

# OPTIMAL POTENTIALS FOR SCHRÖDINGER OPERATORS

G. BUTTAZZO, A. GEROLIN, B. RUFFINI, AND B. VELICHKOV

ABSTRACT. We consider the Schrödinger operator  $-\Delta + V(x)$  on  $H_0^1(\Omega)$ , where  $\Omega$  is a given domain of  $\mathbb{R}^d$ . Our goal is to study some optimization problems where an optimal potential  $V \geq 0$  has to be determined in some suitable admissible classes and for some suitable optimization criteria, like the energy or the Dirichlet eigenvalues.

**Keywords:** Schrödinger operators, optimal potentials, spectral optimization, capacity.

**2010 Mathematics Subject Classification:** 49J45, 35J10, 49R05, 35P15, 35J05.

## 1. INTRODUCTION

In this paper we consider optimization problems of the form

$$\min \left\{ F(V) : V \in \mathcal{V} \right\}, \quad (1.1)$$

for functionals  $F$ , depending on the Schrödinger operator  $-\Delta + V(x)$  with *potential*  $V$  belonging to a prescribed admissible class  $\mathcal{V}$  of Lebesgue measurable functions on a set  $\Omega \subset \mathbb{R}^d$ , which is typically chosen to be a bounded open set or the entire space  $\Omega = \mathbb{R}^d$ . Problems of this type have been studied, for example, by Ashbaugh-Harrell [2], Egnell [15], Essen [16], Harrell [20], Talenti [24] and, more recently, by Carlen-Frank-Lieb [11]. We refer to the monography [22], and to the references therein, for a complete list of references and as a comprehensive guide to the known results about the problem.

In our framework we include very general cost functionals, as for example the following.

*Integral functionals.* Given a function  $f \in L^2(\Omega)$  we consider the solution  $u_V$  to the elliptic PDE

$$-\Delta u + Vu = f \text{ in } \Omega, \quad u \in H_0^1(\Omega).$$

The integral cost functionals we may consider are of the form

$$F(V) = \int_{\Omega} j(x, u_V(x), \nabla u_V(x)) dx,$$

where  $j$  is a suitable integrand that we assume convex in the gradient variable and bounded from below. One may take, for example,

$$j(x, s, z) \geq -a(x) - c|s|^2,$$

with  $a \in L^1(\Omega)$  and  $c$  smaller than the first Dirichlet eigenvalue of the Laplace operator  $-\Delta$  in  $\Omega$ . In particular, the energy  $\mathcal{E}_f(V)$  defined by

$$\mathcal{E}_f(V) = \inf \left\{ \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} V(x) u^2 - f(x) u \right) dx : u \in H_0^1(\Omega) \right\}, \quad (1.2)$$

belongs to this class since, integrating by parts its Euler-Lagrange equation, we have

$$\mathcal{E}_f(V) = -\frac{1}{2} \int_{\Omega} f(x) u_V dx,$$

which corresponds to the integral functional above with

$$j(x, s, z) = -\frac{1}{2}f(x)s.$$

*Spectral functionals.* For every admissible potential  $V \geq 0$  we consider the spectrum  $\Lambda(V)$  of the Schrödinger operator  $-\Delta + V(x)$  on  $H_0^1(\Omega)$ . If  $\Omega$  is bounded or has finite measure, or if the potential  $V$  satisfies some suitable integrability properties, then the operator  $-\Delta + V(x)$  has compact resolvent and so its spectrum  $\Lambda(V)$  is discrete:

$$\Lambda(V) = (\lambda_1(V), \lambda_2(V), \dots),$$

where  $\lambda_k(V)$  are the eigenvalues counted with their multiplicity. The spectral cost functionals we may consider are of the form

$$F(V) = \Phi(\Lambda(V)),$$

for suitable functions  $\Phi : \mathbb{R}^{\mathbb{N}} \rightarrow (-\infty, +\infty]$ . For instance, taking  $\Phi(\Lambda) = \lambda_k$  we obtain

$$F(V) = \lambda_k(V).$$

The class of admissible potentials  $\mathcal{V}$  we consider satisfies an integrability condition, namely

$$\mathcal{V} = \left\{ V : \Omega \rightarrow [0, +\infty] : V \text{ Lebesgue measurable, } \int_{\Omega} \Psi(V) dx \leq 1 \right\}. \quad (1.3)$$

for a suitable function  $\Psi : [0, +\infty] \rightarrow [0, +\infty]$ . It is worth remarking that the requirement  $V \geq 0$  is not, in general, necessary for the well-posedness of problem (1.1), but allowing  $V$  to change sign radically changes the conduct of the problem. An instance of optimization problem for sign-changing potentials can be found in the recent work [11], where the authors study a quantitative stability for the first eigenvalue of the Schrödinger operator. The integrability constraint in (1.3) naturally appears in the following cases.

*Approximation of optimal sets.* In the case of spectral and energy functionals  $F$  as above, the optimization problems related to the Schrödinger operators may be linked to the classical shape optimization theory<sup>1</sup> for problems of the form

$$\min \left\{ F(E) : E \subset \Omega, |E| \leq \text{constant} \right\}.$$

Indeed, if we set  $V_E = 0$  in  $E$  and  $V_E = +\infty$  outside of  $E$ , then the Schrödinger operator  $-\Delta + V_E$  corresponds to the Dirichlet-Laplacian on the set  $E$ . This observation suggests, by one side, that we can approach problem (1.1) by means of techniques developed in the study of more classical shape optimization problems and, on the other hand, that we can approximate the potential  $V_E$ , corresponding to an optimal set  $E$ , by means of potentials that solve (1.1) under suitable constraints. We will show in Section 5 that a good approximation is given by the family of constraints

$$\Psi(V) = e^{-\alpha V}.$$

*Ground states of semilinear equations.* If the cost functional  $F$  is of energy type, as  $F(V) = \lambda_1(V)$ , then the study of the optimization problem

$$\min \left\{ F(V) : V : \Omega \rightarrow [0, +\infty], \int_{\Omega} V^p dx = 1 \right\}$$

---

<sup>1</sup>For an introduction to the theory of shape optimization problems we refer to the papers [8], [9], [10] and to the books [4], [22] and [23]

naturally reduces to the one of ground states of the equation

$$-\Delta\psi + |\psi|^s\psi = \lambda\psi, \quad \psi \in H^1(\Omega) \cap L^{2+s}(\Omega). \quad (1.4)$$

The case  $p > 0$  corresponds to the superlinear case  $s > 0$ , while the case of negative exponent  $p < 0$  corresponds to the sublinear case  $s < 0$ . Indeed, the potential  $V(x) = |\psi(x)|^s$  satisfies an integrability condition inherited from the ground state  $\psi$ . In the superlinear case  $s > 0$ , we have  $V \in L^p(\Omega)$  with  $p = (s+2)/s$ , while in the sublinear case  $s \in (-1, 0)$  we get  $\int_{\Omega} V^{-p} dx < +\infty$  with  $p = -(s+2)/s$ .

The paper is organized as follows. In Section 2 we recall the concepts of capacity measures and  $\gamma$ -convergence together with their main properties. Then we prove some preliminary results which will be exploited in the subsequent sections.

In Section 3 we prove two general results concerning the existence of optimal potentials in a bounded domain  $\Omega \subset \mathbb{R}^d$ . In Theorem 3.1 we deal with constraints  $\mathcal{V}$  which are bounded subsets of  $L^p(\Omega)$ , while Theorem 3.4 deals with the case of admissible classes consisting of suitable subsets of capacity measures.

In Section 3 our assumptions allow to take  $F(V) = -\mathcal{E}_f(V)$  and thus the optimization problem becomes the maximization of  $\mathcal{E}_f$  under the constraint  $\int_{\Omega} V^p dx \leq 1$ . We prove that for  $p \geq 1$ , there exists an optimal potential for the problem

$$\max \left\{ \mathcal{E}_f(V) : \int_{\Omega} V^p dx \leq 1 \right\}.$$

The existence result is sharp in the sense that for  $p < 1$  the maximum cannot be achieved (see Remark 3.11). For the existence issue in the case of a bounded domain, we follow the ideas of Egnell [15], summarized in [22, Chapter 8]. The case  $p = 1$  is particularly interesting and we show that in this case the optimal potentials are of the form

$$V = \frac{f}{M} (\chi_{\omega_+} - \chi_{\omega_-}),$$

where  $\chi_U$  indicates the characteristic function of the set  $U$ ,  $f \in L^2(\Omega)$ ,  $M = \|u_V\|_{L^\infty(\Omega)}$ , and  $\omega_{\pm} = \{u = \pm M\}$ .

In Section 4 we deal with minimization problems of the form

$$\min \left\{ F(V) : \int_{\Omega} \Psi(V) dx \leq 1 \right\}, \quad (1.5)$$

and we prove existence for the problem (1.1) for a large class of functionals  $F$  and of constraints  $\Psi$ , including the particular cases

$$\Psi(s) = s^{-p} \quad \text{and} \quad \Psi(s) = e^{-\alpha s}.$$

These type of constraints are, as far as we know, new in the literature. In the case  $\Psi(s) = s^{-p}$  the equation reduces, as already pointed out, to the sublinear case of (1.4).

In some cases the Schrödinger operator  $-\Delta + V(x)$  is compact even if  $\Omega$  is not bounded (see for instance [5]). This allows to consider spectral optimization problems in unbounded domains as  $\Omega = \mathbb{R}^d$ . We deal with this case in Section 5, where we prove that for  $F = \mathcal{E}_f$  or  $F = \lambda_1$ , there exist solutions to problem (1.5) in  $\mathbb{R}^d$ , with  $\Psi(s) = s^{-p}$ . Moreover, we characterize the optimal potential  $V$  as an explicit function of the solution  $u$  to a quasi-linear PDE of the form (1.4). Thus the qualitative properties of  $u$  immediately translate into qualitative properties for  $V$ . Thanks to this, we prove that, in the case  $F = \mathcal{E}_f$ ,  $1/V$  is compactly supported, provided  $f$  is compactly supported. In the case  $F = \lambda_1$  the same holds and the optimal potential  $V$  is

an (explicit) function of the optimizers of a family of Gagliardo-Nirenberg-Sobolev inequalities (see Remark 5.7).

In the final Section 6 we make some further remarks about the state of the art of spectral optimization for Schrödinger operators on unbounded domains, and we apply the results of Section 5 to get, in Theorem 6.1, the qualitative behavior of the optimal potential for  $F = \lambda_2$  for problem (1.5) with  $\Psi(s) = s^{-p}$ .

## 2. CAPACITARY MEASURES AND $\gamma$ -CONVERGENCE

For a subset  $E \subset \mathbb{R}^d$  its *capacity* is defined by

$$\text{cap}(E) = \inf \left\{ \int_{\mathbb{R}^d} |\nabla u|^2 dx + \int_{\mathbb{R}^d} u^2 dx : u \in H^1(\mathbb{R}^d), u \geq 1 \text{ in a neighborhood of } E \right\}.$$

If a property  $P(x)$  holds for all  $x \in \Omega$ , except for the elements of a set  $E \subset \Omega$  of capacity zero, we say that  $P(x)$  holds *quasi-everywhere* (shortly *q.e.*) in  $\Omega$ , whereas the expression *almost everywhere* (shortly *a.e.*) refers, as usual, to the Lebesgue measure, which we often denote by  $|\cdot|$ .

A subset  $A$  of  $\mathbb{R}^d$  is said to be *quasi-open* if for every  $\varepsilon > 0$  there exists an open subset  $A_\varepsilon$  of  $\mathbb{R}^d$ , with  $A \subset A_\varepsilon$ , such that  $\text{cap}(A_\varepsilon \setminus A) < \varepsilon$ . Similarly, a function  $u : \mathbb{R}^d \rightarrow \mathbb{R}$  is said to be *quasi-continuous* (respectively *quasi-lower semicontinuous*) if there exists a decreasing sequence of open sets  $(A_n)_n$  such that  $\text{cap}(A_n) \rightarrow 0$  and the restriction  $u_n$  of  $u$  to the complement  $A_n^c$  of  $A_n$  is continuous (respectively lower semicontinuous). It is well known (see for instance [18]) that every function  $u \in H^1(\mathbb{R}^d)$  has a quasi-continuous representative  $\tilde{u}$ , which is uniquely defined up to a set of capacity zero, and given by

$$\tilde{u}(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u(y) dy,$$

where  $B_\varepsilon(x)$  denotes the ball of radius  $\varepsilon$  centered at  $x$ . We identify the (a.e.) equivalence class  $u \in H^1(\mathbb{R}^d)$  with the (q.e.) equivalence class of quasi-continuous representatives  $\tilde{u}$ .

We denote by  $\mathcal{M}^+(\mathbb{R}^d)$  the set of positive Borel measures on  $\mathbb{R}^d$  (not necessarily finite or Radon) and by  $\mathcal{M}_{\text{cap}}^+(\mathbb{R}^d) \subset \mathcal{M}^+(\mathbb{R}^d)$  the set of *capacitary measures*, i.e. the measures  $\mu \in \mathcal{M}^+(\mathbb{R}^d)$  such that  $\mu(E) = 0$  for any set  $E \subset \mathbb{R}^d$  of capacity zero. We note that when  $\mu$  is a capacitary measure, the integral  $\int_{\mathbb{R}^d} |u|^2 d\mu$  is well-defined for each  $u \in H^1(\mathbb{R}^d)$ , i.e. if  $\tilde{u}_1$  and  $\tilde{u}_2$  are two quasi-continuous representatives of  $u$ , then  $\int_{\mathbb{R}^d} |\tilde{u}_1|^2 d\mu = \int_{\mathbb{R}^d} |\tilde{u}_2|^2 d\mu$ .

