CONVERGENCE TO EQUILIBRIUM OF GRADIENT FLOWS DEFINED ON PLANAR CURVES

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Abstract

We consider the evolution of open planar curves by the steepest descent flow of a geometric functional, under different boundary conditions. We prove that, if any set of stationary solutions with fixed energy is finite, then a solution of the flow converges to a stationary solution as time goes to infinity. We also present a few applications of this result.

Contents

1. Introduction

The steepest descent flow for the total squared curvature defined on curves has been widely studied in the literature. By virtue of a smoothing effect of the functional, there are various results such that the flow has a smooth solution for all times and subconverges to a (possibly nonunique) stationary solution. However there are few results proving the full convergence of solutions to a stationary state, see for instance [**5**, **10**, **11**, **12**, **13**, **14**]. For the case of closed curves in $\mathbb{R}^n$, it has been recently proved in [**15**] that the L^2-gradient flow for a generalized

Helfrich functional has a solution for any time, and the solution converges to an equilibrium. This strengthens a result obtained in [**4**, **12**]. In [**5**, **10**, **11**] convergence is proved with the aid of an additional constraint, the so-called inextensible condition, while in [**12**, **13**, **14**] it follows from the uniqueness of the equilibrium state.

The purpose of this paper is to prove convergence under a weaker condition, namely that there are only finitely many equilibrium states at each prescribed energy level.

The plan of the paper is the following: in Section 2, we present our method in the case of the gradient flow of a general geometric functional defined on planar curves. In Section 3 we apply the result to the equation

$$\partial_t \gamma = (-2\partial_s^2 \kappa - \kappa^3 + \lambda^2 \kappa)\boldsymbol{\nu}, \tag{1.1}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$ and κ, $\boldsymbol{\nu}$ are respectively the scalar curvature and the unit normal, under typical boundary conditions. More precisely, we discuss in detail the boundary conditions:

(i) $\gamma(0,t) = (0,0)$, $\quad \gamma(1,t) = (R,0)$, $\quad \gamma_s(0,t) = \tau_0$, $\quad \gamma_s(1,t) = \tau_1$
(ii) $\gamma(0,t) = (0,0)$, $\quad \gamma(1,t) = (R,0)$, $\quad \kappa(0,t) = \kappa(1,t) = \alpha$

where τ_0, $\tau_1 \in \mathbb{R}^2$ are given constant unit vectors and $\alpha \in \mathbb{R}$ is a prescribed constant. Condition (i) is usually called *clamped boundary condition* (see [**6**]), and (ii) is referred to as *symmetric Navier boundary condition* (see [**1**, **3**]).

Eventually, Appendix 4 is concerned with the analyticity of certain functions which play an important rôle in the proof of the convergence result, while in Appendix 5 we prove the long time existence of smooth solutions to (1.1) under the boundary condition (ii).

2. Geometric gradient flows

In this section we consider the gradient flow of a general geometric functional $\mathcal{E}(\gamma)$ defined on planar curves $\gamma : I \to \mathbb{R}^2$, which we assume to be bounded from below, that is, $\inf_\gamma \mathcal{E}(\gamma) > -\infty$.

A L^2-gradient flow of $\mathcal{E}$ is a one parameter family of curves $\gamma : I \times [0,\infty) \to \mathbb{R}^2$ such that

$$\partial_t \gamma = -\nabla \mathcal{E}(\gamma) \tag{2.1}$$

and

$$\frac{d}{dt}\mathcal{E}(\gamma(t)) = -\int_\gamma |\nabla \mathcal{E}(\gamma(t))|^2 \, ds, \tag{2.2}$$

where $\nabla\mathcal{E}(\gamma)$ denotes the Euler-Lagrange operator of $\mathcal{E}(\gamma)$, i.e., $\nabla\mathcal{E}(\gamma)$ satisfies

$$\frac{d}{d\varepsilon}\mathcal{E}(\gamma(\cdot)+\varphi(\cdot,\varepsilon))\Big|_{\varepsilon=0} = \int_\gamma \nabla\mathcal{E}(\gamma)\cdot\varphi_\varepsilon\,ds$$

for any $\varphi \in C^\infty((-\varepsilon_0,\varepsilon_0):(C_c^\infty(I))^2)$, where $\varphi_\varepsilon = \varphi_\varepsilon(\cdot,0)$.

Since the curves are open, in order to have uniqueness of the evolution we need to impose a boundary condition $\mathcal{B}(\gamma)=0$ on ∂I. Notice that (2.2) does not follow from (2.1) if the boundary condition given by $\mathcal{B}$ is not *natural* for $\mathcal{E}$, i.e., the flow (2.1) with a boundary condition is not always the L^2-gradient flow for $\mathcal{E}(\gamma)$. Indeed, if γ satisfies (2.1) under an unnatural boundary condition $\tilde{\mathcal{B}}(\gamma)=0$, then it can happen that (2.2) does not hold. Therefore we shall assume the following:

Assumption 2.1 (Compatibility). *The flow* (2.1) *with boundary condition $\mathcal{B}(\gamma)=0$ is a L^2-gradient flow for $\mathcal{E}(\gamma)$.*

