Quasi-static rate-independent evolutions: characterization, existence, approximation and application to fracture mechanics*

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Abstract. We characterize quasi-static rate-independent evolutions, by means of their graph parametrization, in terms of a couple of equations: the first gives stationarity while the second provides the energy balance. An abstract existence result is given for functionals $\mathcal{F}$ of class C^1 in reflexive separable Banach spaces. We provide a couple of constructive proofs of existence which share commune features with the theory of minimizing movements for gradient flows. Moreover, considering a sequence of functionals $\mathcal{F}_n$ a its Γ-limit $\mathcal{F}$ we provide, under suitable assumptions, a convergence result for the associated quasi-static evolutions. Finally, we apply this approach to a phase field model in brittle fracture.

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1 Introduction

In the last few years the analysis of quasi-static rate-independent evolutions has been the object of several important advances. Theoretical and applied results have been developed essentially along two alternatives lines: on the one hand the evolutions by global minima (usually named "energetic evolutions") on the other the evolutions by critical points.

To illustrate the picture in a simple setting, consider a stored energy $\mathcal{E} : [t_0, t_1] \times \mathcal{V} \to \mathbb{R}$, where $[t_0, t_1]$ is the time interval and $\mathcal{V}$ is a separable reflexive Banach space (either finite or infinite dimensional) together with a dissipation functional $\Delta : \mathcal{V} \times \mathcal{V} \to [0, +\infty]$. Given an initial condition $v(0) = v_0$ (with v_0 globally stable) a trajectory $v : [t_0, t_1] \to \mathcal{V}$ is an energetic evolution [12] if the following conditions hold

(S) for every t it holds

$$\mathcal{E}(t, v(t)) \le \mathcal{E}(t, \phi) + \Delta(\phi, v(t)) \quad \text{for every } \phi \in \mathcal{V},$$

(E) for every t it holds

$$\mathcal{E}(t, v(t)) = \mathcal{E}(0, v(0)) - \Delta(v(t), v(0)) + \int_0^t \partial_t \mathcal{E}(s, v(s))\, ds\,.$$

The letters (S) and (E) denote respectively (global) stability and energy balance. Assuming further that there exists a dissipation "potential" $\mathcal{D} : \mathcal{V} \to [0, +\infty)$ such that $\Delta(v, w) = \mathcal{D}(v) - \mathcal{D}(w)$ and introducing the energy functional $\mathcal{F}(t, v) = \mathcal{E}(t, v) + \mathcal{D}(v)$ the (S)-(E) conditions read

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(S) for every t it holds

$$\mathcal{F}(t, v(t)) \le \mathcal{F}(t, \phi) \quad \text{for every } \phi \in \mathcal{V},$$

(E) for every t it holds

$$\mathcal{F}(t, v(t)) = \mathcal{F}(0, v(0)) + \int_0^t \partial_t \mathcal{F}(s, v(s))\, ds.$$

Often a representation of the form $\Delta(v, w) = \mathcal{D}(v) - \mathcal{D}(w)$ holds under some constraint on the admissible increment (it is indeed the case in brittle and cohesive fracture and in associative plasticity), however, to keep the presentation clear, we will not introduce any constraint in the abstract picture; we will see how to deal with a constrained problem in §7. It is important to highlight that in the above definition there are no derivatives of the stored energy $\mathcal{E}$ with respect to the state variable v, thanks to this fact it is possible to prove existence of energetic evolutions under (very) low regularity assumptions on $\mathcal{E}$, including the case of spaces without a vectorial structure, see for instance [6] for an application to fracture. On the other hand, it may happen that the behaviour of energetic solutions is not physically admissible: when a discontinuity occurs the evolution typically "tunnels" under an energy barrier.

Let us turn to evolutions by critical points. In the literature there are several equivalent definitions, most of them formulated in terms of the trajectory $t \mapsto v(t)$, as the BV solutions of [13]. Here we will not adopt exactly this description, preferring a graph parametrization, similar to that of [8]. The idea is to define the evolution by means of a (Lipschitz) parametrization of the extended graph, of the type $\tau \mapsto (t(\tau), v(\tau))$ for $\tau \in [0, T)$. This is a convenient choice to focus on discontinuities, indeed with this parametrization jumps are represented by "vertical parts" of the extended graph, of the form $\tau \mapsto (t, v(\tau))$ for $\tau \in [\tau^-, \tau^+]$ with $t(\tau)$ constant and $v(\tau^-) \neq v(\tau^+)$. Adopting graph parametrization, our goal is to provide a definition which resembles the (S)-(E) formulation above and then to provide an existence result in separable reflexive Banach spaces. Let us see in more detail our definition: let $\tau \mapsto (t(\tau), v(\tau))$ be a Lipschitz map with $t' \ge 0$ and $\|v'\| \le 1$ (t' and v' denote the derivatives with respect to τ). We will say that (t, v) is (a parametrization of) a quasi-static evolution if

(S') for every τ with $t'(\tau) > 0$ it holds

$$\|\partial_v \mathcal{F}(t(\tau), v(\tau))\| = 0,$$

(E') for every τ it holds

$$\mathcal{F}(t(\tau), v(\tau)) = \mathcal{F}(t(0), v(0)) - \int_0^\tau \|\partial_v \mathcal{F}(t(s), v(s))\|\, ds + \int_0^\tau \partial_t \mathcal{F}(t(s), v(s))\, t'(s)\, ds\,,$$

where $\|\partial_v \mathcal{F}(t, v)\|$ is the norm in the dual $\mathcal{V}'$, i.e.

$$\|\partial_v \mathcal{F}(t, v)\| = \max \left\{ \partial_v \mathcal{F}(t, v)[\phi] : \|\phi\| \le 1 \right\}.$$

Here the labels (S') and (E') stand respectively for stationarity and energy balance while the prime symbol suggests the dependence on derivatives of the stored energy. Close to our definition of evolutions by critical points are those of [8, 13, 18] while close to our existence proof are those on minimizing movements for gradient flows, e.g. [7, 15, 17].

Several properties of the evolution follow from this definition. First of all, note that (S') can be written also in the (norm free) form

$$\partial_v \mathcal{F}(t(\tau), v(\tau))[\phi] = 0 \quad \text{for every } \phi \in \mathcal{V}.$$

Next, if $t'(s) > 0$ for every $s \in [\tau_1, \tau_2]$ (and thus there are no jump discontinuities in $[\tau_1, \tau_2]$) then (E') yields

$$\mathcal{F}(t(\tau_2), v(\tau_2)) = \mathcal{F}(t(\tau_1), v(\tau_1)) + \int_{\tau_1}^{\tau_2} \partial_t \mathcal{F}(t(s), v(s))\, t'(s)\, ds\,,$$

which, up to a change of variable, is equivalent to (E). On the contrary, if $t'(s) = 0$ for every $s \in [\tau^-, \tau^+]$ (and thus there is a jump discontinuity at time $t = t(s)$) then (E') reads

$$\mathcal{F}(t, v(\tau^+)) = \mathcal{F}(t, v(\tau^-)) - \int_{\tau^-}^{\tau^+} \| \partial_v \mathcal{F}(t, v(s)) \|\, ds\,,$$

and in particular $\mathcal{F}(t, v(\tau_2)) \le \mathcal{F}(t, v(\tau_1))$ for every $\tau^- \le \tau_1 < \tau_2 \le \tau^+$. Most important, the path $\tau \mapsto v(\tau)$ between $v(\tau^-)$ and $v(\tau^+)$ is a curve of maximal (normalized) slope for the autonomous functional $\mathcal{F}(t, \cdot)$; this property will follow from the optimality of $v'(\tau)$ described hereafter. By the chain rule we can write

$$d_\tau \mathcal{F}(t(\tau), v(\tau)) = \partial_v \mathcal{F}(t(\tau), v(\tau))\, [v'(\tau)] + \partial_t \mathcal{F}(t(\tau), v(\tau)) t'(\tau)\,,$$

and thus for $\tau_1 < \tau_2$

$$\begin{aligned}
\mathcal{F}(t(\tau_2), v(\tau_2)) &= \mathcal{F}(t(\tau_1), v(\tau_1)) + \int_{\tau_1}^{\tau_2} d_\tau \mathcal{F}(t(\tau), v(\tau))\, ds \\
&= \mathcal{F}(t(\tau_1), v(\tau_1)) + \int_{\tau_1}^{\tau_2} \partial_v \mathcal{F}(t(\tau), v(\tau))\, [v'(\tau)]\, d\tau + \int_{\tau_1}^{\tau_2} \partial_t \mathcal{F}(t(\tau), v(\tau))\, t'(\tau)\, d\tau\,.
\end{aligned}$$

On the other hand, by (E') we can write

$$\mathcal{F}(t(\tau_2), v(\tau_2)) = \mathcal{F}(t(\tau_1), v(\tau_1)) - \int_{\tau_1}^{\tau_2} \| \partial_v \mathcal{F}(t(\tau), v(\tau)) \|\, d\tau + \int_{\tau_1}^{\tau_2} \partial_t \mathcal{F}(t(\tau), v(\tau))\, t'(\tau)\, d\tau\,.$$

Therefore, $\partial_v \mathcal{F}(t(\tau), v(\tau))\, [v'(\tau)] = \| \partial_v \mathcal{F}(t(\tau), v(\tau)) \|$ for a.e. τ. Since $\|v'\| \le 1$ it follows that for a.e. τ

$$v'(\tau) \in \operatorname{argmin} \left\{ \partial_v \mathcal{F}(t(\tau), v(\tau))\, [\phi] : \|\phi\| \le 1 \right\}.$$

If $t'(\tau) > 0$ the above property is not of interest since $\partial_v \mathcal{F}(t(\tau), v(\tau)) = 0$ and thus any v' is a minimizer. On the contrary, on jump discontinuities, where $\partial_v \mathcal{F}(t(\tau), v(\tau)) \ne 0$, it says that $v'(\tau)$ is the steepest descent direction. Roughly speaking, on jumps we have a normalized gradient flow

$$v'(\tau) = -\nabla \mathcal{F}(t, v(\tau)) / \| \nabla \mathcal{F}(t, v(\tau)) \|,$$

where $\nabla \mathcal{F}$ is the gradient of $\mathcal{F}$ (with respect to the norm $\| \cdot \|$), τ belongs to the jump interval $[\tau^-, \tau^+]$ and $t = t(\tau)$ is the discontinuity point. To understand the idea behind the normalization of the gradient consider at time t a jump between the equilibrium configurations $v(t^-)$ and $v(t^+)$. In the graph parametrization setting we have $t = t(\tau)$ for $\tau \in [\tau_1, \tau_2]$, $v(t^-) = v(\tau_1)$ and $v(t^+) = v(\tau_2)$. Since $v(t^-)$ and $v(t^+)$ are equilibrium configurations a (non-normalized) gradient flow would provide in general a transition in an infinite interval, say $[\tau_1, +\infty)$, and thus it would not be possible to extend the evolution after the jump time t. On the contrary, a normalized gradient flow allows to find an arc-length parametrization of the curve.

Under suitable conditions on the energy functional, we prove existence of such an evolution both by means of a "parametrized" minimizing movement and by means of a forward Euler scheme; the proofs, based on sequences of incremental problems, do not employ viscosity arguments. The common structure of the existence results could be summarized as: construction of a discrete evolution, discrete stationarity, discrete energy inequality, construction of a continuum interpolation, compactness, convergence of discrete stationarity and energy balance to their to continuum counterparts (E') and (S').

Next, following the scheme of [16], we provide an approximation result of the following type. Consider a sequence of functionals $\mathcal{F}_n$ together with the corresponding quasi-static evolutions, say (t_n, v_n). Assume that $\mathcal{F}_n$ Γ-converge to $\mathcal{F}$ with respect to the strong convergence in t and the weak convergence in v. Under suitable condition on the convergence of $\|\partial_v \mathcal{F}_h\|$ and $\partial_t \mathcal{F}_h$ to $\|\partial_v \mathcal{F}\|$ and $\partial_t \mathcal{F}$ respectively, we show that the quasi-static evolutions (t_n, v_n) converge (up to subsequences) to a quasi-static evolution (t, v) for the Γ-limit $\mathcal{F}$. The proof is based on the same arguments developed for the convergence of discrete stationarity and energy balance, which appear in the existence result. We could consider this result as the analogue of [14], developed for energetic evolutions.

Moreover, we consider the quasi-static evolution with respect to a weaker norm, induced by the continuous immersion of $\mathcal{V}$ in a reflexive separable Banach space $\mathcal{W}$. In this context the existence result developed in the norm of $\mathcal{V}$ is not suitable, we follow instead a "Galerkin approach" approximating the evolution in finite dimensional spaces and then passing to the limit by virtue of the approximation result.

