi,R)(x_0) := \frac{2}{R}\int_{\frac{R}{2}}^{R} \bar{m}(\psi,\phi,r)(x_0)\ \mathrm{d}r. \tag{43}$$

The following remark will be fundamental later in this section.

Remark 3.4. *Let $g \in C^1(\mathbb{R}^{2n})$, $r > 0$ and $c_1, c_2 > 0$ we define:*

$$A_{r,c_1,c_2} := \{y \in \mathbb{R}^{2n} : c_1 r < g(y) < c_2 r\}.$$

Then, for every $f, \psi \in C^1(\mathbb{R}^{2n})$ and $R_1, R_2 \in \mathbb{R}$ with $R_1 < R_2$, using the fact that $\partial_{2n} = \frac{1}{2}(\nabla_1^{\phi(x_0)}\nabla_{n+1}^{\phi} - \nabla_{n+1}^{\phi(x_0)}\nabla_1^{\phi})$ and integrating by part we have:

$$\begin{aligned}
&\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{A_{r,c_1,c_2}} f(y)\partial_{2n}\psi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)dr = \\
=&\frac{1}{2}\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{\{y:g(y)/c_2=r\}} f(y)\nabla_{n+1}^{\phi}\psi(y)\frac{\nabla_1^{\phi(x_0)}g(y)}{|\nabla_E g(y)|}\,\mathrm{d}\mathcal{H}^{2n-1}(y)\,\mathrm{d}r \\
&-\frac{1}{2}\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{\{y:g(y)/c_1=r\}} f(y)\nabla_{n+1}^{\phi}\psi(y)\frac{\nabla_1^{\phi(x_0)}g(y)}{|\nabla_E g(y)|}\,\mathrm{d}\mathcal{H}^{2n-1}(y)\mathrm{d}r \\
&-\frac{1}{2}\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{\{y:g(y)/c_2=r\}} f(y)\nabla_1^{\phi}\psi(y)\frac{\nabla_{n+1}^{\phi(x_0)}g(y)}{|\nabla_E g(y)|}\,\mathrm{d}\mathcal{H}^{2n-1}(y)\mathrm{d}r \\
&+\frac{1}{2}\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{\{y:g(y)/c_1=r\}} f(y)\nabla_1^{\phi}\psi(y)\frac{\nabla_{n+1}^{\phi(x_0)}g(y)}{|\nabla_E g(y)|}\,\mathrm{d}\mathcal{H}^{2n-1}(y)\mathrm{d}r \\
&-\frac{1}{2}\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{A_{r,c_1,c_2}} \nabla_1^{\phi(x_0)}f(y)\nabla_{n+1}^{\phi}\psi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r \\
&+\frac{1}{2}\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{A_{r,c_1,c_2}} \nabla_{n+1}^{\phi(x_0)}f(y)\nabla_1^{\phi}\psi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r,
\end{aligned}$$

where ∇_E denotes the Euclidean gradient. By the coarea formula we infer that:

$$\begin{aligned}
&\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{A_{r,c_1,c_2}} f(y)\partial_{2n}\psi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)dr = \\
=&\frac{1}{2}\int_{A_{r,c_2R_1,c_2R_2}} \frac{g^{\mathcal{Q}-1}(y)}{c_2^{\mathcal{Q}-1}} f(y)\nabla_{n+1}^{\phi}\psi(y)\nabla_1^{\phi(x_0)}g(y)\,\mathrm{d}\mathcal{L}^{2n}(y) \\
&-\frac{1}{2}\int_{A_{r,c_1R_1,c_1R_2}} \frac{g^{\mathcal{Q}-1}(y)}{c_1^{\mathcal{Q}-1}} f(y)\nabla_{n+1}^{\phi}\psi(y)\nabla_1^{\phi(x_0)}g(y)\,\mathrm{d}\mathcal{L}^{2n}(y) \\
&-\frac{1}{2}\int_{A_{r,c_2R_1,c_2R_2}} \frac{g^{\mathcal{Q}-1}(y)}{c_2^{\mathcal{Q}-1}} f(y)\nabla_1^{\phi}\psi(y)\nabla_{n+1}^{\phi(x_0)}g(y)\,\mathrm{d}\mathcal{L}^{2n}(y) \\
&+\frac{1}{2}\int_{A_{r,c_1R_1,c_1R_2}} \frac{g^{\mathcal{Q}-1}(y)}{c_1^{\mathcal{Q}-1}} f(y)\nabla_1^{\phi}\psi(y)\nabla_{n+1}^{\phi(x_0)}g(y)\,\mathrm{d}\mathcal{L}^{2n}(y) \\
&-\frac{1}{2}\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{A_{r,c_1,c_2}} \nabla_1^{\phi(x_0)}f(y)\nabla_{n+1}^{\phi}\psi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r \\
&+\frac{1}{2}\int_{R_1}^{R_2} r^{\mathcal{Q}-1}\int_{A_{r,c_1,c_2}} \nabla_{n+1}^{\phi(x_0)}f(y)\nabla_1^{\phi}\psi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r.
\end{aligned}$$

If in addition $c_1 = 0$ then the integrals on A_{r,c_1R_1,c_1R_2} are not present.

In the following Proposition we will slightly modify the mean formula in Proposition 3.1, which contains derivatives in the direction of the vector fields $\nabla^{\phi(x_0)}$ of ψ, in order to obtain a mean representation formula which contains derivatives with respect to the vector fields ∇^ϕ.

Proposition 3.5. *For every $x_0 \in \omega$ and $R > 0$ such that $\Omega_{\phi(x_0)}(x_0, R) \subset \omega$ we have*

$$\begin{aligned}
&\psi(x_0) - m(\psi, \phi, R)(x_0) = \\
=&\frac{2}{R}\int_{\frac{R}{4}}^{R} f_1\left(\frac{r}{R}\right)\int_{\Omega_{\phi(x_0)}(x_0,r)} \left\langle K_1(x_0, y), \nabla^\phi \psi(y)\right\rangle \, \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r \\
+&\frac{2}{R}\int_{\frac{R}{2}}^{R} \int_{\Omega_{\phi(x_0)}(x_0,r)\setminus\Omega_{\phi(x_0)}(x_0,\frac{r}{2})} \left\langle K_2(x_0, y, r), \nabla^\phi \psi(y)\right\rangle \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r,
\end{aligned}$$

where $f_1 \in C^0([\frac{1}{4}, 1])$ and the vector valued functions K_1 and K_2 are defined in (47) *and* (48) *respectively. Moreover,*

$$|K_1(x_0, y)| \le \tilde{C}_1(L_{\phi,\Omega_{\phi(x_0)}(x_0,R)} + 1)^2 \mathrm{d}^{1-\mathcal{Q}}_{\phi(x_0)}(x_0, y) \quad \forall y \in \Omega_{\phi(x_0)}(x_0, R) \tag{44}$$

and

$$|K_2(x_0, y, r)| \le \tilde{C}_2(L_\phi + 1)^2 \mathrm{d}^{1-\mathcal{Q}}_{\phi(x_0)}(x_0, y),\ \forall y \in \Omega_{\phi(x_0)}(x_0, R) \setminus \Omega_{\phi(x_0)}(x_0, R/2), r \in \left(\frac{R}{2}, R\right) \tag{45}$$

where L_ϕ means $L_{\phi,\Omega_{\phi(x_0)}(x_0,R)\setminus\Omega_{\phi(x_0)}(x_0,R/2)}$ and $\tilde{C}_1, \tilde{C}_2 > 0$ are suitable constants depending only on the homogeneous dimension $\mathcal{Q}$ and on the structure constants C_1 and C_2 in (31).