Given a smooth curve γ we let $s\in[0,\mathcal{L}(\gamma)]$ be the arclength parameter defined as

$$s(x) := \int_0^x |\gamma_x|\,dx \qquad x\in I,$$

where $\mathcal{L}(\gamma)$ is the length of γ

$$\mathcal{L}(\gamma) := s(1) = \int_0^1 |\gamma_x|\,dx.$$

Notice that in the arclength variable s there holds $|\gamma_s(s)|=1$ for all $s\in[0,\mathcal{L}(\gamma)]$. Given a function $f(s)$ defined on γ, we let

$$\|f\|_{L^\infty_\gamma} := \sup_{s\in\mathcal{L}(\gamma)} |f(s)| \qquad \|f\|_{L^2_\gamma} := \left(\int_\gamma f(s)^2\,ds\right)^{\frac{1}{2}}.$$

We shall consider the initial boundary value problem:

$$\text{(2.3)} \qquad \begin{cases} \partial_t\gamma = -\nabla\mathcal{E}(\gamma) & \text{in} \quad I\times(0,\infty), \\ \mathcal{B}(\gamma(x,t))=0 & \text{on} \quad \partial I\times[0,\infty), \\ \gamma(x,0)=\gamma_0(x) & \text{in} \quad I, \end{cases}$$

where $\gamma_0(x): I\to\mathbb{R}^2$ is a smooth planar open curve satisfying the boundary condition $\mathcal{B}(\gamma_0(x))=0$ on ∂I. Regarding the solvability of (2.3), we assume the following:

Assumption 2.2 (**Regularity**). *There exists a smooth solution* $\gamma : I \times [0, +\infty) \to \mathbb{R}^2$ *of* (2.3)*, satisfying*

$$\|\partial_t \gamma(\cdot, t)\|_{L^\infty_{\gamma(t)}} < C \qquad \text{and} \qquad \int_\gamma |\partial_s^m \gamma(t)|^2 \, ds < C \tag{2.4}$$

for any $m \in \mathbb{N}$ *and for any* $t > 0$*, where the constant* C *is independent of* t*. Moreover,* $\|\nabla \mathcal{E}(\gamma)\|_{L^2_\gamma}$ *is continuous in* γ *with respect to the* C^∞*-topology.*

Notice that, as the functional $\mathcal{E}$ is bounded from below, then (2.2) implies the estimate

$$\int_0^{+\infty} \|\partial_t \gamma(\cdot, t)\|^2_{L^2_{\gamma(t)}} \le \mathcal{E}(\gamma_0) - \inf \mathcal{E} \tag{2.5}$$

for any solution γ of (2.3).

Under an additional assumption on $(\mathcal{E}, \mathcal{B})$, we shall prove that a solution of (2.3) converges to a stationary solution as $t \to +\infty$. Let $\mathcal{S}$ be a set of all stationary solutions of (2.3), i.e., the smooth curves $\tilde{\gamma}$ satisfying

$$\begin{cases} \nabla \mathcal{E}(\tilde{\gamma}(x)) = 0 & \text{in} \quad I, \\ \mathcal{B}(\tilde{\gamma}(x)) = 0 & \text{on} \quad \partial I. \end{cases} \tag{2.6}$$

For each $A \in \mathbb{R}$, we define the subset of $\mathcal{S}$

$$\Sigma_A := \{\tilde{\gamma} \in \mathcal{S} \mid \mathcal{E}(\tilde{\gamma}) = A\}.$$

We shall assume the following:

Assumption 2.3. Σ_A *is finite for any* $A \in \mathbb{R}$*.*

We can now state our main result.

Theorem 2.1. *Let* $\gamma(x, t) : I \times [0, \infty) \to \mathbb{R}^2$ *be a solution of* (2.3)*, and suppose that Assumptions* 2.1*,* 2.2 *and* 2.3 *hold. Then, there exists a smooth curve* $\tilde{\gamma} : I \to \mathbb{R}^2$ *satisfying* (2.6) *and such that*

$$\gamma(\cdot, t) \to \tilde{\gamma}(\cdot) \quad \text{as} \quad t \to \infty$$

in the C^∞*-topology.*

We start with a preliminary result.

Lemma 2.1. *Let* $\{t_j\}_{j=1}^\infty$ *be a monotone increasing sequence with* $\inf_{j \in \mathbb{N}} (t_{j+1} - t_j) \ge 0$*. Then, for any* $0 < \varepsilon \le \inf_{j \in \mathbb{N}} (t_{j+1} - t_j)$*, there exists a sequence* $\{t_j^\varepsilon\}_j$ *with* $t_j^\varepsilon \in (t_j, t_j + \varepsilon)$ *such that*

$$\left\|\nabla \mathcal{E}(\gamma(t_j^\varepsilon))\right\|_{L^2_\gamma} \to 0 \quad \text{as} \quad j \to \infty.$$

Proof. Let fix $0 < \varepsilon < \inf_{j \in \mathbb{N}} (t_{j+1} - t_j)$ arbitrarily. Recall that

$$\int_0^\infty \|\nabla \mathcal{E}(\gamma(t))\|^2_{L^2_\gamma} \, dt = - \int_0^\infty \frac{d}{dt} \mathcal{E}(\gamma(t)) \, dt = \Big[\, \mathcal{E}(\gamma(t)) \, \Big]_{t=\infty}^{t=0} < +\infty,$$

so that we have

$$\sum_{j=1}^{\infty} \int_{t_j}^{t_j+\varepsilon} \|\nabla \mathcal{E}(\gamma(t))\|^2_{L^2_\gamma} \, dt < \infty, \tag{2.7}$$

which implies

$$\lim_{j \to \infty} \int_{t_j}^{t_j+\varepsilon} \|\nabla \mathcal{E}(\gamma(t))\|^2_{L^2_\gamma} \, dt = 0. \tag{2.8}$$

The thesis follows directly from (2.8). q.e.d.