For a subset  $\Omega \subset \mathbb{R}^d$ , we define the Sobolev space  $H_0^1(\Omega)$  as

$$H_0^1(\Omega) = \left\{ u \in H^1(\mathbb{R}^d) : u = 0 \text{ q.e. on } \Omega^c \right\}.$$

Alternatively, by using the capacitary measure  $I_\Omega$  defined as

$$I_\Omega(E) = \begin{cases} 0 & \text{if } \text{cap}(E \setminus \Omega) = 0 \\ +\infty & \text{if } \text{cap}(E \setminus \Omega) > 0 \end{cases} \quad \text{for every Borel set } E \subset \mathbb{R}^d, \quad (2.1)$$

the Sobolev space  $H_0^1(\Omega)$  can be defined as

$$H_0^1(\Omega) = \left\{ u \in H^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} |u|^2 dI_\Omega < +\infty \right\}.$$

More generally, for any capacitary measure  $\mu \in \mathcal{M}_{\text{cap}}^+(\mathbb{R}^d)$ , we define the space

$$H_\mu^1 = \left\{ u \in H^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} |u|^2 d\mu < +\infty \right\},$$

which is a Hilbert space when endowed with the norm  $\|u\|_{1,\mu}$ , where

$$\|u\|_{1,\mu}^2 = \int_{\mathbb{R}^d} |\nabla u|^2 dx + \int_{\mathbb{R}^d} u^2 dx + \int_{\mathbb{R}^d} u^2 d\mu.$$

If  $u \notin H_\mu^1$ , then we set  $\|u\|_{1,\mu} = +\infty$ .

For  $\Omega \subset \mathbb{R}^d$ , we define  $\mathcal{M}_{\text{cap}}^+(\Omega)$  as the space of capacitary measures  $\mu \in \mathcal{M}_{\text{cap}}^+(\mathbb{R}^d)$  such that  $\mu(E) = +\infty$  for any set  $E \subset \mathbb{R}^d$  such that  $\text{cap}(E \setminus \Omega) > 0$ . For  $\mu \in \mathcal{M}_{\text{cap}}^+(\mathbb{R}^d)$ , we denote with  $H_\mu^1(\Omega)$  the space  $H_{\mu \vee I_\Omega}^1 = H_\mu^1 \cap H_0^1(\Omega)$ .

**Definition 2.1.** *Given a metric space  $(X, d)$  and sequence of functionals  $J_n : X \rightarrow \mathbb{R} \cup \{+\infty\}$ , we say that  $J_n$   $\Gamma$ -converges to the functional  $J : X \rightarrow \mathbb{R} \cup \{+\infty\}$ , if the following two conditions are satisfied:*

(a) *for every sequence  $x_n$  converging to  $x \in X$ , we have*

$$J(x) \leq \liminf_{n \rightarrow \infty} J_n(x_n);$$

(b) *for every  $x \in X$ , there exists a sequence  $x_n$  converging to  $x$ , such that*

$$J(x) = \lim_{n \rightarrow \infty} J_n(x_n).$$

For all details and properties of  $\Gamma$ -convergence we refer to [12]; here we simply recall that, whenever  $J_n$   $\Gamma$ -converges to  $J$ ,

$$\min_{x \in X} J(x) \leq \liminf_{n \rightarrow \infty} \min_{x \in X} J_n(x).$$

**Definition 2.2.** *We say that the sequence of capacitary measures  $\mu_n \in \mathcal{M}_{\text{cap}}^+(\Omega)$ ,  $\gamma$ -converges to the capacitary measure  $\mu \in \mathcal{M}_{\text{cap}}^+(\Omega)$  if the sequence of functionals  $\|\cdot\|_{1,\mu_n}$   $\Gamma$ -converges to the functional  $\|\cdot\|_{1,\mu}$  in  $L^2(\Omega)$ , i.e. if the following two conditions are satisfied:*

- *for every sequence  $u_n \rightarrow u$  in  $L^2(\Omega)$  we have*

$$\int_{\mathbb{R}^d} |\nabla u|^2 dx + \int_{\mathbb{R}^d} u^2 d\mu \leq \liminf_{n \rightarrow \infty} \left\{ \int_{\mathbb{R}^d} |\nabla u_n|^2 dx + \int_{\mathbb{R}^d} u_n^2 d\mu_n \right\};$$

- *for every  $u \in L^2(\Omega)$ , there exists  $u_n \rightarrow u$  in  $L^2(\Omega)$  such that*

$$\int_{\mathbb{R}^d} |\nabla u|^2 dx + \int_{\mathbb{R}^d} u^2 d\mu = \lim_{n \rightarrow \infty} \left\{ \int_{\mathbb{R}^d} |\nabla u_n|^2 dx + \int_{\mathbb{R}^d} u_n^2 d\mu_n \right\}.$$

If  $\mu \in \mathcal{M}_{\text{cap}}^+(\Omega)$  and  $f \in L^2(\Omega)$  we define the functional  $J_\mu(f, \cdot) : L^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$  by

$$J_\mu(f, u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} u^2 d\mu - \int_{\Omega} f u dx. \quad (2.2)$$

If  $\Omega \subset \mathbb{R}^d$  is a bounded open set,  $\mu \in \mathcal{M}_{\text{cap}}^+(\Omega)$  and  $f \in L^2(\Omega)$ , then the functional  $J_\mu(f, \cdot)$  has a unique minimizer  $u \in H_\mu^1$  that verifies the PDE formally written as

$$-\Delta u + \mu u = f, \quad u \in H_\mu^1(\Omega),$$

and whose precise meaning is given in the weak form

$$\begin{cases} \int_{\Omega} \nabla u \cdot \nabla \varphi dx + \int_{\Omega} u \varphi d\mu = \int_{\Omega} f \varphi dx, & \forall \varphi \in H_\mu^1(\Omega), \\ u \in H_\mu^1(\Omega). \end{cases}$$

The resolvent operator of  $-\Delta + \mu$ , that is the map  $\mathcal{R}_\mu$  that associates to every  $f \in L^2(\Omega)$  the solution  $u \in H_\mu^1(\Omega) \subset L^2(\Omega)$ , is a compact linear operator in  $L^2(\Omega)$  and so, it has a discrete spectrum

$$0 < \dots \leq \Lambda_k \leq \dots \leq \Lambda_2 \leq \Lambda_1.$$

Their inverses  $1/\Lambda_k$  are denoted by  $\lambda_k(\mu)$  and are the eigenvalues of the operator  $-\Delta + \mu$ .

In the case  $f = 1$  the solution will be denoted by  $w_\mu$  and when  $\mu = I_\Omega$  we will use the notation  $w_\Omega$  instead of  $w_{I_\Omega}$ . We also recall (see [4]) that if  $\Omega$  is bounded, then the strong  $L^2$ -convergence of the minimizers  $w_{\mu_n}$  to  $w_\mu$  is equivalent to the  $\gamma$ -convergence of Definition 2.2.

*Remark 2.3.* An important well-known characterization of the  $\gamma$ -convergence is the following: a sequence  $\mu_n$   $\gamma$ -converges to  $\mu$ , if and only if, the sequence of resolvent operators  $\mathcal{R}_{\mu_n}$  associated to  $-\Delta + \mu_n$ , converges (in the strong convergence of linear operators on  $L^2$ ) to the resolvent  $\mathcal{R}_\mu$  of the operator  $-\Delta + \mu$ . A consequence of this fact is that the spectrum of the operator  $-\Delta + \mu_n$  converges (pointwise) to the one of  $-\Delta + \mu$ .

*Remark 2.4.* The space  $\mathcal{M}_{\text{cap}}^+(\Omega)$  endowed with the  $\gamma$ -convergence is metrizable. If  $\Omega$  is bounded, one may take  $d_\gamma(\mu, \nu) = \|w_\mu - w_\nu\|_{L^2}$ . Moreover, in this case, in [14] it is proved that the space  $\mathcal{M}_{\text{cap}}^+(\Omega)$  endowed with the metric  $d_\gamma$  is compact.

**Proposition 2.5.** *Let  $\Omega \subset \mathbb{R}^d$  and let  $V_n \in L^1(\Omega)$  be a sequence weakly converging in  $L^1(\Omega)$  to a function  $V$ . Then the capacitary measures  $V_n dx$   $\gamma$ -converge to  $V dx$ .*

*Proof.* We have to prove that the solutions  $u_n = R_{V_n}(1)$  to

$$\begin{cases} -\Delta u_n + V_n(x)u_n = 1 \\ u \in H_0^1(\Omega) \end{cases}$$

weakly converge in  $H_0^1(\Omega)$  to the solution  $u = R_V(1)$  to

$$\begin{cases} -\Delta u + V(x)u = 1 \\ u \in H_0^1(\Omega), \end{cases}$$

or equivalently that the functionals

$$J_n(u) = \int_\Omega |\nabla u|^2 dx + \int_\Omega V_n(x)u^2 dx$$

$\Gamma$ -converge in  $L^2(\Omega)$  to the functional

$$J(u) = \int_\Omega |\nabla u|^2 dx + \int_\Omega V(x)u^2 dx.$$

The  $\Gamma$ -liminf inequality (Definition 2.1 (a)) is immediate since, if  $u_n \rightarrow u$  in  $L^2(\Omega)$ , we have

$$\int_\Omega |\nabla u|^2 dx \leq \liminf_{n \rightarrow \infty} \int_\Omega |\nabla u_n|^2 dx$$

by the lower semicontinuity of the  $H^1(\Omega)$  norm with respect to the  $L^2(\Omega)$ -convergence, and

$$\int_\Omega V(x)u^2 dx \leq \liminf_{n \rightarrow \infty} \int_\Omega V_n(x)u_n^2 dx$$

by the strong-weak lower semicontinuity theorem for integral functionals (see for instance [7]).

Let us now prove the  $\Gamma$ -limsup inequality (Definition 2.1 (b)) which consists, given  $u \in H_0^1(\Omega)$ , in constructing a sequence  $u_n \rightarrow u$  in  $L^2(\Omega)$  such that

$$\limsup_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^2 dx + \int_{\Omega} V_n(x) u_n^2 dx \leq \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} V(x) u^2 dx. \quad (2.3)$$

For every  $t > 0$  let  $u^t = (u \wedge t) \vee (-t)$ ; then, by the weak convergence of  $V_n$ , for  $t$  fixed we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} V_n(x) |u^t|^2 dx = \int_{\Omega} V(x) |u^t|^2 dx,$$

and

$$\lim_{t \rightarrow +\infty} \int_{\Omega} V(x) |u^t|^2 dx = \int_{\Omega} V(x) |u|^2 dx.$$

Then, by a diagonal argument, we can find a sequence  $t_n \rightarrow +\infty$  such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} V_n(x) |u^{t_n}|^2 dx = \int_{\Omega} V(x) |u|^2 dx.$$

Taking now  $u_n = u^{t_n}$ , and noticing that for every  $t > 0$

$$\int_{\Omega} |\nabla u^t|^2 dx \leq \int_{\Omega} |\nabla u|^2 dx,$$

we obtain (2.3) and so the proof is complete.  $\square$

In the case of weak\* convergence of measures the statement of Proposition 2.5 is no longer true, as the following proposition shows.

**Proposition 2.6.** *Let  $\Omega \subset \mathbb{R}^d$  ( $d \geq 2$ ) be a bounded open set and let  $V, W$  be two functions in the class  $L_+^1(\Omega)$  of nonnegative integrable functions on  $\Omega$  such that  $V \geq W$ . Then, there exists a sequence  $V_n \in L_+^1(\Omega)$ , uniformly bounded in  $L^1(\Omega)$ , such that the sequence of measures  $V_n(x) dx$  converges weakly\* to  $V(x) dx$  and  $\gamma$ -converges to  $W(x) dx$ .*

*Proof.* <sup>2</sup> Without loss of generality we can suppose  $\int_{\Omega} (V - W) dx = 1$ . Let  $\mu_n$  be a sequence of probability measures on  $\Omega$  weakly\* converging to  $(V - W) dx$  and such that each  $\mu_n$  is a finite sum of Dirac masses. For each  $n \in \mathbb{N}$  consider a sequence of positive functions  $V_{n,m} \in L^1(\Omega)$  such that  $\int_{\Omega} V_{n,m} dx = 1$  and  $V_{n,m} dx$  converges weakly\* to  $\mu_n$  as  $m \rightarrow \infty$ . Moreover, we choose  $V_{n,m}$  as a convex combination of functions of the form  $|B_{1/m}|^{-1} \chi_{B_{1/m}(x_j)}$ .

We now prove that for fixed  $n \in \mathbb{N}$ ,  $(V_{n,m} + W) dx$   $\gamma$ -converges, as  $m \rightarrow \infty$ , to  $W dx$  or, equivalently, that the sequence  $w_{W+V_{n,m}}$  converges in  $L^2$  to  $w_W$ , as  $m \rightarrow \infty$ . Indeed, by the weak maximum principle, we have

$$w_{W+I_{\Omega_{m,n}}} \leq w_{W+V_{n,m}} \leq w_W,$$

where  $\Omega_{m,n} = \Omega \setminus \cup_j B_{1/m}(x_j)$  and  $I_{\Omega_{m,n}}$  is as in (2.1).