Proof. We will always denote by C a positive constant depending only on $\mathcal{Q}$ which can be different from line to line. By Proposition 3.1 for all $r \in (0, R)$

$$\begin{aligned}
&\psi(x_0) - \bar{m}(\psi, \phi, r)(x_0) = \\
&= \frac{C}{r^{\mathcal{Q}}}\int_{\frac{r}{2}}^{r} s^{\mathcal{Q}-1}\int_{\Omega_{\phi(x_0)}(x_0,s)} \left\langle \nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0, y), \nabla^{\phi(x_0)}\psi(y)\right\rangle \, \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}s \\
&= \frac{C}{r^{\mathcal{Q}}}\int_{\frac{r}{2}}^{r} s^{\mathcal{Q}-1}\int_{\Omega_{\phi(x_0)}(x_0,s)} \left\langle \nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0, y), \nabla^{\phi}\psi(y)\right\rangle \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}s \\
&+ \frac{C}{r^{\mathcal{Q}}}\int_{\frac{r}{2}}^{r} s^{\mathcal{Q}-1}\int_{\Omega_{\phi(x_0)}(x_0,s)} \nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0, y)(\phi(x_0) - \phi(y))\partial_{2n}\psi(y) \, \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}s.
\end{aligned}$$

Using Remark 3.4 with $g(y) := \Gamma^{\frac{1}{2-\mathcal{Q}}}_{\phi(x_0)}(x_0, y)$ we obtain:

$$\begin{aligned}
\psi(x_0) - \bar{m}(\psi, \phi, r)(x_0) =&\frac{1}{r^{\mathcal{Q}}}\int_{\frac{r}{2}}^{r} s^{\mathcal{Q}-1}\int_{\Omega_{\phi(x_0)}(x_0,s)} \left\langle K_1(x_0, y), \nabla^\phi \psi(y)\right\rangle \, \mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}s \\
&+ \int_{\Omega_{\phi(x_0)}(x_0,r)\setminus\Omega_{\phi(x_0)}(x_0,\frac{r}{2})} \left\langle K_2(x_0, y, r)\nabla^\phi \psi(y)\right\rangle \mathrm{d}\mathcal{L}^{2n}(y)
\end{aligned} \tag{46}$$

where

$$K_1(x_0,y) := C\nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y) - C\nabla_1^{\phi(x_0)}\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0)-\phi(y))e_{n+1} + C\nabla_{n+1}^{\phi(x_0)}\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)(\phi(x_0)-\phi(y))e_1, \tag{47}$$

and

$$K_2(x_0,y,r) := \frac{C}{r^{\mathcal{Q}}}\frac{\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)}{\Gamma_{\phi(x_0)}^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}(x_0,y)}(\phi(x_0)-\phi(y))\nabla_{n+1}^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)e_1 - \frac{C}{r^{\mathcal{Q}}}\frac{\nabla_n^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)}{\Gamma_{\phi(x_0)}^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}(x_0,y)}(\phi(x_0)-\phi(y))\nabla_1^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)e_{n+1}, \tag{48}$$

where e_i is the i-th element of the canonical basis of $\mathbb{R}^{2n-1}$.

Integrating (46) from $\frac{R}{2}$ to R we get

$$\psi(x_0) - m(\psi,\phi,R)(x_0) = = \frac{2}{R}\int_{\frac{R}{2}}^{R}\frac{1}{\rho^{\mathcal{Q}}}\int_{\frac{\rho}{2}}^{\rho} r^{\mathcal{Q}-1}\int_{\Omega_{\phi(x_0)}(x_0,r)}\Big\langle K_1(x_0,y),\nabla^{\phi}\psi(y)\Big\rangle \,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r\mathrm{d}\rho + \frac{2}{R}\int_{\frac{R}{2}}^{R}\int_{\Omega_{\phi(x_0)}(x_0,\rho)\setminus\Omega_{\phi(x_0)}(x_0,\frac{\rho}{2})}\Big\langle K_2(x_0,y,\rho),\nabla^{\phi}\psi(y)\Big\rangle \,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}\rho.$$

Exchanging the order of integration in the first integral and setting:

$$f_1(t) := \frac{2^{1-\mathcal{Q}}-(2t)^{\mathcal{Q}-1}}{1-\mathcal{Q}} \text{ if } t\in[1/4,1/2], \quad f_1(t) := \frac{t^{\mathcal{Q}-1}-1}{1-\mathcal{Q}} \text{ if } t\in[1/2,1],$$

we get the thesis. Finally, the estimates on K_1 and K_2 are direct consequences of (31). □

In order to compare $\bar{m}(\psi,\phi,r)(x)$ and $\bar{m}(\psi,\phi,r)(y)$ when $x\neq y$ we will first express them as integrals on the same sphere:

Lemma 3.6. *For each $x_0\in\omega$ and each $R>0$ such that $\Omega_{\phi(x_0)}(x_0,R)\subset\omega$ there exists a kernel*

$$K_3(0,\tilde{y}) := \frac{\mathcal{Q}}{(\mathcal{Q}-2)(1-\frac{1}{2^{\mathcal{Q}}})}\frac{|\nabla\Gamma(0,\tilde{y})|^2}{\Gamma(0,\tilde{y})^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}}$$

such that

$$\bar{m}(\psi,\phi,R)(x_0) = \frac{1}{R^{\mathcal{Q}}}\int_{\tilde{\Omega}(0,R)\setminus\tilde{\Omega}(0,\frac{R}{2})} K_3(0,\tilde{y})\psi(Exp_{\phi(x_0),x_0}(\tilde{y}))\,\mathrm{d}\mathcal{L}^{2n}(\tilde{y}),$$

where $\tilde{\Omega}(0,R)$ is defined in (38). Moreover, there exist constants $\tilde{C}_3,\tilde{C}_4$ depending only $\mathcal{Q}$ and on the structure constants C_1 and C_2 in (31) such that

$$|K_3(0,\tilde{y})|\le\tilde{C}_3, \qquad |\nabla K_3(0,\tilde{y})|\le\frac{\tilde{C}_4}{\|\tilde{y}\|} \quad \forall\tilde{y}\in\tilde{\Omega}(0,R)\setminus\tilde{\Omega}\Big(0,\frac{R}{2}\Big). \tag{49}$$

Proof. By (37) we have that

$$\Omega_{\psi(x_0)}(x_0,R)=Exp_{\phi(x_0),x_0}(\tilde\Omega(0,R)).$$

So that, by Definition 3.3 and (32) we have:

$$\begin{aligned}&\bar m(\psi,\phi,R)(x_0)=\\&=\frac{C(\mathcal{Q})}{(\mathcal{Q}-2)R^{\mathcal{Q}}}\int_{\Omega_{\phi(x_0)}(x_0,R)\setminus\Omega_{\phi(x_0)}(x_0,\frac{R}{2})}\frac{|\nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)|^2}{\Gamma_{\phi(x_0)}^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}(x_0,y)}\,\psi(y)\,\mathrm{d}\mathcal{L}^{2n}(y)\\&=\frac{C(\mathcal{Q})}{(\mathcal{Q}-2)R^{\mathcal{Q}}}\int_{\tilde\Omega(0,R)\setminus\tilde\Omega(0,\frac{R}{2})}\frac{|\nabla\Gamma(0,\tilde y)|^2}{\Gamma^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}(0,\tilde y)}\psi(Exp_{\phi(x_0),x_0}(\tilde y))\,\mathrm{d}\mathcal{L}^{2n}(\tilde y),\end{aligned}$$

where in the last equality we have applied a change of variables and the fact that the determinant of the Jacobian matrix of $Exp_{\phi(x_0),x_0}$ is equal to 1. Finally, we observe that (49) follows directly from the estimates on Γ in (31). □

Proposition 3.7. *For every $\bar x\in\omega$ there exists $R_0>0$ such that, if $0<R<R_0$ then $\Omega_{\phi(\bar x)}(\bar x,R)\Subset\omega$ and for every $x,x_0\in\Omega_{\phi(\bar x)}(\bar x,R)$ and*

$$\tilde y\in Log_{\phi(x_0),x_0}(\Omega_{\phi(\bar x)}(\bar x,R))\cap Log_{\phi(x),x}(\Omega_{\phi(\bar x)}(\bar x,R))$$

defined

$$\tilde h=\tilde h(x,x_0;\tilde y):=Log_{\phi(x_0),Exp_{\phi(x_0),x_0}(\tilde y)}\Big(Exp_{\phi(x),x}(\tilde y)\Big)\tag{50}$$

and

$$\gamma_{\tilde y}(t):=\exp\Big(t\tilde h\hat\nabla^{\phi(x_0)}\Big)\Big(\exp(\tilde y\hat\nabla^{\phi(x_0)})(x_0)\Big)\quad t\in[0,1],\tag{51}$$

il holds

$$\gamma_{\tilde y}(t)\in\omega\quad\forall t\in[0,1].\tag{52}$$

Moreover, we have

$$\begin{aligned}&\psi(Exp_{\phi(x),x}(\tilde y))-\psi(Exp_{\phi(x_0),x_0}(\tilde y))=\\&=\int_0^1\sum_{i=1}^{2n}(Log_{\phi(x_0),x_0}(x))_i\hat\nabla_i^{\phi(x_0)}\psi(\gamma_{\tilde y}(t))dt+K_4(x,x_0,\tilde y)\int_0^1\partial_{2n}\psi(\gamma_{\tilde y}(t))\mathrm{d}t\end{aligned}\tag{53}$$

where

$$K_4(x,x_0,\tilde y):=2(\phi(x)-\phi(x_0))\tilde y_n-2\sigma(\tilde y,x-x_0).\tag{54}$$

The kernel K_4 is of class C^∞ with respect to $\tilde y$ and the following estimates hold:

$$|K_4(x,x_0,\tilde y)|\le 2(L_{\phi,\Omega_{\phi(\bar x)}(\bar x,R)}+1)\mathrm{d}_\phi(x,x_0)\|\tilde y\|,\tag{55}$$

$$|\nabla K_4(x,x_0,\tilde y)|\le 2(L_{\phi,\Omega_{\phi(\bar x)}(\bar x,R)}+1)\mathrm{d}_\phi(x,x_0),\tag{56}$$

where $x,x_0\in\Omega_{\phi(\bar x)}(\bar x,R)$ and $\tilde y\in Log_{\phi(x_0),x_0}(\Omega_{\phi(\bar x)}(\bar x,R))\cap Log_{\phi(x),x}(\Omega_{\phi(\bar x)}(\bar x,R))$.