The last part of the paper is dedicated to the quasi-static evolution of a phase field approach in brittle fracture [3]. Irreversibility is modelled with a constraint affecting both the formulation and the existence result, these modifications do not change the core of the arguments and allow to prove existence of a quasi-static evolutions that satisfy equilibrium and energy balance. The reader interested in phase field models will find an energetic evolution in [9], an L^2-gradient flow (in the displacement) in [2], an L^1-viscosity solution in [10] and a dynamic visco-elastic evolution in [11]. Besides our interest for the specific application, this example shows that the formulation and the existence result, both with some modifications, are suitable also in the case of problems with irreversibility constraints.

Contents

2 Definitions and statements of the main results

Let $\mathcal{V}$ be a reflexive, separable Banach space, with norm $\|\cdot\|$, and let $[t_0, t_1]$ be a time interval. Let $\mathcal{F} : [t_0, t_1] \times \mathcal{V} \to [0, +\infty)$ be an energy functional. Assume that $\mathcal{F}$ is of class C^1. Within this setting the quasi-static evolutions of interest in this work are characterized by the following definition.

Definition 2.1 *Let $(t, v) : [0, T] \to [t_0, t_1] \times \mathcal{V}$ be a Lipschitz map with $(t(0), v(0)) = (t_0, v_0)$, $t' \geq 0$ and $\|v'\| \leq 1$; (t, v) is (a parametrization of) a quasi-static evolution if*

(S') *the following stationarity condition holds: for every τ with $t'(\tau) > 0$*

$$\|\partial_v \mathcal{F}(t(\tau), v(\tau))\| = 0, \tag{1}$$

(E') *the following energy balance holds: for every τ*

$$\mathcal{F}(t(\tau), v(\tau)) = \mathcal{F}(t_0, v_0) - \int_0^\tau \|\partial_v \mathcal{F}(t(s), v(s))\| \, ds + \int_0^\tau \partial_t \mathcal{F}(t(s), v(s)) \, t'(s) \, ds \,, \tag{2}$$

where $\|\partial_v \mathcal{F}(t, v)\|$ denotes the norm in the dual space $\mathcal{V}'$.

According to the previous Definition if v_0 is an equilibrium configuration the trivial parametrization $\tau \mapsto (t_0, v_0)$ is an admissible quasi-static evolution. Following [8, Lemma 3.2], it is possible to guarantee the existence of a non-trivial solution making a sort of "coercivity condition" on the dissipation functional, which actually does not enter in our picture. In our setting, it is instead reasonable to assume that $\|\partial_v \mathcal{F}(t_0, v_0)\| \neq 0$. Indeed, considering the case $t(\tau) = t_0$ (otherwise there is nothing to show), (E') reads

$$\mathcal{F}(t_0, v(T)) = \mathcal{F}(t_0, v_0) - \int_0^T \|\partial_v \mathcal{F}(t_0, v(\tau))\| \, d\tau.$$

If, by contradiction, $v(\tau) = v_0$ then we would have

$$\mathcal{F}(t_0, v_0) = \mathcal{F}(t_0, v_0) - \int_0^T \|\partial_v \mathcal{F}(t_0, v_0)\| \, d\tau < \mathcal{F}(t_0, v_0).$$

This condition, despite its simplicity, appears also in [7, Example 1.3] in the context of minimizing movements (and indeed we will show existence re-parametrizing the constrained incremental problems suggested in [7]).

Note that the continuity of $\partial_t \mathcal{F}$ and $\partial_v \mathcal{F}$ are enough for integrability in (2) for every Lipschitz parametrization.

To conclude, note that using parametrizations it may happen that the solution is a pure jump, of the form $(t_0, v(\tau))$, where v solves the autonomous normalized gradient flow

$$v'(\tau) = -\nabla \mathcal{F}(t, v(\tau)) / \|\nabla \mathcal{F}(t, v(\tau))\|$$

for a.e. $\tau \in (0, T)$. This is the only possible solution when there are no equilibrium configurations.

In the next two sections we will prove the following existence result by means of both a minimizing movement and an explicit Euler scheme.

Theorem 2.2 *Let $\mathcal{F} : [t_0, t_1] \times \mathcal{V} \to [0, +\infty)$ be of class C^1 with*

$$\mathcal{F}(t, v) \le \liminf_m \mathcal{F}(t_m, v_m) \qquad \text{for } t_m \to t \text{ and } v_m \rightharpoonup v, \tag{3}$$

$$\|\partial_v \mathcal{F}(t, v)\| \le \liminf_m \|\partial_v \mathcal{F}(t_m, v_m)\| \qquad \text{for } t_m \to t \text{ and } v_m \rightharpoonup v, \tag{4}$$

$$\partial_t \mathcal{F}(t, v) = \lim_m \partial_t \mathcal{F}(t_m, v_m) \qquad \text{for } t_m \to t \text{ and } v_m \rightharpoonup v. \tag{5}$$

Moreover assume that there exists a modulus of continuity $\mathcal{C}$ s.t.

$$\|\partial_v \mathcal{F}(t, v) - \partial_v \mathcal{F}(t, w)\| + |\partial_t \mathcal{F}(t, v) - \partial_t \mathcal{F}(t, w)| \le \mathcal{C}(\|v - w\|) \quad \text{uniformly w.r.t. } t \in [0, T]. \tag{6}$$

Under the above hypotheses there exists a quasi-static evolution in the sense of Definition 2.1.

First, note that coercivity of the functional $\mathcal{F}(t, \cdot)$ is not required for existence.

Next, in the spririt of [16, 17], Theorem 2.3 shows the connection between Γ-convergence [5, 4] of energy functionals and the associated quasi-static evolutions. The proof is contained in §5. A typical application of this Theorem are the finite element approximation. Noteworthy, this convergence Theorem could be used also as an existence result (as we will see in the sequel).

Theorem 2.3 *Let $\mathcal{F}_h : [t_0, t_1] \times \mathcal{V}_h \to [0, +\infty)$ of class C^1 and let $\mathcal{V}_h \subset \mathcal{V}$ be endowed with the norm of $\mathcal{V}$. Let (t_h, v_h) be a quasi-static evolution (in the sense of Definition 2.1) for $\mathcal{F}_h$ with initial conditions $(t_0, v_{h,0})$ and with $t'_h \le 1$. Assume that*

$$\mathcal{F}(t, v) \le \liminf_h \mathcal{F}_h(t_h, v_h) \qquad \text{for } t_h \to t \text{ and } v_h \rightharpoonup v \text{ in } \mathcal{V}, \tag{7}$$

$$\|\partial_v \mathcal{F}(t, v)\| \le \liminf_h \|\partial_v \mathcal{F}_h(t_h, v_h)\|_h \qquad \text{for } t_h \to t \text{ and } v_h \rightharpoonup v \text{ in } \mathcal{V}, \tag{8}$$

$$\partial_t \mathcal{F}(t, v) = \lim_h \partial_t \mathcal{F}_h(t_h, v_h) \qquad \text{for } t_h \to t \text{ and } v_h \rightharpoonup v \text{ in } \mathcal{V}, \tag{9}$$

where $\|\partial_v \mathcal{F}_h(t_h, v_h)\|_h$ is the norm in the dual $\mathcal{V}'_h$. Assume also that the initial condition is "well-prepared", i.e. that

$$v_{h,0} \rightharpoonup v_0 \text{ in } \mathcal{V} \text{ and that } \mathcal{F}_h(t_0, v_{h,0}) \to \mathcal{F}(t_0, v_0) \tag{10}$$

and that the power $\partial_t \mathcal{F}_h(t_h(s), v_h(s))$ is uniformly bounded. Then there exists a subsequence (not relabelled) such that $t_h(\tau) \to t(\tau)$ and $v_h(\tau) \rightharpoonup v(\tau)$, for every $\tau \in [0, T)$; the limit (t, v) is a quasi-static evolution for $\mathcal{F}$ (in the sense of Definition 2.1) with initial conditions (t_0, v_0).

Finally, consider a Banach space $\mathcal{V}$ continuously embedded in a Banach space $\mathcal{W}$ and endowed with the norm of $\mathcal{W}$. (The prototype example is the inclusion of H^1 in L^2). Clearly $\partial_v \mathcal{F}(t, v)[\cdot]$ is linear and continuous on $\mathcal{V}$, endowed with the norm $\| \cdot \|_{\mathcal{V}}$ and thus it is represented by an element $\nabla_{\mathcal{V}} \mathcal{F}(t, v)$ of $\mathcal{V}'$. If $\partial_v \mathcal{F}(t, v)[\cdot]$ is also linear and continuous on $\mathcal{V}$, endowed with the norm $\| \cdot \|_{\mathcal{W}}$, it is possible to define also

$$\|\nabla_{\mathcal{W}} \mathcal{F}(t, v)\| = \sup\{\partial_v \mathcal{F}(t, v)[\phi] : \phi \in \mathcal{V}, \ \|\phi\|_{\mathcal{W}} \le 1\}.$$

If $\partial_v \mathcal{F}(t, v)[\cdot]$ is not linear and continuous on $\mathcal{V}$, endowed with the norm $\|\cdot\|_{\mathcal{W}}$, we set $\|\nabla_{\mathcal{W}} \mathcal{F}(t, v)\| = \infty$. Within this framework we can define the quasi-static evolution with respect to the norm $\| \cdot \|_{\mathcal{W}}$ exactly as we did in Definition 2.1.

Definition 2.4 *Let $(t, v) : [0, T] \to [t_0, t_1] \times \mathcal{V}$ be a Lipschitz map with $(t(0), v(0)) = (t_0, v_0)$, $t' \ge 0$ and $\|v'\|_{\mathcal{W}} \le 1$; (t, v) is (a parametrization of) a quasi-static evolution if*

(S') *for every* τ *with* $t'(\tau) > 0$

$$\|\nabla_{\mathcal{W}} \mathcal{F}(t(\tau), v(\tau))\| = 0, \tag{11}$$

(E') *for every* τ

$$\mathcal{F}(t(\tau), v(\tau)) = \mathcal{F}(t_0, v_0) - \int_0^\tau \|\nabla_{\mathcal{W}} \mathcal{F}(t(s), v(s))\| \, ds + \int_0^\tau \partial_t \mathcal{F}(t(s), v(s)) \, t'(s) \, ds. \tag{12}$$

In the weak norm setting the existence result provided by Theorem 2.2 is not really useful since the uniform continuity of the derivatives, with respect to $\| \cdot \|$, is often too restrictive (think of the Dirichlet energy with respect to the L^2-norm). However, using a sequence of finite dimensional approximations together with Theorem 2.3 we will prove in §6 the following existence result.

Theorem 2.5 *Let* $\mathcal{F} : [t_0, t_1] \times \mathcal{V} \to [0, +\infty)$ *satisfy*

$$\mathcal{F}(t, v) \le \liminf_m \mathcal{F}(t_m, v_m) \qquad \text{for } t_m \to t \text{ and } v_m \rightharpoonup v \text{ in } \mathcal{V}, \tag{13}$$

$$\partial_v \mathcal{F}(t, v)[\phi] = \lim_m \partial_v \mathcal{F}(t_m, v_m)[\phi] \qquad \text{for } t_m \to t,\ v_m \rightharpoonup v \text{ in } \mathcal{V} \text{ and } \phi \in \mathcal{V}, \tag{14}$$

$$\partial_t \mathcal{F}(t, v) = \lim_m \partial_t \mathcal{F}(t_m, v_m) \qquad \text{for } t_m \to t \text{ and } v_m \rightharpoonup v \text{ in } \mathcal{V}. \tag{15}$$

Moreover assume that there exists a modulus of continuity $\mathcal{C}$ *s.t.*

$$\|\partial_v \mathcal{F}(t, v) - \partial_v \mathcal{F}(t, w)\|_{\mathcal{V}'} + |\partial_t \mathcal{F}(t, v) - \partial_t \mathcal{F}(t, w)| \le \mathcal{C}(\|v - w\|_{\mathcal{V}}) \quad \text{uniformly w.r.t. } t \in [0, T]. \tag{16}$$

Let $\mathcal{V}$ *be continuously embedded in a separable reflexive Banach space* $\mathcal{W}$*. Assume also that* $\mathcal{F}$ *is coercive with respect to* $\| \cdot \|_{\mathcal{V}}$ *and that for some* A, B *independent of* t *and* v *it holds*

$$|\partial_t \mathcal{F}(t, v)| \le A \, \mathcal{F}(t, v) + B. \tag{17}$$

Then there exists a quasi-static evolution in the sense of Definition 2.4.

