OPTIMAL POTENTIALS FOR SCHRÖDINGER OPERATORS

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Abstract. We consider the Schrödinger operator $-\Delta + V(x)$ on $H^1_0(\Omega)$, where Ω is a given domain of $\mathbb{R}^d$. Our goal is to study some optimization problems where an optimal potential $V \geq 0$ has to be determined in some suitable admissible classes and for some suitable optimization criteria, like the energy or the Dirichlet eigenvalues.

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1. Introduction

In this paper we consider the Schrödinger operator $-\Delta+V(x)$ on $H^1_0(\Omega)$, where Ω is a given domain of $\mathbb{R}^d$. Our goal is to study some optimization problems where an optimal potential $V \geq 0$ has to be determined, for some suitable optimization criteria, among the ones belonging to some admissible classes. The problems we are dealing with are then

$$\min\big\{F(V)\ :\ V\in\mathcal{V}\big\},$$

where F denotes the cost functional and $\mathcal{V}$ the admissible class. The cost functionals we aim to include in our framework are for instance the following.

Integral functionals. Given a right-hand side $f \in L^2(\Omega)$ we consider the solution u_V of the elliptic PDE

$$-\Delta u + Vu = f \text{ in } \Omega, \qquad u \in H^1_0(\Omega).$$

The integral cost functionals we may consider are of the form

$$F(V) = \int_\Omega j\big(x, u_V(x), \nabla u_V(x)\big)\,dx,$$

where j is a suitable integrand that we assume convex in the gradient variable and bounded from below. One may take, for example,

$$j(x,s,z) \geq -a(x) - c|s|^2,$$

with $a \in L^1(\Omega)$ and c smaller than the first Dirichlet eigenvalue of the Laplace operator $-\Delta$ in Ω. In particular, the energy $\mathcal{E}_f(V)$ defined by

$$\mathcal{E}_f(V) = \inf\left\{\int_\Omega \Big(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}V(x)u^2 - f(x)u\Big)\,dx\ :\ u \in H^1_0(\Omega)\right\}, \tag{1.1}$$

belongs to this class since, integrating by parts its Euler-Lagrange equation, we have

$$\mathcal{E}_f(V) = -\frac{1}{2}\int_\Omega f(x)u_V\,dx,$$

which corresponds to the integral functional above with

$$j(x,s,z) = -\frac{1}{2}f(x)s.$$

Spectral functionals. For every admissible potential $V \geq 0$ we consider the spectrum $\Lambda(V)$ of the Schrödinger operator $-\Delta+V(x)$ on $H^1_0(\Omega)$. If Ω is bounded or has finite measure, or if the potential V satisfies some suitable integral properties, the operator $-\Delta+V(x)$ has a compact resolvent and so its spectrum $\Lambda(V)$ is discrete:

$$\Lambda(V) = \big(\lambda_1(V), \lambda_2(V), \dots\big),$$

where $\lambda_k(V)$ are the eigenvalues counted with their multiplicity. The spectral cost functionals we may consider are of the form

$$F(V) = \Phi\big(\Lambda(V)\big),$$

for a suitable function $\Phi : \mathbb{R}^{\mathbb{N}} \to \overline{\mathbb{R}}$. For instance, taking $\Phi(\Lambda) = \lambda_k$ we obtain

$$F(V) = \lambda_k(V).$$

Concerning the admissible classes we deal with, we consider mainly the cases

$$\mathcal{V} = \left\{V \geq 0\ :\ \int_\Omega V^p\,dx \leq 1\right\} \qquad \text{and} \qquad \mathcal{V} = \left\{V \geq 0\ :\ \int_\Omega V^{-p}\,dx \leq 1\right\};$$

in some situations more general admissible classes $\mathcal{V}$ will be considered, see Theorem 3.1 and Theorem 4.1.

In Section 3.1 our assumptions allow to take $F(V) = -\mathcal{E}_f(V)$ and thus the optimization problem becomes the maximization of $\mathcal{E}_f$ under the constraint $\int_\Omega V^p\,dx \leq 1$. We prove that for $p \geq 1$, there exists an optimal potential for the problem

$$\max\left\{\mathcal{E}_f(V)\ :\ \int_\Omega V^p\,dx \leq 1\right\}. \tag{1.2}$$

The existence result is sharp in the sense that for $p < 1$ the maximum cannot be achieved (see Remark 3.11). For the existence issue in the case of a bounded domain, we follow the ideas of Egnell [17], summarized in [13, Chapter 8] (where a complete reference for the problem can also be found). The case $p = 1$ is particularly interesting and we show that in this case the optimal potentials are of the form

$$V_{opt} = \frac{f}{M}\left(\chi_{\omega_+} - \chi_{\omega_-}\right),$$

where χ_U indicates the characteristic function of the set U, $f \in L^2(\Omega)$, $M = \|u_V\|_{L^\infty(\Omega)}$, and $\omega_\pm = \{u = \pm M\}$. In Section 4 we deal with minimization problems of the form

$$\min\big\{F(V)\ :\ \int_\Omega V^{-p}\,dx \leq 1\big\}. \tag{1.3}$$

We prove a general result (Theorem 4.1) establishing the existence of an optimal potential under some mild conditions on the functional F. In particular, we obtain the existence of optimal potentials for a large class of spectral and energy functionals (see Corollary 4.3).

In Section 5 we deal with the case of unbounded domains Ω. precisely, we prove that in the case $\Omega = \mathbb{R}^d$ and $F = \mathcal{E}_f$ or $F = \lambda_1$, the solutions of problem (1.3) exist and are such that $1/V$ is compactly supported, provided f is compactly supported. Finally, in Section 6 we make some further remarks and present some open questions.

2. Capacitary measures and γ-convergence

For a subset $E \subset \mathbb{R}^d$ its *capacity* is defined by

$$\mathrm{cap}(E) = \inf\left\{\int_{\mathbb{R}^d} |\nabla u|^2\,dx + \int_{\mathbb{R}^d} u^2\,dx \;:\; u \in H^1(\mathbb{R}^d),\ u \geq 1 \text{ in a neighborhood of } E\right\}.$$

If a property $P(x)$ holds for all $x \in \Omega$, except for the elements of a set $E \subset \Omega$ of capacity zero, we say that $P(x)$ holds *quasi-everywhere* (shortly *q.e.*) in Ω, whereas the expression *almost everywhere* (shortly *a.e.*) refers, as usual, to the Lebesgue measure, which we often denote by $|\cdot|$.

A subset A of $\mathbb{R}^d$ is said to be *quasi-open* if for every $\varepsilon > 0$ there exists an open subset A_ε of $\mathbb{R}^d$, with $A \subset A_\varepsilon$, such that $\mathrm{cap}(A_\varepsilon \setminus A) < \varepsilon$. Similarly, a function $u : \mathbb{R}^d \to \mathbb{R}$ is said to be *quasi-continuous* (respectively *quasi-lower semicontinuous*) if there exists a decreasing sequence of open sets $(A_n)_n$ such that $\mathrm{cap}(A_n) \to 0$ and the restriction u_n of u to the set A_n^c is continuous (respectively lower semicontinuous). It is well known (see for instance [18]) that every function $u \in H^1(\mathbb{R}^d)$ has a quasi-continuous representative $\widetilde{u}$, which is uniquely defined up to a set of capacity zero, and given by

$$\widetilde{u}(x) = \lim_{\varepsilon\to 0} \frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} u(y)\,dy\,,$$

where $B_\varepsilon(x)$ denotes the ball of radius ε centered at x. We identify the (a.e.) equivalence class $u \in H^1(\mathbb{R}^d)$ with the (q.e.) equivalence class of quasi-continuous representatives $\widetilde{u}$.

We denote by $\mathcal{M}^+(\mathbb{R}^d)$ the set of positive Borel measures on $\mathbb{R}^d$ (not necessarily finite or Radon) and by $\mathcal{M}^+_{\text{cap}}(\mathbb{R}^d) \subset \mathcal{M}^+(\mathbb{R}^d)$ the set of *capacitary measures*, i.e. the measures $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ such that $\mu(E) = 0$ for any set $E \subset \mathbb{R}^d$ of capacity zero. We note that when μ is a capacitary measure, the integral $\int_{\mathbb{R}^d} |u|^2\,d\mu$ is well-defined for each $u \in H^1(\mathbb{R}^d)$, i.e. if $\widetilde{u}_1$ and $\widetilde{u}_2$ are two quasi-continuous representatives of u, then $\int_{\mathbb{R}^d} |\widetilde{u}_1|^2\,d\mu = \int_{\mathbb{R}^d} |\widetilde{u}_2|^2\,d\mu$.

For a subset $\Omega \subset \mathbb{R}^d$, we define the Sobolev space $H^1_0(\Omega)$ as

$$H^1_0(\Omega) = \left\{u \in H^1(\mathbb{R}^d) :\ u = 0 \text{ q.e. on } \Omega^c\right\}. \tag{2.1}$$

Alternatively, by using the capacitary measure I_Ω defined as

$$I_\Omega(E) = \begin{cases} 0 & \text{if } \mathrm{cap}(E \setminus \Omega) = 0 \\ +\infty & \text{if } \mathrm{cap}(E \setminus \Omega) > 0 \end{cases} \qquad \text{for every Borel set } E \subset \mathbb{R}^d, \tag{2.2}$$

the Sobolev space $H^1_0(\Omega)$ can be defined as

$$H^1_0(\Omega) = \left\{u \in H^1(\mathbb{R}^d) \;:\; \int_{\mathbb{R}^d} |u|^2\,dI_\Omega < +\infty\right\}.$$

More generally, for any capacitary measure $\mu \in \mathcal{M}^+_{\text{cap}}(\mathbb{R}^d)$, we define the space

$$H^1_\mu = \left\{u \in H^1(\mathbb{R}^d) \;:\; \int_{\mathbb{R}^d} |u|^2\,d\mu < +\infty\right\},$$

which is a Hilbert space when endowed with the norm $\|u\|_{1,\mu}$, where

$$\|u\|^2_{1,\mu} = \int_{\mathbb{R}^d} |\nabla u|^2\,dx + \int_{\mathbb{R}^d} u^2\,dx + \int_{\mathbb{R}^d} u^2\,d\mu.$$

If $u \notin H^1_\mu$, then we set $\|u\|_{1,\mu} = +\infty$.

For $\Omega \subset \mathbb{R}^d$, we define $\mathcal{M}^+_{\text{cap}}(\Omega)$ as the space of capacitary measures $\mu \in \mathcal{M}^+_{\text{cap}}(\mathbb{R}^d)$ such that $\mu(E) = +\infty$ for any set $E \subset \mathbb{R}^d$ such that $\text{cap}(E \setminus \Omega) > 0$. For $\mu \in \mathcal{M}^+_{\text{cap}}(\mathbb{R}^d)$, we denote with $H^1_\mu(\Omega)$ the space $H^1_{\mu \vee I_\Omega} = H^1_\mu \cap H^1_0(\Omega)$.