---

<sup>2</sup>the idea of this proof was suggested by Dorin Bucur

Since a point has zero capacity in  $\mathbb{R}^d$  ( $d \geq 2$ ) there exists a sequence  $\phi_m \rightarrow 0$  strongly in  $H^1(\mathbb{R}^d)$  with  $\phi_m = 1$  on  $B_{1/m}(0)$  and  $\phi_m = 0$  outside  $B_{1/\sqrt{m}}(0)$ . We have

$$\begin{aligned} \int_{\Omega} |w_W - w_{W+I_{\Omega_{m,n}}}|^2 dx &\leq 2\|w_W\|_{L^\infty} \int_{\Omega} (w_W - w_{W+I_{\Omega_{m,n}}}) dx \\ &= 4\|w_W\|_{L^\infty} (E(W + I_{\Omega_{m,n}}) - E(W)) \\ &\leq 4\|w_W\|_{L^\infty} \left( \int_{\Omega} \frac{1}{2} |\nabla w_m|^2 + \frac{1}{2} W w_m^2 - w_m dx \right. \\ &\quad \left. - \int_{\Omega} \frac{1}{2} |\nabla w_W|^2 + \frac{1}{2} W w_W^2 - w_W dx \right), \end{aligned} \tag{2.4}$$

where  $w_m$  is any function in  $\in H_0^1(\Omega_{m,n})$ . Taking

$$w_m(x) = w_W(x) \prod_j (1 - \phi_m(x - x_j)),$$

since  $\phi_m \rightarrow 0$  strongly in  $H^1(\mathbb{R}^d)$ , it is easy to see that  $w_m \rightarrow w_W$  strongly in  $H^1(\Omega)$  and so, by (2.4),  $w_{W+I_{\Omega_{m,n}}} \rightarrow w_W$  in  $L^2(\Omega)$  as  $m \rightarrow \infty$ . Since the weak convergence of probability measures and the  $\gamma$ -convergence are both induced by metrics, a diagonal sequence argument brings to the conclusion.  $\square$

*Remark 2.7.* When  $d = 1$ , a result analogous to Proposition 2.5 is that any sequence  $(\mu_n)$  weakly\* converging to  $\mu$  is also  $\gamma$ -converging to  $\mu$ . This is an easy consequence of the compact embedding of  $H_0^1(\Omega)$  into the space of continuous functions on  $\Omega$ .

We note that the hypothesis  $V \geq W$  in Proposition 2.6 is necessary. Indeed, we have the following proposition, whose proof is contained in [13, Theorem 3.1] and we report it here for the sake of completeness.

**Proposition 2.8.** *Let  $\mu_n \in \mathcal{M}_{cap}^+(\Omega)$  be a sequence of capacitary and Radon measures weakly\* converging to the measure  $\nu$  and  $\gamma$ -converging to the capacitary measure  $\mu \in \mathcal{M}_{cap}^+(\Omega)$ . Then  $\mu \leq \nu$  in  $\Omega$ .*

*Proof.* We note that it is enough to show that  $\mu(K) \leq \nu(K)$  whenever  $K \subset\subset \Omega$  is a compact set. Let  $u$  be a nonnegative smooth function with compact support in  $\Omega$  such that  $u \leq 1$  in  $\Omega$  and  $u = 1$  on  $K$ ; we have

$$\mu(K) \leq \int_{\Omega} u^2 d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} u^2 d\mu_n = \int_{\Omega} u^2 d\nu \leq \nu(\{u > 0\}).$$

Since  $u$  is arbitrary, we have the conclusion by the Borel regularity of  $\nu$ .  $\square$

### 3. EXISTENCE OF OPTIMAL POTENTIALS IN $L^p(\Omega)$

In this section we consider the optimization problem

$$\min \left\{ F(V) : V : \Omega \rightarrow [0, +\infty], \int_{\Omega} V^p dx \leq 1 \right\}, \tag{3.1}$$

where  $p > 0$  and  $F(V)$  is a cost functional acting on Schrödinger potentials, or more generally on capacitary measures. Typically,  $F(V)$  is the minimum of some functional  $J_V : H_0^1(\Omega) \rightarrow \mathbb{R}$  depending on  $V$ . A natural assumption in this case is the lower semicontinuity of the functional  $F$  with respect to the  $\gamma$ -convergence, that is

$$F(\mu) \leq \liminf_{n \rightarrow \infty} F(\mu_n), \quad \text{whenever } \mu_n \rightarrow_{\gamma} \mu.$$



**Theorem 3.1.** *Let  $F : L^1_+(\Omega) \rightarrow \mathbb{R}$  be a functional, lower semicontinuous with respect to the  $\gamma$ -convergence, and let  $\mathcal{V}$  be a weakly  $L^1(\Omega)$  compact set. Then the problem*

$$\min \{F(V) : V \in \mathcal{V}\},$$

*admits a solution.*

*Proof.* Let  $(V_n)$  be a minimizing sequence in  $\mathcal{V}$ . By the compactness assumption on  $\mathcal{V}$ , we may assume that  $V_n$  tends weakly  $L^1(\Omega)$  to some  $V \in \mathcal{V}$ . By Proposition 2.5, we have that  $V_n$   $\gamma$ -converges to  $V$  and so, by the semicontinuity of  $F$ ,

$$F(V) \leq \liminf_{n \rightarrow \infty} F(V_n),$$

which gives the conclusion. □

*Remark 3.2.* Theorem 3.1 applies for instance to the integral functionals and to the spectral functionals considered in the introduction; it is not difficult to show that they are lower semicontinuous with respect to the  $\gamma$ -convergence.

*Remark 3.3.* In some special cases the solution to (3.1) can be written explicitly in terms of the solution to some partial differential equation on  $\Omega$ . This is the case of the Dirichlet Energy (see Propositions 3.6 and 3.9), and of the first eigenvalue of the Dirichlet Laplacian  $\lambda_1$  (see [21, Chapter 8]).

The compactness assumption on the admissible class  $\mathcal{V}$  for the weak  $L^1(\Omega)$  convergence in Theorem 3.1 is for instance satisfied if  $\Omega$  has finite measure and  $\mathcal{V}$  is a convex closed and bounded subset of  $L^p(\Omega)$ , with  $p > 1$ . When  $\mathcal{V}$  is only bounded in  $L^1(\Omega)$  Theorem 3.1 does not apply, since minimizing sequences may weakly\* converge to a measure. It is then convenient to extend our analysis to the case of functionals defined on capacity measures, in which a result analogous to Theorem 3.1 holds.

**Theorem 3.4.** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded open set and let  $F : \mathcal{M}_{\text{cap}}^+(\Omega) \rightarrow \mathbb{R}$  be a functional lower semicontinuous with respect to the  $\gamma$ -convergence. Then the problem*

$$\min \{F(\mu) : \mu \in \mathcal{M}_{\text{cap}}^+(\Omega), \mu(\Omega) \leq 1\}, \tag{3.2}$$

*admits a solution.*

*Proof.* Let  $(\mu_n)$  be a minimizing sequence. Then, up to a subsequence  $\mu_n$  converges weakly\* to some measure  $\nu$  and  $\gamma$ -converges to some measure  $\mu \in \mathcal{M}_{\text{cap}}^+(\Omega)$ . By Proposition 2.8, we have that  $\mu(\Omega) \leq \nu(\Omega) \leq 1$  and so,  $\mu$  is a solution to (3.2). □

We notice that, since the class of Schrödinger potentials is dense, with respect to the  $\gamma$ -convergence, in the class  $\mathcal{M}_{\text{cap}}^+(\Omega)$  of capacity measures (see [14]), the minimum in (3.2) coincides with

$$\inf \left\{ F(V) : V \geq 0, \int_{\Omega} V \, dx \leq 1 \right\}$$

whenever  $F$  is a  $\gamma$ -continuous cost functional.

The following example shows that the optimal solution to problem (3.2) is not, in general, a function  $V(x)$ , even when the optimization criterion is the energy  $\mathcal{E}_f$  introduced in (1.2). On the other hand, an explicit form for the optimal potential  $V(x)$  will be provided in Proposition 3.9 assuming that the right-hand side  $f$  is in  $L^2(\Omega)$ .

*Example 3.5.* Let  $\Omega = (-1, 1)$  and consider the functional

$$F(\mu) = - \min \left\{ \frac{1}{2} \int_{-1}^1 |u'|^2 dx + \frac{1}{2} \int_{-1}^1 u^2 d\mu - u(0) : u \in H_0^1(-1, 1) \right\}.$$

Then, for any  $\mu$  such that  $\mu(\Omega) \leq 1$ , we have

$$F(\mu) \geq - \min \left\{ \frac{1}{2} \int_{-1}^1 |u'|^2 dx + \frac{1}{2} \left( \sup_{(-1,1)} u \right)^2 - u(0) : u \in H_0^1(-1, 1), u \geq 0 \right\}. \quad (3.3)$$

By a symmetrization argument, the minimizer  $u$  of the right-hand side of (3.3) is radially decreasing; moreover,  $u$  is linear on the set  $u < M$ , where  $M = \sup u$ , and so it is of the form

$$u(x) = \begin{cases} \frac{M}{1-\alpha}x + \frac{M}{1-\alpha}, & x \in [-1, -\alpha], \\ M, & x \in [-\alpha, \alpha], \\ -\frac{M}{1-\alpha}x + \frac{M}{1-\alpha}, & x \in [\alpha, 1], \end{cases}$$

for some  $\alpha \in [0, 1]$ . A straightforward computation gives  $\alpha = 0$  and  $M = 1/3$ . Thus,  $u$  is also the minimizer of

$$F(\delta_0) = - \min \left\{ \frac{1}{2} \int_{-1}^1 |u'|^2 dx + \frac{1}{2} u(0)^2 - u(0) : u \in H_0^1(-1, 1) \right\},$$

and so  $\delta_0$  is the solution to

$$\min \{ F(\mu) : \mu(\Omega) \leq 1 \}.$$

In the rest of this section we consider the particular case  $F(V) = -\mathcal{E}_f(V)$ , in which we can identify the optimal potential through the solution to a nonlinear PDE. Let  $\Omega \subset \mathbb{R}^d$  be a bounded open set and let  $f \in L^2(\Omega)$ . By Theorem 3.1, the problem

$$\min \{ -\mathcal{E}_f(V) : V \in \mathcal{V} \} \quad \text{with} \quad \mathcal{V} = \left\{ V \geq 0, \int_{\Omega} V^p dx \leq 1 \right\}, \quad (3.4)$$

admits a solution, where  $\mathcal{E}_f(V)$  is the energy functional defined in (1.2). We notice that, replacing  $-\mathcal{E}_f(V)$  by  $\mathcal{E}_f(V)$ , makes problem (3.4) trivial, with the only solution  $V \equiv 0$ . Minimization problems for  $\mathcal{E}_f$  will be considered in Section 4 for admissible classes of the form

$$\mathcal{V} = \left\{ V \geq 0, \int_{\Omega} V^{-p} dx \leq 1 \right\}.$$

Analogous results for  $F(V) = -\lambda_1(V)$  were proved in [21, Theorem 8.2.3].

**Proposition 3.6.** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded open set,  $1 < p < \infty$  and  $f \in L^2(\Omega)$ . Then the problem (3.4) has a unique solution*

$$V_p = \left( \int_{\Omega} |u_p|^{2p/(p-1)} dx \right)^{-1/p} |u_p|^{2/(p-1)},$$

where  $u_p \in H_0^1(\Omega) \cap L^{2p/(p-1)}(\Omega)$  is the minimizer of the functional

$$J_p(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \left( \int_{\Omega} |u|^{2p/(p-1)} dx \right)^{(p-1)/p} - \int_{\Omega} u f dx. \quad (3.5)$$

Moreover, we have  $\mathcal{E}_f(V_p) = J_p(u_p)$ .