Proof. Let us fix $\bar{x} \in \omega$ and $0 < R < \bar{R}$ where $\bar{R} := \sup\{R > 0 \mid \Omega_{\phi(\bar{x})}(\bar{x}, R) \Subset \omega\}$. For every $x, x_0 \in \Omega_{\phi(\bar{x})}(\bar{x}, R)$ since $Log_{\phi(x_0),x_0}(x_0) = Log_{\phi(x),x}(x) = 0 \in \mathcal{G}$ then

$$Log_{\phi(x_0),x_0}(\Omega_{\phi(\bar{x})}(\bar{x}, R)) \cap Log_{\phi(x),x}(\Omega_{\phi(\bar{x})}(\bar{x}, R)) \neq \emptyset.$$

By (20) we get

$$Exp_{\phi(x),x}(\tilde{y}) = \Big(x_1 + \tilde{y}_1, \dots, x_{2n-1} + \tilde{y}_{2n-1}, x_{2n} + \tilde{y}_{2n} + 2\tilde{y}_n\phi(x) - \sigma(\tilde{y}, x)\Big),$$
$$Exp_{\phi(x_0),x_0}(\tilde{y}) = \Big(x_{0,1} + \tilde{y}_1, \dots, x_{0,2n-1} + \tilde{y}_{2n-1}, x_{0,2n} + \tilde{y}_{2n} + 2\tilde{y}_n\phi(x_0) - \sigma(\tilde{y}, x_0)\Big),$$

then using (50) and (21) we obtain

$$\tilde{h}_i = (x - x_0)_i \quad i = 1, \dots, 2n-1, \tag{57}$$
$$\tilde{h}_{2n} = (x - x_0)_{2n} - 2\phi(x_0)(x - x_0)_n + 2\tilde{y}_n(\phi(x) - \phi(x_0)) - 2\sigma(\tilde{y}, x - x_0) + \sigma(x, x_0)$$

and calling

$$\tilde{x} := Log_{\phi(x_0),x_0}(x), \tag{58}$$

we realize that

$$\tilde{h} = \tilde{x} + \Big(2\tilde{y}_n(\phi(x) - \phi(x_0)) - 2\sigma(\tilde{y}, \tilde{x})\Big)e_{2n}. \tag{59}$$

By (51) and the Baker-Campbell-Hausdorff formula we have

$$\gamma_{\tilde{y}}(t) = \exp\Big(t\tilde{h}\hat{\nabla}^{\phi(x_0)}\Big)\Big(\exp\Big(\tilde{y}\hat{\nabla}^{\phi(x_0)}\Big)(x_0)\Big)$$
$$= \exp\Big(2t\tilde{y}_n(\phi(x) - \phi(x_0))\partial_{2n} + 2t\sigma(\tilde{y}, \tilde{x})\partial_{2n} - t\sigma(\tilde{y}, \tilde{x})\partial_{2n} + (t\tilde{x} + \tilde{y})\hat{\nabla}^{\phi(x_0)}\Big)(x_0).$$

From this and using (20) we get

$$\begin{aligned}(\gamma_{\tilde{y}}(t))_i &= t(x - x_0)_i + (\tilde{y} + x_0)_i \quad i = 1, \dots, 2n-1\\ (\gamma_{\tilde{y}}(t))_{2n} &= t(x - x_0)_{2n} + (\tilde{y} + x_0)_{2n} + 2t\tilde{y}_n(\phi(x) - \phi(x_0))\\ &\qquad + 2\phi(x_0)\tilde{y}_n + \sigma(t(x - x_0) + x_0, \tilde{y}).\end{aligned} \tag{60}$$

Therefore, the following estimate holds

$$\mathrm{d}_{\phi(x_0)}(x_0, \gamma_{\tilde{y}}(t)) \leq \|\tilde{y}\| + \|\tilde{x}\| + \sqrt{\|\tilde{y}\|\|\tilde{x}\|} + \sqrt{\|\tilde{y}\||\phi(x) - \phi(x_0)|}, \tag{61}$$

where $\tilde{x}$ is as in (58). Indeed, using (23) and (60) we get

$$\mathrm{d}_{\phi(x_0)}(x_0, \gamma_{\tilde{y}}(t)) \leq \Big|(t\tilde{x}_1 + \tilde{y}_1, \dots, t\tilde{x}_{2n-1} + \tilde{y}_{2n-1})\Big|_{\mathbb{R}^{2n-1}} +$$
$$+ \Big|t\tilde{x}_{2n} + \tilde{y}_{2n} + 2t\tilde{y}_n(\phi(x) - \phi(x_0)) + t\sigma(\tilde{x}, \tilde{y})\Big|^{\frac{1}{2}}$$

and (61) follows using the triangle inequality. Since, $x, x_0 \in \Omega_{\phi(\bar{x})}(\bar{x}, R)$ and

$$\tilde{y} \in Log_{\phi(x_0),x_0}(\Omega_{\phi(\bar{x})}(\bar{x}, R)) \cap Log_{\phi(x),x}(\Omega_{\phi(\bar{x})}(\bar{x}, R))$$

then by (58) and (26) we get

$$\|\tilde{x}\| \leq CR, \quad \|\tilde{y}\| \leq CR \quad \text{and} \quad \mathrm{d}_{\phi}(x, x_0) \leq CR \tag{62}$$

for some constant $C = C(L_{\phi,\Omega_{\phi(\bar{x})}(\bar{x},\bar{R})}) > 0$. Finally, by (61), (62) and (31) we conclude that

$$\gamma_{\tilde{y}}(t) \in \Omega_{\phi(x_0)}(x_0, CR) \quad \forall t \in [0,1]$$

for some $C = C(L_{\phi,\Omega_{\phi(\bar{x})}(\bar{x},\bar{R})}) > 0$ and (52) follows taking $R_0 := \max\{\bar{R}, \mathrm{d}_{\phi(x_0)}(x_0, \partial\omega)/C\}$. Since $\psi \in C^\infty(\omega)$ and $\gamma_{\tilde{y}}$ is horizontal with respect to the family of vector fields $\{\hat{\nabla}^{\phi(x_0)}\}$, we obtain

$$\begin{aligned}\psi(Exp_{\phi(x),x}(\tilde{y})) - \psi(Exp_{\phi(x_0),x_0}(\tilde{y})) &= \int_0^1 (\psi \circ \gamma_{\tilde{y}})'(t)\mathrm{d}t \\ &= \sum_{i=1}^{2n} \int_0^1 \tilde{h}_i \hat{\nabla}_i^{\phi(x_0)} \psi(\gamma_{\tilde{y}}(t))\mathrm{d}t,\end{aligned}$$

so that (53) immediately follows using (57). In order to prove (55) it suffices to observe that $\sigma(x - x_0, \tilde{y}) \leq \mathrm{d}_\phi(x, x_0)\|\tilde{y}\|$. Moreover, since $\partial_{\tilde{y}_{2n}} K_4(x, x_0, \tilde{y}) = 0$ it follows that to prove (56) it is enough to estimate the Euclidean gradient of K_4 (with respect to the variable $\tilde{y}$). By a direct computation and using the expression of K_4 in (54) we obtain

$$\begin{aligned}&\partial_{\tilde{y}_i} K_4(x, x_0, \tilde{y}) = -2(x - x_0)_{n+i} \quad \text{if} \ \ i = 1, \dots, n-1, \\ &\partial_{\tilde{y}_n} K_4(x, x_0, \tilde{y}) = 2(\phi(x) - \phi(x_0)), \\ &\partial_{\tilde{y}_i} K_4(x, x_0, \tilde{y}) = 2(x - x_0)_i \quad \ \text{if} \ \ i = n+1, \dots, 2n-1.\end{aligned}$$

Hence $|\nabla K_4(x, x_0, \tilde{y})| \leq 2(L_{\phi,\Omega_{\phi(\bar{x})}(\bar{x},R)} + 1)\mathrm{d}_\phi(x, x_0)$, which is the thesis. □

Let us now prove the following proposition.