Definition 2.1. *Given a metric space (X, d) and sequence of functionals $J_n : X \to \mathbb{R} \cup \{+\infty\}$, we say that J_n Γ-converges to the functional $J : X \to \mathbb{R} \cup \{+\infty\}$, if the following two conditions are satisfied:*

(a) for every sequence x_n converging in to $x \in X$, we have

$$J(x) \le \liminf_{n\to\infty} J_n(x_n);$$

(b) for every $x \in X$, there exists a sequence x_n converging to x, such that

$$J(x) = \lim_{n\to\infty} J_n(x_n).$$

For all details and properties of Γ-convergence we refer to [8]; here we simply recall that, whenever J_n Γ-converges to J,

$$\min_{x\in X} J(x) \le \liminf_{n\to\infty} \min_{x\in X} J_n(x). \tag{2.3}$$

Definition 2.2. *We say that the sequence of capacitary measures $\mu_n \in \mathcal{M}^+_{\text{cap}}(\Omega)$, γ-converges to the capacitary measure $\mu \in \mathcal{M}^+_{\text{cap}}(\Omega)$ if the sequence of functionals $\|\cdot\|_{1,\mu_n}$ Γ-converges to the functional $\|\cdot\|_{1,\mu}$ in $L^2(\Omega)$, i.e. if the following two conditions are satisfied:*

- *for every sequence $u_n \to u$ in $L^2(\Omega)$ we have*

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx + \int_{\mathbb{R}^d} u^2\,d\mu \le \liminf_{n\to\infty} \left\{ \int_{\mathbb{R}^d} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^d} u_n^2\,d\mu_n \right\};$$

- *for every $u \in L^2(\Omega)$, there exists $u_n \to u$ in $L^2(\Omega)$ such that*

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx + \int_{\mathbb{R}^d} u^2\,d\mu = \lim_{n\to\infty} \left\{ \int_{\mathbb{R}^d} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^d} u_n^2\,d\mu_n \right\}.$$

If $\mu \in \mathcal{M}^+_{\text{cap}}(\Omega)$ and $f \in L^2(\Omega)$ we define the functional $J_\mu(f, \cdot) : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$J_\mu(f, u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega u^2\,d\mu - \int_\Omega f u\,dx. \tag{2.4}$$

If $\Omega \subset \mathbb{R}^d$ is a bounded open set, $\mu \in \mathcal{M}^+_{\text{cap}}(\Omega)$ and $f \in L^2(\Omega)$, then the functional $J_\mu(f, \cdot)$ has a unique minimizer $u \in H^1_\mu$ that verifies the PDE formally written as

$$-\Delta u + \mu u = f, \qquad u \in H^1_\mu(\Omega), \tag{2.5}$$

and whose precise meaning is given in the weak form

$$\begin{cases} \displaystyle\int_\Omega \nabla u \cdot \nabla \varphi\,dx + \int_\Omega u\varphi\,d\mu = \int_\Omega f\varphi\,dx, & \forall \varphi \in H^1_\mu(\Omega), \\ u \in H^1_\mu(\Omega). \end{cases}$$

The resolvent operator of $-\Delta + \mu$, that is the map $\mathcal{R}_\mu$ that associates to every $f \in L^2(\Omega)$ the solution $u \in H^1_\mu(\Omega) \subset L^2(\Omega)$, is a compact linear operator in $L^2(\Omega)$ and so, it has a discrete spectrum

$$0 < \cdots \le \Lambda_k \le \cdots \le \Lambda_2 \le \Lambda_1.$$

Their inverses $1/\Lambda_k$ are denoted by $\lambda_k(\mu)$ and are the eigenvalues of the operator $-\Delta + \mu$.

In the case $f = 1$ the solution will be denoted by w_μ and when $\mu = I_\Omega$ we will use the notation w_Ω instead of w_{I_Ω}. We also recall (see [2]) that if Ω is bounded, then the strong L^2-convergence of the minimizers w_{μ_n} to w_μ is equivalent to the γ-convergence of Definition 2.2.

Remark 2.3. An important well known characterization of the γ-convergence is the following: a sequence μ_n γ-converges to μ, if and only if, the sequence of resolvent operators $\mathcal{R}_{\mu_n}$ associated to $-\Delta + \mu_n$, converges (in the strong convergence of linear operators on L^2) to the resolvent $\mathcal{R}_\mu$ of the operator $-\Delta + \mu$. A consequence of this fact is that the spectrum of the operator $-\Delta + \mu_n$ converges (pointwise) to the one of $-\Delta + \mu$.

Remark 2.4. The space $\mathcal{M}^+_{\text{cap}}(\Omega)$ endowed with the γ-convergence is metrizable. If Ω is bounded, one may take $d_\gamma(\mu, \nu) = \|w_\mu - w_\nu\|_{L^2}$. Moreover, in this case, in [10] it is proved that the space $\mathcal{M}^+_{\text{cap}}(\Omega)$ endowed with the metric d_γ is compact.

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^d$ and let $V_n \in L^1(\Omega)$ be a sequence weakly converging in $L^1(\Omega)$ to a function V. Then the capacitary measures $V_n\,dx$ γ-converge to $V\,dx$.*

Proof. We have to prove that the solutions $u_n = R_{V_n}(1)$ of

$$\begin{cases} -\Delta u_n + V_n(x)u_n = 1 \\ u \in H^1_0(\Omega) \end{cases}$$

weakly converge in $H^1_0(\Omega)$ to the solution $u = R_V(1)$ of

$$\begin{cases} -\Delta u + V(x)u = 1 \\ u \in H^1_0(\Omega), \end{cases}$$

or equivalently that the functionals

$$J_n(u) = \int_\Omega |\nabla u|^2\,dx + \int_\Omega V_n(x)u^2\,dx$$

$\Gamma\big(L^2(\Omega)\big)$-converge to the functional

$$J(u) = \int_\Omega |\nabla u|^2\,dx + \int_\Omega V(x)u^2\,dx.$$

The Γ-liminf inequality (Definition 2.1 (a)) is immediate since, if $u_n \to u$ in $L^2(\Omega)$, we have

$$\int_\Omega |\nabla u|^2\,dx \le \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2\,dx$$

by the lower semicontinuity of the $H^1(\Omega)$ norm with respect to the $L^2(\Omega)$-convergence, and

$$\int_\Omega V(x)u^2\,dx \le \liminf_{n\to\infty} \int_\Omega V_n(x)u_n^2\,dx$$

by the strong-weak lower semicontinuity theorem for integral functionals (see for instance [4]).

Let us now prove the Γ-limsup inequality (Definition 2.1 (b)) which consists, given $u \in H^1_0(\Omega)$, in constructing a sequence $u_n \to u$ in $L^2(\Omega)$ such that

$$\limsup_{n\to\infty} \int_\Omega |\nabla u_n|^2\,dx + \int_\Omega V_n(x)u_n^2\,dx \le \int_\Omega |\nabla u|^2\,dx + \int_\Omega V(x)u^2\,dx. \tag{2.6}$$

For every $t > 0$ let $u^t = (u \wedge t) \vee (-t)$; then, by the weak convergence of V_n, for t fixed we have

$$\lim_{n\to\infty} \int_\Omega V_n(x)|u^t|^2\,dx = \int_\Omega V(x)|u^t|^2\,dx,$$

and

$$\lim_{t\to+\infty}\int_\Omega V(x)|u^t|^2\,dx=\int_\Omega V(x)|u|^2\,dx.$$

Then, by a diagonal argument, we can find a sequence $t_n\to+\infty$ such that

$$\lim_{n\to\infty}\int_\Omega V_n(x)|u^{t_n}|^2\,dx=\int_\Omega V(x)|u|^2\,dx.$$

Taking now $u_n=u^{t_n}$, and noticing that for every $t>0$

$$\int_\Omega|\nabla u^t|^2\,dx\le\int_\Omega|\nabla u|^2\,dx,$$

we obtain (2.6) and so the proof is complete. □

In the case of weak* convergence of measures the statement of Proposition 2.5 is no longer true, as the following proposition shows.

Proposition 2.6. *Let $\Omega\subset\mathbb{R}^d$ ($d\ge2$) be a bounded open set and let $V,W\in L^1_+(\Omega)$ be two functions such that $V\ge W$. Then, there is a sequence $V_n\in L^1_+(\Omega)$, uniformly bounded in $L^1(\Omega)$, such that the sequence of measures $V_n(x)\,dx$ converges weakly* to $V(x)\,dx$ and γ-converges to $W(x)\,dx$.*

Proof. [1] Without loss of generality we can suppose $\int_\Omega(V-W)\,dx=1$. Let μ_n be a sequence of probability measures on Ω weakly* converging to $(V-W)\,dx$ and such that each μ_n is a finite sum of Dirac masses. For each $n\in\mathbb{N}$ consider a sequence of positive functions $V_{n,m}\in L^1(\Omega)$ such that $\int_\Omega V_{n,m}\,dx=1$ and $V_{n,m}dx$ converges weakly* to μ_n as $m\to\infty$. Moreover, we choose $V_{n,m}$ as a convex combination of functions of the form $|B_{1/m}|^{-1}\chi_{B_{1/m}(x_j)}$.

We now prove that for fixed $n\in\mathbb{N}$, $(V_{n,m}+W)\,dx$ γ-converges, as $m\to\infty$, to $W\,dx$ or, equivalently, that the sequence $w_{W+V_{n,m}}$ converges in L^2 to w_W, as $m\to\infty$. Indeed, by the weak maximum principle, we have

$$w_{W+I_{\Omega_{m,n}}}\le w_{W+V_{n,m}}\le w_W,$$

where $\Omega_{m,n}=\Omega\setminus\cup_jB_{1/m}(x_j)$ and $I_{\Omega_{m,n}}$ is as in (2.2).