*Proof.* We first show that we have

$$\max_{V \in \mathcal{V}} \min_{u \in H_0^1(\Omega)} \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} u^2 V - uf \right) dx \leq \min_{u \in H_0^1(\Omega)} \max_{V \in \mathcal{V}} \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} u^2 V - uf \right) dx, \quad (3.6)$$

where the maximums are taken over all positive functions  $V \in L^p(\Omega)$  with  $\int_{\Omega} V^p dx \leq 1$ . For a fixed  $u \in H_0^1(\Omega)$ , the maximum on the right-hand side (if finite) is achieved for a function  $V$  such that  $\Lambda p V^{p-1} = u^2$ , where  $\Lambda$  is a Lagrange multiplier. By the condition  $\int_{\Omega} V^p dx = 1$  we obtain that the maximum is achieved for

$$V = \left( \int_{\Omega} |u|^{2p/(p-1)} dx \right)^{-1/p} |u|^{2/(p-1)}.$$

Substituting in (3.6), we obtain

$$\max \{ \mathcal{E}_f(V) : V \in \mathcal{V} \} \leq \min \{ J_p(u) : u \in H_0^1(\Omega) \}.$$

Let  $u_n$  be a minimizing sequence for  $J_p$ . Since  $\inf J_p \leq 0$ , we can assume  $J_p(u_n) \leq 0$  for each  $n \in \mathbb{N}$ . Thus, we have

$$\frac{1}{2} \int_{\Omega} |\nabla u_n|^2 dx + \frac{1}{2} \left( \int_{\Omega} |u_n|^{2p/(p-1)} dx \right)^{(p-1)/p} \leq \int_{\Omega} u_n f dx \leq C \|f\|_{L^2(\Omega)} \|\nabla u_n\|_{L^2}, \quad (3.7)$$

where  $C$  is a constant depending on  $\Omega$ . Thus we obtain

$$\int_{\Omega} |\nabla u_n|^2 dx + \left( \int_{\Omega} |u_n|^{2p/(p-1)} dx \right)^{(p-1)/p} \leq 4C^2 \|f\|_{L^2(\Omega)}^2, \quad (3.8)$$

and so, up to subsequence  $u_n$  converges weakly in  $H_0^1(\Omega)$  and  $L^{2p/(p-1)}(\Omega)$  to some  $u_p \in H_0^1(\Omega) \cap L^{2p/(p-1)}(\Omega)$ . By the semicontinuity of the  $L^2$ -norm of the gradient and the  $L^{2p/(p-1)}$ -norm and the fact that  $\int_{\Omega} f u_n dx \rightarrow \int_{\Omega} f u_p dx$ , as  $n \rightarrow \infty$ , we have that  $u_p$  is a minimizer of  $J_p$ . By the strict convexity of  $J_p$ , we have that  $u_p$  is unique. Moreover, by (3.7) and (3.8),  $J_p(u_p) > -\infty$ . Writing down the Euler-Lagrange equation for  $u_p$ , we obtain

$$-\Delta u_p + \left( \int_{\Omega} |u_p|^{2p/(p-1)} dx \right)^{-1/p} |u_p|^{2/(p-1)} u_p = f.$$

Setting

$$V_p = \left( \int_{\Omega} |u_p|^{2p/(p-1)} dx \right)^{-1/p} |u_p|^{2/(p-1)},$$

we have that  $\int_{\Omega} V_p^p dx = 1$  and  $u_p$  is the solution to

$$-\Delta u_p + V_p u_p = f.$$

In particular, we have  $J_p(u_p) = \mathcal{E}_f(V_p)$  and so  $V_p$  solves (3.4). The uniqueness of  $V_p$  follows by the uniqueness of  $u_p$  and the equality case in the Hölder inequality

$$\int_{\Omega} u^2 V dx \leq \left( \int_{\Omega} V^p dx \right)^{1/p} \left( \int_{\Omega} |u|^{2p/(p-1)} dx \right)^{(p-1)/p} \leq \left( \int_{\Omega} |u|^{2p/(p-1)} dx \right)^{(p-1)/p}.$$

□

When the functional  $F$  is  $-\mathcal{E}_f$ , then the existence result holds also in the case  $p = 1$ . Before we give the proof of this fact in Proposition 3.9, we need some preliminary results. We also note that the analogous results were obtained in the case  $F = -\lambda_1$  (see [21, Theorem 8.2.4]) and in the case  $F = -\mathcal{E}_f$ , where  $f$  is a positive function (see [13]).

*Remark 3.7.* Let  $u_p$  be the minimizer of  $J_p$ , defined in (3.5). By (3.8), we have the estimate

$$\|\nabla u_p\|_{L^2(\Omega)} + \|u_p\|_{L^{2p/(p-1)}(\Omega)} \leq 2\sqrt{2}C\|f\|_{L^2(\Omega)},$$

where  $C$  is the constant from (3.7). Moreover, we have  $u_p \in H_{loc}^2(\Omega)$  and for each open set  $\Omega' \subset\subset \Omega$ , there is a constant  $C$  not depending on  $p$  such that

$$\|u_p\|_{H^2(\Omega')} \leq C(f, \Omega').$$

Indeed,  $u_p$  satisfies the PDE

$$-\Delta u + c|u|^\alpha u = f, \quad (3.9)$$

with  $c > 0$  and  $\alpha = 2/(p-1)$ , and standard elliptic regularity arguments (see [17, Section 6.3]) give that  $u \in H_{loc}^2(\Omega)$ . To show that  $\|u_p\|_{H^2(\Omega')}$  is bounded independently of  $p$  we apply the Nirenberg operator  $\partial_k^h u = \frac{u(x+he_k) - u(x)}{h}$  on both sides of (3.9), and multiplying by  $\phi^2 \partial_k^h u$ , where  $\phi$  is an appropriate cut-off function which equals 1 on  $\Omega'$ , we have

$$\int_{\Omega} \phi^2 |\nabla \partial_k^h u|^2 dx + \int_{\Omega} \nabla(\partial_k^h u) \nabla(\phi^2) \partial_k^h u dx + c(\alpha+1) \int_{\Omega} \phi^2 |u|^\alpha |\partial_k^h u|^2 dx = - \int f \partial_k^h (\phi^2 \partial_k^h u) dx,$$

for all  $k = 1, \dots, d$ . Some straightforward manipulations now give

$$\|\nabla^2 u\|_{L^2(\Omega')}^2 \leq \sum_{k=1}^d \int_{\Omega} \phi^2 |\nabla \partial_k u|^2 dx \leq C(\Omega') (\|f\|_{L^2(\{\phi^2 > 0\})} + \|\nabla u\|_{L^2(\Omega)}).$$

**Lemma 3.8.** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded open set and let  $f \in L^2(\Omega)$ . Consider the functional  $J_1 : L^2(\Omega) \rightarrow \mathbb{R}$  defined by*

$$J_1(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \|u\|_{\infty}^2 - \int_{\Omega} u f dx, \quad (3.10)$$

*Then,  $J_p$   $\Gamma$ -converges in  $L^2(\Omega)$  to  $J_1$ , as  $p \rightarrow 1$ , where  $J_p$  is defined in (3.5).*

*Proof.* Let  $v_n \in L^2(\Omega)$  be a sequence of positive functions converging in  $L^2$  to  $v \in L^2(\Omega)$  and let  $\alpha_n \rightarrow +\infty$ . Then, we have that

$$\|v\|_{L^\infty(\Omega)} \leq \liminf_{n \rightarrow \infty} \|v_n\|_{L^{\alpha_n}(\Omega)}. \quad (3.11)$$

In fact, suppose first that  $\|v\|_{L^\infty} = M < +\infty$  and let  $\omega_\varepsilon = \{v > M - \varepsilon\}$ , for some  $\varepsilon > 0$ . Then, we have

$$\liminf_{n \rightarrow \infty} \|v_n\|_{L^{\alpha_n}(\Omega)} \geq \lim_{n \rightarrow \infty} |\omega_\varepsilon|^{(1-\alpha_n)/\alpha_n} \int_{\omega_\varepsilon} v_n dx = |\omega_\varepsilon|^{-1} \int_{\omega_\varepsilon} v dx \geq M - \varepsilon,$$

and so, letting  $\varepsilon \rightarrow 0$ , we have  $\liminf_{n \rightarrow \infty} \|v_n\|_{L^{\alpha_n}(\Omega)} \geq M$ . If  $\|v\|_{L^\infty} = +\infty$ , then setting  $\omega_k = \{v > k\}$ , for any  $k \geq 1$ , and arguing as above, we obtain (3.11).

Let  $u_n \rightarrow u$  in  $L^2(\Omega)$ . Then, by the semicontinuity of the  $L^2$  norm of the gradient and (3.11) and the continuity of the term  $\int_{\Omega} u f dx$ , we have

$$J_1(u) \leq \liminf_{n \rightarrow \infty} J_{p_n}(u_n),$$

for any decreasing sequence  $p_n \rightarrow 1$ . On the other hand, for any  $u \in L^2$ , we have  $J_{p_n}(u) \rightarrow J_1(u)$  as  $n \rightarrow \infty$  and so, we have the conclusion.  $\square$

**Proposition 3.9.** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded open set and  $f \in L^2(\Omega)$ . Then there is a unique solution to problem (3.4) with  $p = 1$ , given by*

$$V_1 = \frac{1}{M} (\chi_{\omega_+} f - \chi_{\omega_-} f),$$

where  $M = \|u_1\|_{L^\infty(\Omega)}$ ,  $\omega_+ = \{u_1 = M\}$ ,  $\omega_- = \{u_1 = -M\}$ , being  $u_1 \in H_0^1(\Omega) \cap L^\infty(\Omega)$  the unique minimizer of the functional  $J_1$ , defined in (3.10). In particular,  $\int_{\omega_+} f dx - \int_{\omega_-} f dx = M$ ,  $f \geq 0$  on  $\omega_+$  and  $f \leq 0$  on  $\omega_-$ .

*Proof.* For any  $u \in H_0^1(\Omega)$  and any  $V \geq 0$  with  $\int_\Omega V dx \leq 1$  we have

$$\int_\Omega u^2 V dx \leq \|u\|_\infty^2 \int_\Omega V dx \leq \|u\|_\infty^2,$$

where for sake of simplicity, we write  $\|\cdot\|_\infty$  instead of  $\|\cdot\|_{L^\infty(\Omega)}$ . Arguing as in the proof of Proposition 3.6, we obtain the inequalities

$$\begin{aligned} \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \frac{1}{2} \int_\Omega u^2 V dx - \int_\Omega u f dx &\leq J_1(u), \\ \max \left\{ \mathcal{E}_f(V) : \int_\Omega V \leq 1 \right\} &\leq \min \{ J_1(u) : u \in H_0^1(\Omega) \}. \end{aligned}$$

As in (3.7), we have that a minimizing sequence of  $J_1$  is bounded in  $H_0^1(\Omega) \cap L^\infty(\Omega)$  and thus by semicontinuity there is a minimizer  $u_1 \in H_0^1(\Omega) \cap L^\infty(\Omega)$  of  $J_1$ , which is also unique, by the strict convexity of  $J_1$ . Let  $u_p$  denotes the minimizer of  $J_p$  as in Proposition 3.6. Then, by Remark 3.7, we have that the family  $u_p$  is bounded in  $H_0^1(\Omega)$  and in  $H^2(\Omega')$  for each  $\Omega' \subset\subset \Omega$ . Then, we have that each sequence  $u_{p_n}$  has a subsequence converging weakly in  $L^2(\Omega)$  to some  $u \in H_{loc}^2(\Omega) \cap H_0^1(\Omega)$ . By Lemma 3.8, we have  $u = u_1$  and so,  $u_1 \in H_{loc}^2(\Omega) \cap H_0^1(\Omega)$ . Thus  $u_{p_n} \rightarrow u_1$  in  $L^2(\Omega)$ .

Let us define  $M = \|u_1\|_\infty$  and  $\omega = \omega_+ \cup \omega_-$ . We claim that  $u_1$  satisfies, on  $\Omega$  the PDE

$$-\Delta u + \chi_\omega f = f. \quad (3.12)$$

Indeed, setting  $\Omega_t = \Omega \cap \{|u| < t\}$  for  $t > 0$ , we compute the variation of  $J_1$  with respect to any function  $\varphi \in H_0^1(\Omega_{M-\varepsilon})$ . Namely we consider functions of the form  $\varphi = \psi w_\varepsilon$  where  $w_\varepsilon$  is the solution to  $-\Delta w_\varepsilon = 1$  on  $\Omega_{M-\varepsilon}$ , and  $w_\varepsilon = 0$  on  $\partial\Omega_{M-\varepsilon}$ . Thus we obtain that  $-\Delta u_1 = f$  on  $\Omega_{M-\varepsilon}$  and letting  $\varepsilon \rightarrow 0$  we conclude, thanks to the Monotone Convergence Theorem, that

$$-\Delta u_1 = f \quad \text{on } \Omega_M = \Omega \setminus \omega.$$

Moreover, since  $u_1 \in H_{loc}^2(\Omega)$ , we have that  $\Delta u_1 = 0$  on  $\omega$  and so, we obtain (3.12). Since  $u_1$  is the minimizer of  $J_1$ , we have that for each  $\varepsilon \in \mathbb{R}$ ,  $J_1((1 + \varepsilon)u_1) - J_1(u_1) \geq 0$ . Taking the derivative of this difference at  $\varepsilon = 0$ , we obtain

$$\int_\Omega |\nabla u_1|^2 dx + M^2 = \int_\Omega f u_1 dx.$$

By (3.12), we have  $\int_\Omega |\nabla u_1|^2 dx = \int_{\Omega \setminus \omega} f u_1 dx$  and so

$$M = \int_{\omega_+} f dx - \int_{\omega_-} f dx.$$

Setting  $V_1 := \frac{1}{M} (\chi_{\omega_+} f - \chi_{\omega_-} f)$ , we have that  $\int_\Omega V_1 dx = 1$ ,  $-\Delta u_1 + V_1 u_1 = f$  in  $H^{-1}(\Omega)$  and

$$J_1(u_1) = \frac{1}{2} \int_\Omega |\nabla u_1|^2 dx + \frac{1}{2} \int_\Omega u_1^2 V_1 dx - \int_\Omega u_1 f dx.$$

We are left to prove that  $V_1$  is admissible, i.e.  $V_1 \geq 0$ . To do this, consider  $w_\varepsilon$  the energy function of the quasi-open set  $\{u < M - \varepsilon\}$  and let  $\varphi = w_\varepsilon \psi$  where  $\psi \in C_c^\infty(\mathbb{R}^d)$ ,  $\psi \geq 0$ . Since  $\varphi \geq 0$ , we get that

$$0 \leq \lim_{t \rightarrow 0^+} \frac{J_1(u_1 + t\varphi) - J_1(u_1)}{t} = \int_{\Omega} \langle \nabla u_1, \nabla \varphi \rangle dx - \int_{\Omega} f\varphi dx.$$

This inequality holds for any  $\psi$  so that, integrating by parts, we obtain

$$-\Delta u_1 - f \geq 0$$

almost everywhere on  $\{u_1 < M - \varepsilon\}$ . In particular, since  $\Delta u_1 = 0$  almost everywhere on  $\omega_- = \{u = -M\}$ , we obtain that  $f \leq 0$  on  $\omega_-$ . Arguing in the same way, and considering test functions supported on  $\{u_1 \geq -M + \varepsilon\}$ , we can prove that  $f \geq 0$  on  $\omega_+$ . This implies  $V_1 \geq 0$  as required.  $\square$

*Remark 3.10.* Under some additional assumptions on  $\Omega$  and  $f$  one can obtain some more precise regularity results for  $u_1$ . In fact, in [15, Theorem A1] it was proved that if  $\partial\Omega \in C^2$  and if  $f \in L^\infty(\Omega)$  is positive, then  $u_1 \in C^{1,1}(\overline{\Omega})$ .