Note that if $\partial_v \mathcal{F}$ is weakly* continuous, i.e. if for every $t_m \to t$ and $v_m \rightharpoonup v$ in $\mathcal{V}$

$$\partial_v \mathcal{F}(t_m, v_m)[\phi] \to \partial_v \mathcal{F}(t, v)[\phi] \qquad \text{for every } \phi \in \mathcal{V}$$

then by Banach-Steinhaus Theorem we have

$$\|\partial_v \mathcal{F}(t, v)\| \le \liminf_m \|\partial_v \mathcal{F}_\varepsilon(t_m, v_m)\|.$$

Before proving the above results, it is fair to mention that the statements are far from being sharp, for instance, (5) could be replaced by a limsup condition, the uniform continuity of (6) could be made time dependent while conditions (7)-(9) could include the energy excess, as in [16]. Indeed, in this work our goal is to present a scheme and the essential ingredients, rather then providing a general framework.

3 Existence by a parametrized minimizing movement

3.1 Incremental problem

In this section we prove the existence result following the discretization used in [8]. Let $\Delta\tau_n \to 0^+$. For every $n \in \mathbb{N}$ we define a discrete in time evolution by means of constrained incremental minimization problems. Set the initial conditions $t_{n,0} = t_0$ and $v_{n,0} = v_0$; known $t_{n,k} < t_1$ and $v_{n,k}$ define $t_{n,k+1}$ and $v_{n,k+1}$ as

$$\begin{cases} v_{n,k+1} \in \operatorname{argmin}\{\mathcal{F}_\varepsilon(t_{n,k}, v) : \|v - v_{n,k}\| \leq \Delta\tau_n\}, \\ t_{n,k+1} = t_{n,k} + \big(\Delta\tau_n - \|v_{n,k+1} - v_{n,k}\|\big). \end{cases} \tag{18}$$

Existence of a minimizer follows by the direct method of the calculus of variations: weak compactness of the closed ball $\|v - v_{n,k}\| \leq \Delta\tau_n$ is a consequence of $\mathcal{V}$ being reflexive while the lower semi-continuity of $\mathcal{F}(t_{n,k}, \cdot)$ is assumed in (3). (Note that a similar incremental problem, actually without parametrization, appears in [7, Example 1.3]). Let $\bar{k}_n = \sup\{k : t_{n,k} < t_1\}$ where $\bar{k}_n \in \mathbb{N} \cup \{\infty\}$ (the incremental construction can be finite or infinite). Note that $0 \leq t_{n,k+1} - t_{n,k} < \Delta\tau_n$ and thus $\bar{k}_n \geq (t_1 - t_0)/\Delta\tau_n$.

Next, let $\tau_{n,k} = k\Delta\tau_n$ for $0 \leq k \leq \bar{k}_n$ and denote $T_n = \bar{k}_n\Delta\tau_n \geq (t_1 - t_0)$. Now we consider the affine interpolation of $t_{n,k}$ and $v_{n,k}$ in the points $\tau_{n,k}$; in this way we define the discrete evolutions $(t_n, v_n) : [0, T_n) \to [t_0, t_1] \times \mathcal{V}$. It is important to remark that the map (t_n, v_n) is Lipschitz continuous with $t'_n \geq 0$ and $t'_n + \|v'_n\| = 1$ a.e. in $[0, T_n)$.

Now, let us see the two properties which will provide the base to get (S') and (E') in the limit as $\Delta\tau_n \to 0^+$.

Proposition 3.1 *If $t_{n,k+1} > t_{n,k}$ then $v_{n,k+1}$ satisfies the equilibrium condition*

$$\|\partial_v \mathcal{F}(t_{n,k}, v_{n,k+1})\| = 0. \tag{19}$$

Proof. If $t_{n,k+1} > t_{n,k}$ then by (18) we have $\|v_{n,k+1} - v_{n,k}\| < \Delta\tau_n$. Since $v_{n,k+1}$ is a minimizer the Euler-Lagrange equation (19) holds. ∎

Proposition 3.2 *The following incremental energy estimate holds*

$$\begin{aligned} \mathcal{F}(t_{n,k+1}, v_{n,k+1}) \leq\; & \mathcal{F}(t_{n,k}, v_{n,k}) - \int_{\tau_{n,k}}^{\tau_{n,k+1}} \|\partial_v \mathcal{F}(t_{n,k}, v_n(\tau))\|\, d\tau \\ & + \int_{\tau_{n,k}}^{\tau_{n,k+1}} \partial_t \mathcal{F}(t_n(\tau), v_n(\tau))\, t'_n(\tau)\, d\tau + 3\,\mathcal{C}(\|v_{n,k} - v_{n,k+1}\|)\,\Delta\tau_n, \end{aligned} \tag{20}$$

where $\mathcal{C}(\cdot)$ is the modulus of continuity appearing in (6).

Proof. Given $(t_{n,k},\, v_{n,k})$ let

$$\phi_{n,k} \in \operatorname{argmin}\{\partial_v \mathcal{F}(t_{n,k}, v_{n,k})[\phi] : \|\phi\| \leq 1\}.$$

Now, we write

$$\begin{aligned} \mathcal{F}(t_{n,k+1}, v_{n,k+1}) &= \mathcal{F}(t_{n,k}, v_{n,k+1}) + \int_{t_{n,k}}^{t_{n,k+1}} \partial_t \mathcal{F}(t, v_{n,k+1})\, dt \\ &\leq \mathcal{F}(t_{n,k}, v_{n,k} + \Delta\tau_n \phi_{n,k}) + \int_{\tau_{n,k}}^{\tau_{n,k+1}} \partial_t \mathcal{F}(t_n(\tau), v_{n,k+1})\, t'_n(\tau)\, d\tau \\ &= \mathcal{F}(t_{n,k}, v_{n,k}) + \int_{\tau_{n,k}}^{\tau_{n,k+1}} \partial_v \mathcal{F}(t_{n,k}, v_{n,k} + (\tau - \tau_{n,k})\phi_{n,k})[\phi_{n,k}]\, d\tau\, + \\ &\quad + \int_{\tau_{n,k}}^{\tau_{n,k+1}} \partial_t \mathcal{F}(t_n(\tau), v_{n,k+1})\, t'_n(\tau)\, d\tau\,. \end{aligned}$$

We consider separately the two integrals on the right hand side.

For every $\tau \in [\tau_{n,k}, \tau_{n,k+1}]$ by (6) we can write

$$\begin{aligned}\partial_v \mathcal{F}(t_{n,k}, v_{n,k} + (\tau - \tau_{n,k})\phi_{n,k})[\phi_{n,k}] &\le \partial_v \mathcal{F}(t_{n,k}, v_{n,k})[\phi_{n,k}] + \mathcal{C}(\|v_{n,k} - v_{n,k+1}\|) \\ &= -\|\partial_v \mathcal{F}(t_{n,k}, v_{n,k})\| + \mathcal{C}(\|v_{n,k} - v_{n,k+1}\|) \\ &= -\|\partial_v \mathcal{F}(t_{n,k}, v_n(\tau))\| + 2\,\mathcal{C}(\|v_{n,k} - v_{n,k+1}\|)\,.\end{aligned}$$

Thus

$$\begin{aligned}\int_{\tau_{n,k}}^{\tau_{n,k+1}} \partial_v \mathcal{F}(t_{n,k}, v_{n,k} + (\tau - \tau_{n,k})\phi_{n,k})[\phi_{n,k}]\, d\tau \le &- \int_{\tau_{n,k}}^{\tau_{n,k+1}} \|\partial_v \mathcal{F}(t_{n,k}, v_n(\tau))\|\, d\tau \\ &+ 2\,\mathcal{C}(\|v_{n,k} - v_{n,k+1}\|)\,\Delta\tau_n.\end{aligned}$$

Similarly, again by (6), for every $\tau \in [\tau_{n,k}, \tau_{n,k+1}]$

$$\partial_t \mathcal{F}(t_n(\tau), v_{n,k+1}) \le \partial_t \mathcal{F}(t_n(\tau), v_n(\tau)) + \mathcal{C}(\|v_{n,k} - v_{n,k+1}\|).$$

As $0 \le t_n' \le 1$ we get

$$\begin{aligned}\int_{\tau_{n,k}}^{\tau_{n,k+1}} \partial_t \mathcal{F}(t_n(\tau), v_{n,k+1})\, t_n'(\tau)\, d\tau \le &\int_{\tau_{n,k}}^{\tau_{n,k+1}} \partial_t \mathcal{F}(t_n(\tau), v_n(\tau))\, t_n'(\tau)\, d\tau \\ &+ \mathcal{C}(\|v_{n,k} - v_{n,k+1}\|)\,\Delta\tau_n\end{aligned}$$

which concludes the proof. ■

3.2 Compactness and convergence

Proposition 3.3 *Let $(t_n, v_n) : [0, T_n) \to [t_0, t_1] \times \mathcal{V}$ be given by §3.1. Let $0 < T < \liminf_n T_n$. There exists a subsequence (not relabelled) such that $t_n \overset{*}{\rightharpoonup} t$ in $W^{1,\infty}(0,T)$ and $v_n \overset{*}{\rightharpoonup} v$ in $W^{1,\infty}(0,T;\mathcal{V})$. In particular $t_n(\tau_n) \to t(\tau)$ and $v_n(\tau_n) \rightharpoonup v(\tau)$ in $\mathcal{V}$ if $\tau_n \to \tau$. Moreover $0 \le t' \le 1$ and $\|v'\| \le 1$ a.e. in $(0,T)$.*

Proof. As $T_n \ge (t_1 - t_0)$ we have $\liminf_n T_n > 0$. Then, for n sufficiently large $T_n > T$ and we can consider (t_n, v_n) to be defined in $(0,T)$. Since $|t_n'(\tau)| \le 1$ there exists a subsequence (not relabelled) with $t_n \overset{*}{\rightharpoonup} t$ in $W^{1,\infty}(0,T)$. As t_n is non-decreasing the limit t is non-decreasing, moreover we have $0 \le t' \le 1$.

Being $\mathcal{V}$ reflexive and separable its dual $\mathcal{V}'$ is reflexive and separable, thus the space $L^1(0,T;\mathcal{V}')$ is separable and its dual is $L^\infty(0,T;\mathcal{V})$. The sequence v_n' is bounded in $L^\infty(0,T;\mathcal{V})$ and thus there exists a subsequence (not relabelled) such that $v_n' \overset{*}{\rightharpoonup} v'$ in $L^\infty(0,T;\mathcal{V})$. Denoting by $\langle\, , \rangle$ the duality pairing between $\mathcal{V}$ and $\mathcal{V}'$ and by $(\, ,)$ the corresponding duality between $L^1(0,T;\mathcal{V}')$ and $L^\infty(0,T;\mathcal{V})$, we can write

$$\langle v_n(\tau_n), \psi \rangle = \langle v_0 + \int_0^{\tau_n} v_n'(s)\, ds, \psi \rangle = \langle v_0, \psi \rangle + \int_0^{\tau_n} \langle v_n'(s), \psi \rangle\, ds$$

from which it follows that $\langle v_n(\tau_n), \psi \rangle \to \langle v(\tau), \psi \rangle$ for every $\psi \in \mathcal{V}'$. By the lower semi-continuity of the norm with respect to weak* convergence in $L^\infty(0,T;\mathcal{V})$ it follows that $\|v'(\tau)\| \le 1$ for a.e. $\tau \in (0,T)$. ■

Theorem 3.4 *Let (t_n, v_n) and (t, v) be as in Proposition 3.3; then (t, v) is a quasi-static evolution, i.e. it satisfies $(t(0), v(0)) = (t_0, v_0)$, $t' \ge 0$, $\|v'\| \le 1$. Moreover*

(S') *for every* $\tau > 0$ *with* $t'(\tau) > 0$ *it holds*

$$\|\partial_v \mathcal{F}(t(\tau), v(\tau))\| = 0,$$

(E') *for every* τ *it holds*

$$\mathcal{F}(t(\tau), v(\tau)) = \mathcal{F}(t_0, v_0) - \int_0^\tau \|\partial_v \mathcal{F}(t(s), v(s))\| \, ds + \int_0^\tau \partial_t \mathcal{F}(t(s), v(s))\, t'(s)\, ds.$$

Proof. If $t'(\tau) > 0$ then $t(\tau) < t(\tau + \delta)$ for every $\delta > 0$, further, since t_n converges to t pointwise, for n sufficiently large there exists an index k (depending on n) such that $\tau < \tau_{n,k} < \tau + \delta$ and $t_{n,k} = t_n(\tau_{n,k}) < t_n(\tau_{n,k+1}) = t_{n,k+1}$. Since $t_{n,k} < t_{n,k+1}$ by Lemma 3.1 we get

$$\|\partial_v \mathcal{F}(t_{n,k}, v_{n,k+1})\| = 0.$$

By the arbitrarity of δ it follows that there exists a sequence k_n with $\tau_{n,k_n} \to \tau$ such that $t_{n,k_n} < t_{n,k_n+1}$ and $\|\partial_v \mathcal{F}(t_{n,k_n}, v_{n,k_n+1})\| = 0$. By Proposition 3.3 we known that $t_{n,k_n} = t_n(\tau_{n,k_n}) \to t(\tau)$ and that $v_{n,k_n+1} = v_n(\tau_{n,k_n+1}) \rightharpoonup v(\tau)$ weakly in $\mathcal{V}$. Then, by (4)

$$\|\partial_v \mathcal{F}(t(\tau), v(\tau))\| \le \liminf_n \|\partial_v \mathcal{F}(t_{n,k_n}, v_{n,k_n+1})\| = 0$$

which is (S').