We now prove Theorem 2.1.

Proof. To begin with, remark that Assumption 2.2 and Lemma 2.1 imply that the solution γ subconverges to a stationary solution $\tilde{\gamma}$ as $t \to \infty$. Indeed, by Lemma 2.1, one can find a sequence $\{t_j\}$ with $t_j \to \infty$ such that

$$\|\nabla \mathcal{E}(\gamma(t_j))\|_{L^2_\gamma} \to 0 \quad \text{as} \quad t_j \to \infty. \tag{2.9}$$

Since Assumption 2.2 allows us to apply Arzelà-Ascoli's theorem to the family of planar open curves $\gamma(t_j)$, we see that there exists a subsequence $\{t_{j_k}\} \subset \{t_j\}$ such that

$$\gamma(t_{j_k}) \to \tilde{\gamma} \quad \text{as} \quad t_{j_k} \to \infty \tag{2.10}$$

in the C^∞-topology. Combining (2.9) with the definition of the L^2-gradient flow (2.1)-(2.2), we observe that the limit $\tilde{\gamma}$ is independent of t and satisfies

$$\nabla \mathcal{E}(\tilde{\gamma}) = 0 \quad \text{on} \quad I. \tag{2.11}$$

We shall prove Theorem 2.1 by contradiction. Suppose not, there exist sequences $\{t^1_j\}_j$, $\{t^2_j\}_j$ and stationary solutions $\tilde{\gamma}_1$, $\tilde{\gamma}_2 \in \mathcal{S}$ such that

$$\gamma(t^1_j) \to \tilde{\gamma}_1, \quad \gamma(t^2_j) \to \tilde{\gamma}_2 \tag{2.12}$$

as $j \to \infty$. We may assume that $\{t^1_j\}_j$ and $\{t^2_j\}_j$ are monotone increasing sequences. Let

$$A = \mathcal{E}(\tilde{\gamma}_1) = \mathcal{E}(\tilde{\gamma}_2).$$

Thanks to Assumption 2.3, the set Σ_A is finite. On the other hand, for each curves $\tilde{\gamma}_n$, $\tilde{\gamma}_m \in \Sigma_A$, there exists a constant $\delta_{nm} > 0$ such that

$$\mathrm{dist}_{\mathcal{H}}\left(\tilde{\gamma}_n, \tilde{\gamma}_m\right) > \delta_{nm},$$

where $\mathrm{dist}_{\mathcal{H}}\left(\cdot, \cdot\right)$ denotes the Hausdorff distance defined as follows:

$$\mathrm{dist}_{\mathcal{H}}\left(\gamma, \Gamma\right) = \max \left\{ \sup_{u \in Im(\gamma)} \inf_{v \in Im(\Gamma)} |u - v|, \sup_{v \in Im(\Gamma)} \inf_{u \in Im(\gamma)} |u - v| \right\}.$$

Since Σ_A is finite, there exists a constant $\delta_* > 0$ such that

$$\min_{\tilde{\gamma}_n, \tilde{\gamma}_m \in \Sigma_A} \mathrm{dist}_{\mathcal{H}}\left(\tilde{\gamma}_n, \tilde{\gamma}_m\right) = \delta_*. \tag{2.13}$$

Let $\delta = \delta_*/2$. Then, for any $\tilde{\gamma}_n$, $\tilde{\gamma}_m \in \Sigma_A$, it holds that

$$\mathcal{O}_{\mathcal{H}}(\tilde{\gamma}_n, \delta) \cap \mathcal{O}_{\mathcal{H}}(\tilde{\gamma}_m, \delta) = \emptyset, \tag{2.14}$$

where

$$\mathcal{O}_{\mathcal{H}}(\tilde{\gamma}, \delta) = \{\gamma \mid \mathrm{dist}_{\mathcal{H}}\left(\tilde{\gamma}, \gamma\right) < \delta\}.$$

It follows from (2.12) that there exists $J \in \mathbb{N}$ such that

$$\gamma(t_j^1) \in \mathcal{O}_{\mathcal{H}}(\tilde{\gamma}_1, \delta), \quad \gamma(t_j^2) \in \mathcal{O}_{\mathcal{H}}(\tilde{\gamma}_2, \delta) \tag{2.15}$$

for any $j \geq J$. Up to a subsequence, we may assume that it holds that

$$t_j^1 < t_j^2 < t_{j+1}^1$$

for any $j \geq J$. Then, by (2.13), (2.14), (2.15), and the continuity of $\mathrm{dist}_{\mathcal{H}}$, we see that there exists a monotone increasing sequence $\{t_j^3\}_j$ such that

$$d(t_j^3) > \delta \tag{2.16}$$

for any $j \in \mathbb{N}$, where $d(t) := \min_{\tilde{\gamma} \in \Sigma_A} \mathrm{dist}_{\mathcal{H}}\left(\gamma(t), \tilde{\gamma}\right)$. Up to a subsequence, we may also assume that $\inf_{j \in \mathbb{N}}\left(t_{j+1}^3 - t_j^3\right) > 0$.