Since a point has zero capacity in $\mathbb{R}^d$ ($d\ge2$) there exists a sequence $\phi_m\to0$ strongly in $H^1(\mathbb{R}^d)$ with $\phi_m=1$ on $B_{1/m}(0)$ and $\phi_m=0$ outside $B_{1/\sqrt{m}}(0)$. We have

$$\begin{aligned}\int_\Omega|w_W-w_{W+I_{\Omega_{m,n}}}|^2\,dx&\le2\|w_W\|_{L^\infty}\int_\Omega(w_W-w_{W+I_{\Omega_{m,n}}})\,dx\\&=4\|w_W\|_{L^\infty}\big(E(W+I_{\Omega_{m,n}})-E(W)\big)\\&\le4\|w_W\|_{L^\infty}\bigg(\int_\Omega\frac12|\nabla w_m|^2+\frac12Ww_m^2-w_m\,dx\\&\qquad-\int_\Omega\frac12|\nabla w_W|^2+\frac12Ww_W^2-w_W\,dx\bigg),\end{aligned}\tag{2.7}$$

where w_m is any function in $\in H^1_0(\Omega_{m,n})$. Taking

$$w_m(x)=w_W(x)\prod_j(1-\phi_m(x-x_j)),$$

since $\phi_m\to0$ strongly in $H^1(\mathbb{R}^d)$, it is easy to see that $w_m\to w_W$ strongly in $H^1(\Omega)$ and so, by (2.7), $w_{W+I_{\Omega_{m,n}}}\to w_W$ in $L^2(\Omega)$ as $m\to\infty$. Since the weak convergence of probability

[1] the idea of this proof was suggested by Dorin Bucur

measures and the γ-convergence are both induced by metrics, a diagonal sequence argument brings to the conclusion. □

Remark 2.7. When $d = 1$, a result analogous to Proposition 2.5 is that any sequence (μ_n) weakly* converging to μ is also γ-converging to μ. This is an easy consequence of the compact embedding of $H^1_0(\Omega)$ into the space of continuous functions on Ω.

We note that the hypothesis $V \geq W$ in Proposition 2.6 is necessary. Indeed, we have the following proposition, whose proof is contained in [9, Theorem 3.1] and we report it here for the sake of completeness.

Proposition 2.8. *Let $\mu_n \in \mathcal{M}^+_{cap}(\Omega)$ be a sequence of capacitary Radon measures weakly* converging to the measure ν and γ-converging to the capacitary measure $\mu \in \mathcal{M}^+_{cap}(\Omega)$. Then $\mu \leq \nu$ in Ω.*

Proof. We note that it is enough to show that $\mu(K) \leq \nu(K)$ whenever $K \subset\subset \Omega$ is a compact set. Let u be a nonnegative smooth function with compact support in Ω such that $u \leq 1$ in Ω and $u = 1$ on K; we have

$$\mu(K) \leq \int_\Omega u^2\,d\mu \leq \liminf_{n\to\infty} \int_\Omega u^2\,d\mu_n = \int_\Omega u^2\,d\nu \leq \nu\left(\{u > 0\}\right).$$

Since u is arbitrary, we have the conclusion by the Borel regularity of ν. □

3. Existence of optimal potentials in $L^p(\Omega)$

In this section we consider the optimization problem

$$\min\left\{F(V)\ :\ V : \Omega \to [0, +\infty],\ \int_\Omega V^p\,dx \leq 1\right\}, \tag{3.1}$$

where $p > 0$ and $F(V)$ is a cost functional depending on the solution of some partial differential equation on Ω. Typically, $F(V)$ is the minimum of some functional $J_V : H^1_0(\Omega) \to \mathbb{R}$ depending on V. A natural assumption in this case is the lower semicontinuity of the functional F with respect to the γ-convergence, that is

$$F(\mu) \leq \liminf_{n\to\infty} F(\mu_n), \qquad \text{whenever } \mu_n \to_\gamma \mu. \tag{3.2}$$

Theorem 3.1. *Let $F : L^1_+(\Omega) \to \mathbb{R}$ be a functional, lower semicontinuous with respect to the γ-convergence, and let $\mathcal{V}$ be a weakly $L^1(\Omega)$ compact set. Then the problem*

$$\min\left\{F(V)\ :\ V \in \mathcal{V}\right\}, \tag{3.3}$$

admits a solution.

Proof. Let (V_n) be a minimizing sequence in $\mathcal{V}$. By the compactness assumption on $\mathcal{V}$, we may assume that V_n tends weakly $L^1(\Omega)$ to some $V \in \mathcal{V}$. By Proposition 2.5, we have that V_n γ-converges to V and so, by the semicontinuity of F,

$$F(V) \leq \liminf_{n\to\infty} F(V_n),$$

which gives the conclusion. □

Remark 3.2. Theorem 3.1 applies for instance to the integral functionals and to the spectral functionals considered in the introduction; it is not difficult to show that they are lower semicontinuous with respect to the γ-convergence.

Remark 3.3. In some special cases the solution of (3.1) can be written explicitly in terms of the solution of some partial differential equation on Ω. This is the case of the Dirichlet Energy, that we discuss in Subsection 3.1, and of the first eigenvalue of the Dirichlet Laplacian λ_1 (see [12, Chapter 8]).

The compactness assumption on the admissible class $\mathcal{V}$ for the weak $L^1(\Omega)$ convergence in Theorem 3.1 is for instance satisfied if Ω has finite measure and $\mathcal{V}$ is a convex closed and bounded subset of $L^p(\Omega)$, with $p \geq 1$. In the case of measures an analogous result holds.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and let $F : \mathcal{M}^+_{\text{cap}}(\Omega) \to \mathbb{R}$ be a functional lower semicontinuous with respect to the γ-convergence. Then the problem*

$$\min\left\{F(\mu) \ : \ \mu \in \mathcal{M}^+_{\text{cap}}(\Omega),\ \mu(\Omega) \leq 1\right\}, \tag{3.4}$$

admits a solution.

Proof. Let (μ_n) be a minimizing sequence. Then, up to a subsequence μ_n converges weakly* to some measure ν and γ-converges to some measure $\mu \in \mathcal{M}^+_{\text{cap}}(\Omega)$. By Proposition 2.8, we have that $\mu(\Omega) \leq \nu(\Omega) \leq 1$ and so, μ is a solution of (3.4). □

The following example shows that the optimal solution of problem (3.4) is not, in general, a function $V(x)$, even when the optimization criterion is the energy $\mathcal{E}_f$ introduced in (1.1). On the other hand, an explicit form for the optimal potential $V(x)$ will be provided in Proposition 3.9 assuming that the right-hand side f is in $L^2(\Omega)$.

Example 3.5. Let $\Omega = (-1, 1)$ and consider the functional

$$F(\mu) = -\min\left\{\frac{1}{2}\int_\Omega |u'|^2\,dx + \frac{1}{2}\int_\Omega u^2\,d\mu - u(0) \ : \ u \in H^1_0(\Omega)\right\}.$$

Then, for any μ such that $\mu(\Omega) \leq 1$, we have

$$F(\mu) \geq -\min\left\{\frac{1}{2}\int_\Omega |u'|^2\,dx + \frac{1}{2}\big(\sup_\Omega u\big)^2 - u(0) \ : \ u \in H^1_0(\Omega),\ u \geq 0\right\}. \tag{3.5}$$

By a symmetrization argument, the minimizer u of the right-hand side of (3.5) is radially decreasing; moreover, u is linear on the set $u < M$, where $M = \sup u$, and so it is of the form

$$u(x) = \begin{cases} \frac{M}{1-\alpha}x + \frac{M}{1-\alpha}, & x \in [-1, -\alpha], \\ M, & x \in [-\alpha, \alpha], \\ -\frac{M}{1-\alpha}x + \frac{M}{1-\alpha}, & x \in [\alpha, 1], \end{cases} \tag{3.6}$$

for some $\alpha \in [0, 1]$. A straightforward computation gives $\alpha = 0$ and $M = 1/3$. Thus, u is also the minimizer of

$$F(\delta_0) = -\min\left\{\frac{1}{2}\int_\Omega |u'|^2\,dx + \frac{1}{2}u(0)^2 - u(0) \ : \ u \in H^1_0(\Omega)\right\},$$

and so δ_0 is the solution of

$$\min\left\{F(\mu) \ : \ \mu(\Omega) \leq 1\right\}.$$

3.1. **Minimization problems in L^p concerning the Dirichlet Energy functional.** Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and let $f \in L^2(\Omega)$. By Theorem 3.1, the problem

$$\min\{-\mathcal{E}_f(V)\ :\ V \in \mathcal{V}\} \qquad \text{with} \qquad \mathcal{V} = \left\{V \geq 0,\ \int_\Omega V^p\,dx \leq 1\right\}, \tag{3.7}$$

admits a solution, where $\mathcal{E}_f(V)$ is the energy functional defined in (1.1). We notice that, replacing $-\mathcal{E}_f(V)$ by $\mathcal{E}_f(V)$, makes problem (3.7) trivial, with the only solution $V \equiv 0$. Minimization problems for $\mathcal{E}_f$ will be considered in Section 4 for admissible classes of the form

$$\mathcal{V} = \left\{V \geq 0,\ \int_\Omega V^{-p}\,dx \leq 1\right\}.$$

Analogous results for $F(V) = -\lambda_1(V)$ were proved in [12, Theorem 8.2.3].

Proposition 3.6. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $1 < p < \infty$ and $f \in L^2(\Omega)$. Then the problem* (3.7) *has a unique solution*

$$V_p = \left(\int_\Omega |u_p|^{2p/(p-1)}\,dx\right)^{-1/p} |u_p|^{-1+(p+1)/(p-1)},$$

where $u_p \in H_0^1(\Omega) \cap L^{2p/(p-1)}(\Omega)$ is the minimizer of the functional

$$J_p(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\left(\int_\Omega |u|^{2p/(p-1)}\,dx\right)^{(p-1)/p} - \int_\Omega uf\,dx. \tag{3.8}$$

Moreover, we have $\mathcal{E}_f(V_p) = J_p(u_p)$.