Lemma 3.8. *Let $\bar{x} \in \omega$ and $R_0 > 0$ be as Proposition 3.7. For each $0 < R < R_0$, $x, x_0 \in \Omega_{\phi(\bar{x})}(\bar{x}, R)$ and*

$$\tilde{y} \in \omega_{x,x_0;\bar{x}} := Log_{\phi(x_0),x_0}(\Omega_{\phi(\bar{x})}(\bar{x}, R)) \cap Log_{\phi(x),x}(\Omega_{\phi(\bar{x})}(\bar{x}, R)),$$

let us denote by $\gamma_{\tilde{y}}(t)$ the curve introduced in Proposition 3.7. Then the function

$$\begin{aligned}H : [0,1] \times \omega_{x,x_0;\bar{x}} &\longrightarrow [0,1] \times \omega \\ (t, \tilde{y}) &\mapsto (t, \gamma_{\tilde{y}}(t))\end{aligned}$$

has inverse function $(t, \tilde{F}(z,t))$ and the map $z \to (t, \tilde{F}(z,t))$ is C^∞ and its Jacobian matrix has determinant equal to 1.

Proof. Using (60), (20) and setting $(t, z) := (t, \gamma_{\tilde{y}}(t))$, $\tilde{F}$ can be expressed as

$$\begin{aligned}\tilde{F}_i(z,t) =& (z - x_0)_i - t(x - x_0)_i \quad i = 1, \dots, 2n-1, \\ \tilde{F}_{2n}(z,t) =& (z - x_0)_{2n} - t(x - x_0)_{2n} - 2t((z - x_0)_n - t(x - x_0)_n)(\phi(x) - \phi(x_0)) + \\ & - 2\phi(x_0)((z - x_0)_n - t(x - x_0)_n) + \sigma(z, t(x - x_0) + x_0).\end{aligned} \tag{63}$$

In particular it is clear from (63) that $\tilde{F}$ is of class C^∞ as a function of the variable z and that the Jacobian determinant of $z \to \tilde{F}(z,t)$ is equal to 1 for each $t \in [0,1]$. □

Lemma 3.9. *Let $g \in C^\infty(\mathbb{R}^n)$ and $\tilde{F}(z,t)$ as in Lemma 3.8 then*

(64)
$$\nabla^{\phi(x_0)}_{z_i}(g(\tilde{F}(z,t)) = \begin{cases} (\nabla_{\tilde{y}_i} g)(\tilde{F}(z,t)) & i = 1,\dots,n-1, \\ (\nabla_{\tilde{y}_n} g)(\tilde{F}(z,t)) - 2t(\phi(x)-\phi(x_0))(\partial_{\tilde{y}_{2n}} g)(\tilde{F}(z,t)) & i = n, \\ (\nabla_{\tilde{y}_i} g)(\tilde{F}(z,t)) & i = n+1,\dots,2n-1, \end{cases}$$

where $(\nabla_1,\dots,\nabla_{2n-1})$ is the family of vector fields defined in (27).

Proof. Let us start computing $\nabla^{\phi(x_0)}_{z_i}(g(\tilde{F}(z,t)))$ with $i = 1,\dots,n-1$, that is

$$\Big(\partial_{z_i} - z_{i+n}\partial_{z_{2n}}\Big)(g(\tilde{F}(z,t))). \tag{65}$$

To this end, we calculate

$$\partial_{z_i}(g(\tilde{F}(z,t))) \quad \text{and} \quad \partial_{z_{2n}}(g(\tilde{F}(z,t))).$$

By the explicit expression of $\tilde{F}(z,t)$ we obtain:

$$\partial_{z_i}(g(\tilde{F}(z,t))) = (\partial_{\tilde{y}_i} g)(\tilde{F}(z,t)) + (\partial_{\tilde{y}_{2n}} g)(\tilde{F}(z,t))\partial_{z_i}\tilde{F}_{2n}(z,t), \tag{66}$$

$$\partial_{z_{2n}}(g(\tilde{F}(z,t))) = (\partial_{\tilde{y}_{2n}} g)(\tilde{F}(z,t)), \tag{67}$$

hence by (65), (66) and (67) we get:

$$\begin{aligned}\nabla^{\phi(x_0)}_{z_i}(g(\tilde{F}(z,t))) =& \Big(\partial_{\tilde{y}_i} g - \tilde{F}_{i+n}(z,t)\partial_{\tilde{y}_{2n}} g\Big)(\tilde{F}(z,t)) + \\ &+ \Big(\tilde{F}_{i+n}(z,t) - z_{i+n} + \partial_{z_i}\tilde{F}_{2n}(z,t)\Big)\partial_{\tilde{y}_{2n}} g(\tilde{F}(z,t)).\end{aligned}$$

Since

$$\begin{aligned}\tilde{F}_i(z,t) =& (z-x_0)_i - t(x-x_0)_i \quad i = 1,\dots,2n-1 \\ \tilde{F}_{2n}(z,t) =& (z-x_0)_{2n} - t(x-x_0)_{2n} - 2t((z-x_0)_n - t(x-x_0)_n)(\phi(x)-\phi(x_0)) + \\ & - 2\phi(x_0)((z-x_0)_n - t(x-x_0)_n) + \sigma(z, t(x-x_0)+x_0)\end{aligned} \tag{68}$$

this implies

$$\nabla^{\phi(x_0)}_{z_i}(g(\tilde{F}(z,t))) = (\nabla_{\tilde{y}_i} g)(\tilde{F}(z,t)).$$

The computations for $\nabla^{\phi(x_0)}_{z_i}(g(\tilde{F}(z,t)))$ when $i = n+1,\dots,2n-1$ are similar.

Finally, let us compute $\nabla^{\phi(x_0)}_{z_n}(g(\tilde{F}(z,t)))$. By definition:

$$\nabla^{\phi(x_0)}_{z_n}(g(\tilde{F}(z,t))) = \Big(\partial_{z_n} + 2\phi(x_0)\partial_{z_{2n}}\Big)(g(\tilde{F}(z,t))) \tag{69}$$

and since

$$\partial_{z_n}(g(\tilde{F}(z,t))) = (\partial_{\tilde{y}_n} g)(\tilde{F}(z,t)) - 2[t(\phi(x)-\phi(x_0)) + \phi(x_0)](\partial_{\tilde{y}_{2n}} g)(\tilde{F}(z,t)) \tag{70}$$

by (69), (67) and (70) we get:

$$\nabla^{\phi(x_0)}_{z_n} g(\tilde{F}(z,t)) = (\nabla_{\tilde{y}_n} g)(\tilde{F}(z,t)) - 2t(\phi(x)-\phi(x_0))(\partial_{\tilde{y}_{2n}} g)(\tilde{F}(z,t)).$$

□

Proposition 3.10. *For every $t \in [0,1], c_1, c_2 > 0$ and $r > 0$ let us define*

$$D_{t,c_1,c_2,r} := \{z \in \mathbb{R}^{2n} : c_1 r \leq K(\tilde{F}(z,t)) < c_2 r\}, \tag{71}$$

where K is as in (39). Let $\bar{x} \in \omega$ and $R_0 > 0$ be as in Proposition 3.7, then for every $0 < R < R_0$ and $x, x_0 \in \Omega_{\phi(\bar{x})}(\bar{x}, R)$ with $x \neq x_0$ it holds:

$$\begin{aligned} m(\psi,\phi,R)(x) - m(\psi,\phi,R)(x_0) =& \frac{2}{R}\int_0^1 \int_{\frac{R}{2}}^R \int_{D_{t,\frac{1}{2},1,r}} < K_5(x,x_0,t,z,r), \nabla^\phi \psi(z) > \mathrm{d}\mathcal{L}^{2n}(z)\mathrm{d}r\mathrm{d}t \\ &+ \int_0^1 \int_{D_{t,\frac{1}{2},1,R}} < K_6(x,x_0,t,z,R), \nabla^\phi \psi(z) > \mathrm{d}\mathcal{L}^{2n}(z)\mathrm{d}t \\ &- \int_0^1 \int_{D_{t,\frac{1}{4},\frac{1}{2},R}} < K_7(x,x_0,t,z,R), \nabla^\phi \psi(z) > \mathrm{d}\mathcal{L}^{2n}(z)\mathrm{d}t \end{aligned}$$