For the proof of (E') it is more convenient to have τ as the integration variable, so we will show that for every s

$$\mathcal{F}(t(s), v(s)) = \mathcal{F}(t_0, v_0) - \int_0^s \|\partial_v \mathcal{F}(t(\tau), v(\tau))\| \, d\tau + \int_0^s \partial_t \mathcal{F}(t(\tau), v(\tau))\, t'(\tau)\, d\tau\,.$$

For every $n \in \mathbb{N}$ let $k \in \mathbb{N}$ (depending on n) such that $s \in [\tau_{n,k}, \tau_{n,k+1})$. Iterating the incremental energy estimate of Lemma 3.2 yields

$$\begin{aligned} \mathcal{F}(t_{n,k}, v_{n,k}) \,\le\, & \mathcal{F}(t_0, v_0) - \sum_{m=0}^{k-1} \int_{\tau_{n,m}}^{\tau_{n,m+1}} \|\partial_v \mathcal{F}(t_{n,m}, v_n(\tau))\| \, d\tau \\ & + \int_0^{\tau_{n,k}} \partial_t \mathcal{F}(t_n(\tau), v_n(\tau))\, t_n'(\tau)\, d\tau + 3\, \mathcal{C}(\Delta\tau_n) T. \end{aligned}$$

Taking the limsup we get

$$\begin{aligned} \limsup_n \mathcal{F}(t_{n,k}, v_{n,k}) \,\le\, & \mathcal{F}(t_0, v_0) - \liminf_n \sum_{m=0}^{k-1} \int_{\tau_{n,m}}^{\tau_{n,m+1}} \|\partial \mathcal{F}(t_{n,m}, v_n(\tau))\| \, d\tau \\ & + \limsup_n \int_0^{\tau_{n,k}} \partial_t \mathcal{F}(t_n(\tau), v_n(\tau))\, t_n'(\tau)\, d\tau. \end{aligned}$$

Given τ, let $\tau_{n,m} \le \tau < \tau_{n,m+1}$, since $t_{n,m} = t(\tau_{n,m}) \to t(\tau)$ and $v_n(\tau) \rightharpoonup v(\tau)$ by (4) we get

$$\|\partial_v \mathcal{F}(t(\tau), v(\tau))\| \le \liminf_n \|\partial_v \mathcal{F}(t_{n,m}, v_n(\tau))\|$$

and by Fatou's Lemma we get

$$\int_0^s \|\partial_v \mathcal{F}(t(\tau), v(\tau))\| \, d\tau \le \liminf_n \sum_{m=0}^{k-1} \int_{\tau_{n,m}}^{\tau_{n,m+1}} \|\partial_v \mathcal{F}(t_{n,m}, v_n(\tau))\| \, d\tau.$$

By (5) we known that $\partial_t \mathcal{F}(t_n(\tau), v_n(\tau))$ converge to $\partial_t \mathcal{F}(t(\tau), v(\tau))$. Since $\|v_n'\| \le 1$ it follows that $\|v_n(\tau) - v_0\| \le T$ for every $\tau \in (0,T)$. Then (6) implies that $\partial_t \mathcal{F}(t_n(\cdot), v_n(\cdot))$ is uniformly bounded from above; therefore by dominated convergence $\partial_t \mathcal{F}(t_n(\cdot), v_n(\cdot))$ converge to $\partial_t \mathcal{F}(t(\cdot), v(\cdot))$ strongly in $L^1(0,T)$. We already known that $t_n' \overset{*}{\rightharpoonup} t'$ in $L^\infty(0,T)$. As a consequence

$$\lim_n \int_0^{\tau_{n,k}} \partial_t \mathcal{F}(t_n(\tau), v_n(\tau))\, t_n'(\tau)\, d\tau = \int_0^s \partial_t \mathcal{F}(t(\tau), v(\tau))\, t'(\tau)\, d\tau.$$

Therefore

$$\limsup_n \mathcal{F}(t_{n,k}, v_{n,k}) \le \mathcal{F}(t_0, v_0) - \int_0^\tau \|\partial_v \mathcal{F}(t(\tau), v(\tau))\|\, d\tau + \int_0^\tau \partial_t \mathcal{F}(t(\tau), v(\tau))\, t'(\tau)\, d\tau.$$

Since $\|v'\| \le 1$ we can write

$$-\|\partial_v \mathcal{F}(t(\tau), v(\tau))\| \le \partial_v \mathcal{F}(t(\tau), v(\tau))[v'(\tau)]$$

and by the chain rule

$$\begin{aligned} \limsup_n \mathcal{F}(t_{n,k}, v_{n,k}) &\le \mathcal{F}(t_0, v_0) - \int_0^\tau \|\partial_v \mathcal{F}(t(\tau), v(\tau))\|\, d\tau + \int_0^\tau \partial_t \mathcal{F}(t(\tau), v(\tau))\, t'(\tau)\, d\tau \\ &\le \mathcal{F}(t_0, v_0) + \int_0^s \partial_v \mathcal{F}(t(\tau), v(\tau))[v'(\tau)]\, d\tau + \int_0^s \partial_t \mathcal{F}(t(\tau), v(\tau))\, t'(\tau) d\tau \\ &\le \mathcal{F}(t_0, v_0) + \int_0^s d_\tau \mathcal{F}(t(\tau), v(\tau))\, d\tau = \mathcal{F}(t(s), v(s))\,. \end{aligned} \tag{21}$$

Thus $\limsup_n \mathcal{F}(t_{n,k}, v_{n,k}) \le \mathcal{F}(t(s), v(s))$. On the other hand, $t_{n,k} \to t(s)$ and $v_{n,k} \rightharpoonup v(s)$ by Proposition 3.3, thus by (3) we can write $\mathcal{F}(t(s), v(s)) \le \liminf_n \mathcal{F}(t_{n,k}, v_{n,k})$. It follows that

$$\limsup_n \mathcal{F}(t_{n,k}, v_{n,k}) = \lim_n \mathcal{F}(t_{n,k}, v_{n,k}) = \mathcal{F}(t(s), v(s)) \tag{22}$$

and all the inequalities in (21) becomes equality, which gives (E'). ■

As a by-product of the previous results we get the convergence of energies, stated in the next Corollary.

Corollary 3.5 $\mathcal{F}(t_n(\tau), v_n(\tau)) \to \mathcal{F}(t(\tau), v(\tau))$ *for every* $\tau \in [0,T]$.

Proof. For $\tau \in [\tau_{n,k}, \tau_{n,k+1})$ both $\|v_n(\tau) - v_{n,k}\| \le \Delta\tau_n$ and $|t_n(\tau) - t_{n,k}| \le \Delta\tau_n$ therefore by (22) together with the uniform continuity of $\mathcal{F}$ we get the pointwise convergence of the energy. ■

Finally, note that, without further assumptions, it is not obvious that the limit (t,v) (provided by Proposition 3.3) is not the trivial evolution $(t(\tau), v(\tau)) = (t_0, v_0)$. For instance, consider the energy

$$\mathcal{F}(t_0, v) = \begin{cases} 0 & \text{if } \|v\| \le 1, \\ (\|v\| - 1)^2 & \text{otherwise.} \end{cases}$$

The unit ball is the set of minimizers of this functional. If $\Delta\tau_n \le 1$ and if $v_0 = 0$ it is possible to choose a sequence $v_{n,k}$ with $\|v_{n,k}\| \le \Delta\tau_n$ and with $\|v_{n,k+1} - v_{n,k}\| = \Delta\tau_n$. As a consequence $t_{n,k} = t_0$ and $v_{n,k} \to v_0$, therefore the limit is the trivial solution. In the applications irreversibility conditions are usually helpful to rule out this solution, see [8].

4 Existence by a forward Euler scheme

4.1 Incremental problem

In applications and in numerical simulations sometimes it is not feasible to minimize (even locally) the energy of a non-convex function, as in the previous scheme, while it is more feasible to employ a forward scheme, based on descent directions. Let us see how to define the evolution in this setting.

Let $\Delta\tau_n$ be positive with $\Delta\tau_n \to 0$ and denote again by τ the parametrization variable. As before, we will provide a sequence converging to (the parametrization of) a quasi-static evolution, according to Definition 2.1. Let the initial conditions (for $\tau_{n,0} = 0$) be $t_{n,0} = t_0$ and $v_{n,0} = v_0$. Given $t_{n,k}$ and $v_{n,k}$ we will employ a further sequence, denoted by $v_{n,k,i}$ for $i \in \mathbb{N}$, in order to define $v_{n,k+1}$ and then $t_{n,k+1}$. For convenience we will also introduce a sequence $\tau_{n,k,i}$ which actually depends on $v_{n,k,i}$. Let $v_{n,k,0} = v_{n,k}$ and $\tau_{n,k,0} = \tau_{n,k}$; given $v_{n,k,i}$ and $\tau_{n,k,i}$ consider the steepest descent direction

$$\phi_{n,k,i} \in \operatorname{argmin}\{\partial_v \mathcal{F}(t_{n,k}, v_{n,k,i})[\phi] : \|\phi\| \le 1\}. \tag{23}$$

Existence of $\phi_{n,k,i}$ is straightforward. Uniqueness holds for instance when $\partial_v \mathcal{F}(t_{n,k}, v_{n,k,i}) \neq 0$ and $\mathcal{V}$ is uniformly convex: indeed, if $\partial_v \mathcal{F}(t_{n,k}, v_{n,k,i}) \neq 0$ then $\|\phi_{n,k,i}\| = 1$, thus if

$$\zeta, \xi \in \operatorname{argmin}\{\partial_v \mathcal{F}(t_{n,k-1}, v_{n,k-1})[\phi] : \|\phi\| \le 1\}$$

then by linearity $(\zeta + \xi)/2$ is a minimizer and thus $\|(\zeta + \xi)/2\| = 1$; the uniform convexity of $\mathcal{V}$ implies that $\zeta = \xi$. On the contrary, if $\partial_v \mathcal{F}(t_{n,k}, v_{n,k,i})[\phi] = 0$ then every ϕ is a minimizers, in this case it is convenient to choose again a direction $\phi_{n,k,i}$ with $\|\phi_{n,k,i}\| = 1$ in order to avoid any trouble in the following construction. Of course, in any case

$$\partial_v \mathcal{F}(t_{n,k}, v_{n,k,i})[\phi_{n,k,i}] = -\|\partial_v \mathcal{F}(t_{n,k}, v_{n,k,i})\|.$$

Once the direction $\phi_{n,k,i}$ is found the step $s_{n,k,i}$ is given by a gradient descent in the direction $\phi_{n,k,i}$. To this end, for $s \in [0, \Delta\tau_n - \tau_{n,k,i}]$, let us introduce the function

$$f(s) = \mathcal{F}(t_{n,k}, v_{n,k,i} + s\phi_{n,k,i})$$

and the associated ODE

$$\begin{cases} s'(\cdot) = (-f'(s(\cdot)))_+, \\ s(0) = 0, \end{cases} \tag{24}$$

where $(\cdot)_+$ denotes the positive part (the independent variable in the ODE is purely auxiliary and has no physical meaning). Since

$$f'(s) = \partial_v \mathcal{F}(t_{n,k}, v_{n,k,i} + s\phi_{n,k,i})[\phi_{n,k,i}]$$

it follows by the assumptions on $\partial_v \mathcal{F}$ that the right hand side is continuous and therefore there exists a solution s of the ODE (the solution is unique if the right hand side is Lipschitz continuous). By definition, the solution s is positive, non-decreasing and bounded from above by $\Delta\tau_n - \tau_{n,k,i}$. In particular it makes sense to take $s_{n,k,i} = \sup s(\cdot) \le (\Delta\tau_n - \tau_{n,k,i})$ and then to define

$$v_{n,k,i+1} = v_{n,k,i} + s_{n,k,i}\ \phi_{n,k,i},$$

$$\tau_{n,k,i+1} = \tau_{n,k,i} + s_{n,k,i} = \tau_{n,k,i} + \|v_{n,k,i+1} - v_{n,k,i}\| \le \Delta\tau_n.$$