Here we claim that the function $d(t)$ is Lipschitz continuous on $(0, +\infty)$. Remark that (2.4) gives us that there exists a constant $C > 0$ such that

$$\sup_{x \in I} |\partial_t \gamma(x, t)| < C \tag{2.17}$$

for any $t > 0$. Let x, $y \in I$ fix arbitrarily. Then the fact (2.17) yields that

$$\begin{aligned} ||\gamma(x, t_1) - \tilde{\gamma}(y)| - |\gamma(x, t_2) - \tilde{\gamma}(y)|| &\leq |\gamma(x, t_1) - \gamma(x, t_2)| \\ &\leq \int_{t_2}^{t_1} |\partial_t \gamma(x, t)|\, dt < C\,|t_1 - t_2|\,. \end{aligned}$$

Combining the estimate with the definition of the Hausdorff distance, we obtain

$$|d(t_1) - d(t_2)| < C\,|t_1 - t_2|\,.$$

Thus the function $d(t)$ is C-Lipschitz, in particular uniform continuous, on $(0, +\infty)$. Then it follows from (2.16) that there exists $0 < \varepsilon < \inf_{j\in\mathbb{N}} (t^3_{j+1} - t^3_j)$ such that

$$\operatorname{dist}_{\mathcal{H}}(\gamma(t), \Sigma_A) \geq \frac{\delta}{2} \tag{2.18}$$

for any $j \in \mathbb{N}$ and any $t \in [t^3_j, t^3_j + \varepsilon]$. The inequality (2.18) implies that, for any $\{t^\varepsilon_j\}_j$ with $t^\varepsilon_j \in [t^3_j, t^3_j + \varepsilon]$, $\gamma(t^\varepsilon_j)$ does not converges to any stationary solution as $j \to \infty$. However, by Assumption 2.2 and Lemma 2.1, we can find a sequence $\{\tilde{t}_j\}$ with $\tilde{t}_j \in (t^3_j, t^3_j + \varepsilon]$ such that $\gamma(\tilde{t}_j)$ converges to a stationary solution, which gives a contradiction. q.e.d.

3. Applications

In this section, we apply Theorem 2.1 to the geometric equation

$$\partial_t\gamma = (-2\partial_s^2\kappa - \kappa^3 + \lambda^2\kappa)\boldsymbol{\nu}, \tag{3.1}$$

where κ and $\boldsymbol{\nu}$ denote respectively the scalar curvature and the unit normal vector with the direction of the curvature, and λ is a non-zero constant. Throughout the section we assume that $\gamma(x, t) : I \times [0, \infty) \to \mathbb{R}^2$ are fixed at the boundary, i.e.,

$$\gamma(0, t) = (0, 0), \quad \gamma(1, t) = (R, 0) \quad \text{on} \quad [0, \infty), \tag{3.2}$$

where $R > 0$ is a given constant.

Here we prepare several notations. In what follows let us set

$$f(\kappa) = \kappa^3 - \lambda^2\kappa.$$

From the Euler-Lagrange equation

$$2\partial_s^2\kappa + \kappa^3 - \lambda^2\kappa = 0,$$

we obtain the relation

$$\left(\frac{d\kappa}{ds}\right)^2 + F(\kappa) = E, \tag{3.3}$$

where E is an arbitral constant and $F' = f$, i.e., F is given by

$$F(\kappa) = \frac{1}{4}\kappa^4 - \frac{\lambda^2}{2}\kappa^2.$$

Let $\kappa_M(E)$ and $\kappa_m(E)$ be solutions of $F(\kappa) = E$ as follows:

$$\kappa_M(E) = \sqrt{\lambda^2 + \sqrt{\lambda^4 + 4E}} \qquad \text{for} \quad E \in (-\tfrac{\lambda^4}{4}, \infty),$$
$$\kappa_m(E) = \begin{cases} -\kappa_M(E) & \text{for} \quad E \in (0, \infty), \\ \sqrt{\lambda^2 - \sqrt{\lambda^4 + 4E}} & \text{for} \quad E \in (-\tfrac{\lambda^4}{4}, 0]. \end{cases}$$

If there is no fear of confusion, we write κ_M and κ_m instead of $\kappa_M(E)$ and $\kappa_m(E)$. Let us set

$$L(E) = 2 \int_{\kappa_m(E)}^{\kappa_M(E)} \frac{d\kappa}{\sqrt{E - F(\kappa)}}.$$

Lemma 3.1. *Let* (κ, E) *be a pair satisfying*

$$\begin{cases} \left(\dfrac{d\kappa}{ds} \right)^2 + F(\kappa) = E, \\ \kappa(0) = 0. \end{cases}$$

Then it holds that

$$\int_0^{L(E)} \kappa(s)^2 \, ds \to \infty \quad \textit{as} \quad E \to \infty. \tag{3.4}$$