for suitable kernels K_5, K_6, K_7 defined in (76), (77) and (78) respectively. Moreover, there are positive constants $\tilde{C}_5$, $\tilde{C}_6$ independent of L_ϕ such that

$$|K_5(x,x_0,t,z,r)| \leq \tilde{C}_5 (L_\phi + 1)^2 \frac{\mathrm{d}_{\phi(x_0)}(x_0,x)}{r^{\mathcal{Q}}} \quad \text{on } D_{t,\frac{1}{2},1,r}, \forall t \in [0,1], \tag{72}$$

$$|K_6(x,x_0,t,z,R)| \leq \tilde{C}_6 (L_\phi + 1)^2 \frac{\mathrm{d}_{\phi(x_0)}(x_0,x)}{R\|\tilde{F}(z,t)\|^{\mathcal{Q}-1}} \quad \text{on } D_{t,\frac{1}{2},1,R}, \forall t \in [0,1], \tag{73}$$

$$|K_7(x,x_0,t,z,R)| \leq \tilde{C}_6 (L_\phi + 1)^2 \frac{\mathrm{d}_{\phi(x_0)}(x_0,x)}{R\|\tilde{F}(z,t)\|^{\mathcal{Q}-1}} \quad \text{on } D_{t,\frac{1}{4},\frac{1}{2},R}, \forall t \in [0,1]. \tag{74}$$

Proof. By Lemma 3.6 for every $0 < r < R_0$ such that $\Omega_{\phi(x)}(x,r), \Omega_{\phi(x_0)}(x_0,r) \Subset \omega$, we have

$$\begin{aligned} &\bar{m}(\psi,\phi,r)(x) - \bar{m}(\psi,\phi,r)(x_0) = \\ &= \frac{1}{r^{\mathcal{Q}}} \int_{\tilde{\Omega}(0,r)\setminus\tilde{\Omega}(0,\frac{r}{2})} K_3(0,\tilde{y}) \Big(\psi(Exp_{\phi(x),x}(\tilde{y})) - \psi(Exp_{\phi(x_0),x_0}(\tilde{y}))\Big) \mathrm{d}\mathcal{L}^{2n}(\tilde{y}) \end{aligned}$$

by Proposition 3.7

$$\begin{aligned} &= \frac{1}{r^{\mathcal{Q}}} \int_{\tilde{\Omega}(0,r)\setminus\tilde{\Omega}(0,\frac{r}{2})} K_3(0,\tilde{y}) \int_0^1 < Log_{\phi(x_0),x_0}(x), \hat{\nabla}^{\phi(x_0)} \psi(\gamma_{\tilde{y}}(t)) > \mathrm{d}t\mathrm{d}\mathcal{L}^{2n}(\tilde{y}) \\ &+ \frac{1}{r^{\mathcal{Q}}} \int_{\tilde{\Omega}(0,r)\setminus\tilde{\Omega}(0,\frac{r}{2})} K_3(0,\tilde{y}) \int_0^1 K_4(x,x_0,\tilde{y}) \partial_{2n} \psi(\gamma_{\tilde{y}}(t)) \mathrm{d}t\mathrm{d}\mathcal{L}^{2n}(\tilde{y}). \end{aligned}$$

The change of variable $z = \gamma_{\tilde{y}}(t)$, changes $\tilde{\Omega}(0,r) \setminus \tilde{\Omega}(0,\frac{r}{2})$ in the set $D_{t,\frac{1}{2},1,r}$ and the inverse mapping has Jacobian determinant equal to 1. Hence:

(75)

$$\begin{aligned} &m(\psi,\phi,R)(x) - m(\psi,\phi,R)(x_0) = \\ &= \frac{2}{R}\int_0^1 \int_{\frac{R}{2}}^R \frac{1}{r^{\mathcal{Q}}} \int_{D_{t,\frac{1}{2},1,r}} K_3(0,\tilde{F}(z,t)) < Log_{\phi(x_0),x_0}(x), \hat{\nabla}^{\phi(x_0)}\psi(z) > \mathrm{d}\mathcal{L}^{2n}(z)\mathrm{d}r\mathrm{d}t + \\ &+ \frac{2}{R}\int_0^1 \int_{\frac{R}{2}}^R \frac{1}{r^{\mathcal{Q}}} \int_{D_{t,\frac{1}{2},1,r}} K_3(0,\tilde{F}(z,t))K_4(x,x_0,\tilde{F}(z,t))\partial_{2n}\psi(z)\mathrm{d}\mathcal{L}^{2n}(z)\mathrm{d}r\mathrm{d}t. \end{aligned}$$

Now applying Remark 3.4 we get the thesis calling:

$$\begin{aligned} K_5(x,x_0,t,z,r) :=& \frac{1}{r^{\mathcal{Q}}} K_3(0,\tilde{F}(z,t)) Log_{\phi(x_0),x_0}(x) + \\ &+ \frac{1}{2r^{\mathcal{Q}}} \nabla_1^{\phi(x_0)} \Big(K_3(0,\tilde{F}(z,t))K_4(x,x_0,\tilde{F}(z,t)) \Big) e_{n+1} + \\ &+ \frac{1}{2r^{\mathcal{Q}}} \nabla_{n+1}^{\phi(x_0)} \Big(K_3(0,\tilde{F}(z,t))K_4(x,x_0,\tilde{F}(z,t)) \Big) e_1; \end{aligned} \tag{76}$$

$$\begin{aligned} K_6(x,x_0,t,z,R) :=& \frac{1}{R} \frac{\nabla_1^{\phi(x_0)} K(\tilde{F}(z,t))}{K^{\mathcal{Q}}(\tilde{F}(z,t))} K_3(0,\tilde{F}(z,t))K_4(x,x_0,\tilde{F}(z,t)) e_{n+1} - \\ &- \frac{1}{R} \frac{\nabla_{n+1}^{\phi(x_0)} K(\tilde{F}(z,t))}{K^{\mathcal{Q}}(\tilde{F}(z,t))} K_3(0,\tilde{F}(z,t))K_4(x,x_0,\tilde{F}(z,t)) e_1; \end{aligned} \tag{77}$$

$$\begin{aligned} K_7(x,x_0,t,z,R) :=& -\frac{1}{R} \frac{\nabla_1^{\phi(x_0)} K(\tilde{F}(z,t))}{2^{\mathcal{Q}} K^{\mathcal{Q}}(\tilde{F}(z,t))} K_3(0,\tilde{F}(z,t))K_4(x,x_0,\tilde{F}(z,t)) e_{n+1} \\ &+ \frac{1}{R} \frac{\nabla_{n+1}^{\phi(x_0)} K(\tilde{F}(z,t))}{2^{\mathcal{Q}} K^{\mathcal{Q}}(\tilde{F}(z,t))} K_3(0,\tilde{F}(z,t))K_4(x,x_0,\tilde{F}(z,t)) e_1, \end{aligned} \tag{78}$$

where as usual e_i denotes the i-th element of the canonical basis of $\mathbb{R}^{2n-1}$. To prove (72) we observe that by Lemma 3.9

$$\begin{aligned} &K_5(x,x_0,t,z,r) = \frac{1}{r^Q} K_3(0,\tilde{F}(z,t)) Log_{\phi(x_0),x_0}(x) \\ &+ \frac{1}{2r^Q} \Big((\nabla_1 K_3)(0,\tilde{F}(z,t))K_4(x,x_0,\tilde{F}(z,t)) + (\nabla_1 K_4)(x,x_0,\tilde{F}(z,t))K_3(0,\tilde{F}(z,t)) \Big) e_{n+1} \\ &+ \frac{1}{2r^Q} \Big((\nabla_{n+1} K_3)(0,\tilde{F}(z,t))K_4(x,x_0,\tilde{F}(z,t)) + (\nabla_{n+1} K_4)(x,x_0,\tilde{F}(z,t))K_3(0,\tilde{F}(z,t)) \Big) e_1 \end{aligned}$$

hence using (49), (55) and (56) we get

$$|K_5(x,x_0,t,z,r)| \leq \frac{\tilde{C}_3}{r^Q} \mathrm{d}_{\phi(x_0)}(x_0,x) + 2\frac{\tilde{C}_4(L_\phi+1)\mathrm{d}_\phi(x,x_0)}{r^Q} + 2\frac{\tilde{C}_3(L_\phi+1)\mathrm{d}_\phi(x,x_0)}{r^Q}$$

and the conclusion follows using (24). Finally, (73) and (74) are direct consequences of (31),(49), (55) and Lemma 3.9. □

4. Poincaré inequality

Aim of this section is to prove Theorem 1.2. The Poincaré inequality we prove here is partially inspired to the Sobolev type inequality for vector fields with non regular coefficients contained in [15] and successively extended to a more general class of vector fields in [40]. The key point in our strategy is to establish a representation formula for intrinsic Lipschitz continuous functions. To this end we use Theorem 2.5 and the representation formula proved in Theorem 3.10 for C^∞ functions.