Proof. We first show that we have

$$\max_{V\in\mathcal{V}}\ \min_{u\in H_0^1(\Omega)} \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + u^2V - uf\right)dx \leq \min_{u\in H_0^1(\Omega)}\ \max_{V\in\mathcal{V}} \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + u^2V - uf\right)dx, \tag{3.9}$$

where the maximums are taken over all positive functions $V \in L^p(\Omega)$. For a fixed $u \in H_0^1(\Omega)$, the maximum on the right-hand side (if finite) is achieved for a function V such that $\Lambda p V^{p-1} = u^2$, where Λ is a Lagrange multiplier. By the condition $\int_\Omega V^p\,dx = 1$ we obtain that the maximum is achieved for

$$V = \left(\int_\Omega |u|^{\frac{2p}{p-1}}\,dx\right)^{1/p} |u|^{\frac{2}{p-1}}.$$

Substituting in (3.9), we obtain

$$\max\{\mathcal{E}_f(V)\ :\ V \in \mathcal{V}\} \leq \min\left\{J_p(u):\ u \in H_0^1(\Omega)\right\}. \tag{3.10}$$

Let u_n be a minimizing sequence for J_p. Since $\inf J_p \leq 0$, we can assume $J_p(u_n) \leq 0$ for each $n \in \mathbb{N}$. Thus, we have

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx + \frac{1}{2}\left(\int_\Omega |u_n|^{2p/(p-1)}\,dx\right)^{(p-1)/p} \leq \int_\Omega u_n f\,dx \leq C\|f\|_{L^2(\Omega)}\|\nabla u_n\|_{L^2}, \tag{3.11}$$

where C is a constant depending on Ω. Thus we obtain

$$\int_\Omega |\nabla u_n|^2\,dx + \left(\int_\Omega |u_n|^{2p/(p-1)}\,dx\right)^{(p-1)/p} \leq 4C^2\|f\|^2_{L^2(\Omega)}, \tag{3.12}$$

and so, up to subsequence u_n converges weakly in $H_0^1(\Omega)$ and $L^{2p/(p-1)}(\Omega)$ to some $u_p \in H_0^1(\Omega) \cap L^{2p/(p-1)}(\Omega)$. By the semicontinuity of the L^2-norm of the gradient and the $L^{\frac{2p}{p-1}}$-norm and the fact that $\int_\Omega fu_n\,dx \to \int_\Omega fu_p\,dx$, as $n \to \infty$, we have that u_p is a minimizer of J_p. By the strict convexity of J_p, we have that u_p is unique. Moreover, by (3.11) and (3.12), $J_p(u_p) > -\infty$. Writing down the Euler-Lagrange equation for u_p, we obtain

$$-\Delta u_p + \left(\int_\Omega |u_p|^{2p/(p-1)}\,dx\right)^{-1/p} |u_p|^{2/(p-1)} u_p = f.$$

Setting

$$V_p = \left(\int_\Omega |u_p|^{2p/(p-1)}\,dx\right)^{-1/p} |u_p|^{2/(p-1)},$$

we have that $\int_\Omega V_p^p\,dx = 1$ and u_p is the solution of

$$-\Delta u_p + V_p u_p = f. \tag{3.13}$$

In particular, we have $J_p(u_p) = \mathcal{E}_p(V_p)$ and so V_p solves (3.7). The uniqueness of V_p follows by the uniqueness of u_p and the equality case in the Hölder inequality

$$\int_\Omega u^2 V\,dx \le \left(\int_\Omega V^p\,dx\right)^{1/p} \left(\int_\Omega |u|^{2p/(p-1)}\,dx\right)^{(p-1)/p} \le \left(\int_\Omega |u|^{2p/(p-1)}\,dx\right)^{(p-1)/p}.$$

□

When the functional F is the energy $\mathcal{E}_f$, the existence result holds also in the case $p = 1$. Before we give the proof of this fact in Proposition 3.9, we need some preliminary results. We also note that the analogous results were obtained in the case $F = -\lambda_1$ (see [12, Theorem 8.2.4]) and in the case $F = -\mathcal{E}_f$, where f is a positive function (see [9]).

Remark 3.7. Let u_p be the minimizer of J_p, defined in (3.8). By (3.12), we have the estimate

$$\|\nabla u_p\|_{L^2(\Omega)} + \|u_p\|_{L^{2p/(p-1)}(\Omega)} \le 4C^2\|f\|_{L^2(\Omega)}, \tag{3.14}$$

where C is the constant from (3.11). Moreover, we have $u_p \in H^2_{loc}(\Omega)$ and for each open set $\Omega' \subset\subset \Omega$, there is a constant C not depending on p such that

$$\|u_p\|_{H^2(\Omega')} \le C(f, \Omega').$$

Indeed, u_p satisfies the PDE

$$-\Delta u + c|u|^\alpha u = f, \tag{3.15}$$

with $c > 0$ and $\alpha = 2/(p-1)$, and standard elliptic regularity arguments (see [11, Section 6.3]) give that $u \in H^2_{loc}(\Omega)$. To show that $\|u_p\|_{H^2(\Omega')}$ is bounded independently of p we apply the Nirenberg operator $\partial_k^h u = \frac{u(x+he_k)-u(x)}{h}$ on both sides of (3.15), and multiplying by $\phi^2\partial_k^h u$, where ϕ is an appropriate cut-off function which equals 1 on Ω', we have

$$\begin{aligned}\int_\Omega \phi^2|\nabla\partial_k^h u|^2\,dx + \int_\Omega \nabla(\partial_k^h u)\cdot\nabla(\phi^2)\partial_k^h u\,dx + c(\alpha+1)\int_\Omega \phi^2|u|^\alpha|\partial_k^h u|^2\,dx \\ = -\int f\partial_k^h(\phi^2\partial_k^h u)\,dx,\end{aligned} \tag{3.16}$$

for all $k = 1, \ldots, d$. Some straightforward manipulations now give

$$\|\nabla^2 u\|^2_{L^2(\Omega')} \le \sum_{k=1}^d \int_\Omega \phi^2|\nabla\partial_k u|^2\,dx \le C(\Omega')\left(\|f\|_{L^2(\{\phi^2>0\})} + \|\nabla u\|_{L^2(\Omega)}\right). \tag{3.17}$$

Lemma 3.8. *Let $\Omega \subset \mathbb{R}^d$ be an open set and $f \in L^2(\Omega)$. Consider the functional $J_1 : L^2(\Omega) \to \mathbb{R}$ defined by*

$$J_1(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\|u\|_\infty^2 - \int_\Omega uf\,dx, \tag{3.18}$$

Then, J_p Γ-converges in $L^2(\Omega)$ to J_1, as $p \to 1$, where J_p is defined in (3.8).

Proof. Let $v_n \in L^2(\Omega)$ be a sequence of positive functions converging in L^2 to $v \in L^2(\Omega)$ and let $\alpha_n \to +\infty$. Then, we have that

$$\|v\|_{L^\infty(\Omega)} \le \liminf_{n\to\infty} \|v_n\|_{L^{\alpha_n}(\Omega)}. \tag{3.19}$$

In fact, suppose first that $\|v\|_{L^\infty} = M < +\infty$ and let $\omega_\varepsilon = \{v > M - \varepsilon\}$, for some $\varepsilon > 0$. Then, we have

$$\liminf_{n\to\infty} \|v_n\|_{L^{\alpha_n}(\Omega)} \ge \lim_{n\to\infty} |\omega_\varepsilon|^{(1-\alpha_n)/\alpha_n} \int_{\omega_\varepsilon} v_n\,dx = |\omega_\varepsilon|^{-1}\int_{\omega_\varepsilon} v\,dx \ge M - \varepsilon,$$

and so, letting $\varepsilon \to 0$, we have $\liminf_{n\to\infty} \|v_n\|_{L^{\alpha_n}(\Omega)} \le M$. If $\|v\|_{L^\infty} = +\infty$, then setting $\omega_k = \{v > k\}$, for any $k \ge 1$, and arguing as above, we obtain (3.19).
Let $u_n \to u$ in $L^2(\Omega)$. Then, by the semicontinuity of the L^2 norm of the gradient and (3.19) and the continuity of the term $\int_\Omega uf\,dx$, we have

$$J_1(u) \le \liminf_{n\to\infty} J_{p_n}(u_n), \tag{3.20}$$

for any decreasing sequence $p_n \to 1$. On the other hand, for any $u \in L^2$, we have $J_{p_n}(u) \to J_1(u)$ as $n \to \infty$ and so, we have the conclusion. □

Proposition 3.9. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and $f \in L^2(\Omega)$. Then there is a unique solution of problem* (3.7) *with $p = 1$, given by*

$$V_1 = \frac{1}{M}\left(\chi_{\omega_+} f - \chi_{\omega_-} f\right),$$

where $M = \|u_1\|_{L^\infty(\Omega)}$, $\omega_+ = \{u_1 = M\}$, $\omega_- = \{u_1 = -M\}$, being $u_1 \in H^1_0(\Omega) \cap L^\infty(\Omega)$ the unique minimizer of the functional J_1, defined in (3.18). *In particular, $\int_{\omega_+} f\,dx - \int_{\omega_-} f\,dx = M$, $f \ge 0$ on ω_+ and $f \le 0$ on ω_-.*

Proof. For any $u \in H^1_0(\Omega)$ and any $V \ge 0$ with $\int_\Omega V\,dx \le 1$ we have

$$\int_\Omega u^2 V\,dx \le \|u\|_\infty^2 \int_\Omega V\,dx \le \|u\|_\infty^2,$$

where for sake of simplicity, we write $\|\cdot\|_\infty$ instead of $\|\cdot\|_{L^\infty(\Omega)}$. Arguing as in the proof of Proposition 3.6, we obtain the inequalities

$$\frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega u^2 V\,dx - \int_\Omega uf\,dx \le J_1(u),$$
$$\max\left\{\mathcal{E}_f(V) \;:\; \int_\Omega V \le 1\right\} \le \min\left\{J_1(u) \;:\; u \in H^1_0(\Omega)\right\}.$$

As in (3.11), we have that a minimizing sequence of J_1 is bounded in $H^1_0(\Omega) \cap L^\infty(\Omega)$ and thus by semicontinuity there is a minimizer $u_1 \in H^1_0(\Omega) \cap L^\infty(\Omega)$ of J_1, which is also unique, by the strict convexity of J_1. Let u_p denotes the minimizer of J_p as in Proposition 3.6. Then, by Remark 3.7, we have that the family u_p is bounded in $H^1_0(\Omega)$ and in $H^2(\Omega')$ for each $\Omega' \subset\subset \Omega$. Then, we have that each sequence u_{p_n} has a subsequence converging weakly in $L^2(\Omega)$ to some

$u \in H^2_{loc}(\Omega) \cap H^1_0(\Omega)$. By Lemma 3.8, we have $u = u_1$ and so, $u_1 \in H^2_{loc}(\Omega) \cap H^1_0(\Omega)$. Thus $u_{p_n} \to u_1$ in $L^2(\Omega)$.