Proof. Since it holds that

$$L(E) = 4 \int_0^{\kappa_M(E)} \frac{d\kappa}{\sqrt{E - F(\kappa)}} \quad \text{for} \quad E > 0, \tag{3.5}$$

it is sufficient to prove that

$$\int_0^{\frac{L(E)}{4}} \kappa(s)^2 \, ds \to \infty \quad \text{as} \quad E \to \infty. \tag{3.6}$$

Since $\kappa_M(E) \to \infty$ as $E \to \infty$, it holds that

$$\sqrt{2}\,|\lambda| < \kappa_M(E)$$

for sufficiently large E, where $\sqrt{2}\,|\lambda|$ is a solution of $F(\kappa) = 0$. Then we have

$$\begin{aligned} \int_0^{\frac{\kappa_M}{2}} \frac{d\kappa}{\sqrt{E - F(\kappa)}} &= \int_0^{\sqrt{2}|\lambda|} \frac{d\kappa}{\sqrt{E - F(\kappa)}} + \int_{\sqrt{2}|\lambda|}^{\frac{\kappa_M}{2}} \frac{d\kappa}{\sqrt{E - F(\kappa)}} \\ &< \frac{\sqrt{2}\,|\lambda|}{\sqrt{E}} + \frac{\kappa_M/2 - \sqrt{2}\,|\lambda|}{\sqrt{E - F(\kappa_*)}}, \end{aligned} \tag{3.7}$$

where $\kappa_* \in (\sqrt{2}\,|\lambda|\,, \kappa_M/2)$. On the other hand, it holds that

$$\int_{\frac{\kappa_M}{2}}^{\kappa_M} \frac{d\kappa}{\sqrt{E - F(\kappa)}} > \frac{\kappa_M/2}{\sqrt{E - F(\kappa_*)}}. \tag{3.8}$$

If $E > 1$, then we find

$$\frac{\kappa_M/2}{\sqrt{E - F(\kappa_*)}} - \left\{ \frac{\sqrt{2}\,|\lambda|}{\sqrt{E}} + \frac{\kappa_M/2 - \sqrt{2}\,|\lambda|}{\sqrt{E - F(\kappa_*)}} \right\} > 0. \tag{3.9}$$

Combining (3.7)-(3.8) with (3.9), we observe that

$$\int_{\frac{\kappa_M}{2}}^{\kappa_M} \frac{d\kappa}{\sqrt{E - F(\kappa)}} > \int_0^{\frac{\kappa_M}{2}} \frac{d\kappa}{\sqrt{E - F(\kappa)}}$$

for sufficiently large E. Set

$$\mathfrak{L}_1 = \int_0^{\frac{\kappa_M}{2}} \frac{d\kappa}{\sqrt{E - F(\kappa)}}, \qquad \mathfrak{L}_2 = \int_{\frac{\kappa_M}{2}}^{\kappa_M} \frac{d\kappa}{\sqrt{E - F(\kappa)}}.$$

By virtue of (3.5), we see that

$$\int_0^{\frac{L(E)}{4}} \kappa(s)^2 \, ds > \int_{\mathfrak{L}_1}^{\frac{L(E)}{4}} \kappa(s)^2 \, ds > \frac{\mathfrak{L}_2 {\kappa_M}^2}{4} > \frac{{\kappa_M}^3}{8\sqrt{E}}. \tag{3.10}$$

Here we used (3.8). Since it holds that

$$\lim_{E \to \infty} \frac{\kappa_M(E)^2}{2\sqrt{E}} = \lim_{E \to \infty} \frac{\lambda^2 + \sqrt{\lambda^4 + 4E}}{2\sqrt{E}} \to 1,$$

the estimate (3.10) implies (3.6). q.e.d.

3.1. Clamped boundary condition. Recently C.-C. Lin considered a motion of open curves in $\mathbb{R}^n$ with boundary points fixed. Although he considered the problem for any $n \geq 2$ ([**6**]), we restrict the dimension $n = 2$. The motion is governed by the geometric evolution equation (3.1) with the boundary condition (3.2) and

$$\gamma_s(0, t) = \tau_0, \quad \gamma_s(1, t) = \tau_1, \tag{3.11}$$

where τ_0, $\tau_1 \in \mathbb{R}^2$ are prescribed unit vectors. The boundary condition (3.2)-(3.11) is called the *clamped boundary condition.* One can verify that Assumption 2.1 holds, i.e., the flow (3.1) with the clamped boundary condition is a L^2-gradient flow for the functional

$$\mathcal{E}_\lambda(\gamma) = \int_\gamma (\kappa^2 + \lambda^2) \, ds. \tag{3.12}$$

The functional is well known as the modified total squared curvature.

Let $\gamma_0 : I \to \mathbb{R}^2$ be a smooth planar open curve satisfying the following:

$$\gamma_0(0) = (0,0),\ \gamma_0(1) = (R,0),\ \gamma_{0s}(0) = \tau_0,\ \gamma_{0s}(1) = \tau_1.$$

For such curve γ_0, we consider the following initial boundary value problem:

$$\begin{cases} \partial_t \gamma = (-2\partial_s^2 \kappa - \kappa^3 + \lambda^2 \kappa)\boldsymbol{\nu} & \text{in} \quad I \times [0,\infty), \\ \gamma(0,t) = (0,0),\ \ \gamma(1,t) = (R,0), & \\ \gamma_s(0,t) = \tau_0,\ \ \gamma_s(1,t) = \tau_1 & \text{in} \quad [0,\infty), \\ \gamma(x,0) = \gamma_0(x) & \text{in} \quad I. \end{cases} \tag{3.13}$$

The purpose of this subsection is to prove a convergence of a solution of (3.13) to an equilibrium as $t \to \infty$. Regarding the problem (3.13), C.-C. Lin obtained the following result:

Proposition 3.1. ([**6**]) *For any prescribed constant $\lambda \neq 0$ and smooth initial curve γ_0 with finite length, there exists a global smooth solution γ of* (3.13)*. Moreover, after reparametrization by arc length, the family of curves $\{\gamma(t)\}$ subconverges to γ_∞, which is an equilibrium of the energy functional* (3.12)*.*

It follows from the proof of Proposition 3.1 that Assumption 2.2 holds.