In all this section we denote by ω an open and bounded subset of $\mathbb{R}^{2n}$ with $n \geq 2$ and by ϕ an intrinsic Lipschitz function defined on ω with Lipschitz constant equal to L_ϕ.

Let $\psi \in W_\phi$ and let $\{\psi_k\}_{k\in\mathbb{N}}$, $\{\phi_k\}_{k\in\mathbb{N}}$ smooth functions on ω which satisfy conditions $(i)-(iv)$ of Definition 1.1. We denote by $\mathrm{d}_{\phi_k(x_0)}$ the distance introduced in (23), by $\Gamma_{\phi_k(x_0)}$ the fundamental solution of the operator $\mathcal{L}_{\phi_k(x_0)}$ defined in (32) and by $\Omega_{\phi_k(x_0)}(x_0, r)$ the superlevel set of $\Gamma_{\phi_k(x_0)}$ defined in (33).

Lemma 4.1. *Let $x_0 \in \omega$ and $R > 0$ such that $\Omega_{\phi(x_0)}(x_0, R) \subset \omega$. Then*

(i) $\chi_{\Omega_{\phi_k(x_0)}(x_0,R)} \to \chi_{\Omega_{\phi(x_0)}(x_0,R)}$ *uniformly in ω as $k \to +\infty$;*

(ii) $\bar{m}(\psi_k, \phi_k, R)(x_0) \to \bar{m}(\psi, \phi, R)(x_0)$ *uniformly in $R > 0$ as $k \to +\infty$.*

Here χ_A denotes the characteristic function of A.

Proof. We recall that

$$\tilde{\Omega}(0, R) = \left\{\tilde{y} \in \mathbb{R}^{2n} \mid \Gamma(0, \tilde{y}) > R^{2-\mathcal{Q}}\right\},$$

then, by (37), for each $k \in \mathbb{N}$ we have:

$$\Omega_{\phi_k(x_0)}(x_0, R) = Exp_{\phi_k(x_0),x_0}(\tilde{\Omega}(0, R)).$$

Using the explicit form of $Exp_{\phi_k(x_0),x_0}$ and $Exp_{\phi(x_0),x_0}$ stated in (20) we easily conclude that $(Exp_{\phi_k(x_0),x_0})_{k\in\mathbb{N}}$ uniformly converges to $Exp_{\phi(x_0),x_0}$ in ω as $k \to +\infty$. In order to prove (i) we observe that it is sufficient to prove that for all $\epsilon > 0$ there exists $\bar{k} = \bar{k}(\epsilon) > 0$ such that for all $k > \bar{k}$

$$\Omega_{\phi(x_0)}(x_0, R) \subseteq (\Omega_{\phi_k(x_0)}(x_0, R))_\epsilon \tag{79}$$

where

$$(\Omega_{\phi_k(x_0)}(x_0, R))_\epsilon := \{y \in \omega \mid \mathrm{d}_{\phi_k(x_0)}(\partial\Omega_{\phi_k(x_0)}(x_0, R), y) < \epsilon\}. \tag{80}$$

For simplify the notation we define

$$E_k(\tilde{\Omega}(0, R)) := Exp_{\phi_k(x_0),x_0}(\tilde{\Omega}(0, R)), \quad E(\tilde{\Omega}(0, R)) := Exp_{\phi(x_0),x_0}(\tilde{\Omega}(0, R)).$$

Suppose by contradiction that there exists $\epsilon > 0$ such that for every $\bar{k}$ there are $k > \bar{k}$ and $y_k \in E(\tilde{\Omega}(0, R))$ such that $y_k \notin E_k(\tilde{\Omega}(0, R))_\epsilon$. Then, there exist $(k_j)_j$, $k_j \to +\infty$ as $j \to +\infty$ and $(x_{k_j})_j$ in $\tilde{\Omega}(0, R)$ such that $E(x_{k_j}) \notin E_{k_j}(\tilde{\Omega}(0, R))_\epsilon$. So that, the distance between $E(x_{k_j})$ and $E_{k_j}(x_{k_j})$ is greater than ϵ and this is absurd being E_k uniformly convergent to E. Then, (79) follows and hence (i).

To prove (ii) we observe that by Definition 3.3:

$$\lim_{k\to+\infty}\bar{m}(\psi_k,\phi_k,R)(x_0)=$$

$$=\lim_{k\to+\infty}\frac{C(\mathcal{Q})}{\mathcal{Q}-2}\frac{1}{R^{\mathcal{Q}}}\int_\omega \frac{|\nabla^{\phi_k(x_0)}\Gamma_{\phi_k(x_0)}(x_0,y)|^2}{\Gamma_{\phi_k(x_0)}(x_0,y)^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}}\psi_k(y)\,\chi_{\Omega_{\phi_k(x_0)}(x_0,R)\setminus\Omega_{\phi_k(x_0)}(x_0,\frac{R}{2})}(y)\,\mathrm{d}\mathcal{L}^{2n}(y).$$

By (31) and (32)

$$\lim_{k\to+\infty}\frac{|\nabla^{\phi_k(x_0)}\Gamma_{\phi_k(x_0)}(x_0,y)|^2}{\Gamma_{\phi_k(x_0)}(x_0,y)^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}}=\frac{|\nabla^{\phi(x_0)}\Gamma_{\phi(x_0)}(x_0,y)|^2}{\Gamma_{\phi(x_0)}(x_0,y)^{2(\mathcal{Q}-1)/(\mathcal{Q}-2)}}\le C\quad\forall y\ne x_0 \tag{81}$$

therefore, (ii) follows from (i) and the fact that $\psi_k\to\psi$ in $L^1_{loc}(\omega)$. □

In what follows we prove that the representation formulas obtained in Proposition 3.5 and in Proposition 3.10 for C^∞ functions still hold if ϕ is intrinsic Lipschitz and $\psi\in W_\phi(\omega)$.

Lemma 4.2. *Let ϕ be a Lipschitz continuous function and $\psi\in W_\phi(\omega)$. For each $x_0\in\omega$ and each $R>0$ such that $\Omega_{\phi(x_0)}(x_0,R)\subset\omega$, the following formula holds:*

$$\phi(x_0)=m(\psi,\phi,R)(x_0)+I_R(x_0)$$

where $m(\psi,\phi,R)(x_0)$ is as in (43) and

$$\begin{aligned}I_R(x_0):=&\frac{2}{R}\int_{\frac{R}{4}}^R f_1\left(\frac{r}{R}\right)\int_{\Omega_{\phi(x_0)}(x_0,r)}\left\langle K_1(x_0,y),\nabla^\phi\psi(y)\right\rangle\,\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r\\&+\frac{2}{R}\int_{\frac{R}{2}}^R\int_{\Omega_{\phi(x_0)}(x_0,r)\setminus\Omega_{\phi(x_0)}(x_0,\frac{r}{2})}\left\langle K_2(x_0,y,r),\nabla^\phi\psi(y)\right\rangle\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}r,\end{aligned}\tag{82}$$

where K_1,K_2 are as in Proposition 3.5. Let $\bar{x}\in\omega$ and $R_0>0$ be as in Proposition 3.7, then for every $0<R<R_0$ and $x,x_0\in\Omega_{\phi(\bar{x})}(\bar{x},R)$ with $x\ne x_0$ it holds:

$$\begin{aligned}&m(\psi,\phi,R)(x)-m(\psi,\phi,R)(x_0)=\\&=\frac{2}{R}\int_0^1\int_{\frac{R}{2}}^R\int_{D_{t,\frac{1}{2},1,r}}<K_5(x,x_0,t,z,r),\nabla^\phi\psi(z)>\mathrm{d}\mathcal{L}^{2n}(z)\mathrm{d}r\mathrm{d}t\\&+\int_0^1\int_{D_{t,\frac{1}{2},1,R}}<K_6(x,x_0,t,z,R),\nabla^\phi\psi(z)>\mathrm{d}\mathcal{L}^{2n}(z)\mathrm{d}t\\&-\int_0^1\int_{D_{t,\frac{1}{4},\frac{1}{2},R}}<K_7(x,x_0,t,z,R),\nabla^\phi\psi(z)>\mathrm{d}\mathcal{L}^{2n}(z)\mathrm{d}t,\end{aligned}\tag{83}$$

where $D_{t,c_1,c_2,r}$ is as in (71) *and K_5,K_6,K_7 are as in* (76),(77) *and* (78) *respectively and they satisfy the same estimates proved in Proposition 3.10 with possibly different constants.*