Let us define $M = \|u_1\|_\infty$ and $\omega = \omega_+ \cup \omega_-$. We claim that u_1 satisfies, on Ω the PDE

$$-\Delta u + \chi_\omega f = f. \tag{3.21}$$

Indeed, setting $\Omega_t = \Omega \cap \{|u| < t\}$ for $t > 0$, we compute the variation of J_1 with respect to any function $\varphi \in H^1_0(\Omega_{M-\varepsilon})$. Namely we consider functions of the form $\varphi = \psi w_\varepsilon$ where w_ε is the solution of $-\Delta w_\varepsilon = 1$ on $\Omega_{M-\varepsilon}$, and $w_\varepsilon = 0$ on $\partial\Omega_{M-\varepsilon}$. Thus we obtain that $-\Delta u_1 = f$ on $\Omega_{M-\varepsilon}$ and letting $\varepsilon \to 0$ we conclude, thanks to the Monotone Convergence Theorem, that

$$-\Delta u_1 = f \qquad \text{on } \Omega_M = \Omega \setminus \omega.$$

Moreover, since $u_1 \in H^2_{loc}(\Omega)$, we have that $\Delta u_1 = 0$ on ω and so, we obtain (3.21).
Since u_1 is the minimizer of J_1, we have that for each $\varepsilon \in \mathbb{R}$, $J_1((1+\varepsilon)u_1) - J_1(u_1) \geq 0$. Taking the derivative of this difference at $\varepsilon = 0$, we obtain

$$\int_\Omega |\nabla u_1|^2\,dx + M^2 = \int_\Omega f u_1\,dx. \tag{3.22}$$

By (3.21), we have $\int_\Omega |\nabla u_1|^2\,dx = \int_{\Omega\setminus\omega} f u_1\,dx$ and so

$$M = \int_{\omega_+} f\,dx - \int_{\omega_-} f\,dx. \tag{3.23}$$

Setting $V_1 := \frac{1}{M}\left(\chi_{\omega_+} f - \chi_{\omega_-} f\right)$, we have that $\int_\Omega V_1\,dx = 1$, $-\Delta u_1 + V_1 u_1 = f$ in $H^{-1}(\Omega)$ and

$$J_1(u_1) = \frac{1}{2}\int_\Omega |\nabla u_1|^2\,dx + \frac{1}{2}\int_\Omega u_1^2 V_1\,dx - \int_\Omega u_1 f\,dx.$$

We are left to prove that V_1 is admissible, i.e. $V_1 \geq 0$. To do this, consider w_ε the energy function of the quasi-open set $\{u < M - \varepsilon\}$ and let $\varphi = w_\varepsilon \psi$ where $\psi \in C^\infty_c(\mathbb{R}^d)$, $\psi \geq 0$. Since $\varphi \geq 0$, we get that

$$0 \leq \lim_{t\to 0^+} \frac{J_1(u_1 + t\varphi) - J_1(u_1)}{t} = \int_\Omega \langle \nabla u_1, \nabla\varphi\rangle\,dx - \int_\Omega f\varphi\,dx.$$

This inequality holds for any ψ so that, integrating by parts, we obtain

$$-\Delta u_1 - f \geq 0$$

almost everywhere on $\{u_1 < M - \varepsilon\}$. In particular, since $\Delta u_1 = 0$ almost everywhere on $\omega_- = \{u = -M\}$, we obtain that $f \leq 0$ on ω_-. Arguing in the same way, and considering test functions supported on $\{u_1 \geq -M + \varepsilon\}$, we can prove that $f \geq 0$ on ω_+. This implies $V_1 \geq 0$ as required. □

Remark 3.10. Under some additional assumptions on Ω and f one can obtain some more precise regularity results for u_1. In fact, in [17, Theorem A1] it was proved that if $\partial\Omega \in C^2$ and if $f \in L^\infty(\Omega)$ is positive, then $u_1 \in C^{1,1}(\overline{\Omega})$.

Remark 3.11. In the case $p < 1$ problem (3.7) does not admit, in general, a solution, even for regular f and Ω. We give a counterexample in dimension one, which can be easily adapted to higher dimensions.

Let $\Omega = (0,1)$, $f = 1$, and let $x_{n,k} = k/n$ for any $n \in \mathbb{N}$ and $k = 1, \dots, n-1$. We define the (capacitary) measures

$$\mu_n = \sum_{k=1}^{n-1} +\infty\,\delta_{x_{n,k}},$$

where δ_x is the Dirac measure at the point x. Let w_n be the minimizer of the functional $J_{\mu_n}(1,\cdot)$, defined in (2.4). Then w_n vanishes at $x_{n,k}$, for $k = 1,\dots,n-1$, and so we have

$$\mathcal{E}(\mu_n) = n\,\min\left\{\frac{1}{2}\int_0^{1/n}|u'|^2\,dx - \int_0^{1/n}u\,dx\ :\ u\in H_0^1(0,1/n)\right\} = -\frac{C}{n^2},$$

where $C>0$ is a constant.

For any fixed n and j, let V_j^n be the sequence of positive functions such that $\int_0^1|V_j^n|^p\,dx = 1$, defined by

$$V_j^n = C_n\sum_{k=1}^{n-1} j^{1/p}\chi_{\left[\frac{k}{n}-\frac{1}{j},\frac{k}{n}+\frac{1}{j}\right]} < \sum_{k=1}^{n-1} I_{\left[\frac{k}{n}-\frac{1}{j},\frac{k}{n}+\frac{1}{j}\right]}, \tag{3.24}$$

where C_n is a constant depending on n and I is as in (2.2). By the compactness of the γ-convergence, we have that, up to a subsequence, $V_j^n\,dx$ γ-converges to some capacitary measure μ as $j\to\infty$. On the other hand it is easy to check that $\sum_{k=1}^{n-1} I_{\left[\frac{k}{n}-\frac{1}{j},\frac{k}{n}+\frac{1}{j}\right]}(x)$ γ-converges to μ_n as $j\to\infty$. By (3.24), we have that $\mu\le\mu_n$. In order to show that $\mu=\mu_n$ it is enough to check that each nonnegative function $u\in H_0^1((0,1))$, for which $\int u^2\,d\mu<+\infty$, vanishes at $x_{n,k}$ for $k=1,\dots,n-1$. Suppose that $u(k/n)>0$. By the definition of the γ-convergence, there is a sequence $u_j\in H_0^1(\Omega)=H^1_{V_j^n}(\Omega)$ such that $u_j\to u$ weakly in $H_0^1(\Omega)$ and $\int u_j^2V_j^n\,dx\le C$, for some constant C not depending on $j\in\mathbb{N}$. Since u_j are uniformly 1/2-Hölder continuous, we can suppose that $u_j\ge\varepsilon>0$ on some interval I containing k/n. But then for j large enough I contains $[k/n-1/j,k/n+1/j]$ so that

$$C\ge\int_0^1 u_j^2V_j^n\,dx\ge\int_{k/n-1/j}^{k/n+1/j}u_j^2V_j^n\,dx\ge 2C_n\varepsilon^2 j^{1/p-1},$$

which is a contradiction for $p<1$. Thus, we have that $\mu=\mu_n$ and so V_j^n γ-converges to μ_n as $j\to\infty$. In particular, $\mathcal{E}(\mu_n)=\lim_{j\to\infty}\mathcal{E}_1(V_j^n)$ and since the left-hand side converges to zero as $n\to\infty$, we can choose a diagonal sequence $V_{j_n}^n$ such that $\mathcal{E}(V_{j_n}^n)\to 0$ as $n\to\infty$. Since there is no admissible functional V such that $\mathcal{E}_1(V)=0$, we have the conclusion.

4. Existence of optimal potentials for unbounded constraints

In this section we consider the optimization problem

$$\min\left\{F(V)\ :\ V\in\mathcal{V}\right\}, \tag{4.1}$$

where $\mathcal{V}$ is an admissible class of nonnegative Borel functions on the bounded open set $\Omega\subset\mathbb{R}^d$ and F is a cost functional on the family of capacitary measures $\mathcal{M}^+_{\mathrm{cap}}(\Omega)$. The admissible classes we study depend on a function $\Psi:[0,+\infty]\to[0,+\infty]$

$$\mathcal{V}=\left\{V:\Omega\to[0,+\infty]\ :\ V\text{ Lebesgue measurable, }\int_\Omega\Psi(V)\,dx\le 1\right\}.$$

Theorem 4.1. *Let $\Omega\subset\mathbb{R}^d$ be a bounded open set and $\Psi:[0,+\infty]\to[0,+\infty]$ a strictly decreasing function with Ψ^{-1} convex. Then, for any functional $F:\mathcal{M}^+_{\mathrm{cap}}(\Omega)\to\mathbb{R}$ which is increasing and lower semicontinuous with respect to the γ-convergence, the problem (4.1) has a solution.*

Proof. Let $V_n\in\mathcal{V}$ be a minimizing sequence for problem (4.1). Then, $v_n:=\Psi(V_n)$ is a bounded sequence in $L^1(\Omega)$ and so, up to a subsequence, v_n converges weakly* to some measure ν. We will prove that $V:=\Psi^{-1}(\nu_a)$ is a solution of (4.1), where ν_a denotes the density of the absolutely continuous part of ν with respect to the Lebesgue measure. Clearly $V\in\mathcal{V}$ and so it remains to prove that $F(V) \le \liminf_n F(V_n)$. In view of Remark 2.4, we can suppose that, up to a subsequence, V_n γ-converges to a capacitary measure $\mu \in \mathcal{M}^+_{\mathrm{cap}}(\Omega)$. We claim that the following inequalities hold true:

$$F(V) \le F(\mu) \le \liminf_{n\to\infty} F(V_n). \tag{4.2}$$

In fact, the second inequality in (4.2) is the lower semicontinuity of F with respect to the γ-convergence, while the first needs a more careful examination. By the definition of γ-convergence, we have that for any $u \in H^1_0(\Omega)$, there is a sequence $u_n \in H^1_0(\Omega)$ which converges to u in $L^2(\Omega)$ and is such that

$$\begin{aligned}
\int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 d\mu &= \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx + \int_\Omega u_n^2 V_n \,dx \\
&= \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx + \int_\Omega u_n^2 \Psi^{-1}(v_n)\,dx \\
&\ge \int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 \Psi^{-1}(\nu_a)\,dx \\
&= \int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 V\,dx,
\end{aligned} \tag{4.3}$$

where the inequality in (4.3) is due to strong-weak* lower semicontinuity of integral functionals (see for instance [4]). Thus, for any $u \in H^1_0(\Omega)$, we have

$$\int_\Omega u^2\,d\mu \ge \int_\Omega u^2 V\,dx,$$

which gives $V \le \mu$. Since F is increasing, we obtain the first inequality in (4.2) and so the conclusion. □

Remark 4.2. The condition on the function Ψ in Theorem 4.1 is satisfied for instance by the following functions:

(1) $\Psi(x) = x^{-p}$, for any $p > 0$;
(2) $\Psi(x) = e^{-\alpha x}$, for any $\alpha > 0$.