*Remark 3.11.* In the case  $p < 1$  problem (3.4) does not admit, in general, a solution, even for regular  $f$  and  $\Omega$ . We give a counterexample in dimension one, which can be easily adapted to higher dimensions.

Let  $\Omega = (0, 1)$ ,  $f = 1$ , and let  $x_{n,k} = k/n$  for any  $n \in \mathbb{N}$  and  $k = 1, \dots, n-1$ . We define the capacitary measures  $\mu_n = I_{\Omega \setminus K_n}$  where  $K_n = \{k/n : k = 1, \dots, n-1\}$  and  $I_{\Omega \setminus K_n}$  is defined in (2.1). Let  $w_n$  be the minimizer of the functional  $J_{\mu_n}(1, \cdot)$ , defined in (2.2). Then  $w_n$  vanishes at  $x_{n,k}$ , for  $k = 1, \dots, n-1$ , and so we have

$$\mathcal{E}(\mu_n) = n \min \left\{ \frac{1}{2} \int_0^{1/n} |u'|^2 dx - \int_0^{1/n} u dx : u \in H_0^1(0, 1/n) \right\} = -\frac{C}{n^2},$$

where  $C > 0$  is a constant.

For any fixed  $n$  and  $j$ , let  $V_j^n$  be the sequence of positive functions such that  $\int_0^1 |V_j^n|^p dx = 1$ , defined by

$$V_j^n = C_n \sum_{k=1}^{n-1} j^{1/p} \chi_{\left[\frac{k}{n} - \frac{1}{j}, \frac{k}{n} + \frac{1}{j}\right]} < \sum_{k=1}^{n-1} I_{\Omega \setminus \left[\frac{k}{n} - \frac{1}{j}, \frac{k}{n} + \frac{1}{j}\right]}, \quad (3.13)$$

where  $C_n$  is a constant depending on  $n$ , and  $I$  is as in (2.1). By the compactness of the  $\gamma$ -convergence, we have that, up to a subsequence,  $V_j^n dx$   $\gamma$ -converges to some capacitary measure  $\mu$  as  $j \rightarrow \infty$ . On the other hand it is easy to check that  $\sum_{k=1}^{n-1} I_{\Omega \setminus \left[\frac{k}{n} - \frac{1}{j}, \frac{k}{n} + \frac{1}{j}\right]}$   $\gamma$ -converges to  $\mu_n$  as  $j \rightarrow \infty$ . By (3.13), we have that  $\mu \leq \mu_n$ . In order to show that  $\mu = \mu_n$  it is enough to check that each nonnegative function  $u \in H_0^1((0, 1))$ , for which  $\int u^2 d\mu < +\infty$ , vanishes at  $x_{n,k}$  for  $k = 1, \dots, n-1$ . Suppose that  $u(k/n) > 0$ . By the definition of the  $\gamma$ -convergence, there is a sequence  $u_j \in H_0^1(\Omega) = H_{V_j^n}^1(\Omega)$  such that  $u_j \rightarrow u$  weakly in  $H_0^1(\Omega)$  and  $\int u_j^2 V_j^n dx \leq C$ , for some constant  $C$  not depending on  $j \in \mathbb{N}$ . Since  $u_j$  are uniformly  $1/2$ -Hölder continuous, we can suppose that  $u_j \geq \varepsilon > 0$  on some interval  $A$  containing  $k/n$ . But then for  $j$  large enough  $A$  contains  $[k/n - 1/j, k/n + 1/j]$  so that

$$C \geq \int_0^1 u_j^2 V_j^n dx \geq \int_{k/n-1/j}^{k/n+1/j} u_j^2 V_j^n dx \geq 2C_n \varepsilon^2 j^{1/p-1},$$

which is a contradiction for  $p < 1$ . Thus, we have that  $\mu = \mu_n$  and so  $V_j^n$   $\gamma$ -converges to  $\mu_n$  as  $j \rightarrow \infty$ . In particular,  $\mathcal{E}(\mu_n) = \lim_{j \rightarrow \infty} \mathcal{E}_1(V_j^n)$  and since the left-hand side converges to zero as  $n \rightarrow \infty$ , we can choose a diagonal sequence  $V_{j_n}^n$  such that  $\mathcal{E}(V_{j_n}^n) \rightarrow 0$  as  $n \rightarrow \infty$ . Since there is no admissible functional  $V$  such that  $\mathcal{E}_1(V) = 0$ , we have the conclusion.

#### 4. EXISTENCE OF OPTIMAL POTENTIALS FOR UNBOUNDED CONSTRAINTS

In this section we consider the optimization problem

$$\min \{F(V) : V \in \mathcal{V}\}, \quad (4.1)$$

where  $\mathcal{V}$  is an admissible class of nonnegative Borel functions on the bounded open set  $\Omega \subset \mathbb{R}^d$  and  $F$  is a cost functional on the family of capacitary measures  $\mathcal{M}_{\text{cap}}^+(\Omega)$ . The admissible classes we study depend on a function  $\Psi : [0, +\infty) \rightarrow [0, +\infty)$

$$\mathcal{V} = \left\{ V : \Omega \rightarrow [0, +\infty) : V \text{ Lebesgue measurable, } \int_{\Omega} \Psi(V) dx \leq 1 \right\}.$$

**Theorem 4.1.** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded open set and let  $\Psi : [0, +\infty) \rightarrow [0, +\infty)$  be an injective function satisfying the condition*

$$\text{there exist } p > 1 \text{ such that the function } s \mapsto \Psi^{-1}(s^p) \text{ is convex.} \quad (4.2)$$

*Then, for any functional  $F : \mathcal{M}_{\text{cap}}^+(\Omega) \rightarrow \mathbb{R}$  which is increasing and lower semicontinuous with respect to the  $\gamma$ -convergence, the problem (4.1) has a solution, provided the admissible set  $\mathcal{V}$  is nonempty.*

*Proof.* Let  $V_n \in \mathcal{V}$  be a minimizing sequence for problem (4.1). Then,  $v_n := (\Psi(V_n))^{1/p}$  is a bounded sequence in  $L^p(\Omega)$  and so, up to a subsequence,  $v_n$  converges weakly in  $L^p(\Omega)$  to some function  $v$ . We will prove that  $V := \Psi^{-1}(v^p)$  is a solution to (4.1). Clearly  $V \in \mathcal{V}$  and so it remains to prove that  $F(V) \leq \liminf_n F(V_n)$ . In view of the compactness of the  $\gamma$ -convergence on the class  $\mathcal{M}_{\text{cap}}^+(\Omega)$  of capacitary measures (see Section 2), we can suppose that, up to a subsequence,  $V_n$   $\gamma$ -converges to a capacitary measure  $\mu \in \mathcal{M}_{\text{cap}}^+(\Omega)$ . We claim that the following inequalities hold true:

$$F(V) \leq F(\mu) \leq \liminf_{n \rightarrow \infty} F(V_n). \quad (4.3)$$

In fact, the second inequality in (4.3) is the lower semicontinuity of  $F$  with respect to the  $\gamma$ -convergence, while the first needs a more careful examination. By the definition of  $\gamma$ -convergence, we have that for any  $u \in H_0^1(\Omega)$ , there is a sequence  $u_n \in H_0^1(\Omega)$  which converges to  $u$  in  $L^2(\Omega)$  and is such that

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} u^2 d\mu &= \lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^2 dx + \int_{\Omega} u_n^2 V_n dx \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^2 dx + \int_{\Omega} u_n^2 \Psi^{-1}(v_n^p) dx \\ &\geq \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} u^2 \Psi^{-1}(v^p) dx \\ &= \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} u^2 V dx, \end{aligned} \quad (4.4)$$

where the inequality in (4.4) is due to strong-weak lower semicontinuity of integral functionals (see for instance [7]), which follows by assumption (4.2). Thus, for any  $u \in H_0^1(\Omega)$ , we have

$$\int_{\Omega} u^2 d\mu \geq \int_{\Omega} u^2 V dx,$$

which gives  $V \leq \mu$ . Since  $F$  is assumed to be monotone increasing, we obtain the first inequality in (4.3) and so the conclusion.  $\square$

*Remark 4.2.* The condition on the function  $\Psi$  in Theorem 4.1 is satisfied for instance by the following functions:

- (1)  $\Psi(s) = s^{-p}$ , for any  $p > 0$ ;
- (2)  $\Psi(s) = e^{-\alpha s}$ , for any  $\alpha > 0$ .

In some special cases, the solution to the optimization problem (4.1) can be computed explicitly through the solution to some PDE, as in Proposition 3.6. This occurs for instance when  $F = \lambda_1$  or when  $F = \mathcal{E}_f$ , with  $f \in L^2(\Omega)$ . We note that, by the variational formulation

$$\lambda_1(V) = \min \left\{ \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} u^2 V dx : u \in H_0^1(\Omega), \int_{\Omega} u^2 dx = 1 \right\},$$

we can rewrite problem (4.1) as

$$\begin{aligned} \min \left\{ \min_{\|u\|_2=1} \left\{ \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} u^2 V dx \right\} : V \geq 0, \int_{\Omega} \Psi(V) dx \leq 1 \right\} \\ = \min_{\|u\|_2=1} \left\{ \min \left\{ \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} u^2 V dx : V \geq 0, \int_{\Omega} \Psi(V) dx \leq 1 \right\} \right\}. \end{aligned} \quad (4.5)$$

One can compute that, if  $\Psi$  is differentiable with  $\Psi'$  invertible, then the second minimum in (4.5) is achieved for

$$V = (\Psi')^{-1}(\Lambda_u u^2), \quad (4.6)$$

where  $\Lambda_u$  is a constant such that  $\int_{\Omega} \Psi((\Psi')^{-1}(\Lambda_u u^2)) dx = 1$ . Thus, the solution to the problem on the right hand side of (4.5) is given through the solution to

$$\min \left\{ \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} u^2 (\Psi')^{-1}(\Lambda_u u^2) dx : u \in H_0^1(\Omega), \int_{\Omega} u^2 dx = 1 \right\}. \quad (4.7)$$

Analogously, we obtain that the optimal potential for the Dirichlet Energy  $\mathcal{E}_f$  is given by (4.6), where this time  $u$  is a solution to

$$\min \left\{ \int_{\Omega} \frac{1}{2} |\nabla u|^2 dx + \int_{\Omega} \frac{1}{2} u^2 (\Psi')^{-1}(\Lambda_u u^2) dx - \int_{\Omega} f u dx : u \in H_0^1(\Omega) \right\}. \quad (4.8)$$

Thus we obtain the following result.

**Corollary 4.3.** *Under the assumptions of Theorem 4.1, for the functionals  $F = \lambda_1$  and  $F = \mathcal{E}_f$  there exists a solution to (4.1) given by  $V = (\Psi')^{-1}(\Lambda_u u^2)$ , where  $u \in H_0^1(\Omega)$  is a minimizer of (4.7), in the case  $F = \lambda_1$ , and of (4.8), in the case  $F = \mathcal{E}_f$ .*

*Example 4.4.* If  $\Psi(x) = x^{-p}$  with  $p > 0$ , the optimal potentials for  $\lambda_1$  and  $\mathcal{E}_f$  are given by

$$V = \left( \int_{\Omega} |u|^{2p/(p+1)} dx \right)^{1/p} u^{-2/(p+1)},$$

where  $u$  is the minimizer of (4.7) and (4.8), respectively. We also note that, in this case

$$\int_{\Omega} u^2 (\Psi')^{-1}(\Lambda_u u^2) dx = \left( \int_{\Omega} |u|^{2p/(p+1)} dx \right)^{(1+p)/p}.$$



*Example 4.5.* If  $\Psi(x) = e^{-\alpha x}$  with  $\alpha > 0$ , the optimal potentials for  $\lambda_1$  and  $\mathcal{E}_f$  are given by

$$V = \frac{1}{\alpha} \left( \log \left( \int_{\Omega} u^2 dx \right) - \log(u^2) \right),$$

where  $u$  is the minimizer of (4.7) and (4.8), respectively. We also note that, in this case

$$\int_{\Omega} u^2 (\Psi')^{-1}(\Lambda_u u^2) dx = \frac{1}{\alpha} \left( \int_{\Omega} u^2 dx \int_{\Omega} \log(u^2) dx - \int_{\Omega} u^2 \log(u^2) dx \right).$$

## 5. OPTIMIZATION PROBLEMS IN UNBOUNDED DOMAINS

In this section we consider optimization problems for which the domain region is the entire Euclidean space  $\mathbb{R}^d$ . General existence results, in the case when the design region  $\Omega$  is unbounded, are hard to achieve since most of the cost functionals are not semicontinuous with respect to the  $\gamma$ -convergence in these domains. For example, it is not hard to check that if  $\mu$  is a capacitary measure, infinite outside the unit ball  $B_1$ , then, for every  $x_n \rightarrow \infty$ , the sequence of translated measures  $\mu_n = \mu(\cdot + x_n)$   $\gamma$ -converges to the capacitary measure

$$I_{\emptyset}(E) = \begin{cases} 0, & \text{if } \text{cap}(E) = 0, \\ +\infty, & \text{if } \text{cap}(E) > 0. \end{cases}$$

Thus increasing and translation invariant functionals are never lower semicontinuous with respect to the  $\gamma$ -convergence. In some special cases, as the Dirichlet Energy or the first eigenvalue of the Dirichlet Laplacian, one can obtain existence results by a more direct methods, as those in Proposition 3.6.