Proof. By Definition 1.1 there are $\{\psi_k\}_{k\in\mathbb{N}}$, $\{\phi_k\}_{k\in\mathbb{N}}$ sequences of smooth functions defined on ω satisfying conditions $(i)-(iv)$. By Proposition 3.5 and 3.10, the thesis is true for every ϕ_k,ψ_k as above. Passing to the limit as in the previous proposition, it holds true also for the limit functions ϕ and ψ. □

As it is well known (see for example [25], [26]) the key step for the proof of a Poincaré inequality, is a representation formula as the one proved in Lemma 4.2, which is indeed equivalent to the Poincaré inequality itself. For further applications, we note that we can now derive this formula on any family of balls, equivalent to the superlevels $\Omega_{\phi(\bar{x})}(\bar{x},R)$, which can be $\Omega_{\phi(\bar{x})}(\bar{x},R)$ or $U_\phi(\bar{x},R)$, defined respectively in (33) and (35).

Precisely, let us denote with $B_\phi(\bar{x},R)$ a family of spheres with centre in $\bar{x}$ and radius R, equivalent to the family $\Omega_{\phi(\bar{x})}(\bar{x},R)$. Let us denote $\psi_{B_\phi(\bar{x},R)}$ the mean of ψ on the set $B_\phi(\bar{x},R)$ with respect to the Lebesgue measure, i.e.

$$\psi_{B_\phi(\bar{x},R)} := \frac{1}{\mathcal{L}^{2n}(B_\phi(\bar{x},R))}\int_{B_\phi(\bar{x},R)} \psi(x)\ \mathrm{d}\mathcal{L}^{2n}(x)\,. \tag{84}$$

We will prove

Theorem 4.3. *For every $\phi \in Lip(\omega)$ and $\psi \in W_\phi(\omega)$ there exist positive constants C_1, C_2 with $C_2 > 1$ (depending continuosly on the Lipschitz constant L_ϕ of ϕ) such that*

$$\int_{B_\phi(\bar{x},R)} |\psi(y) - \psi_{B_\phi(\bar{x},R)}|\ \mathrm{d}\mathcal{L}^{2n}(y) \le C_1\, R \int_{B_\phi(\bar{x},C_2R)} |\nabla^\phi\psi(y)|\ \mathrm{d}\mathcal{L}^{2n}(y), \tag{85}$$

for every $B_\phi(\bar{x},C_2R) \subset \omega$.

Our representation formula can be stated as follows:

Proposition 4.4. *Let $\phi : \omega \to \mathbb{R}$ be an intrinsic Lipschitz function and $\psi : \omega \to \mathbb{R}$ be a W_ϕ function. Let $\bar{x} \in \omega$ and $R_0 > 0$ be as in Proposition 3.7. Let $0 < R < R_0$, $x, x_0 \in B_\phi(\bar{x},R)$. Then there are $C, \tilde{C}_1, \tilde{C}_2$, with $C > 1$ and $\tilde{C}_1, \tilde{C}_2 > 0$ depending only on $L_{\phi,\omega}, \mathcal{Q}$ and the structure constants in (31) such that $B_\phi(\bar{x},CR) \Subset \omega$ and*

$$\begin{aligned} &|\psi(x_0) - \psi_{B_\phi(\bar{x},R)}| \le \\ &\le \tilde{C}_1 \int_{B_\phi(\bar{x},CR)} \mathrm{d}_\phi^{1-\mathcal{Q}}(x_0,y)|\nabla^\phi\psi(y)|\mathrm{d}\mathcal{L}^{2n}(y) + \\ &+ \frac{\tilde{C}_2}{\mathcal{L}^{2n}(B_\phi(\bar{x},R))}\int_{B_\phi(\bar{x},CR)}\int_{B_\phi(\bar{x},CR)} \mathrm{d}_\phi^{1-\mathcal{Q}}(x,y)|\nabla^\phi\psi(y)|\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}\mathcal{L}^{2n}(x). \end{aligned} \tag{86}$$

Proof. By Lemma 4.2 for each $x, x_0 \in B_\phi(\bar{x},R)$ we have:

$$\begin{aligned} \psi(x_0) &= m(\psi,\phi,R)(x_0) + I_R(x_0),\\ \psi(x) &= m(\psi,\phi,R)(x) + I_R(x), \end{aligned}$$

hence

$$\psi(x_0) - \psi(x) = m(\psi,\phi,R)(x_0) - m(\psi,\phi,R)(x) + I_R(x_0) - I_R(x). \tag{87}$$

Integrating equation (87) with respect to the variable x on a sphere $B_\phi(\bar{x},R)$ and recalling the definition of $\psi_{B_\phi(\bar{x},R)}$ we get:

$$\begin{aligned} \psi(x_0) - \psi_{B_\phi(\bar{x},R)} =& \frac{1}{\mathcal{L}^{2n}(B_\phi(\bar{x},R))}\int_{B_\phi(\bar{x},R)} m(\psi,\phi,R)(x_0) - m(\psi,\phi,R)(x)\ \mathrm{d}\mathcal{L}^{2n}(x)\\ &+ \frac{1}{\mathcal{L}^{2n}(B_\phi(\bar{x},R))}\int_{B_\phi(\bar{x},R)} I_R(x_0) - I_R(x)\ \mathrm{d}\mathcal{L}^{2n}(x). \end{aligned}$$

Hence:
(88)
$$|\psi(x_0) - \psi_{B_\phi(\bar{x},R)}| \le \frac{1}{\mathcal{L}^{2n}(B_\phi(\bar{x},R))} \int_{B_\phi(\bar{x},R)} \Big| m(\psi,\phi,R)(x_0) - m(\psi,\phi,R)(x) \Big| \, \mathrm{d}\mathcal{L}^{2n}(x) + |I_R(x_0)| + \frac{1}{\mathcal{L}^{2n}(B_\phi(\bar{x},R))} \int_{B_\phi(\bar{x},R)} |I_R(x)| \, \mathrm{d}\mathcal{L}^{2n}(x).$$

Now, by Lemma 4.2, we have:

$$\begin{aligned} |m(\psi,\phi,R)(x_0) - &m(\psi,\phi,R)(x)| \\ &\le \frac{2}{R} \int_0^1 \int_{\frac{R}{2}}^R \int_{D_{t,\frac{1}{2},1,r}} | < K_5(x,x_0,t,z,r), \nabla^\phi \psi(z) > | \mathrm{d}\mathcal{L}^{2n}(z) \mathrm{d}r \mathrm{d}t \\ &+ \int_0^1 \int_{D_{t,\frac{1}{2},1,R}} | < K_6(x,x_0,t,z,R), \nabla^\phi \psi(z) > | \mathrm{d}\mathcal{L}^{2n}(z) \mathrm{d}t \\ &+ \int_0^1 \int_{D_{t,\frac{1}{4},\frac{1}{2},R}} | < K_7(x,x_0,t,z,R), \nabla^\phi \psi(z) > | \mathrm{d}\mathcal{L}^{2n}(z) \mathrm{d}t. \end{aligned}$$

We claim that there exists $C = C(L_\phi) > 1$ such that for all $r \in (R/2, R)$, $t \in [0,1]$ it holds

$$D_{t,\frac{1}{2},1,r} \subseteq \Omega_{\phi(x_0)}(x_0, CR) \Subset \omega. \tag{89}$$

To this end let us fix $t \in [0,1]$ and $r \in (R/2, R)$ then for each $\tilde{y} \in \tilde{\Omega}(0,r) \setminus \tilde{\Omega}(0,r/2)$ we have

$$\frac{r}{2} < \|\tilde{y}\| < r \le R \tag{90}$$

and, by (61), it also holds

$$\mathrm{d}_{\phi(x_0)}(x_0, \gamma_{\tilde{y}}(t)) \le \|\tilde{y}\| + \|\tilde{x}\| + \sqrt{\|\tilde{y}\|\|\tilde{x}\|} + \sqrt{\|\tilde{y}\||\phi(x) - \phi(x_0)|}. \tag{91}$$

Since $x, x_0 \in \Omega_{\phi(\bar{x})}(\bar{x}, R)$ by (24) and (31) we have

$$\mathrm{d}_\phi(x, x_0) \le CR \quad \text{and} \quad \|\tilde{x}\| \le CR \tag{92}$$

for some $C = C(L_\phi) > 1$. Using (90), (91) and (92) we immediately get (89) with possibly smaller R.