4.1. **Optimal potentials for the Dirichlet Energy and the first eigenvalue of the Dirichlet Laplacian.** In some special cases, the solution of the optimization problem (4.1) can be computed explicitly through the solution of some PDE, as in Subsection 3.1. This occurs for instance when $F = \lambda_1$ or when $F = \mathcal{E}_f$, with $f \in L^2(\Omega)$. We note that, by the variational formulation

$$\lambda_1(V) = \min\left\{\int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 V\,dx \;:\; u \in H^1_0(\Omega),\ \int_\Omega u^2\,dx = 1\right\}, \tag{4.4}$$

we can rewrite problem (4.1) as

$$\begin{aligned}
&\min\left\{\min_{\|u\|_2=1}\left\{\int_\Omega |\nabla u|^2\,dx + \int_\Omega u^2 V\,dx\right\} \;:\; V \ge 0,\ \int_\Omega \Psi(V)\,dx \le 1\right\} \\
&\quad = \min_{\|u\|_2=1}\left\{\min\left\{\int_\Omega |\nabla u|^2\,dx + \int_\Omega u^2 V\,dx \;:\; V \ge 0,\ \int_\Omega \Psi(V)\,dx \le 1\right\}\right\}.
\end{aligned} \tag{4.5}$$

One can compute that, if Ψ is differentiable with Ψ' invertible, then the second minimum in (4.5) is achieved for

$$V = (\Psi')^{-1}(\Lambda_u u^2), \tag{4.6}$$

where Λ_u is a constant such that $\int_\Omega \Psi\left((\Psi')^{-1}(\Lambda_u u^2)\right)\,dx = 1$. Thus, the solution of the problem on the right hand side of (4.5) is given through the solution of

$$\min\left\{\int_\Omega |\nabla u|^2\,dx + \int_\Omega u^2(\Psi')^{-1}(\Lambda_u u^2)\,dx:\ u\in H^1_0(\Omega),\ \int_\Omega u^2\,dx = 1\right\}. \tag{4.7}$$

Analogously, we obtain that the optimal potential for the Dirichlet Energy $\mathcal{E}_f$ is given by (4.6), where this time u is a solution of

$$\min\left\{\int_\Omega \frac{1}{2}|\nabla u|^2\,dx + \int_\Omega \frac{1}{2}u^2(\Psi')^{-1}(\Lambda_u u^2)\,dx - \int_\Omega fu\,dx\ :\ u\in H^1_0(\Omega)\right\}. \tag{4.8}$$

Thus we obtain the following result.

Corollary 4.3. *Under the assumptions of Theorem 4.1, for the functionals $F=\lambda_1$ and $F=\mathcal{E}_f$ there exists a solution of (4.1) given by $V=(\Psi')^{-1}(\Lambda_u u^2)$, where $u\in H^1_0(\Omega)$ is a minimizer of (4.7), in the case $F=\lambda_1$, and of (4.8), in the case $F=\mathcal{E}_f$.*

Example 4.4. If $\Psi(x)=x^{-p}$ with $p>0$, the optimal potentials for λ_1 and $\mathcal{E}_f$ are given by

$$V=\left(\int_\Omega |u|^{2p/(p+1)}\,dx\right)^{1/p} u^{-2/(p+1)}, \tag{4.9}$$

where u is the minimizer of (4.7) and (4.8), respectively. We also note that, in this case

$$\int_\Omega u^2(\Psi')^{-1}(\Lambda_u u^2)\,dx = \left(\int_\Omega |u|^{2p/(p+1)}\,dx\right)^{(1+p)/p}.$$

Example 4.5. If $\Psi(x)=e^{-\alpha x}$ with $\alpha>0$, the optimal potentials for λ_1 and $\mathcal{E}_f$ are given by

$$V=\frac{1}{\alpha}\left(\log\left(\int_\Omega u^2\,dx\right) - \log\left(u^2\right)\right), \tag{4.10}$$

where u is the minimizer of (4.7) and (4.8), respectively. We also note that, in this case

$$\int_\Omega u^2(\Psi')^{-1}(\Lambda_u u^2)\,dx = \frac{1}{\alpha}\left(\int_\Omega u^2\,dx\int_\Omega \log\left(u^2\right)\,dx - \int_\Omega u^2\log\left(u^2\right)\,dx\right).$$

5. Optimization problems in unbounded domains

In this section we consider optimization problems for which the domain region is the entire Euclidean space $\mathbb{R}^d$. General existence results, in the case when the design region Ω is unbounded, are hard to achieve since most of the cost functionals are not semicontinuous with respect to the γ-convergence in these domains. For example, it is not hard to check that if μ is a capacitary measure, infinite outside the unit ball B_1, then, for every $x_n\to\infty$, the sequence of translated measures $\mu_n=\mu(\cdot+x_n)$ γ-converges to the capacitary measure

$$I_\emptyset(E)=\begin{cases}0, & \text{if } \operatorname{cap}(E)=0,\\ +\infty, & \text{if } \operatorname{cap}(E)>0.\end{cases}$$

Thus increasing and translation invariant functionals are never lower semicontinuous with respect to the γ-convergence. In some special cases, as the Dirichlet Energy or the first eigenvalue of the Dirichlet Laplacian, one can obtain existence results by more direct methods, as those in Proposition 3.6.

For a potential $V\ge 0$ and a function $f\in L^q(\mathbb{R}^d)$, we define the Dirichlet energy as

$$\mathcal{E}_f(V)=\inf\left\{\int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla u|^2+\frac{1}{2}V(x)u^2 - f(x)u\right)dx\ :\ u\in C^\infty_c(\mathbb{R}^d)\right\}. \tag{5.1}$$

In some cases it is convenient to work with the space $\dot H^1(\mathbb{R}^d)$, obtained as the closure of $C_c^\infty(\mathbb{R}^d)$ with respect to the L^2 norm of the gradient, instead of the classical Sobolev space $H^1(\mathbb{R}^d)$. We recall that if $d\ge 3$, the Gagliardo-Nirenberg-Sobolev inequality

$$\|u\|_{L^{2d/(d-2)}}\le C_d\|\nabla u\|_{L^2},\qquad \forall u\in \dot H^1(\mathbb{R}^d), \tag{5.2}$$

holds, while in the cases $d\le 2$, we have respectively

$$\|u\|_{L^\infty}\le\left(\frac{r+2}{2}\right)^{2/(r+2)}\|u\|_{L^r}^{r/(r+2)}\|u'\|_{L^2}^{2/(r+2)},\qquad \forall r\ge 1,\ \forall u\in\dot H^1(\mathbb{R}); \tag{5.3}$$

$$\|u\|_{L^{r+2}}\le\left(\frac{r+2}{2}\right)^{2/(r+2)}\|u\|_{L^r}^{r/(r+2)}\|\nabla u\|_{L^2}^{2/(r+2)},\qquad \forall r\ge 1,\ \forall u\in\dot H^1(\mathbb{R}^2). \tag{5.4}$$

5.1. **Optimal potentials in $L^p(\mathbb{R}^d)$.** In this section we consider optimization problems for the Dirichlet energy $\mathcal{E}_f$ among potentials $V\ge 0$ satisfying a constraint of the form $\|V\|_{L^p}\le 1$. We note that the results in this section hold in a generic unbounded domain Ω. Nevertheless, for sake of simplicity, we restrict our attention to the case $\Omega=\mathbb{R}^d$.

Proposition 5.1. *Let $p>1$ and let q be in the interval with end-points $a=2p/(p+1)$ and $b=\max\{1,2d/(d+2)\}$ (with a included for every $d\ge1$, and b included for every $d\ne 2$). Then, for every $f\in L^q(\mathbb{R}^d)$, there is a unique solution of the problem*

$$\max\left\{\mathcal{E}_f(V)\ :\ V\ge 0,\ \int_{\mathbb{R}^d}V^p\,dx\le 1\right\}. \tag{5.5}$$

Proof. Arguing as in Proposition 3.6, we have that for $p>1$ the optimal potential V_p is given by

$$V_p=\left(\int_{\mathbb{R}^d}|u_p|^{2p/(p-1)}\,dx\right)^{-1/p}|u_p|^{2/(p-1)}, \tag{5.6}$$

where u_p is the solution of the problem

$$\min\left\{\frac12\int_{\mathbb{R}^d}|\nabla u|^2\,dx+\frac12\left(\int_{\mathbb{R}^d}|u|^{2p/(p-1)}\,dx\right)^{(p-1)/p}-\int_{\mathbb{R}^d}uf\,dx\ :\ u\in\dot H^1(\mathbb{R}^d)\cap L^{2p/(p-1)}(\mathbb{R}^d)\right\}. \tag{5.7}$$

Thus, it is enough to prove that there exists a solution of (5.7). For a minimizing sequence u_n we have

$$\frac12\int_{\mathbb{R}^d}|\nabla u_n|^2\,dx+\frac12\left(\int_{\mathbb{R}^d}|u_n|^{2p/(p-1)}\,dx\right)^{(p-1)/p}\le\int_{\mathbb{R}^d}u_nf\,dx\le C\|f\|_{L^q}\|u_n\|_{L^{q'}}. \tag{5.8}$$

Suppose that $d\ge3$. Interpolating q' between $2p/(p-1)$ and $2d/(d-2)$ and using the Gagliardo-Nirenberg-Sobolev inequality (5.2), we obtain that there is a constant C, depending only on p,d and f, such that

$$\frac12\int_{\mathbb{R}^d}|\nabla u_n|^2\,dx+\frac12\left(\int_{\mathbb{R}^d}|u_n|^{2p/(p-1)}\,dx\right)^{(p-1)/p}\le C.$$

Thus we can suppose that u_n converges weakly in $\dot H^1(\mathbb{R}^d)$ and in $L^{2p/(p-1)}(\mathbb{R}^d)$ and so, the problem (5.7) has a solution. In the case $d\le 2$, the claim follows since, by using (5.3), (5.4) and interpolation, we can still estimate $\|u_n\|_{L^{q'}}$ by means of $\|\nabla u_n\|_{L^2}$ and $\|u_n\|_{L^{2p/(p-1)}}$. $\square$

Repeating the argument of Subsection 3.1, one obtains an existence result for (5.5) in the case $p = 1$, too.

Proposition 5.2. *Let $f \in L^q(\mathbb{R}^d)$, where $q \in [1, \frac{2d}{d+2}]$, if $d \geq 3$, and $q = 1$, if $d = 1, 2$. Then there is a unique solution V_1 of problem* (5.5) *with $p = 1$, which is given by*

$$V_1 = \frac{f}{M}\left(\chi_{\omega_+} - \chi_{\omega_-}\right),$$

where $M = \|u_1\|_{L^\infty(\mathbb{R}^d)}$, $\omega_+ = \{u_1 = M\}$, $\omega_- = \{u_1 = -M\}$, and u_1 is the unique minimizer of

$$\min\left\{\frac{1}{2}\int_{\mathbb{R}^d}|\nabla u|^2\,dx + \frac{1}{2}\|u\|_{L^\infty}^2 - \int_{\mathbb{R}^d} uf\,dx :\ u \in \dot{H}^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\right\}. \tag{5.9}$$

In particular, $\int_{\omega_+} f\,dx - \int_{\omega_-} f\,dx = M$, $f \geq 0$ on ω_+ and $f \leq 0$ on ω_-.

We note that, when $p = 1$, the support of the optimal potential V_1 is contained in the support of the function f. This is not the case if $p > 1$, as the following example shows.