For a potential  $V \geq 0$  and a function  $f \in L^q(\mathbb{R}^d)$ , we define the Dirichlet energy  $\mathcal{E}_f(V)$  as in (1.2). In some cases it is convenient to work with the space  $\dot{H}^1(\mathbb{R}^d)$ , obtained as the closure of  $C_c^\infty(\mathbb{R}^d)$  with respect to the  $L^2$  norm of the gradient, instead of the classical Sobolev space  $H^1(\mathbb{R}^d)$ . In fact, since the energy only contains the term  $|\nabla u|^2$ , its minimizers are not necessarily in  $L^2(\mathbb{R}^d)$ . We recall that if  $d \geq 3$ , the Gagliardo-Nirenberg-Sobolev inequality

$$\|u\|_{L^{2d/(d-2)}} \leq C_d \|\nabla u\|_{L^2}, \quad \forall u \in \dot{H}^1(\mathbb{R}^d), \quad (5.1)$$

holds, while in the cases  $d \leq 2$ , we have respectively

$$\|u\|_{L^\infty} \leq \left( \frac{r+2}{2} \right)^{2/(r+2)} \|u\|_{L^r}^{r/(r+2)} \|u'\|_{L^2}^{2/(r+2)}, \quad \forall r \geq 1, \forall u \in \dot{H}^1(\mathbb{R}); \quad (5.2)$$

$$\|u\|_{L^{r+2}} \leq \left( \frac{r+2}{2} \right)^{2/(r+2)} \|u\|_{L^r}^{r/(r+2)} \|\nabla u\|_{L^2}^{2/(r+2)}, \quad \forall r \geq 1, \forall u \in \dot{H}^1(\mathbb{R}^2). \quad (5.3)$$

**5.1. Optimal potentials in  $L^p(\mathbb{R}^d)$ .** In this section we consider the maximization problems for the Dirichlet energy  $\mathcal{E}_f$  among potentials  $V \geq 0$  satisfying a constraint of the form  $\|V\|_{L^p} \leq 1$ . We note that the results in this section hold in a generic unbounded domain  $\Omega$ . Nevertheless, for sake of simplicity, we restrict our attention to the case  $\Omega = \mathbb{R}^d$ .

**Proposition 5.1.** *Let  $p > 1$  and let  $q$  be in the interval with end-points  $a = 2p/(p+1)$  and  $b = \max\{1, 2d/(d+2)\}$  (with  $a$  included for every  $d \geq 1$ , and  $b$  included for every  $d \neq 2$ ). Then, for every  $f \in L^q(\mathbb{R}^d)$ , there is a unique solution to the problem*

$$\max \left\{ \mathcal{E}_f(V) : V \geq 0, \int_{\mathbb{R}^d} V^p dx \leq 1 \right\}. \quad (5.4)$$

*Proof.* Arguing as in Proposition 3.6, we have that for  $p > 1$  the optimal potential  $V_p$  is given by

$$V_p = \left( \int_{\mathbb{R}^d} |u_p|^{2p/(p-1)} dx \right)^{-1/p} |u_p|^{2/(p-1)}, \quad (5.5)$$

where  $u_p$  is the solution to the problem

$$\min \left\{ \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{1}{2} \left( \int_{\mathbb{R}^d} |u|^{2p/(p-1)} dx \right)^{(p-1)/p} - \int_{\mathbb{R}^d} u f dx : \right. \\ \left. u \in \dot{H}^1(\mathbb{R}^d) \cap L^{2p/(p-1)}(\mathbb{R}^d) \right\}. \quad (5.6)$$

Thus, it is enough to prove that there exists a solution to (5.6). For a minimizing sequence  $u_n$  we have

$$\frac{1}{2} \int_{\mathbb{R}^d} |\nabla u_n|^2 dx + \frac{1}{2} \left( \int_{\mathbb{R}^d} |u_n|^{2p/(p-1)} dx \right)^{(p-1)/p} \leq \int_{\mathbb{R}^d} u_n f dx \leq C \|f\|_{L^q} \|u_n\|_{L^{q'}}.$$

Suppose that  $d \geq 3$ . Interpolating  $q'$  between  $2p/(p-1)$  and  $2d/(d-2)$  and using the Gagliardo-Nirenberg-Sobolev inequality (5.1), we obtain that there is a constant  $C$ , depending only on  $p, d$  and  $f$ , such that

$$\frac{1}{2} \int_{\mathbb{R}^d} |\nabla u_n|^2 dx + \frac{1}{2} \left( \int_{\mathbb{R}^d} |u_n|^{2p/(p-1)} dx \right)^{(p-1)/p} \leq C.$$

Thus we can suppose that  $u_n$  converges weakly in  $\dot{H}^1(\mathbb{R}^d)$  and in  $L^{2p/(p-1)}(\mathbb{R}^d)$  and so, the problem (5.6) has a solution. In the case  $d \leq 2$ , the claim follows since, by using (5.2), (5.3) and interpolation, we can still estimate  $\|u_n\|_{L^{q'}}$  by means of  $\|\nabla u_n\|_{L^2}$  and  $\|u_n\|_{L^{2p/(p-1)}}$ .  $\square$

Repeating the arguments of Propositions 3.6 and 3.9, one obtains an existence result for (5.4) in the case  $p = 1$ , too.

**Proposition 5.2.** *Let  $f \in L^q(\mathbb{R}^d)$ , where  $q \in [1, \frac{2d}{d+2}]$ , if  $d \geq 3$ , and  $q = 1$ , if  $d = 1, 2$ . Then there is a unique solution  $V_1$  to problem (5.4) with  $p = 1$ , which is given by*

$$V_1 = \frac{f}{M} (\chi_{\omega_+} - \chi_{\omega_-}),$$

where  $M = \|u_1\|_{L^\infty(\mathbb{R}^d)}$ ,  $\omega_+ = \{u_1 = M\}$ ,  $\omega_- = \{u_1 = -M\}$ , and  $u_1$  is the unique minimizer of

$$\min \left\{ \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{1}{2} \|u\|_{L^\infty}^2 - \int_{\mathbb{R}^d} u f dx : u \in \dot{H}^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \right\}.$$

In particular,  $\int_{\omega_+} f dx - \int_{\omega_-} f dx = M$ ,  $f \geq 0$  on  $\omega_+$  and  $f \leq 0$  on  $\omega_-$ .

We note that, when  $p = 1$ , the support of the optimal potential  $V_1$  is contained in the support of the function  $f$ . This is not the case if  $p > 1$ , as the following example shows.

*Example 5.3.* Let  $f = \chi_{B(0,1)}$  and  $p > 1$ . By our previous analysis we know that there exist a solution  $u_p$  to problem (5.6) and a solution  $V_p$  to problem (5.4) given by (5.5). We note that  $u_p$  is positive, radially decreasing and satisfies the equation

$$-u''(r) - \frac{d-1}{r} u'(r) + C u^\alpha = 0, \quad r \in (1, +\infty),$$

where  $\alpha = 2p/(p-1) > 2$  and  $C$  is a positive constant. Thus, we have that

$$u_p(r) = k r^{2/(1-\alpha)},$$

where  $k$  is an explicit constant depending on  $C$ ,  $d$  and  $\alpha$ . In particular, we have that  $u_p$  is not compactly supported on  $\mathbb{R}^d$  (see Figure 1).

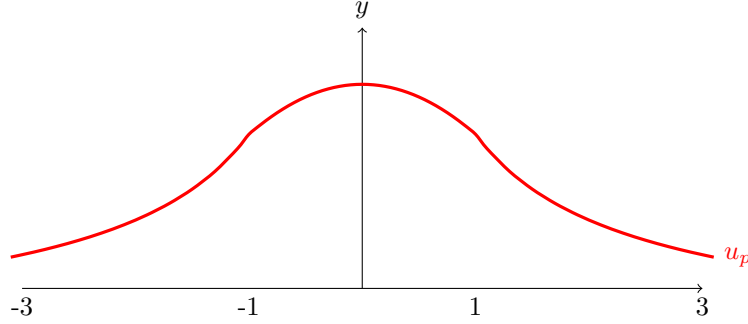


FIGURE 1. The solution  $u_p$  to problem (5.6), with  $p > 1$  and  $f = \chi_{B(0,1)}$  does not have a compact support.

**5.2. Optimal potentials with unbounded constraint.** In this subsection we consider the problems

$$\min \left\{ \mathcal{E}_f(V) : V \geq 0, \int_{\mathbb{R}^d} V^{-p} dx \leq 1 \right\}, \quad (5.7)$$

$$\min \left\{ \lambda_1(V) : V \geq 0, \int_{\mathbb{R}^d} V^{-p} dx \leq 1 \right\}, \quad (5.8)$$

for  $p > 0$  and  $f \in L^q(\mathbb{R}^d)$ . We will see in Proposition 5.4 that in order to have existence for (5.7) the parameter  $q$  must satisfy some constraint, depending on the value of  $p$  and on the dimension  $d$ . Namely, we need  $q$  to satisfy the following conditions

$$\begin{aligned} q &\in \left[ \frac{2d}{d+2}, \frac{2p}{p-1} \right], \text{ if } d \geq 3 \text{ and } p > 1, \\ q &\in \left[ \frac{2d}{d+2}, +\infty \right], \text{ if } d \geq 3 \text{ and } p \leq 1, \\ q &\in \left( 1, \frac{2p}{p-1} \right], \text{ if } d = 2 \text{ and } p > 1, \\ q &\in (1, +\infty], \text{ if } d = 2 \text{ and } p \leq 1, \\ q &\in \left[ 1, \frac{2p}{p-1} \right], \text{ if } d = 1 \text{ and } p > 1, \\ q &\in [1, +\infty], \text{ if } d = 1 \text{ and } p \leq 1. \end{aligned} \quad (5.9)$$

We say that  $q = q(p, d) \in [1, +\infty]$  is *admissible* if it satisfies (5.9). Note that  $q = 2$  is admissible for any  $d \geq 1$  and any  $p > 0$ .

**Proposition 5.4.** *Let  $p > 0$  and  $f \in L^q(\mathbb{R}^d)$ , where  $q$  is admissible in the sense of (5.9). Then the minimization problem (5.7) has a solution  $V_p$  given by*

$$V_p = \left( \int_{\mathbb{R}^d} |u_p|^{2p/(p+1)} dx \right)^{1/p} |u_p|^{-2/(1+p)}, \quad (5.10)$$

where  $u_p$  is a minimizer of

$$\min \left\{ \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{1}{2} \left( \int_{\mathbb{R}^d} |u|^{2p/(p+1)} dx \right)^{(p+1)/p} - \int_{\mathbb{R}^d} u f dx : \right. \\ \left. u \in \dot{H}^1(\mathbb{R}^d), |u|^{2p/(p+1)} \in L^1(\mathbb{R}^d) \right\}. \quad (5.11)$$

Moreover, if  $p \geq 1$ , then the functional in (5.11) is convex, its minimizer is unique and so is the solution to (5.7).

*Proof.* By means of (5.1), (5.2) and (5.3), and thanks to the admissibility of  $q$ , we get the existence of a solution to (5.11) through an interpolation argument similar to the one used in the proof of Proposition 5.1. The existence of an optimal potential follows by the same argument as in Corollary 4.3.  $\square$

In Example 5.3, we showed that the optimal potentials for (5.4), may be supported on the whole  $\mathbb{R}^d$ . The analogous question for the problem (5.7) is whether the optimal potentials given by (5.10) have a bounded set of finiteness  $\{V_p < +\infty\}$ . In order to answer this question, it is sufficient to study the support of the solutions  $u_p$  to (5.11), which solve the equation

$$-\Delta u + C_p |u|^{-2/(p+1)} u = f, \quad (5.12)$$

where  $C_p > 0$  is a constant depending on  $p$ .