Note that with this definition

$$\|v_{n,k,i+1} - v_{n,k,i}\| = \tau_{n,k,i+1} - \tau_{n,k,i} = s_{n,k,i}. \tag{25}$$

Moreover, thank to the positive part in the ODE the energy $f(s)$ is non-increasing in $[0, s_{n,k,i}]$. Next, let us define

$$v_{n,k+1} = \lim_i v_{n,k,i} = v_{n,k} + \sum_{i=0}^{\infty} (v_{n,k,i+1} - v_{n,k,i}) \quad \text{and} \quad \bar\tau_{n,k} = \lim_i \tau_{n,k,i} = \tau_{n,k} + \sum_{i=0}^{\infty} s_{n,k,i}.$$

Note that the sequence $\tau_{n,k,i}$ is non-decreasing, with respect to i, and bounded from above by $\Delta\tau_n$, thus the limit exists and is bounded by the same constant. Moreover, the limit of $v_{n,k,i}$ exists because

$$\sum_{i=0}^{\infty} \| v_{n,k,i+1} - v_{n,k,i} \| = \sum_{i=0}^{\infty} s_{n,k,i} = \bar\tau_{n,k} - \tau_{n,k}\,.$$

In particular $\|v_{n,k+1} - v_{n,k}\| \le \Delta\tau_n$. Finally, let

$$t_{n,k+1} = t_{n,k} + (\Delta\tau_n - \bar\tau_{n,k}). \tag{26}$$

As in the previous section, let $\bar k_n = \sup\{k : t_{n,k} < t_1\}$ and denote $T_n = \bar k_n \Delta\tau_n \ge (t_1 - t_0)$. Now we define the sequences $v_n : [0, T_n) \to \mathcal{V}$ and $t_n : [0, T_n) \to [t_0, t_1]$. In the subinterval $[\tau_{n,k}, \bar\tau_{n,k})$ we define v_n to be the piecewise affine interpolation of $v_{n,k,i}$ in the points $\tau_{n,k,i}$ while in the subinterval $[\bar\tau_{n,k}, \tau_{n,k+1}]$ we set $v_n = v_{n,k+1}$. The definition of t_n is somehow complementary: in the subinterval $[\tau_{n,k}, \bar\tau_{n,k})$ we define $t_n = t_{n,k}$ while in the subinterval $[\bar\tau_{n,k}, \tau_{n,k+1}]$ we take the affine interpolation of $t_{n,k}$ and $t_{n,k+1}$. In this way in the subinterval $[\tau_{n,k}, \bar\tau_{n,k})$ we have $\|v_n'\| = 1$, thank to (25), and $t_n' = 0$; in the subinterval $[\bar\tau_{n,k}, \tau_{n,k+1}]$ we have $v_n' = 0$ and $t_n' = 1$, thank to (26). Therefore we still have $t_n' + \|v_n'\| = 1$.

Proposition 4.1 *If $t_{n,k+1} > t_{n,k}$ then $v_{n,k+1}$ satisfies the equilibrium condition*

$$\|\partial_v \mathcal{F}(t_{n,k}, v_{n,k+1})\| = 0. \tag{27}$$

Proof. If $t_{n,k+1} > t_{n,k}$ then $\bar\tau_{n,k} < \Delta\tau_n$. If $\bar\tau_{n,k} = \tau_{n,k}$ then $v_{n,k+1} = v_{n,k}$ and $\partial_v \mathcal{F}(t_{n,k}, v_{n,k+1})\,[\phi_{n,k,i}] = 0$ by (24), therefore by (23)

$$\partial_v \mathcal{F}(t_{n,k}, v_{n,k+1})\,[\phi] = 0 \quad \text{for every } \phi \in \mathcal{V}\,.$$

Otherwise, if $0 < \bar\tau_{n,k} < \Delta\tau_n$ then $v_{n,k} = \lim_i v_{n,k,i}$ and $s_{n,k,i} < \Delta\tau_n - \tau_{n,k,i}$ for every i, hence again by (24)

$$\partial_v \mathcal{F}(t_{n,k}, v_{n,k,i+1})\,[\phi_{n,k,i}] = 0\,.$$

By the uniform continuity (6) of $\partial_v \mathcal{F}$ it follows that

$$\|\partial_v \mathcal{F}(t_{n,k}, v_{n,k,i})\| = |\partial_v \mathcal{F}(t_{n,k}, v_{n,k,i})\,[\phi_{n,k,i}]| \le \mathcal{C}(\|v_{n,k,i+1} - v_{n,k,i}\|).$$

Then, by the lower semi-continuity (4) of $\|\partial_v \mathcal{F}\|$ and by the convergence of $v_{n,k,i}$ we get

$$\|\partial_v \mathcal{F}(t_{n,k}, v_{n,k+1})\| \le \liminf_i \|\partial_v \mathcal{F}(t_{n,k}, v_{n,k,i})\| \le \liminf_i \mathcal{C}(\|v_{n,k,i+1} - v_{n,k,i}\|) = 0,$$

which is (27). ■

Proposition 4.2 *The following incremental energy estimate holds*

$$\begin{aligned} \mathcal{F}(t_{n,k+1}, v_{n,k+1}) \le\ & \mathcal{F}(t_{n,k}, v_{n,k}) - \int_{\tau_{n,k}}^{\tau_{n,k+1}} \|\partial_v \mathcal{F}(t_{n,k}, v_{n,k})\|\, d\tau \\ & + \int_{\tau_{n,k}}^{\tau_{n,k+1}} \partial_t \mathcal{F}(t_n(\tau), v_n(\tau))\, t_n'(\tau)\, d\tau + 3\,\mathcal{C}(\|v_{n,k} - v_{n,k+1}\|)\,\Delta\tau_n, \end{aligned} \tag{28}$$

where $\mathcal{C}(\cdot)$ is the modulus of continuity appearing in (6).

Proof. Arguing as in the proof of Proposition 3.2 we obtain

$$\begin{aligned}\mathcal{F}(t_{n,k+1},v_{n,k+1}) &= \mathcal{F}(t_{n,k},v_{n,k+1}) + \int_{t_{n,k}}^{t_{n,k+1}} \partial_t\mathcal{F}(t,v_{n,k+1})\,dt \\ &= F(t_{n,k},v_{n,k}) + \int_{\tau_{n,k}}^{\tau_{n,k+1}} \partial_v\mathcal{F}(t_{n,k},v_n(\tau))[v_n'(\tau)]\,d\tau + \\ &\quad + \int_{\tau_{n,k}}^{\tau_{n,k+1}} \partial_t\mathcal{F}(t_n(\tau),v_{n,k+1})\,t_n'(\tau)\,d\tau\end{aligned}$$

but this time we cannot use minimality for $v_{n,k+1}$. Thus, for $\tau \in [\tau_{n,k,i},\tau_{n,k,i+1}]$ we have $v_n'(\tau) = \phi_{n,k,i} \in \operatorname{argmin}\{\partial_v\mathcal{F}(t_{n,k},v_{n,k,i})[\phi] : \|\phi\| = 1\}$ and then by (6) we can write

$$\begin{aligned}\partial_v\mathcal{F}(t_{n,k},v_n(\tau))[v_n'(\tau)] &\le \partial_v\mathcal{F}(t_{n,k},v_{n,k,i})[\phi_{n,k,i}] + \mathcal{C}(\|v_{n,k,i+1}-v_n(\tau)\|) \\ &\le -\|\partial_v\mathcal{F}(t_{n,k},v_{n,k,i})\| + \mathcal{C}(\|v_{n,k,i+1}-v_n(\tau)\|) \\ &\le -\|\partial_v\mathcal{F}(t_{n,k},v_n(\tau))\| + 2\,\mathcal{C}(\|v_{n,k,i+1}-v_n(\tau)\|).\end{aligned}$$

Then

$$\begin{aligned}\int_{\tau_{n,k,i}}^{\tau_{n,k,i+1}} \partial_v\mathcal{F}(t_{n,k},v_n(\tau))[v_n'(\tau)]\,d\tau &\le -\int_{\tau_{n,k,i}}^{\tau_{n,k,i+1}} \|\partial_v\mathcal{F}(t_{n,k},v_n(\tau))\|\,d\tau \\ &\quad +2\,\mathcal{C}(\|v_{n,k,i+1}-v_{n,k,i}\|)\,|\tau_{n,k,i+1}-\tau_{n,k,i}|\end{aligned}$$

and hence in the subinterval $[\tau_{n,k},\bar\tau_{n,k}]$ we have

$$\begin{aligned}\int_{\tau_{n,k}}^{\bar\tau_{n,k}} \partial_v\mathcal{F}(t_{n,k},v_n(\tau))[v_n'(\tau)]\,d\tau &\le -\int_{\tau_{n,k}}^{\bar\tau_{n,k}} \|\partial_v\mathcal{F}(t_{n,k},v_n(\tau))\|\,d\tau \\ &\quad +2\,\mathcal{C}(\|v_{n,k+1}-v_{n,k}\|)\,|\bar\tau_{n,k}-\tau_{n,k}|.\end{aligned}$$

In the subinterval $[\bar\tau_{n,k},\tau_{n,k+1}]$ (if it is not a single point) we have by Proposition 4.1

$$\partial_v\mathcal{F}(t_{n,k},v_n(\tau))[v_n'(\tau)] = \|\partial_v\mathcal{F}(t_{n,k},v_{n,k+1})\| = 0.$$

Therefore in the whole interval $[\tau_{n,k},\tau_{n,k+1}]$ we can write

$$\begin{aligned}\int_{\tau_{n,k}}^{\tau_{n,k+1}} \partial_v\mathcal{F}(t_{n,k},v_n(\tau))[v_n'(\tau)]\,d\tau &\le -\int_{\tau_{n,k}}^{\tau_{n,k+1}} \|\partial_v\mathcal{F}(t_{n,k},v_n(\tau))\|\,d\tau \\ &\quad +2\,\mathcal{C}(\|v_{n,k+1}-v_{n,k}\|)\,\Delta\tau_n.\end{aligned}$$

Finally, again by (6)

$$\begin{aligned}\int_{\tau_{n,k}}^{\tau_{n,k+1}} \partial_t\mathcal{F}(t_n(\tau),v_{n,k+1})\,t_n'(\tau)\,d\tau &\le \int_{\tau_{n,k}}^{\tau_{n,k+1}} \partial_t\mathcal{F}(t_n(\tau),v_n(\tau))\,t_n'(\tau)\,d\tau \\ &\quad +\mathcal{C}(\|v_{n,k+1}-v_{n,k}\|)\,\Delta\tau_n,\end{aligned}$$

which concludes the proof. ■

4.2 Compactness and convergence

Arguing exactly as in §3.2 we can prove the next results; this is possible thanks to the fact that the proof of Proposition 3.3, Theorem 3.4 and Corollary 3.5 depend only on Proposition 3.1 and 3.2.