Example 5.3. Let $f = \chi_{B(0,1)}$ and $p > 1$. By our previous analysis we know that there exist a solution u_p for problem (5.7) and a solution V_p for problem (5.5) given by (5.6). We note that u_p is positive, radially decreasing and satisfies the equation

$$-u''(r) - \frac{d-1}{r}u'(r) + Cu^\alpha = 0, \qquad r \in (1, +\infty),$$

where $\alpha = 2p/(p-1) > 2$ and C is a positive constant. Thus, we have that

$$u_p(r) = kr^{2/(1-\alpha)},$$

where k is an explicit constant depending on C, d and α. In particular, we have that u_p is not compactly supported on $\mathbb{R}^d$ (see Figure 1).

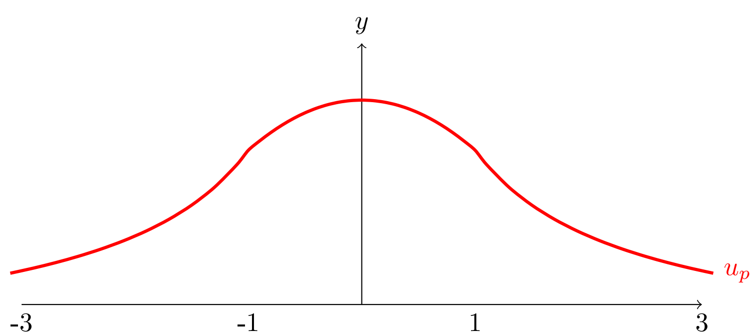

Figure 1. The solution u_p of problem (5.7), with $p > 1$ and $f = \chi_{B(0,1)}$ does not have a compact support.

5.2. **Optimal potentials with unbounded constraint.** In this subsection we consider the problems

$$\min\left\{\mathcal{E}_f(V)\ :\ V \geq 0,\ \int_{\mathbb{R}^d} V^{-p}\,dx \leq 1\right\}, \tag{5.10}$$

$$\min\left\{\lambda_1(V)\ :\ V \geq 0,\ \int_{\mathbb{R}^d} V^{-p}\,dx \leq 1\right\}, \tag{5.11}$$

for $p > 0$ and $f \in L^q(\mathbb{R}^d)$. We will see in Proposition 5.4 that in order to have existence for (5.10) the parameter q must satisfy some constraint, depending on the value of p and on the dimension d. Namely, we need q to satisfy the following conditions

$$\begin{aligned} q \in [\frac{2d}{d+2}, \frac{2p}{p-1}],\ &\text{if } d \geq 3 \text{ and } p > 1,\\ q \in [\frac{2d}{d+2}, +\infty],\ &\text{if } d \geq 3 \text{ and } p \leq 1,\\ q \in (1, \frac{2p}{p-1}],\ &\text{if } d = 2 \text{ and } p > 1,\\ q \in (1, +\infty],\ &\text{if } d = 2 \text{ and } p \leq 1,\\ q \in [1, \frac{2p}{p-1}],\ &\text{if } d = 1 \text{ and } p > 1,\\ q \in [1, +\infty],\ &\text{if } d = 1 \text{ and } p \leq 1. \end{aligned} \tag{5.12}$$

We say that $q = q(p, d) \in [1, +\infty]$ is *admissible* if it satisfy (5.12). Note that $q = 2$ is admissible for any $d \geq 1$ and any $p > 0$.

Proposition 5.4. *Let $p > 0$ and $f \in L^q(\mathbb{R}^d)$, where q is admissible in the sense of* (5.12). *Then the minimization problem* (5.10) *has a solution V_p given by*

$$V_p = \left(\int_{\mathbb{R}^d} |u_p|^{2p/(p+1)}\, dx\right)^{1/p} |u_p|^{-2/(1+p)}, \tag{5.13}$$

where u_p is a minimizer of

$$\min\left\{\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2\, dx + \frac{1}{2}\left(\int_{\mathbb{R}^d} |u|^{2p/(p+1)}\, dx\right)^{(p+1)/p} - \int_{\mathbb{R}^d} uf\, dx : \tag{5.14}\right.$$
$$\left. u \in \dot{H}^1(\mathbb{R}^d),\ |u|^{2p/(p+1)} \in L^1(\mathbb{R}^d)\right\}.$$

Moreover, if $p \geq 1$, then the functional in (5.14) *is convex, its minimizer is unique and so is the solution of* (5.10).

Proof. By means of (5.2), (5.3) and (5.4), and thanks to the admissibility of q, we get the existence of a solution of (5.14) through an interpolation argument similar to the one used in the proof of Proposition 5.1. The existence of an optimal potential follows by the same argument as in Subsection 4.1. □

In Example 5.3, we showed that the optimal potentials for (5.5), may be supported on the whole $\mathbb{R}^d$. The analogous question for the problem (5.10) is whether the optimal potentials given by (5.13) have a bounded set of finiteness $\{V_p < +\infty\}$. In order to answer this question, it is sufficient to study the support of the solutions u_p of (5.14), which solve the equation

$$-\Delta u + C_p |u|^{-2/(p+1)} u = f, \tag{5.15}$$

where $C_p > 0$ is a constant depending on p.

Proposition 5.5. *Let $p > 0$ and let $f \in L^q(\mathbb{R}^d)$, for $q > d/2$, be a nonnegative function with a compact support. Then every solution u_p of problem* (5.14) *has a compact support.*

Proof. With no loss of generality we may assume that f is supported in the unit ball of $\mathbb{R}^d$. We first prove the result when f is radially decreasing. In this case u_p is also radially decreasing and nonnegative. Let v be the function defined by $v(|x|) = u_p(x)$. Thus v satisfies the equation

$$\begin{cases} -v'' - \dfrac{d-1}{r} v' + C_p v^s = 0 \qquad r \in (1, +\infty), \\ v(1) = u_p(1), \end{cases} \tag{5.16}$$

where $s = (p-1)/(p+1)$ and $C_p > 0$ is a constant depending on p. Since $v \geq 0$ and $v' \leq 0$, we have that v is convex. Moreover, since

$$\int_1^{+\infty} v^2 r^{d-1}\,dr < +\infty, \qquad \int_1^{+\infty} |v'|^2 r^{d-1}\,dr < +\infty,$$

we have that v, v' and v'' vanish at infinity. Multiplying (5.16) by v' we obtain

$$\left(\frac{v'(r)^2}{2} - C_p \frac{v(r)^{s+1}}{s+1}\right)' = -\frac{d-1}{r} v'(r)^2 \leq 0.$$

Thus the function $v'(r)^2/2 - C_p v(r)^{s+1}/(s+1)$ is decreasing and vanishing at infinity and thus nonnegative. Thus we have

$$-v'(r) \geq C v(r)^{(s+1)/2}, \; r \in (1, +\infty), \tag{5.17}$$

where $C = \big(2C_p/(s+1)\big)^{1/2}$. Arguing by contradiction, suppose that v is strictly positive on $(1, +\infty)$. Dividing both sides of (5.17) and integrating, we have

$$-v(r)^{(1-s)/2} \geq Ar + B,$$

where $A = 2C/(1-s)$ and B is determined by the initial datum $v(1)$. This cannot occur, since the left hand side is negative, while the right hand side goes to $+\infty$, as $r \to +\infty$.

We now prove the result for a generic compactly supported and nonnegative $f \in L^q(\mathbb{R}^d)$. Since the solution u_p of (5.14) is nonnegative and is a weak solution of (5.15), we have that on each ball $B_R \subset \mathbb{R}^d$, $u_p \leq u$, where $u \in H^1(B_R)$ is the solution of

$$-\Delta u = f \text{ in } B_R, \qquad u = u_p \text{ on } \partial B_R.$$

Since $f \in L^{d/2}(\mathbb{R}^d)$, by [19, Theorem 9.11] and a standard bootstrap argument on the integrability of u, we have that u is continuous on $B_{R/2}$. As a consequence, u_p is locally bounded in $\mathbb{R}^d$. In particular, it is bounded since $u_p \wedge M$, where $M = \|u_p\|_{L^\infty(B_1)}$, is a better competitor than u_p in (5.14). Let w be a radially decreasing minimizer of (5.14) with $f = \chi_{B_1}$. Thus w is a solution of the PDE

$$-\Delta w + C_p w^s = \chi_{B_1},$$

in $\mathbb{R}^d$, where C_p is as in (5.16). Then, the function $w_t(x) = t^{2/(1-s)} w(x/t)$ is a solution of the equation

$$-\Delta w_t + C_p w_t^s = t^{2s/(1-s)} \chi_{B_t}.$$

Since u_p is bounded, there exists some $t \geq 1$ large enough such that $w_t \geq u_p$ on the ball B_t. Moreover, w_t minimizes (5.14) with $f = t^{2s/(1-s)} \chi_{B_t}$ and so $w_t \geq u_p$ on $\mathbb{R}^d$ (otherwise $w_t \wedge u_p$ would be a better competitor in (5.14) than w_p). The conclusion follows since, by the first step of the proof, w_t has compact support. ☐

The problems (5.11) and (5.10) are similar both in the questions of existence and the qualitative properties of the solutions.

Proposition 5.6. *For every $p > 0$ there is a solution of the problem* (5.11) *given by*

$$V_p = \left(\int_{\mathbb{R}^d} |u_p|^{2p/(p+1)}\,dx\right)^{1/p} |u_p|^{-2/(1+p)}, \tag{5.18}$$

where u_p is a radially decreasing minimizer of

$$\min\left\{\int_{\mathbb{R}^d} |\nabla u|^2\,dx + \left(\int_{\mathbb{R}^d} |u|^{2p/(p+1)}\,dx\right)^{(p+1)/p} \;:\; u \in H^1(\mathbb{R}^d),\ \int_{\mathbb{R}^d} u^2\,dx = 1\right\}. \tag{5.19}$$

Moreover, u_p has a compact support, hence the set $\{V_p < +\infty\}$ is a ball of finite radius in $\mathbb{R}^d$.

Proof. Let us first show that the minimum in (5.19) is achieved. Let $u_n \in H^1(\mathbb{R}^d)$ be a minimizing sequence of positive functions normalized in L^2. Note that by the Pólya-Szegö inequality we may assume that each of these functions is radially decreasing in $\mathbb{R}^d$ and so we will use the identification $u_n = u_n(r)$. In order to prove that the minimum is achieved it is enough to show that the sequence u_n converges in $L^2(\mathbb{R}^d)$. Indeed, since u_n is a radially decreasing minimizing sequence, there exists $C > 0$ such that for each $r > 0$ we have

$$u_n(r)^{2p/(p+1)} \le \frac{1}{|B_r|}\int_{B_r} u_n^{2p/(p+1)}\,dx \le \frac{C}{r^d}.$$

Thus, for each $R > 0$, we obtain

$$\int_{B_R^c} u_n^2\,dx \le C_1 \int_R^{+\infty} r^{-d(p+1)/p}\, r^{d-1}\,dr = C_2 R^{-1/p}, \tag{5.20}$$

where C_1 and C_2 do not depend on n and R. Since the sequence u_n is bounded in $H^1(\mathbb{R}^d)$, it converges locally in $L^2(\mathbb{R}^d)$ and, by (5.20), this convergence is also strong in $L^2(\mathbb{R}^d)$. Thus, we obtain the existence of a radially symmetric and decreasing solution u_p of (5.19) and so, of an optimal potential V_p given by (5.18).