**Proposition 5.5.** *Let  $p > 0$  and let  $f \in L^q(\mathbb{R}^d)$ , for  $q > d/2$ , be a nonnegative function with a compact support. Then every solution  $u_p$  to problem (5.11) has a compact support.*

*Proof.* With no loss of generality we may assume that  $f$  is supported in the unit ball of  $\mathbb{R}^d$ . We first prove the result when  $f$  is radially decreasing. In this case  $u_p$  is also radially decreasing and nonnegative. Let  $v$  be the function defined by  $v(|x|) = u_p(x)$ . Thus  $v$  satisfies the equation

$$\begin{cases} -v'' - \frac{d-1}{r} v' + C_p v^s = 0 & r \in (1, +\infty), \\ v(1) = u_p(1), \end{cases} \quad (5.13)$$

where  $s = (p-1)/(p+1)$  and  $C_p > 0$  is a constant depending on  $p$ . Since  $v \geq 0$  and  $v' \leq 0$ , we have that  $v$  is convex. Moreover, since

$$\int_1^{+\infty} v^2 r^{d-1} dr < +\infty, \quad \int_1^{+\infty} |v'|^2 r^{d-1} dr < +\infty,$$

we have that  $v$ ,  $v'$  and  $v''$  vanish at infinity. Multiplying (5.13) by  $v'$  we obtain

$$\left( \frac{v'(r)^2}{2} - C_p \frac{v(r)^{s+1}}{s+1} \right)' = -\frac{d-1}{r} v'(r)^2 \leq 0.$$

Thus the function  $v'(r)^2/2 - C_p v(r)^{s+1}/(s+1)$  is decreasing and vanishing at infinity and thus nonnegative. Thus we have

$$-v'(r) \geq C v(r)^{(s+1)/2}, \quad r \in (1, +\infty), \quad (5.14)$$

where  $C = (2C_p/(s+1))^{1/2}$ . Arguing by contradiction, suppose that  $v$  is strictly positive on  $(1, +\infty)$ . Dividing both sides of (5.14) and integrating, we have

$$-v(r)^{(1-s)/2} \geq Ar + B,$$

where  $A = 2C/(1-s)$  and  $B$  is determined by the initial datum  $v(1)$ . This cannot occur, since the left hand side is negative, while the right hand side goes to  $+\infty$ , as  $r \rightarrow +\infty$ .

We now prove the result for a generic compactly supported and nonnegative  $f \in L^q(\mathbb{R}^d)$ . Since the solution  $u_p$  to (5.11) is nonnegative and is a weak solution to (5.12), we have that on each ball  $B_R \subset \mathbb{R}^d$ ,  $u_p \leq u$ , where  $u \in H^1(B_R)$  is the solution to

$$-\Delta u = f \text{ in } B_R, \quad u = u_p \text{ on } \partial B_R.$$

Since  $f \in L^q(\mathbb{R}^d)$  with  $q > d/2$ , by [19, Theorem 9.11] and a standard bootstrap argument on the integrability of  $u$ , we have that  $u$  is continuous on  $B_{R/2}$ . As a consequence,  $u_p$  is locally bounded in  $\mathbb{R}^d$ . In particular, it is bounded since  $u_p \wedge M$ , where  $M = \|u_p\|_{L^\infty(B_1)}$ , is a better competitor than  $u_p$  in (5.11). Let  $w$  be a radially decreasing minimizer of (5.11) with  $f = \chi_{B_1}$ . Thus  $w$  is a solution to the PDE

$$-\Delta w + C_p w^s = \chi_{B_1},$$

in  $\mathbb{R}^d$ , where  $C_p$  is as in (5.13). Then, the function  $w_t(x) = t^{2/(1-s)}w(x/t)$  is a solution to the equation

$$-\Delta w_t + C_p w_t^s = t^{2s/(1-s)}\chi_{B_t}.$$

Since  $u_p$  is bounded, there exists some  $t \geq 1$  large enough such that  $w_t \geq u_p$  on the ball  $B_t$ . Moreover,  $w_t$  minimizes (5.11) with  $f = t^{2s/(1-s)}\chi_{B_t}$  and so  $w_t \geq u_p$  on  $\mathbb{R}^d$  (otherwise  $w_t \wedge u_p$  would be a better competitor in (5.11) than  $w_p$ ). The conclusion follows since, by the first step of the proof,  $w_t$  has compact support.  $\square$

The problems (5.8) and (5.7) are similar both in the questions of existence and the qualitative properties of the solutions.

**Proposition 5.6.** *For every  $p > 0$  there is a solution to the problem (5.8) given by*

$$V_p = \left( \int_{\mathbb{R}^d} |u_p|^{2p/(p+1)} dx \right)^{1/p} |u_p|^{-2/(1+p)}, \quad (5.15)$$

where  $u_p$  is a radially decreasing minimizer of

$$\min \left\{ \int_{\mathbb{R}^d} |\nabla u|^2 dx + \left( \int_{\mathbb{R}^d} |u|^{2p/(p+1)} dx \right)^{(p+1)/p} : u \in H^1(\mathbb{R}^d), \int_{\mathbb{R}^d} u^2 dx = 1 \right\}. \quad (5.16)$$

Moreover,  $u_p$  has a compact support, hence the set  $\{V_p < +\infty\}$  is a ball of finite radius in  $\mathbb{R}^d$ .

*Proof.* Let us first show that the minimum in (5.16) is achieved. Let  $u_n \in H^1(\mathbb{R}^d)$  be a minimizing sequence of positive functions normalized in  $L^2$ . Note that by the Pólya-Szegő inequality we may assume that each of these functions is radially decreasing in  $\mathbb{R}^d$  and so we will use the identification  $u_n = u_n(r)$ . In order to prove that the minimum is achieved it is enough to show that the sequence  $u_n$  converges in  $L^2(\mathbb{R}^d)$ . Indeed, since  $u_n$  is a radially decreasing minimizing sequence, there exists  $C > 0$  such that for each  $r > 0$  we have

$$u_n(r)^{2p/(p+1)} \leq \frac{1}{|B_r|} \int_{B_r} u_n^{2p/(p+1)} dx \leq \frac{C}{r^d}.$$

Thus, for each  $R > 0$ , we obtain

$$\int_{B_R^c} u_n^2 dx \leq C_1 \int_R^{+\infty} r^{-d(p+1)/p} r^{d-1} dr = C_2 R^{-1/p}, \quad (5.17)$$

where  $C_1$  and  $C_2$  do not depend on  $n$  and  $R$ . Since the sequence  $u_n$  is bounded in  $H^1(\mathbb{R}^d)$ , it converges locally in  $L^2(\mathbb{R}^d)$  and, by (5.17), this convergence is also strong in  $L^2(\mathbb{R}^d)$ . Thus, we obtain the existence of a radially symmetric and decreasing solution  $u_p$  to (5.16) and so, of an optimal potential  $V_p$  given by (5.15).

We now prove that the support of  $u_p$  is a ball of finite radius. By the radial symmetry of  $u_p$  we can write it in the form  $u_p(x) = u_p(|x|) = u_p(r)$ , where  $r = |x|$ . With this notation,  $u_p$  satisfies the equation:

$$-u_p'' - \frac{d-1}{r}u_p' + C_p u_p^s = \lambda u_p,$$

where  $s = (p-1)/(p+1) < 1$  and  $C_p > 0$  is a constant depending on  $p$ . Arguing as in Proposition 5.5, we obtain that, for  $r$  large enough,

$$-u_p'(r) \geq \left( \frac{C_p}{s+1} u_p(r)^{s+1} - \frac{\lambda}{2} u_p(r)^2 \right)^{1/2} \geq \left( \frac{C_p}{2(s+1)} u_p(r)^{s+1} \right)^{1/2},$$

where, in the last inequality, we used the fact that  $u_p(r) \rightarrow 0$ , as  $r \rightarrow \infty$ , and  $s+1 < 2$ . Integrating both sides of the above inequality, we conclude that  $u_p$  has a compact support. In Figure 2 we show the case  $d = 1$  and  $f = \chi_{(-1,1)}$ .  $\square$

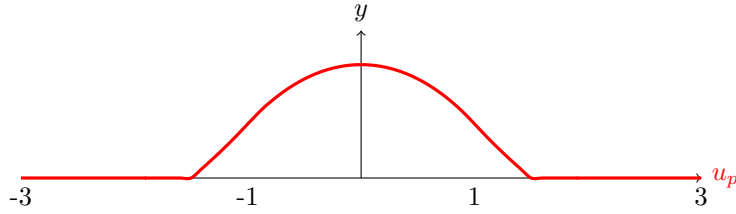


FIGURE 2. The solution  $u_p$  to problem (5.11), with  $p > 1$  and  $f = \chi_{(-1,1)}$ .

*Remark 5.7.* We note that the solution  $u_p \in H^1(\mathbb{R}^d)$  to (5.16) is the function for which the best constant  $C$  in the interpolated Gagliardo-Nirenberg-Sobolev inequality

$$\|u\|_{L^2(\mathbb{R}^d)} \leq C \|\nabla u\|_{L^2(\mathbb{R}^d)}^{d/(d+2p)} \|u\|_{L^{2p/(p+1)}(\mathbb{R}^d)}^{2p/(d+2p)} \quad (5.18)$$

is achieved. Indeed, for any  $u \in H^1(\mathbb{R}^d)$  and any  $t > 0$ , we define  $u_t(x) := t^{d/2}u(tx)$ . Thus, we have that  $\|u\|_{L^2} = \|u_t\|_{L^2}$ , for any  $t > 0$ . Moreover, up to a rescaling, we may assume that the function  $g : (0, +\infty) \rightarrow \mathbb{R}$ , defined by

$$\begin{aligned} g(t) &= \int_{\mathbb{R}^d} |\nabla u_t|^2 dx + \left( \int_{\mathbb{R}^d} |u_t|^{2p/(p+1)} dx \right)^{(p+1)/p} \\ &= t^2 \int_{\mathbb{R}^d} |\nabla u|^2 dx + t^{-d/p} \left( \int_{\mathbb{R}^d} |u|^{2p/(p+1)} dx \right)^{(p+1)/p}, \end{aligned}$$

achieves its minimum in the interval  $(0, +\infty)$  and, moreover, we have

$$\min_{t \in (0, +\infty)} g(t) = C \left( \int_{\mathbb{R}^d} |\nabla u|^2 dx \right)^{d/(d+2p)} \left( \int_{\mathbb{R}^d} |u|^{\frac{2p}{p+1}} dx \right)^{2(p+1)/(d+2p)},$$

where  $C$  is a constant depending on  $p$  and  $d$ . In the case  $u = u_p$ , the minimum of  $g$  is achieved for  $t = 1$  and so, we have that  $u_p$  is a solution also to

$$\min \left\{ \left( \int_{\mathbb{R}^d} |\nabla u|^2 dx \right)^{d/(d+2p)} \left( \int_{\mathbb{R}^d} |u|^{2p/(p+1)} dx \right)^{2(p+1)/(d+2p)} : u \in H^1(\mathbb{R}^d), \int_{\mathbb{R}^d} u^2 dx = 1 \right\},$$

which is just another form of (5.18).

*Remark 5.8.* We conclude this section with a remark about the constraint  $\Psi(s) = e^{-\alpha s}$ . This type of constraint may be used to approximate shape optimization problems, in which the main unknown is a domain  $\Omega$ , i.e. the potential  $V = I_\Omega$  is the capacitary measure of  $\Omega$ . To get an example of this fact we recall the problem

$$\min \left\{ \mathcal{E}_f(V) + \Lambda \int_{\Omega} e^{-\alpha V} dx : V \in \mathcal{B}(\Omega) \right\}, \quad (5.19)$$

where  $\Lambda$  is a Lagrange multiplier and  $\mathcal{B}(\Omega)$  is the class of nonnegative Borel measurable functions on  $\Omega$ . As before, we note that the problem (5.19) is equivalent to

$$\min \left\{ \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} V u^2 - f u + \Lambda e^{-\alpha V} \right) dx : u \in H_0^1(\Omega), V \in \mathcal{B}(\Omega) \right\}. \quad (5.20)$$

Fixing  $u \in H_0^1(\Omega)$  and minimizing in  $V \in \mathcal{B}(\Omega)$  leads to the problem

$$\min \left\{ \int_{\Omega} V u^2 dx + \Lambda \int_{\Omega} e^{-\alpha V} dx : V \in \mathcal{B}(\Omega) \right\},$$

whose solution  $V$  satisfies

$$u^2 - \Lambda \alpha e^{-\alpha V} = 0 \quad \text{on } \{V(x) > 0\}.$$

We note that if  $u^2 \geq \Lambda \alpha$ , then necessarily  $V = 0$ . On the other hand, if  $u^2 < \Lambda \alpha$ , then by the optimality of  $V$ , we have  $V > 0$ . Finally, we get

$$V(x) = 0 \vee \left( -\frac{1}{\alpha} \log \frac{u^2}{\Lambda \alpha} \right).$$

Substituting in (5.20), we obtain the problem

$$\begin{aligned} \min_{u \in H_0^1(\Omega)} \left\{ \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2\alpha} \int_{\{u^2 < \Lambda \alpha\}} u^2 \log \left( \frac{u^2}{\Lambda \alpha} \right) dx \right. \\ \left. - \int_{\Omega} f u dx + \Lambda |\{u^2 \geq \Lambda \alpha\}| + \frac{1}{\alpha} \int_{\{u^2 < \Lambda \alpha\}} u^2 dx \right\}, \end{aligned}$$

or, equivalently,

$$\begin{aligned} \min_{u \in H_0^1(\Omega)} \left\{ \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2\alpha} \int_{\{u^2 \leq \Lambda \alpha\}} u^2 \log \left( \frac{u^2}{\Lambda \alpha} \right) dx \right. \\ \left. - \int_{\Omega} f u dx + \Lambda |\{u^2 > \Lambda \alpha\}| + \frac{1}{\alpha} \int_{\{u^2 \leq \Lambda \alpha\}} u^2 dx \right\}. \end{aligned} \quad (5.21)$$

Note that the second term is actually positive and so, by a standard variational argument, we have that the problem (5.21) has a solution  $u_\alpha \in H_0^1(\Omega)$ . Moreover, on the quasi-open set  $u^2 > \Lambda \alpha$ , we have  $-\Delta u = f$ . Let  $J_\alpha$  be the functional in (5.21), i.e.

$$J_\alpha(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{\alpha} \int_{\{u^2 \leq \Lambda \alpha\}} u^2 \left[ 1 - \frac{1}{2} \log \left( \frac{u^2}{\Lambda \alpha} \right) \right] dx + \Lambda |\{u^2 > \Lambda \alpha\}|.$$

Then  $J_\alpha$   $\Gamma$ -converges in  $L^2(\Omega)$ , as  $\alpha \rightarrow 0$ , to the functional

$$J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \Lambda|\{u \neq 0\}|.$$

Note that this implies the convergence of the optimal potentials  $V_\alpha$  for (5.19) to a limit potential of the form

$$V(x) = \begin{cases} +\infty & \text{if } u(x) = 0 \\ 0 & \text{if } u(x) \neq 0, \end{cases}$$

where  $u$  is a solution to the limit problem

$$\min \left\{ \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} f u dx + \Lambda|\{u \neq 0\}| : u \in H_0^1(\Omega) \right\}.$$

This limit problem is indeed a shape optimization problem written in terms of the state function  $u$ , and several results on the regularity of the optimal domains are known (see for instance [1], [3]).