Proposition 4.3 *Let $(t_n, v_n) : [0, T_n) \to [t_0, t_1] \times \mathcal{V}$ be given by §4.1. Let $0 < T < \liminf_n T_n$. There exists a subsequence (not relabelled) such that $t_n \overset{*}{\rightharpoonup} t$ in $W^{1,\infty}(0,T)$ and $v_n \overset{*}{\rightharpoonup} v$ in $W^{1,\infty}(0,T;\mathcal{V})$. In particular $t_n(\tau_n) \to t(\tau)$ and $v_n(\tau_n) \rightharpoonup v(\tau)$ in $\mathcal{V}$ if $\tau_n \to \tau$. Moreover $0 \le t' \le 1$ and $\|v'\| \le 1$ a.e. in $(0,T)$.*

Theorem 4.4 *Let (t_n, v_n) and (t, v) be as in Proposition 4.3; then (t, v) is (a parametrization of) a quasi-static evolution, i.e. it satisfies $(t(0), v(0)) = (t_0, v_0)$, $t' \ge 0$, $\|v'\| \le 1$. Moreover*

(S') for every $\tau > 0$ with $t'(\tau) > 0$ it holds

$$\|\partial_v \mathcal{F}(t(\tau), v(\tau))\| = 0,$$

(E') for every τ it holds

$$\mathcal{F}(t(\tau), v(\tau)) = \mathcal{F}(t_0, v_0) - \int_0^\tau \|\partial_v \mathcal{F}(t(s), v(s))\| \, ds + \int_0^\tau \partial_t \mathcal{F}(t(s), v(s)) \, t'(s) \, ds \, .$$

Corollary 4.5 *$\mathcal{F}(t_n(\tau), v_n(\tau)) \to \mathcal{F}(t(\tau), v(\tau))$ for every $\tau \in [0, T]$.*

5 Proof of the approximation result

In this section we will prove Theorem 2.3. Let $(t_h, v_h) : [0, T) \to [t_0, t_1] \times \mathcal{V}$ be a Lipschitz map with $(t_h(0), v_h(0)) = (t_0, v_{h,0})$, $0 \le t'_h \le 1$ and $\|v'_h\| \le 1$, and such that

(S'_h) for every τ with $t'_h(\tau) > 0$ it holds

$$\|\partial_v \mathcal{F}_h(t_h(\tau), v_h(\tau))\|_h = 0, \tag{29}$$

(E'_h) for every τ it holds

$$\mathcal{F}_h(t_h(\tau), v_h(\tau)) = \mathcal{F}_h(t_0, v_0) - \int_0^\tau \|\partial_v \mathcal{F}_h(t_h(s), v_h(s))\|_h \, ds + \int_0^\tau \partial_t \mathcal{F}_h(t_h(s), v_h(s)) \, t'_h(s) \, ds. \tag{30}$$

The "sequence" (t_h, v_h) is weakly compact in $W^{1,\infty}(0,T) \times W^{1,\infty}(0,T;\mathcal{V})$, therefore by the arguments of Proposition 3.3 there exists a subsequence (not relabelled) such that $t_h \overset{*}{\rightharpoonup} t$ in $W^{1,\infty}(0,T)$ and $v_h \overset{*}{\rightharpoonup} v$ in $W^{1,\infty}(0,T;\mathcal{V})$, $t_h(\tau_h) \to t(\tau)$ and $v_h(\tau_h) \rightharpoonup v(\tau)$ if $\tau_h \to \tau$, $0 \le t' \le 1$ and $\|v'\| \le 1$ a.e. in $(0,T)$. In particular $t_h(\tau) \to t(\tau)$, $v_h(\tau) \rightharpoonup v(\tau)$ for every $\tau \in [0,T)$.

Now, to check that the limit (t, v) satisfies conditions (S') and (E') we follow the same arguments used in the proof of Theorem 3.4, roughly speaking replacing (t_n, v_n) with (t_h, v_h) and conditions (3)-(5) with (7)-(9).

Let τ with $t'(\tau) > 0$. Since t_h converge to t pointwise there exists a sequence $\tau_h \to \tau$ such that $t'_h(\tau_h) > 0$ and thus $\|\partial_v \mathcal{F}_h(t_h(\tau_h), v(\tau_h))\|_h = 0$ by (S'_h). As $\tau_h \to \tau$ we also have $v_h(\tau_h) \rightharpoonup v(\tau)$ in $\mathcal{V}$. Therefore, thanks to (8) we get

$$\|\partial_v \mathcal{F}(t(\tau), v(\tau))\| \le \liminf_h \|\partial_v \mathcal{F}(t_h(\tau_h), v(\tau_h))\|_h = 0.$$

It remains to show that (E') follows from (E'_h). Taking the limsup in (30) yields

$$\limsup_h \mathcal{F}_h(t_h(\tau), v_h(\tau)) \le \lim_h \mathcal{F}_h(t_0, v_{h,0}) - \liminf_h \int_0^\tau \|\partial_v \mathcal{F}_h(t_h(s), v_h(s))\| \, ds$$
$$+ \limsup_h \int_0^\tau \partial_t \mathcal{F}_h(t_h(s), v_h(s)) \, t'_h(s) \, ds.$$

By (10) we known that

$$\lim_h \mathcal{F}_h(t_0, v_{h,0}) = \mathcal{F}(t_0, v_0).$$

As $t_h(s) \to t(s)$ and $v_h(s) \rightharpoonup v(s)$ by (8) we have $\|\partial_v \mathcal{F}(t(s), v(s))\| \le \liminf_h \|\partial_v \mathcal{F}_h(t_h(s), v_h(s))\|_h$ and then by Fatou's Lemma

$$\int_0^\tau \|\partial_v \mathcal{F}(t(s), v(s))\| \, ds \le \liminf_h \int_0^\tau \|\partial_v \mathcal{F}_h(t_h(s), v_h(s))\|_h \, ds.$$

By (9) we have $\partial_t \mathcal{F}(t(s), v(s)) = \lim_h \partial_t \mathcal{F}_h(t_h(s), v_h(s)) \le C$ where (by assumption) the upper bound C is uniform. By dominated convergence it follows that $\partial_t \mathcal{F}_h(t_h(\cdot), v_h(\cdot)) \to \partial_t \mathcal{F}(t(\cdot), v(\cdot))$ strongly in $L^1(0,T)$. Since $t'_h \overset{*}{\rightharpoonup} t'$ in $L^\infty(0,T)$ we get

$$\lim_h \int_0^\tau \partial_t \mathcal{F}_h(t_h(s), v_h(s)) \, t'_h(s) \, ds = \int_0^\tau \partial_t \mathcal{F}(t(s), v(s)) \, t'(s) \, ds.$$

In conclusion,

$$\limsup_h \mathcal{F}_h(t_h(\tau), v_h(\tau)) \le \mathcal{F}(t_0, v_0) - \int_0^\tau \|\partial_v \mathcal{F}(t(s), v(s))\| \, ds + \int_0^\tau \partial_t \mathcal{F}(t(s), v(s)) \, t'(s) \, ds. \quad (31)$$

Since $\|v'\| \le 1$ by the chain rule we deduce again that

$$\mathcal{F}(t_0, v_0) - \int_0^\tau \|\partial_v \mathcal{F}(t(s), v(s))\| \, ds + \int_0^\tau \partial_t \mathcal{F}(t(s), v(s)) \, t'(s) \, ds \le \mathcal{F}(t(\tau), v(\tau))$$

and thus $\limsup_h \mathcal{F}_h(t_h(\tau), v_h(\tau)) \le \mathcal{F}(t(\tau), v(\tau))$. The liminf inequality

$$\mathcal{F}(t(\tau), v(\tau)) \le \liminf_h \mathcal{F}_h(t_h(\tau), v_h(\tau))$$

is provided by (7). Therefore

$$\limsup_h \mathcal{F}_h(t_h(\tau), v_h(\tau)) = \lim_h \mathcal{F}_h(t_h(\tau), v_h(\tau)) = \mathcal{F}(t(\tau), v(\tau)).$$

As a consequence from (31) we get (E'). Note also that, in the language of Γ-convergence, $(t_h(\tau), v_h(\tau))$ is a recovery sequence for every $\tau \in [0,T)$.

6 Existence in a weaker norm

In this section we will prove the existence result stated in Theorem 2.5. We will not follow the proofs of the previous existence Theorems; we will use instead a "Galerkin proof" approximating the evolution in finite dimensional spaces. To this end, let $\mathcal{V}_h$ be a monotone sequence of finite dimensional subspaces of $\mathcal{V}$ with $\cup_h \mathcal{V}_h$ dense in $\mathcal{V}$. Let $\mathcal{F}_h$ be the restriction to $\mathcal{V}_h$ of the energy functional $\mathcal{F}$. Being $\mathcal{V}_h \subset \mathcal{V} \subset \mathcal{W}$ and being $\mathcal{V}_h$ finite dimensional, the norms $\|\cdot\|_{\mathcal{V}}$ and $\|\cdot\|_{\mathcal{W}}$ are equivalent in $\mathcal{V}_h$ as it is for the weak and strong topology.

First of all let us prove the existence of the discrete evolutions. As $\mathcal{F}_h = \mathcal{F}$ in $\mathcal{V}_h$ it turns out that the energy functionals $\mathcal{F}_h : [t_0, t_1] \times \mathcal{V}_h \to [0, +\infty)$ are of class C^1 with respect to $\| \cdot \|_{\mathcal{V}}$ and thus with respect to $\| \cdot \|_{\mathcal{W}}$. For $v \in \mathcal{V}_h$ denote by $\nabla_{\mathcal{V}} \mathcal{F}_h(t, v)$ and by $\nabla_{\mathcal{W}} \mathcal{F}_h(t, v)$ the gradients in the first and second norm respectively. Clearly, for $v \in \mathcal{V}_h$

$$\|\nabla_{\mathcal{W}} \mathcal{F}_h(t, v)\| = \max\{\partial_v \mathcal{F}_h(t, v)[\phi] : \phi \in \mathcal{V}_h,\ \|\phi\|_{\mathcal{W}} \le 1\}.$$

Since $\mathcal{F}_h$ is of class C^1 in $[t_0, t_1] \times \mathcal{V}_h$ it is easy to check that

$$\mathcal{F}_h(t, v) \le \liminf_m \mathcal{F}_h(t_m, v_m) \qquad \text{for } t_m \to t \text{ and } v_m \to v, \tag{32}$$

$$\|\nabla_{\mathcal{W}} \mathcal{F}_h(t, v)\| \le \liminf_m \|\nabla_{\mathcal{W}} \mathcal{F}_h(t_m, v_m)\| \qquad \text{for } t_m \to t \text{ and } v_m \to v, \tag{33}$$

$$\partial_t \mathcal{F}_h(t, v) = \lim_m \partial_t \mathcal{F}_h(t_m, v_m) \qquad \text{for } t_m \to t \text{ and } v_m \to v. \tag{34}$$

Note that in the finite dimensional setting weak and strong convergence coincide. Now, let us check that there exists a modulus of continuity $\mathcal{C}_h$ such that in $[t_0, t_1] \times \mathcal{V}_h$ we have

$$\|\nabla_{\mathcal{W}} \mathcal{F}_h(t, v) - \nabla_{\mathcal{W}} \mathcal{F}_h(t, w)\| + |\partial_t \mathcal{F}_h(t, v) - \partial_t \mathcal{F}_h(t, w)| \le \mathcal{C}_h(\|v - w\|_{\mathcal{W}}). \tag{35}$$

Note that, being $\mathcal{F}_h = \mathcal{F}$ on $\mathcal{V}_h$, for $v \in \mathcal{V}_h$ we have

$$\|\partial_v \mathcal{F}(t, v)\| = \max\{\partial_v \mathcal{F}(t, v)[\phi] : \phi \in \mathcal{V},\ \|\phi\|_{\mathcal{V}} \le 1\} \ge \|\nabla_{\mathcal{V}} \mathcal{F}_h(t, v)\|,$$

and then, by equivalence of norms, we can write

$$C_h \|\nabla_{\mathcal{W}} \mathcal{F}_h(t, v)\| \le \|\nabla_{\mathcal{V}} \mathcal{F}_h(t, v)\| \le \|\partial_v \mathcal{F}(t, v)\|\,.$$

Therefore, again by by the equivalence of norms, from

$$\|\partial_v \mathcal{F}(t, v) - \partial_v \mathcal{F}(t, w)\| + |\partial_t \mathcal{F}(t, v) - \partial_t \mathcal{F}(t, w)| \le \mathcal{C}(\|v - w\|_{\mathcal{V}})$$

we get (35). Now, let $v_{h,0} \in \mathcal{V}_h$ with $v_{h,0} \to v_0$ in $\mathcal{V}$. Thank to (32)-(35) we can invoke Theorem 2.2 which provides the existence of a quasi-static evolution: there exists a map $(t_h, v_h) : [0, T_h] \to [t_0, t_1] \times \mathcal{V}_h$ with $(t_h(0), v_h(0)) = (t_0, v_{h,0})$, $0 \le t'_h \le 1$ and $\|v'_h\|_{\mathcal{W}} \le 1$ and such that

(S'_h) for every τ with $t'_h(\tau) > 0$

$$\|\nabla_{\mathcal{W}} \mathcal{F}_h(t_h(\tau), v_h(\tau))\| = 0, \tag{36}$$

(E'_h) for every τ

$$\begin{aligned} \mathcal{F}_h(t_h(\tau), v_h(\tau)) = \mathcal{F}_h(t_0, v_{i,0}) &- \int_0^\tau \|\nabla_{\mathcal{W}} \mathcal{F}_h(t_h(s), v_h(s))\|\, ds \\ &+ \int_0^\tau \partial_t \mathcal{F}_h(t_h(s), v_h(s))\, t'_h(s)\, ds. \end{aligned} \tag{37}$$