We now prove that the support of u_p is a ball of finite radius. By the radial symmetry of u_p we can write it in the form $u_p(x) = u_p(|x|) = u_p(r)$, where $r = |x|$. With this notation, u_p satisfies the equation:

$$-u_p'' - \frac{d-1}{r}u_p' + C_p u_p^s = \lambda u_p,$$

where $s = (p-1)/(p+1) < 1$ and $C_p > 0$ is a constant depending on p. Arguing as in Proposition 5.5, we obtain that, for r large enough,

$$-u_p'(r) \ge \left(\frac{C_p}{s+1}u_p(r)^{s+1} - \frac{\lambda}{2}u_p(r)^2\right)^{1/2} \ge \left(\frac{C_p}{2(s+1)}u_p(r)^{s+1}\right)^{1/2},$$

where, in the last inequality, we used the fact that $u_p(r) \to 0$, as $r \to \infty$, and $s + 1 < 2$. Integrating both sides of the above inequality, we conclude that u_p has a compact support. In Figure 2 we show the case $d = 1$ and $f = \chi_{(-1,1)}$. □

Remark 5.7. We note that the solution $u_p \in H^1(\mathbb{R}^d)$ of (5.19) is the function for which the best constant C in the interpolated Gagliardo-Nirenberg-Sobolev inequality

$$\|u\|_{L^2(\mathbb{R}^d)} \le C\|\nabla u\|_{L^2(\mathbb{R}^d)}^{d/(d+2p)}\|u\|_{L^{2p/(p+1)}(\mathbb{R}^d)}^{2p/(d+2p)} \tag{5.21}$$

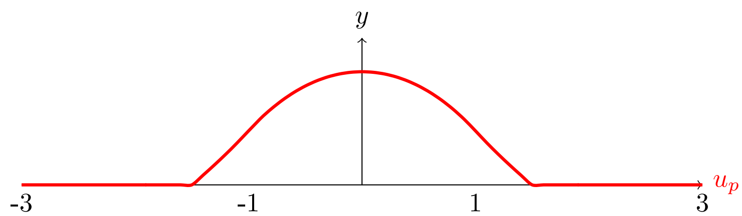

FIGURE 2. The solution u_p of problem (5.14), with $p > 1$ and $f = \chi_{(-1,1)}$.

is achieved. Indeed, for any $u \in H^1(\mathbb{R}^d)$ and any $t > 0$, we define $u_t(x) := t^{d/2}u(tx)$. Thus, we have that $\|u\|_{L^2(\mathbb{R}^d)} = \|u_t\|_{L^2(\mathbb{R}^d)}$, for any $t > 0$. Moreover, up to a rescaling, we may assume that the function $g : (0, +\infty) \to \mathbb{R}$, defined by

$$\begin{aligned} g(t) &= \int_{\mathbb{R}^d} |\nabla u_t|^2 \, dx + \left(\int_{\mathbb{R}^d} |u_t|^{2p/(p+1)} \, dx \right)^{(p+1)/p} \\ &= t^2 \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + t^{-d/p} \left(\int_{\mathbb{R}^d} |u|^{2p/(p+1)} \, dx \right)^{(p+1)/p}, \end{aligned}$$

achieves its minimum in the interval $(0, +\infty)$ and, moreover, we have

$$\min_{t \in (0,+\infty)} g(t) = C \left(\int_{\mathbb{R}^d} |\nabla u|^2 \, dx \right)^{d/(d+2p)} \left(\int_{\mathbb{R}^d} |u|^{\frac{2p}{p+1}} \, dx \right)^{2(p+1)/(d+2p)},$$

where C is a constant depending on p and d. In the case $u = u_p$, the minimum of g is achieved for $t = 1$ and so, we have that u_p is a solution also of

$$\min \left\{ \left(\int_{\mathbb{R}^d} |\nabla u|^2 \, dx \right)^{d/(d+2p)} \left(\int_{\mathbb{R}^d} |u|^{2p/(p+1)} \, dx \right)^{2(p+1)/(d+2p)} \ : \ u \in H^1(\mathbb{R}^d), \ \int_{\mathbb{R}^d} u^2 \, dx = 1 \right\},$$

which is just another form of (5.21).

6. Further remarks and open questions

We recall (see [3]) that the injection $H^1_V(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$ is compact whenever the potential V satisfies $\int_{\mathbb{R}^d} V^{-p} \, dx < +\infty$ for some $0 < p \leq 1$. In this case the spectrum of the Schrödinger operator $-\Delta + V$ is discrete and we denote by $\lambda_k(V)$ its eigenvalues. The existence of an optimal potential for spectral optimization problems of the form

$$\min \left\{ \lambda_k(V) \ : \ V \geq 0, \ \int_{\mathbb{R}^d} V^{-p} \, dx \leq 1 \right\}, \tag{6.1}$$

for general $k \in \mathbb{N}$, cannot be deduced by the direct methods used in Subsection 5.2. In this last section we make the following conjectures:

Conjecture 1) For every $k \geq 1$, there is a solution V_k of the problem (6.1).
Conjecture 2) The set of finiteness $\{V_k < +\infty\}$, of the optimal potential V_k, is bounded.

In what follows, we prove an existence result in the case $k = 2$. We first recall that, by Proposition 5.6, there exists optimal potential V_p, for λ_1, such that the set of finiteness $\{V_p < +\infty\}$ is a ball. Thus, we have a situation analogous to the Faber-Krahn inequality, which states that the minimum

$$\min \left\{ \lambda_1(\Omega) \ : \ \Omega \subset \mathbb{R}^d, \ |\Omega| = c \right\}, \tag{6.2}$$

is achieved for the ball of measure c. We recall that, starting from (6.2), one may deduce, by a simple argument (see for instance [12]), the Krahn-Szegö inequality, which states that the minimum

$$\min\Big\{\lambda_2(\Omega)\ :\ \Omega\subset\mathbb{R}^d,\ |\Omega|=c\Big\},\tag{6.3}$$

is achieved for a disjoint union of equal balls. In the case of potentials one can find two optimal potentials for λ_1 with disjoint sets of finiteness and then apply the argument from the proof of the Krahn-Szegö inequality. In fact, we have the following result.

Proposition 6.1. *There exists an optimal potential, solution of* (6.1) *with* $k=2$. *Moreover, any optimal potential is of the form* $\min\{V_1,V_2\}$*, where* V_1 *and* V_2 *are optimal potentials for* λ_1 *which have disjoint sets of finiteness* $\{V_1<+\infty\}\cap\{V_2<+\infty\}=\emptyset$ *and are such that* $\int_{\mathbb{R}^d}V_1^{-p}\,dx=\int_{\mathbb{R}^d}V_2^{-p}\,dx=1/2$.

Proof. Given V_1 and V_2 as above, we prove that for every $V:\mathbb{R}^d\to[0,+\infty]$ with $\int_{\mathbb{R}^d}V^{-p}\,dx=1$, we have

$$\lambda_2(\min\{V_1,V_2\})\le\lambda_2(V).$$

Indeed, let u_2 be the second eigenfunction of $-\Delta+V$. We first suppose that u_2 changes sign on $\mathbb{R}^d$ and consider the functions $V_+=\sup\{V,\infty_{\{u_2\le0\}}\}$ and $V_-=\sup\{V,\infty_{\{u_2\ge0\}}\}$ where, for any measurable $A\subset\mathbb{R}^d$, we set

$$\infty_A(x)=\begin{cases}+\infty, & x\in A,\\ 0, & x\notin A.\end{cases}$$

We note that

$$1=\int_{\mathbb{R}^d}V^{-p}\,dx=\int_{\mathbb{R}^d}V_+^{-p}\,dx+\int_{\mathbb{R}^d}V_-^{-p}\,dx.$$

Moreover, on the sets $\{u_2>0\}$ and $\{u_2<0\}$, the following equations are satisfied:

$$-\Delta u_2^++V_+u_2^+=\lambda_2(V)u_2^+,\qquad -\Delta u_2^-+V_-u_2^-=\lambda_2(V)u_2^-,$$

and so, multiplying respectively by u_2^+ and u_2^-, we obtain that

$$\lambda_2(V)\ge\lambda_1(V_+),\qquad \lambda_2(V)\ge\lambda_1(V_-),\tag{6.4}$$

where we have equalities if, and only if, u_2^+ and u_2^- are the first eigenfunctions corresponding to $\lambda_1(V_+)$ and $\lambda_1(V_-)$. Let now $\widetilde V_+$ and $\widetilde V_-$ be optimal potentials for λ_1 corresponding to the constraints

$$\int_{\mathbb{R}^d}\widetilde V_+^{-p}\,dx=\int_{\mathbb{R}^d}V_+^{-p}\,dx,\qquad \int_{\mathbb{R}^d}\widetilde V_-^{-p}\,dx=\int_{\mathbb{R}^d}V_-^{-p}\,dx.$$

By Proposition 5.6, the sets of finiteness of $\widetilde V_+$ and $\widetilde V_-$ are compact, hence we may assume (up to translations) that they are also disjoint. By the monotonicity of λ_1, we have

$$\max\{\lambda_1(V_1),\lambda_1(V_2)\}\le\max\{\lambda_1(\widetilde V_+),\lambda_1(\widetilde V_-)\},$$

and so we obtain

$$\lambda_2(\min\{V_1,V_2\})\le\max\{\lambda_1(\widetilde V_+),\lambda_1(\widetilde V_-)\}\le\max\{\lambda_1(V_+),\lambda_1(V_-)\}\le\lambda_2(V),$$

as required. If u_2 does not change sign, then we consider $V_+=\sup\{V,\infty_{\{u_2=0\}}\}$ and $V_-=\sup\{V,\infty_{\{u_1=0\}}\}$, where u_1 is the first eigenfunction of $-\Delta+V$. Then the claim follows by the same argument as above. $\square$

For more general cost functionals $F(V)$, the question if the optimization problem

$$\min\left\{F(V)\ :\ V\ge 0,\ \int_{\mathbb{R}^d} V^p\,dx\le 1\right\}$$

admits a solution is, as far as we know, open.

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