Let $\mathcal{S}$ be a set of all stationary solutions of (3.13), i.e., all open curves satisfying

$$\begin{cases} 2\partial_s^2 \kappa + \kappa^3 - \lambda^2 \kappa = 0 \quad \text{in} \quad I, \\ \gamma(0) = (0,0),\ \gamma(1) = (R,0),\ \gamma_s(0) = \tau_0,\ \gamma_s(1) = \tau_1. \end{cases} \tag{3.14}$$

We denote Σ_A a subset of $\mathcal{S}$ defined by

$$\Sigma_A = \{\tilde{\gamma} \in \mathcal{S} \mid \mathcal{E}_\lambda(\tilde{\gamma}) = A\}.$$

In order to apply Theorem 2.1 to the problem (3.13), we prove that Assumption 2.3 holds, i.e., the set Σ_A is finite for any $A \in \mathbb{R}$.

Lemma 3.2. *The set Σ_A is finite for each $A \in \mathbb{R}$.*

Proof. Suppose not, there exist a constant A and a sequence of planar open curves $\{\gamma_n\}_{n=1}^\infty \subset \Sigma_A$. Let fix a family of planar curves $\gamma(s,E)$ such that

$$\left(\frac{d\kappa}{ds}\right)^2 + F(\kappa) = E, \quad \gamma(0,E) = (0,0), \quad \gamma_s(0,E) = \tau_0,$$

and

$$\gamma(s,E_n) = \gamma_n \quad \text{on} \quad [0, \mathcal{L}(\gamma_n)],$$

where $\mathcal{L}(\gamma_n)$ denotes the length of γ_n. Remark that $\gamma(s,E)$ is analytic in s and E on $\mathbb{R}\times(-\lambda^4/4,\infty)$. In particular, letting $s_n=\mathcal{L}(\gamma_n)$, we have

$$\gamma(s_n,E_n)=(R,0),\quad \gamma_s(s_n,E_n)=\tau_1. \tag{3.15}$$

If $E_n\to\infty$ as $n\to\infty$, then Lemma 4.4 implies

$$L(E_n)\to 0 \quad \text{as} \quad n\to\infty \tag{3.16}$$

and Lemma 3.1 yields that

$$\int_0^{L(E_n)} \kappa_n^2\,ds\to\infty \quad \text{as} \quad n\to\infty, \tag{3.17}$$

where $\kappa_n=\kappa(s,E_n)$. Although (3.16)-(3.17) yields that $\mathcal{E}_\lambda(\gamma(\cdot,E_n))\to\infty$ as $n\to\infty$, this contradicts $\mathcal{E}_\lambda(\gamma(\cdot,E_n))=A$. Thus there exists a constant E^* such that $E_n<E^*$, i.e., $\{E_n\}_{n=1}^\infty$ is bounded sequence. Moreover the fact $\{\gamma_n\}_{n=1}^\infty\subset\Sigma_A$ implies $R\le s_n\le A/\lambda^2$, i.e., $\{s_n\}_{n=1}^\infty$ is also bounded sequence. Hence there exist subsequences $\{E_{n_j}\}_{j=1}^\infty\subset\{E_n\}_{n=1}^\infty$ and $\{s_{n_j}\}_{j=1}^\infty\subset\{s_n\}_{n=1}^\infty$ and constants E_∞ and s_∞ such that $E_{n_j}\to E_\infty$ and $s_{n_j}\to s_\infty$ as $j\to\infty$. In the following we write $\{E_n\}_{n=1}^\infty$ and $\{s_n\}_{n=1}^\infty$ instead of $\{E_{n_j}\}_{j=1}^\infty$ and $\{s_{n_j}\}_{j=1}^\infty$ for short.

We prove that there exist a neighborhood U of E_∞ and a function $s:U\to\mathbb{R}$ such that, for any $E\in U$,

$$\gamma(s(E),E)=(R,0),\quad \gamma_s(s(E),E)=\tau_1. \tag{3.18}$$

If $\tau_1\cdot\boldsymbol{e}_1\neq 0$, then we define a function $\Phi:\mathbb{R}\times(-\lambda^4/4,\infty)\to\mathbb{R}$ as $\Phi(s,E)=\gamma_1(s,E)$, where $\boldsymbol{e}_1=(1,0)$ and $\gamma=(\gamma_1,\gamma_2)$. Since $\Phi(s_\infty,E_\infty)=R$ and $\Phi_s(s_\infty,E_\infty)=\tau_1\cdot\boldsymbol{e}_1\neq 0$, the implicit function theorem yields that there exist a neighborhood U of E_∞ and a function $s:U\to\mathbb{R}$ such that, for any $E\in U$,