Next, let us prove compactness. As already observed in Proposition 3.3, we have $T_h \ge (t_1 - t_0)$. Let $0 < T < \liminf_h T_h$. Since t_h is bounded in $W^{1,\infty}(0,T)$ and v_h is bounded in $W^{1,\infty}(0,T;\mathcal{W})$ it follows (by the arguments of Proposition 3.3) that there exists a subsequence (not relabelled) such that $t_h \overset{*}{\rightharpoonup} t$ in $W^{1,\infty}(0,T)$ and $v_h \overset{*}{\rightharpoonup} v$ in $W^{1,\infty}(0,T;\mathcal{W})$. Moreover, for every $\tau \in [0,T]$ we have $t_h(\tau) \to t(\tau)$ and $v_h(\tau) \rightharpoonup v(\tau)$ in $\mathcal{W}$. Clearly $0 \le t' \le 1$ and $\|v'\|_{\mathcal{W}} \le 1$ a.e. in $(0,T)$. Now, let us see that $v_h(\tau) \rightharpoonup v(\tau)$ in the weak topology of $\mathcal{V}$. By the chain rule, for every $\tau_1 \le \tau_2$ we can write

$$\mathcal{F}_h(t_h(\tau_2), v_h(\tau_2)) = \mathcal{F}_h(t(\tau_1), v(\tau_1)) + \int_{\tau_1}^{\tau_2} \mathcal{F}'_h(t_h(\tau), v_h(\tau))\, d\tau$$

where $\mathcal{F}_h'$ denotes the (total) derivative with respect to τ. At the same time by (E_h') we have

$$\mathcal{F}_h(t_h(\tau_2), v_h(\tau_2)) \le \mathcal{F}_h(t(\tau_1), v(\tau_1)) + \int_{\tau_1}^{\tau_2} \partial_t \mathcal{F}_h(t_h(\tau), v_h(\tau))\, t_h'(\tau)\, d\tau.$$

Since $t_h' \le 1$ and by (17) it follows that for a.e. τ it holds

$$\mathcal{F}_h'(t_h(\tau), v_h(\tau)) \le \partial_t \mathcal{F}_h(t_h(\tau), v_h(\tau))\, t_h'(\tau) \le A\mathcal{F}_h(t_h(\tau), v_h(\tau)) + B \tag{38}$$

(for A, B independent of i). By Gronwall Lemma

$$\mathcal{F}(t_h(\tau), v_h(\tau)) = \mathcal{F}_h(t_h(\tau), v_h(\tau)) \le C(\mathcal{F}_h(t_0, v_{i,0}) + 1)\, e^{A\tau} \le C'.$$

and hence $\mathcal{F}_h(t_h(\tau), v_h(\tau))$ is bounded, uniformly with respect to $\tau \in [0,T]$ and $i \in \mathbb{N}$. By the coercivity of $\mathcal{F}(t,\cdot)$ it follows that $\|v_h(\tau)\|_{\mathcal{V}}$ is bounded. We already known that for every τ we have $v_h(\tau) \rightharpoonup v(\tau)$ in the weak topology of $\mathcal{W}$. Since $v_h(\tau)$ is bounded in $\mathcal{V}$, which is reflexive, for every subsequence $v_{h_k}(\tau)$ there exists a further subsequence (not relabelled) such that $v_{h_k}(\tau) \rightharpoonup z$ in the weak topology of $\mathcal{V}$. Since $\mathcal{V}$ is continuously embedded in $\mathcal{W}$ it follows that $v_{h_k}(\tau) \rightharpoonup z$ in the weak topology of $\mathcal{W}$ and thus $z = v(\tau)$. As a consequence the whole sequence $v_h(\tau) \rightharpoonup v(\tau)$ in $\mathcal{V}$.

Finally, let us see the convergence of (S_h') and (E_h'). In order to pass to the limit in (36) it is sufficient to show that for every τ

$$\|\nabla_{\mathcal{W}} \mathcal{F}(t(\tau), v(\tau))\| \le \liminf_h \|\nabla_{\mathcal{W}} \mathcal{F}_h(t_h(\tau), v_h(\tau))\|.$$

For $h \ge h'$ and $\phi_{h'} \in \mathcal{V}_{h'}$ with $\|\phi_{h'}\|_{\mathcal{W}} \le 1$ we can write

$$\begin{aligned}\|\nabla_{\mathcal{W}} \mathcal{F}_h(t_h(\tau), v_h(\tau))\| &= \max\{\partial_v \mathcal{F}(t_h(\tau), v_h(\tau))[\phi] : \phi \in \mathcal{V}_h,\ \|\phi\|_{\mathcal{W}} \le 1\} \\ &\ge \partial_v \mathcal{F}(t_h(\tau), v_h(\tau))[\phi_{h'}].\end{aligned}$$

Since $v_h(\tau) \rightharpoonup v(\tau)$ in $\mathcal{V}$ we can use (14) to get

$$\liminf_h \|\nabla_{\mathcal{W}} \mathcal{F}_h(t_h(\tau), v_h(\tau))\| \ge \partial_v \mathcal{F}(t(\tau), v(\tau))[\phi_{h'}] \qquad \text{for every } \phi_{h'} \in \mathcal{V}_{h'}.$$

Since $\cup_{h'} \mathcal{V}_{h'}$ is dense in $\mathcal{V}$ it is enough to take the supremum with respect to $\phi_{h'} \in \mathcal{V}_{h'}$ with $\|\phi_{h'}\|_{\mathcal{W}} \le 1$. Note that $\partial_v \mathcal{F}(t_h(\tau), v_h(\tau))[\phi_{h'}] = \partial_v \mathcal{F}_h(t_h(\tau), v_h(\tau))[\phi_{h'}]$ is measurable, therefore its pointwise limit $\partial_v \mathcal{F}(t(\tau), v(\tau))[\phi_{h'}]$ and then the supremum $\|\nabla_{\mathcal{W}} \mathcal{F}(t(\tau), v(\tau))\|$ are measurable, and actually integrable.

Following the arguments of §5, to pass to the limit in (37) it is sufficient to show that

$$\mathcal{F}(t(\tau), v(\tau)) \le \liminf_h \mathcal{F}_h(t_h(\tau), v_h(\tau)), \tag{39}$$

$$\limsup_h \mathcal{F}_h(t_0, v_{i,0}) \le \mathcal{F}(t_0, v_0), \tag{40}$$

$$\lim_h \partial_t \mathcal{F}_h(t_h(\cdot), v_h(\cdot)) = \partial_t \mathcal{F}(t(\cdot), v(\cdot)) \text{ in } L^1(0,T). \tag{41}$$

Remembering that $\mathcal{F}_h$ is just the restriction of $\mathcal{F}$ on $\mathcal{V}_h$ the first and the third condition follow respectively from (13) and (15) together with the uniform bound (38). The second is instead a direct consequence of the fact that $v_{i,0} \to v_0$ strongly in $\mathcal{V}$ and thus the initial datum is well prepared.

7 A phase field model for brittle fracture

7.1 Energy

Let Ω be an open, bounded Lipschitz set in $\mathbb{R}^2$. Let $\partial_D\Omega \subset \partial\Omega$ with $\mathcal{H}^1(\partial_D\Omega) > 0$. For $p > 2$ let $g \in W^{1,p}(\Omega, \mathbb{R}^2)$ and let the space of admissible displacement be

$$\mathcal{U} = \{u \in H^1(\Omega, \mathbb{R}^2) : u = g \text{ in } \partial_D\Omega\}.$$

For the moment we assume that the phase-field variable v belongs to the space $\mathcal{V} = H^1(\Omega)$ and in particular we do not impose the bounds $0 \leq v \leq 1$. For $\varepsilon > 0$ and $0 < \eta_\varepsilon = o(\varepsilon)$ the energy functional $F_\varepsilon(t, \cdot, \cdot) : \mathcal{U} \times \mathcal{V} \to [0, +\infty)$ is the Ambrosio-Tortorelli energy [1] for linear elasticity:

$$F_\varepsilon(t, u, v) = t^2 \int_\Omega (v^2 + \eta_\varepsilon) W(Du)\, dx + G_c \int_\Omega (v-1)^2/4\varepsilon + \varepsilon|\nabla v|^2\, dx,$$

where $W(Du) = Du : \mathbf{C}Du/2$ is the linear elastic energy density and G_c is the fracture toughness. Thank to the linearity of the density W the energy $F_\varepsilon(t, \cdot, \cdot)$ is associated to the proportional boundary condition $u(t) = tg$.

It is convenient to introduce also a notation for the elastic energy and the dissipation potential, respectively

$$E_\varepsilon(t, u, v) = t^2 \int_\Omega (v^2 + \eta_\varepsilon) W(Du)\, dx,$$

$$\mathcal{D}_\varepsilon(v) = G_c \int_\Omega (v-1)^2/4\varepsilon + \varepsilon|\nabla v|^2\, dx.$$

Since we are concerned with quasi-static evolutions, we can "condense" the energy considering only the displacement at equilibrium, to this end let $u(v)$ be the unique minimizer of $E_\varepsilon(t, \cdot, v)$ and denote

$$\mathcal{E}_\varepsilon(t, v) = E_\varepsilon(t, u(v), v) = \min\{E_\varepsilon(t, u, v) : u \in \mathcal{U}\},$$

$$\mathcal{F}_\varepsilon(t, v) = \mathcal{E}_\varepsilon(t, v) + \mathcal{D}_\varepsilon(v) = \min\{F_\varepsilon(t, u, v) : u \in \mathcal{U}\}.$$

Finally, note that, if $v(t)$ is a trajectory, the dissipation (rate of dissipated energy) is given by

$$d_t\mathcal{D}_\varepsilon(v(t)) = d_v\mathcal{D}(v(t))[\dot{v}(t)] = 2G_c \int_\Omega (v(t)-1)\dot{v}(t)/4\varepsilon + \varepsilon\nabla v(t) \cdot \nabla\dot{v}(t)\, dx.$$

In particular the dissipation depends on the state v and is linear with respect to $\dot{v}$ (the latter is indeed always the case when there exists a dissipation potential).

In the evolution, the irreversibility of the crack is given by the monotonicity constraint $v(t_2) \leq v(t_1)$ if $t_2 \geq t_1$. Hence, given $v \in \mathcal{V}$ the set of admissible variations is the cone

$$\Phi = \{v \in \mathcal{V} : v \leq 0\}.$$

In analogy with the un-constrained case let us define

$$-|\partial_v\mathcal{F}_\varepsilon(t, v)| = \min\{\partial_v\mathcal{F}_\varepsilon(t, v)[\phi] : \phi \in \Phi, \|\phi\| \leq 1\}.$$

Since $\phi = 0$ is an admissible variation we always have $-|\partial_v\mathcal{F}_\varepsilon(t, v)| \leq 0$ (which is the reason for the minus sign). Note also that Φ is weakly closed, and that it is not restrictive to choose $\|\phi\| = 1$ in the definition of the slope.

The energy $\mathcal{F}_\varepsilon$ and its derivatives have been studied in detail in the recent work [10], the interested reader will find there the proof of the following Lemmas which provides the properties, corresponding to (3)-(5), employed in the existence result.

Lemma 7.1 *The energy functional $\mathcal{F}_\varepsilon$ is of class C^1 with*

$$\partial_t \mathcal{F}_\varepsilon(t,v) = 2t \int_\Omega (v^2+\eta_\varepsilon)\, Du(v) : \mathbf{C} Dg\, dx,$$

$$\partial_v \mathcal{F}_\varepsilon(t,v)\,[\phi] = 2\int_\Omega v\phi\, W(Du(v))\,dx + 2\,G_c \int_\Omega (v-1)\phi/4\varepsilon + \varepsilon \nabla v\cdot\nabla\phi\,dx.$$

Proof. See Lemma 2.7 in [10]. ■

Lemma 7.2 *The functional $\mathcal{F}_\varepsilon : [t_0,t_1]\times\mathcal{V} \to [0,+\infty)$ is of class C^1 and*

$$\mathcal{F}_\varepsilon(t,v) \le \liminf_m \mathcal{F}_\varepsilon(t_m,v_m) \qquad \text{for } t_m \to t \text{ and } v_m \rightharpoonup v. \tag{42}$$

There exists a constant C such that

$$|\partial_v\mathcal{F}_\varepsilon(t,v)[\phi] - \partial_v\mathcal{F}_\varepsilon(t,w)[\phi]| + |\partial_t\mathcal{F}_\varepsilon(t,v) - \partial_t\mathcal{F}_\varepsilon(t,w)| \le C\|v-w\| \tag{43}$$

for every $t\in[t_0,t_1]$ and for every $\phi\in\Phi$ with $\|\phi\|\le 1$. Moreover,

$$|\partial_v\mathcal{F}_\varepsilon(t,v)| \le \liminf_m |\partial_v\mathcal{F}_\varepsilon(t_m,v_m)| \qquad \text{for } t_m\to t \text{ and } v_m \rightharpoonup v \tag{44}$$

and

$$\partial_t\mathcal{F}_\varepsilon(t,v) = \lim_m \partial_t\mathcal{F}_\varepsilon(t_m,v_m) \qquad \text{for } t_m\to t \text{ and } v_m\rightharpoonup v. \tag{45}$$

Proof. For (42), (44) and (45) see Corollary 2.9 in [10]. For the Lipschitz continuity of $\partial_t\mathcal{F}_\varepsilon$ and $\partial_v\mathcal{E}_\varepsilon$ see respectively (2.28) and (2.36) in [10] while the continuity of $\partial_v\mathcal{D}_\varepsilon$ is standard. ■

7.2 Evolution in the H^1-norm

We use the implicit scheme of §3. Let $\Delta\tau_n \to 0^+$. Given the initial conditions $t_{n,0}=t_0$ and $v_{n,0}=v_0\le 1$ and known $t_{n,k}<t_1$ and $v_{n,k}$, the updates $v_{n,k+1}$ and $t_{n,k+1}$ are defined by

$$\begin{cases} v_{n,k+1} \in \operatorname{argmin}\{\mathcal{F}_\varepsilon(t_{n,k},v) : v\in\mathcal{V},\ v\le v_{n,k},\ \|v-v_{n,k}\|\le\Delta\tau_n\}, \\ t_{n,k+1} = t_{n,k} + \big(\Delta\tau_n - \|v_{n,k+1}-v_{n,k}\|\big). \end{cases} \tag{46}$$

As in §3 let $\bar{k}_n = \sup\{k : t_{n,k}<t_1\}$; let $\tau_{n,k}=k\Delta\tau_n$ for $0\le k\le\bar{k}_n$ and let $T_n=\bar{k}_n\Delta\tau_n \ge (t_1-t_0)$. Next, we define the affine interpolations $(t_n,v_n):[0,T_n)\to[t_0,t_1]\times\mathcal{V}$ which are Lipschitz continuous and satisfy $t_n'\ge 0$ and $t_n'+\|v_n'\|=1$ a.e. in $(0,T_n)$.