By Lemma 4.2 we know that the estimates for K_5, K_6, K_7 proved in Proposition 3.10 also hold for $\phi \in Lip(\omega)$. Hence, by (72) and (90) for each $z \in D_{t,\frac{1}{2},1,r}$ and $t \in [0,1]$ we have

$$|K_5(x,x_0,t,z,r)| \le \tilde{C} \frac{\mathrm{d}_{\phi(x_0)}(x_0,x)}{r^{\mathcal{Q}}} \le \tilde{C} \frac{\mathrm{d}_{\phi(x_0)}(x_0,x)}{\|\tilde{F}(z,t)\|^{\mathcal{Q}}}$$

for some $\tilde{C} = \tilde{C}(L_\phi) > 0$. Moreover, using (89) and (92) we get

$$\mathrm{d}_{\phi(x_0)}(x_0,x) = \|\tilde{x}\| \le CR \le 2Cr \le C\|\tilde{F}(z,t)\|$$

for a suitable constant $C = C(L_\phi) > 0$. Then

$$|K_5(x,x_0,t,z,r)| \le C \frac{1}{\|\tilde{F}(z,t)\|^{\mathcal{Q}-1}}.$$

Moreover, by (89), $z \in \Omega_{\phi(x_0)}(x_0, CR)$ and

$$0 < \mathrm{d}_{\phi(x_0)}(x_0, z) \leq CR \leq 2Cr \leq C\|\tilde{F}(z,t)\|.$$

So that

$$|K_5(x, x_0, t, z, r)| \leq C\mathrm{d}_{\phi(x_0)}(x_0, z)^{1-\mathcal{Q}}. \tag{93}$$

Analogously, we can prove that there exists $C = C(L_\phi) > 0$ such that

$$|K_6(x, x_0, t, z, r)|, |K_7(x, x_0, t, z, r)| \leq C\mathrm{d}_{\phi(x_0)}(x_0, z)^{1-\mathcal{Q}}. \tag{94}$$

In conclusion we proved that:

$$\begin{aligned} &|m(\psi, \phi, R)(x_0) - m(\psi, \phi, R)(x)| \leq \\ &\qquad \leq \tilde{C}_1 \int_{\Omega_{\phi(x_0)}\left(x_0, CR\right)} \mathrm{d}^{1-Q}_{\phi(x_0)}(x_0, z)|\nabla^\phi \psi(z)|\mathrm{d}\mathcal{L}^{2n}(z). \end{aligned} \tag{95}$$

Furthermore, by Lemma 4.2, (44) and (45) we have:

$$|I_R(x_0)| \leq \tilde{C}_2 L_\phi \int_{\Omega_{\phi(x_0)}(x_0,R)} \mathrm{d}^{1-Q}_{\phi(x_0)}(x_0, y)|\nabla^\phi \psi(y)|\mathrm{d}\mathcal{L}^{2n}(y), \tag{96}$$

$$|I_R(x)| \leq \tilde{C}_2 L_\phi \int_{\Omega_{\phi(x)}(x,R)} \mathrm{d}^{1-Q}_{\phi(x)}(x, y)|\nabla^\phi \psi(y)|\mathrm{d}\mathcal{L}^{2n}(y). \tag{97}$$

Finally, since the integrals can be extended on the sphere $B_\phi(\bar{x}, CR)$ which are equivalent to $\Omega_{\phi(\bar{x})}(\bar{x}, R)$, and by Remark 2.6 we can replace $\mathrm{d}^{1-Q}_{\phi(x)}(x, y)$ with $\mathrm{d}^{1-Q}_{\phi}(x, y)$, then the thesis follows by (88), (95), (96) and (97). $\square$

The proof of Theorem 4.3 follows from the representation formula in Proposition 4.4. We recall it here for reader convenience, providing a proof on any family of spheres equivalent to $\Omega_{\phi(x)}$. Hence in particular on the spheres U_ϕ we obtain Theorem 1.2.

Proof of Theorem 4.3. As in the previous proof we denote by $C, \tilde{C}$ positive constants depending only on $L_\phi, \mathcal{Q}$ and the constants defined in (31) which could be different from line to line. Integrating both members of (86) on $B_\phi(\bar{x}, R)$ we get

$$\begin{aligned} &\int_{B_\phi(\bar{x},R)} |\psi(x_0) - \psi_{B_\phi(\bar{x},R)}|\mathrm{d}\mathcal{L}^{2n}(x_0) \leq \\ &\leq \tilde{C}_1 \int_{B_\phi(\bar{x},CR)} \int_{B_\phi(\bar{x},CR)} \mathrm{d}^{1-Q}_\phi(x_0, y)|\nabla^\phi \psi(y)|\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}\mathcal{L}^{2n}(x_0) \\ &+ \tilde{C}_2 \int_{B_\phi(\bar{x},CR)} \int_{B_\phi(\bar{x},CR)} \mathrm{d}^{1-Q}_\phi(x, y)|\nabla^\phi \psi(y)|\mathrm{d}\mathcal{L}^{2n}(y)\mathrm{d}\mathcal{L}^{2n}(x). \end{aligned} \tag{98}$$

This implies:

$$\int_{B_\phi(\bar{x},R)} |\psi(x_0) - \psi_{B_\phi(\bar{x},R)}| \mathrm{d}\mathcal{L}^{2n}(x_0) \leq \tag{99}$$

$$\leq C \int_{B_\phi(\bar{x},CR)} |\nabla^\phi \psi(y)| \Big(\int_{B_\phi(\bar{x},CR)} \mathrm{d}_\phi^{1-Q}(y,x_0) \mathrm{d}\mathcal{L}^{2n}(x_0) \Big) \mathrm{d}\mathcal{L}^{2n}(y),$$

where

$$\int_{B_\phi(\bar{x},CR)} \mathrm{d}_\phi^{1-\mathcal{Q}}(y,x_0)\ \mathrm{d}\mathcal{L}^{2n}(x_0) \leq \tilde{C}R,$$

since the space is homogeneous of dimension Q. Finally, using (99) we get (85).

□

By the approximation result in Theorem 2.5 we can chose $\psi = \phi$ and get the Proof of Corollary 1.3.

We point out, or reader's convenience, the relation between Poincaré's inequality and p-p-Poincaré and Sobolev inequalities. By Corollary 1.3 and the doubling condition for the balls $U_\phi(\bar{x}, r)$ proved in [27], the following corollary directly follows applying Theorem 13.1 in the monograph [32]:

Corollary 4.5 (p-p-Poincaré and Sobolev inequalities)**.** *Let $\phi : \omega \to \mathbb{R}$ be an intrinsic Lipschitz function and $p > 1$. Then there exist positive constants C_1, C_2 with $C_2 > 1$ (depending on the Lipschitz constant L_ϕ of ϕ) such that*

$$\left(\frac{1}{\mathcal{L}^{2n}(U_\phi(\bar{x},r))} \int_{U_\phi(\bar{x},r)} |\phi(y) - \phi_{U_\phi(\bar{x},r)}|^p \mathrm{d}\mathcal{L}^{2n}(y) \right)^{1/p} \leq$$

$$\leq C_1\, r \left(\frac{1}{\mathcal{L}^{2n}(U_\phi(\bar{x},C_2 r))} \int_{U_\phi(\bar{x},C_2 r)} |\nabla^\phi \phi(y)|^p \mathrm{d}\mathcal{L}^{2n}(y) \right)^{1/p},$$

for every $U_\phi(\bar{x}, C_2\, r) \subset \omega$. Moreover, there is a constant $p^* > p$ such that

$$\left(\frac{1}{\mathcal{L}^{2n}(U_\phi(\bar{x},r))} \int_{U_\phi(\bar{x},r)} |\phi(y)|^{p^*} \mathrm{d}\mathcal{L}^{2n}(y) \right)^{1/p^*} \leq$$

$$\leq C_1\, r \left(\frac{1}{\mathcal{L}^{2n}(U_\phi(\bar{x},C_2 r))} \int_{U_\phi(\bar{x},C_2 r)} |\nabla^\phi \phi(y)|^p \mathrm{d}\mathcal{L}^{2n}(y) \right)^{1/p}.$$

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