$$\gamma_1(s(E),E)=R. \tag{3.19}$$

It follows from (3.15) and (3.19) that $s(E_n)=s_n$ holds for any $n\in\mathbb{N}$. Moreover the analyticity of γ implies that $s(E)$ is analytic on U. Combining the analyticity of $s(E)$ with

$$\gamma(s(E_n),E_n)=(R,0),\quad \gamma_s(s(E_n),E_n)=\tau_1,$$

we observe that $s(E)$ satisfies (3.18) on U. If $\tau_1\cdot\boldsymbol{e}_1=0$, then it is sufficient to define a function $\Phi(s,E)$ as $\Phi(s,E)=\gamma_2(s,E)$.

Let us define a function $d:(-\lambda^4/4,\infty)\to\mathbb{R}$ as

$$d(E)=\min_{s\in\mathbb{R}}|\gamma(s,E)-(R,0)|^2.$$

Remark that the function $d(E)$ is analytic and $d(E)=0$ on U. We claim that $d(E)$ is analytic on $(-\lambda^4/4,\infty)$. Suppose that there exists a maximal open set $V \supset U$ such that $d(E)$ is analytic on V. Then we see that $d(E) > 0$ in ∂V. For, if $d(E) = 0$ in ∂V, then the similar argument as above yields that $d(E)$ is analytic on a neighborhood of ∂V. This contradicts that V is maximal. On the other hand, since $d(E)$ is analytic and $d(E) = 0$ on $U \subset V$, we observe that $d(E) = 0$ on V. Therefore $d(E)$ is analytic on $(-\lambda^4/4, \infty)$.

Since $d(E) = 0$ on U, the analyticity yields that $d(E) = 0$ for any $E \in (-\lambda^4/4, \infty)$. Thus there exists an extension $s(E)$ such that

$$\gamma(s(E), E) = (R, 0) \quad \text{for all} \quad E \in (-\lambda^4/4, \infty),$$

where we still denote the extension as $s(E)$, for short.

We claim that $s(E)$ is analytic on $U \cup (E_\infty, \infty)$. Suppose not, there exists a constant $\tilde{E}$ such that $s(E)$ is not extended analytically for $E \geq \tilde{E}$. Then it holds that

$$s(E) \to \infty \quad \text{as} \quad E \nearrow \tilde{E}. \tag{3.20}$$

Since

$$\mathcal{E}_\lambda(\gamma(\cdot, E)) = \int_0^{s(E)} \kappa(s, E)\, ds + \lambda^2 s(E) \quad \text{for any} \quad E \in U \cup (E_\infty, \tilde{E}),$$

(3.20) is equivalent to

$$\mathcal{E}_\lambda(\gamma(\cdot, E)) \to \infty \quad \text{as} \quad E \nearrow \tilde{E}.$$

This contradicts that $\mathcal{E}_\lambda(\gamma(\cdot, E)) = A$ for any $E \in U \cup (E_\infty, \tilde{E})$. Therefore we see that $s(E)$ is extended analytically on $U \cup (E_\infty, \infty)$.

We now obtain a contradiction. Since the analyticity of $s(E)$ implies that (3.18) holds for all $E \in U \cup (E_\infty, \infty)$, it follows that $\gamma(s, E) \in \Sigma_A$ for all $E \in U \cup (E_\infty, \infty)$, i.e.,

$$\mathcal{E}_\lambda(\gamma(\cdot, E)) = A \quad \text{for any} \quad E \in U \cup (E_\infty, \infty). \tag{3.21}$$

However (3.21) contradicts that

$$\mathcal{E}_\lambda(\gamma(\cdot, E)) \to \infty \quad \text{as} \quad E \to \infty.$$

We complete the proof. q.e.d.

Lemma 3.2 implies that one can apply Theorem 2.1 to (3.13). Then the following result is proved.

Theorem 3.1. *Let* $\lambda \neq 0$. *Let* γ *be a smooth solution of* (3.13) *obtained by Proposition* 3.1. *Then, as* $t \to \infty$, *the solution* γ *converges to a solution of* (3.14) *in the* C^∞*-topology.*

3.2. Zero curvature boundary condition. In this subsection, we impose that the curvature of $\gamma(x,t)$ is zero at the boundary of I, i.e.

$$\kappa(0,t) = \kappa(1,t) = 0 \quad \text{in} \quad (0,\infty). \tag{3.22}$$

We shall consider the initial value problem for (3.1) with the boundary conditions (3.2)-(3.22)

$$\begin{cases} \partial_t \gamma = (-2\partial_s^2 \kappa - \kappa^3 + \lambda^2 \kappa)\boldsymbol{\nu} & \text{in} \quad I \times [0,\infty), \\ \gamma(0,t) = (0,0), \ \gamma(1,t) = (R,0), & \\ \kappa(0,t) = \kappa(1,t) = 0 & \text{in} \quad [0,\infty), \\ \gamma(x,0) = \gamma_0(x) & \text{in} \quad I. \end{cases} \tag{3.23}$$

Remark that γ_0 is a smooth planar curve satisfying

$$\left|{\gamma_0}'(x)\right| \equiv 1, \ \gamma_0(0) = (0,0), \ \gamma_0(1) = (R,0), \ \kappa_0(0) = \kappa_0(1) = 0. \tag{3.24}$$

The purpose of this subsection is applying Theorem 2.1 to the problem (3.23) and proveing that the solution $\gamma(x,t)$ converges to a stationary solution as $t \to \infty$.