Proposition 7.3 *If $t_{n,k+1}>t_{n,k}$ then $v_{n,k+1}$ satisfies the equilibrium condition*

$$|\partial_v\mathcal{F}_\varepsilon(t(\tau),v(\tau))| = 0. \tag{47}$$

Proof. Since $\|v_{n,k+1}-v_{n,k}\|<\Delta\tau_n$, the minimality of $v_{n,k+1}$ implies that

$$\lim_{h\to 0^+} \frac{\mathcal{F}_\varepsilon(t,v_{n,k+1}+h\phi) - \mathcal{F}_\varepsilon(t,v_{n,k+1})}{h} = 0,$$

which gives $\partial_v\mathcal{F}_\varepsilon(t_{n,k},v_{n,k+1})\,[\phi]=0$ for every $\phi\in\Phi$. ■

Proposition 7.4 *The following incremental energy estimate holds*

$$\mathcal{F}_\varepsilon(t_{n,k+1}, v_{n,k+1}) \le \mathcal{F}_\varepsilon(t_{n,k}, v_{n,k}) - \int_{\tau_{n,k}}^{\tau_{n,k+1}} |\partial_v \mathcal{F}_\varepsilon(t_{n,k}, v_n(\tau))| \, d\tau + \int_{\tau_{n,k}}^{\tau_{n,k+1}} \partial_t \mathcal{F}_\varepsilon(t_n(\tau), v_n(\tau)) \, t_n'(\tau) \, d\tau + 3C \|v_{n,k} - v_{n,k+1}\| \, \Delta\tau_n, \qquad (48)$$

where C is the Lipschitz constant appearing in (43).

Proof. We can argue exactly as in the proof of Theorem 3.2, replacing the norm $\|\partial \mathcal{F}_\varepsilon(t,v)\|$ with the slope $|\partial_v \mathcal{F}_\varepsilon(t,v)|$ and using (43). ∎

Proposition 7.5 *Let $(t_n, v_n) : [0, T_n) \to [t_0, t_1] \times \mathcal{V}$ be given as above. Let $0 < T < \liminf_n T_n$. There exists a subsequence (not relabelled) such that $t_n \overset{*}{\rightharpoonup} t$ in $W^{1,\infty}(0,T)$ and $v_n \overset{*}{\rightharpoonup} v$ in $W^{1,\infty}(0,T;\mathcal{V})$. In particular $t_n(\tau_n) \to t(\tau)$ and $v_n(\tau_n) \rightharpoonup v(\tau)$ in $\mathcal{V}$ if $\tau_n \to \tau$. Moreover $0 \le t' \le 1$, $v' \le 0$ and $\|v'\| \le 1$ a.e. in $(0,T)$.*

Proof. It is sufficient to follow the proof of Proposition 3.3. ∎

Theorem 7.6 *There exists (a parametrization of) an evolution $(t,v) : [0,T] \to [t_0, t_1] \times \mathcal{V}$ such that $(t(0), v(0)) = (t_0, v_0)$, $t' \ge 0$ and $\|v'\| \le 1$, $v' \le 0$; moreover*

(S') for every τ with $t'(\tau) > 0$ it holds

$$|\partial_v \mathcal{F}_\varepsilon(t(\tau), v(\tau))| = 0, \qquad (49)$$

(E') for every τ it holds

$$\mathcal{F}_\varepsilon(t(\tau), v(\tau)) = \mathcal{F}_\varepsilon(t_0, v_0) - \int_0^\tau |\partial_v \mathcal{F}_\varepsilon(t(s), v(s))| \, ds + \int_0^\tau \partial_t \mathcal{F}_\varepsilon(t(s), v(s)) \, t'(s) \, ds \,. \qquad (50)$$

Proof. In order to prove (S') and (E') it is sufficient again to follow step by step the proof of Theorem 3.4, replacing $\|\partial_v \mathcal{F}_\varepsilon(t,v)\|$ with the slope $|\partial_v \mathcal{F}_\varepsilon(t,v)|$ and using (44) and (45). ∎

As a by-product we get also the convergence of the energies and then the strong convergence of the phase field variable.

Corollary 7.7 *$\mathcal{F}_\varepsilon(t_n(\tau), v_n(\tau)) \to \mathcal{F}_\varepsilon(t(\tau), v(\tau))$ and then $v_n(\tau) \to v(\tau)$ strongly in $H^1(\Omega)$*

Proof. The convergence of the energy follows from Corollary 3.5. For the strong convergence of the phase field variable it is instead enough to observe that $\mathcal{E}_\varepsilon(t_n(\tau), v_n(\tau)) \to \mathcal{E}_\varepsilon(t(\tau), v(\tau))$ if $t_n(\tau) \to t(\tau)$ and $v_n(\tau) \rightharpoonup v(\tau)$. Since $\mathcal{F}_\varepsilon(t_n(\tau), v_n(\tau)) \to \mathcal{F}_\varepsilon(t(\tau), v(\tau))$ it follows that $\mathcal{D}_\varepsilon(v_n(\tau)) \to \mathcal{D}_\varepsilon(v(\tau))$ from which the strong convergence of $v_n(\tau)$ ∎

7.3 Evolution in the L^2-norm

To prove the existence of an evolution with respect to the L^2-norm we will follow the proof of Theorem 2.5. With the properties listed in §7.1 it is easy to check that he hypotheses of Theorem 2.5 are fulfilled, in particular (14) holds, i.e.

$$\partial_v \mathcal{F}_\varepsilon(t,v)[\phi] = \lim_m \partial_v \mathcal{F}_\varepsilon(t_m, v_m)[\phi] \qquad \text{for } t_m \to t,\ v_m \rightharpoonup v \text{ in } \mathcal{V} \text{ and } \phi \in \mathcal{V}. \tag{51}$$

It remains to take into account the irreversibility constraint, the phase field constraint $0 \le v \le 1$ and the boundary condition. It is actually enough to make few changes in the proof, contained in §6.

Let $\mathcal{V}_h$ be as in §6 and let $\mathcal{F}_{\varepsilon,h}$ be the restriction of $\mathcal{F}_\varepsilon$ to $\mathcal{V}_h$. For $v \in \mathcal{V}_h$ let us introduce the slope

$$|\nabla_{L^2} \mathcal{F}_{\varepsilon,h}(t,v)| = \max\{\partial_v \mathcal{F}_{\varepsilon,h}(t,v)[\phi] : \phi \in \mathcal{V}_h,\ \phi \le 0,\ \|\phi\|_{L^2} \le 1\}.$$

Using (51) we have, as in Lemma 7.2, that there exists a constant C_h such that

$$|\nabla_{L^2} \mathcal{F}_{\varepsilon,h}(t,v) - \nabla_{L^2} \mathcal{F}_{\varepsilon,h}(t,w)| + |\partial_t \mathcal{F}_\varepsilon(t,v) - \partial_t \mathcal{F}_\varepsilon(t,w)| \le C_h \|v-w\|_{L^2} \tag{52}$$

for every $t \in [t_0,t_1]$ and for every $v, w \in \mathcal{V}_h$. Moreover,

$$|\nabla_{L^2} \mathcal{F}_{\varepsilon,h}(t,v)| \le \liminf_m |\nabla_{L^2} \mathcal{F}_{\varepsilon,h}(t_m,v_m)| \qquad \text{for } t_m \to t \text{ and } v_m \to v. \tag{53}$$

As a consequence we can employ the minimizing movement to define a discrete evolution in each space $\mathcal{V}_h$: given $t_{h,n,0} = t_0$ and $0 \le v_{h,n,0} = v_{h,0} \le 1$ and known $t_{h,n,k} < t_1$ and $v_{h,n,k}$, the incremental problem for $v_{h,n,k+1}$ and $t_{h,n,k+1}$ is given by

$$\begin{cases} v_{h,n,k+1} \in \operatorname{argmin} \{\mathcal{F}_{\varepsilon,h}(t_{h,n,k}, v) : v \in \mathcal{V}_h,\ v \le v_{n,k},\ \|v - v_{h,n,k}\|_{L^2} \le \Delta\tau_n\}, \\ t_{h,n,k+1} = t_{h,n,k} + \big(\Delta\tau_n - \|v_{h,n,k+1} - v_{h,n,k}\|\big). \end{cases}$$

Let us see that $v_{h,n,k+1} \ge 0$ even if this constraint is not explicitly imposed in the incremental problem: by a simple truncation argument for every $v \in \mathcal{V}$ with $v \le w$ we have $\mathcal{F}_\varepsilon(t,v) \ge \mathcal{F}_\varepsilon(t,\bar v)$ and $\|w - v\|_{L^2} \ge \|w - \bar v\|_{L^2}$ for $\bar v = \max\{v, 0\}$ (it is interesting to note that in the H^1-norm in general it is not true that $\|w - v\|_{H^1} \ge \|w - \bar v\|_{H^1}$).

Then, using (52)-(53) and following step by step the previous section we obtain for every $\mathcal{V}_h$ a discrete evolution $(t_h, v_h) : [0,T] \to [t_0,t_1] \times \mathcal{V}_h$ such that $(t_h(0), v_h(0)) = (t_0, v_{h,0})$, $t'_h \ge 0$, $\|v'_h\| \le 1$, $v'_h \le 0$ and $0 \le v_h \le v_{h,0}$, moreover

(S'_h) for every τ with $t'_h(\tau) > 0$

$$|\nabla_{L^2} \mathcal{F}_{\varepsilon,h}(t_h(\tau), v_h(\tau))| = 0,$$

(E'_h) for every τ

$$\begin{aligned} \mathcal{F}_{\varepsilon,h}(t_h(\tau), v_h(\tau)) = \mathcal{F}_{\varepsilon,h}(t_0, v_{i,0}) &- \int_0^\tau |\nabla_{L^2} \mathcal{F}_{\varepsilon,h}(t_h(s), v_h(s))|\, ds \\ &+ \int_0^\tau \partial_t \mathcal{F}_{\varepsilon,h}(t_h(s), v_h(s))\, t'_h(s)\, ds. \end{aligned}$$

The final step consists in passing to the limit with respect to h. To this end we will follow the last part of §6. Compactness does not present any particular difficulty and provides (up to subsequences) a limit parametrization $(t,v) : [0,T] \to [t_0,t_1] \times \mathcal{V}$ such that $0 \le t' \le 1$, $v' \le 0$ and $\|v'\|_{L^2} \le 1$ a.e. in $(0,T)$. It is important to remark that (up to subsequences) $v_h(\tau) \rightharpoonup v(\tau)$

weakly in H^1 and thus strongly in L^2. As a consequence the limit evolution will still satisfy also the constraint $0 \leq v(\tau) \leq v_0$. Next, to pass to the limit in (S'_h) it is enough to check that

$$|\nabla_{L^2}\mathcal{F}_\varepsilon(t(\tau), v(\tau))| \leq \liminf_h |\nabla_{L^2}\mathcal{F}_{\varepsilon,h}(t_h(\tau), v_h(\tau))|.$$

Once again, it is enough to follow §6.

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