## 6. FURTHER REMARKS ON THE OPTIMAL POTENTIALS FOR SPECTRAL FUNCTIONALS

We recall (see [5]) that the injection  $H_V^1(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$  is compact whenever the potential  $V$  satisfies  $\int_{\mathbb{R}^d} V^{-p} dx < +\infty$  for some  $0 < p < 1$ . In this case the spectrum of the Schrödinger operator  $-\Delta + V(x)$  is discrete and we denote by  $\lambda_k(V)$  its eigenvalues. The existence of an optimal potential for spectral optimization problems of the form

$$\min \left\{ \lambda_k(V) : V \geq 0, \int_{\mathbb{R}^d} V^{-p} dx \leq 1 \right\}, \quad (6.1)$$

was proved in [6], for any  $k \in \mathbb{N}$  and for  $p \in (0, 1)$ . This result cannot be deduced by the direct methods used in Subsection 5.2 and is based on a combination of a concentration-compactness argument and a fine estimate on the diameter of the set of finiteness  $\{V_k < +\infty\}$  of the optimal potential  $V_k$ . We note that the existence of an optimal potential in the case  $k > 2$  and  $p = 1$  is still an open question.

In the case  $k = 2$  the idea from Subsection 5.2 can still be applied to prove an existence result for (6.1) and to explicitly characterize the optimal potential. We first recall that, by Proposition 5.6, there exists optimal potential  $V_p$ , for  $\lambda_1$ , such that the set of finiteness  $\{V_p < +\infty\}$  is a ball. Thus, we have a situation analogous to the Faber-Krahn inequality, which states that the minimum

$$\min \left\{ \lambda_1(\Omega) : \Omega \subset \mathbb{R}^d, |\Omega| = c \right\}, \quad (6.2)$$

is achieved for the ball of measure  $c$ . We recall that, starting from (6.2), one may deduce, by a simple argument (see for instance [21]), the Krahn-Szegö inequality, which states that the minimum

$$\min \left\{ \lambda_2(\Omega) : \Omega \subset \mathbb{R}^d, |\Omega| = c \right\},$$

is achieved for a disjoint union of equal balls. In the case of potentials one can find two optimal potentials for  $\lambda_1$  with disjoint sets of finiteness and then apply the argument from the proof of the Krahn-Szegö inequality. In fact, we have the following result.

**Theorem 6.1.** *There exists an optimal potential, solution to (6.1) with  $k = 2$  and  $p \in (0, 1]$ . Moreover, any optimal potential is of the form  $\min\{V_1, V_2\}$ , where  $V_1$  and  $V_2$  are optimal potentials for  $\lambda_1$  which have disjoint sets of finiteness  $\{V_1 < +\infty\} \cap \{V_2 < +\infty\} = \emptyset$  and are such that  $\int_{\mathbb{R}^d} V_1^{-p} dx = \int_{\mathbb{R}^d} V_2^{-p} dx = 1/2$ .*



*Proof.* Given  $V_1$  and  $V_2$  as above, we prove that for every  $V : \mathbb{R}^d \rightarrow [0, +\infty]$  with  $\int_{\mathbb{R}^d} V^{-p} dx = 1$ , we have

$$\lambda_2(\min\{V_1, V_2\}) \leq \lambda_2(V).$$

Indeed, let  $u_2$  be the second eigenfunction of  $-\Delta + V(x)$ . We first suppose that  $u_2$  changes sign on  $\mathbb{R}^d$  and consider the functions  $V_+ = \sup\{V, \infty_{\{u_2 \leq 0\}}\}$  and  $V_- = \sup\{V, \infty_{\{u_2 \geq 0\}}\}$  where, for any measurable  $A \subset \mathbb{R}^d$ , we set

$$\infty_A(x) = \begin{cases} +\infty, & x \in A, \\ 0, & x \notin A. \end{cases}$$

We note that

$$1 \geq \int_{\mathbb{R}^d} V^{-p} dx \geq \int_{\mathbb{R}^d} V_+^{-p} dx + \int_{\mathbb{R}^d} V_-^{-p} dx.$$

Moreover, on the sets  $\{u_2 > 0\}$  and  $\{u_2 < 0\}$ , the following equations are satisfied:

$$-\Delta u_2^+ + V_+ u_2^+ = \lambda_2(V) u_2^+, \quad -\Delta u_2^- + V_- u_2^- = \lambda_2(V) u_2^-,$$

and so, multiplying respectively by  $u_2^+$  and  $u_2^-$ , we obtain that

$$\lambda_2(V) \geq \lambda_1(V_+), \quad \lambda_2(V) \geq \lambda_1(V_-),$$

where we have equalities, if and only if,  $u_2^+$  and  $u_2^-$  are the first eigenfunctions corresponding to  $\lambda_1(V_+)$  and  $\lambda_1(V_-)$ . Let now  $\tilde{V}_+$  and  $\tilde{V}_-$  be optimal potentials for  $\lambda_1$  corresponding to the constraints

$$\int_{\mathbb{R}^d} \tilde{V}_+^{-p} dx = \int_{\mathbb{R}^d} V_+^{-p} dx, \quad \int_{\mathbb{R}^d} \tilde{V}_-^{-p} dx = \int_{\mathbb{R}^d} V_-^{-p} dx.$$

By Proposition 5.6, the sets of finiteness of  $\tilde{V}_+$  and  $\tilde{V}_-$  are compact, hence we may assume (up to translations) that they are also disjoint. By the monotonicity of  $\lambda_1$ , we have

$$\max\{\lambda_1(V_1), \lambda_1(V_2)\} \leq \max\{\lambda_1(\tilde{V}_+), \lambda_1(\tilde{V}_-)\},$$

and so, we obtain

$$\lambda_2(\min\{V_1, V_2\}) \leq \max\{\lambda_1(\tilde{V}_+), \lambda_1(\tilde{V}_-)\} \leq \max\{\lambda_1(V_+), \lambda_1(V_-)\} \leq \lambda_2(V),$$

as required. If  $u_2$  does not change sign, then we consider  $V_+ = \sup\{V, \infty_{\{u_2=0\}}\}$  and  $V_- = \sup\{V, \infty_{\{u_1=0\}}\}$ , where  $u_1$  is the first eigenfunction of  $-\Delta + V(x)$ . Then the claim follows by the same argument as above.  $\square$

**Acknowledgements.** This work is part of the project 2010A2TFX2 “*Calcolo delle Variazioni*” funded by the Italian Ministry of Research and University. The first, second and third authors are members of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

## REFERENCES

- [1] H.W. ALT, L.A. CAFFARELLI: *Existence and regularity for a minimum problem with free boundary*. J. Reine Angew. Math., 325 (1981), 105–144.
- [2] M.S. ASHBAUGH, E.M. HARRELL: *Maximal and minimal eigenvalues and their associated nonlinear equations*. J. Math. Phys. **28** (8) (1987), 1770–1786.
- [3] T. BRIANÇON, J. LAMBOLEY: *Regularity of the optimal shape for the first eigenvalue of the Laplacian with volume and inclusion constraints*. Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), no. 4, 1149–1163.
- [4] D. BUCUR, G. BUTTAZZO: *Variational Methods in Shape Optimization Problems*. Progress in Nonlinear Differential Equations **65**, Birkhäuser Verlag, Basel (2005).

- [5] D. BUCUR, G. BUTTAZZO: *On the characterization of the compact embedding of Sobolev spaces*. Calc. Var., **44** (3) (2012), 455–475.
- [6] D. BUCUR, G. BUTTAZZO, B. VELICHKOV: *Spectral optimization problems for potentials and measures*. Preprint available at <http://cvgmt.sns.it>.
- [7] G. BUTTAZZO: *Semicontinuity, relaxation and integral representation in the calculus of variations*. Pitman Research Notes in Mathematics **207**, Longman, Harlow (1989).
- [8] G. BUTTAZZO: *Spectral optimization problems*. Rev. Mat. Complut., **24** (2) (2011), 277–322.
- [9] G. BUTTAZZO, G. DAL MASO: *Shape optimization for Dirichlet problems: relaxed formulation and optimality conditions*. Appl. Math. Optim., **23** (1991), 17–49.
- [10] G. BUTTAZZO, G. DAL MASO: *An existence result for a class of shape optimization problems*. Arch. Rational Mech. Anal., **122** (1993), 183–195.
- [11] E.A. CARLEN, R.L. FRANK, E.H. LIEB: *Stability estimates for the lowest eigenvalue of a Schrödinger operator*. Geom. Funct. Anal., (to appear).
- [12] G. DAL MASO: *An Introduction to  $\Gamma$ -convergence*. Birkhäuser Verlag, Basel (1993).
- [13] G. BUTTAZZO, N. VARCHON, H. ZOUBAIRI: *Optimal measures for elliptic problems*. Ann. Mat. Pura Appl., **185** (2) (2006), 207–221.
- [14] G. DAL MASO, U. MOSCO: *Wiener’s criterion and  $\Gamma$ -convergence*. Appl. Math. Optim., **15** (1987), 15–63.
- [15] H. EGNELL: *Extremal properties of the first eigenvalue of a class of elliptic eigenvalue problems*. Ann. Sc. Norm. Super. Pisa Cl. Sci., **14** (1) (1987), 1–48.
- [16] M. ESSEN: *On estimating eigenvalues of a second order linear differential operator*. General inequalities **5**, Oberwolfach (1986), 347–366; Internat. Schriftenreihe Numer. Math. **80** (1987).
- [17] L. EVANS: *Partial Differential Equations*. American Mathematical Society, Providence (1998).
- [18] L. EVANS, R. GARIEPY: *Measure Theory and Fine Properties of Functions*. Studies in Advanced mathematics, Crc Press, Boca Raton (1991).
- [19] D. GILBARG, N.S. TRUDINGER: *Elliptic partial differential equations of second order*. Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin (2001).
- [20] E.M. HARRELL: *Hamiltonian operators with maximal eigenvalues*. J. Math. Phys. **25** (1) (1984), 48–51; **27** (1) (1986), 419.
- [21] A. HENROT: *Minimization problems for eigenvalues of the Laplacian*. J. Evol. Equ., **3** (3) (2003), 443–461.
- [22] A. HENROT: *Extremum Problems for Eigenvalues of Elliptic Operators*. Frontiers in Mathematics, Birkhäuser Verlag, Basel (2006).
- [23] A. HENROT, M. PIERRE: *Variation et Optimisation de Formes. Une Analyse Géométrique*. Mathématiques & Applications **48**, Springer-Verlag, Berlin (2005).
- [24] G. TALENTI: *Estimates for eigenvalues of Sturm-Liouville problems*. General inequalities **4** (Oberwolfach, 1983), International Series of Numerical Mathematics **71**, Birkhäuser Verlag, Basel (1984), 341–350.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PISA, LARGO B. PONTECORVO 5, 56126 PISA, ITALY  
*E-mail address:* `buttazzo@dm.unipi.it`

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PISA, LARGO B. PONTECORVO 5, 56126 PISA, ITALY  
*E-mail address:* `gerolin@mail.dm.unipi.it`

SCUOLA NORMALE SUPERIORE DI PISA, PIAZZA DEI CAVALIERI 7, 56126 PISA, ITALY  
*E-mail address:* `berardo.ruffini@sns.it`

SCUOLA NORMALE SUPERIORE DI PISA, PIAZZA DEI CAVALIERI 7, 56126 PISA, ITALY  
*E-mail address:* `b.velichkov@sns.it`