Regarding Assumption 2.1, it is easy to check that the flow (3.1) with the boundary condition (3.2)-(3.22) is the L^2-gradient flow for the functional $\mathcal{E}_\lambda$ (see Section 5).

By the proof of the following Proposition, we see that Assumption 2.2 holds.

Proposition 3.2. ([**9**]) *Let $\gamma_0(x)$ be a planar curve satisfying* (3.24). *Then there exist a family of smooth planar curves $\gamma(x,t) : I \times [0,\infty) \to \mathbb{R}^2$ satisfying* (3.23). *Moreover, there exist sequence $\{t_j\}_{j=1}^{\infty}$ and a smooth curve $\tilde{\gamma} : I \to \mathbb{R}^2$ such that $\gamma(\cdot, t_j)$ converges to $\tilde{\gamma}(\cdot)$ as $t_j \to \infty$ up to a reparametrization. Moreover the curve $\tilde{\gamma}$ satisfies*

$$\begin{cases} 2\partial_s^2 \tilde{\kappa} + \tilde{\kappa}^3 - \lambda^2 \tilde{\kappa} = 0 \quad \textit{in} \quad I, \\ \tilde{\gamma}(0) = (0,0), \ \tilde{\gamma}(1) = (R,0), \ \tilde{\kappa}(0) = \tilde{\kappa}(1) = 0. \end{cases} \tag{3.25}$$

Let $\mathcal{S}$ be a set of all stationary solutions, i.e., a set of all planar open curves satisfying (3.25). And for each $A \in \mathbb{R}$, let us define the set $\widetilde{\Sigma}_A$ of $\mathcal{S}$ as follows:

$$\widetilde{\Sigma}_A = \{\tilde{\gamma} \in \mathcal{S} \mid \mathcal{E}_\lambda(\tilde{\gamma}) \leq A\}.$$

By making use of Lemma 3.1, we prove that the set $\widetilde{\Sigma}_A$ is finite for any $A \in \mathbb{R}$:

Lemma 3.3. *The set $\widetilde{\Sigma}_A$ is finite for any $A \in \mathbb{R}$.*

Proof. To begin with, we identify $\tilde{\gamma} \in \mathcal{S}$ with $\mathcal{R}\tilde{\gamma} \in \mathcal{S}$, where $\mathcal{R} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If $\widetilde{\Sigma}_A$ is not finite, there exists a sequence $\{\tilde{\gamma}_n\}_{n=1}^{\infty} \subset \widetilde{\Sigma}_A$. Then there exists a constant $E_n \geq 0$ such that $(\tilde{\kappa}_n, E_n)$ satisfies (3.3) for each $n \in \mathbb{N}$. If $E_n \to \infty$ as $n \to \infty$, then Lemma 3.1 implies that

$$\int_{\tilde{\gamma}_n} \tilde{\kappa}_n^2 \, ds \to \infty.$$

This contradicts $\{\tilde{\gamma}_n\}_{n=1}^{\infty} \subset \widetilde{\Sigma}_A$. Thus there exists a constant E^* such that $E_n < E^*$ holds for any $n \in \mathbb{N}$. Moreover, if $E_n \to 0$ as $n \to \infty$, then Lemma 4.1 implies that $L(E_n) \to \infty$ as $n \to \infty$. Then we observe that $\mathcal{E}_\lambda(\tilde{\gamma}_n) \to \infty$ as $n \to \infty$. This also contradicts $\{\tilde{\gamma}_n\}_{n=1}^{\infty} \subset \widetilde{\Sigma}_A$. Hence there exists a positive constant $E_* > 0$ such that $E_* < E_n$ for all $n \in \mathbb{N}$. Since $\{E_n\}_{n=1}^{\infty}$ is a bounded sequence, there exist a constant $E_* \leq E_\infty \leq E^*$ and a subsequence $\{E_{n_k}\}_{k=1}^{\infty} \subset \{E_n\}_{n=1}^{\infty}$ such that $E_{n_k} \to E_\infty$ as $k \to \infty$. By the definition of $\{\tilde{\gamma}_n\}_{n=1}^{\infty}$ and $\{E_n\}_{n=1}^{\infty}$, there exists $\tilde{\gamma}_\infty \in \widetilde{\Sigma}_A$ such that $(\kappa_\infty, E_\infty)$ satisfies (3.3). Here we define a function $d = d(E)$ for a planar open curves with the pair (κ, E) satisfying (3.3) as

$$d(E) = |\gamma(L(E)) - \gamma(0)|\,.$$

In Lemma 4.1, we shall prove that $L(E)$ is analytic on $(0, \infty)$. Since $\gamma(s)$ depends on s analytically, the analyticity of $L(E)$ implies that $d(E)$ is analytic. Since $\tilde{\gamma}_n \in \mathcal{S}$, there exists a number $N_n \in \mathbb{N}$ such that

$$d(E_n) = \frac{R}{N_n}.$$

In particular, there exists a number $N_\infty \in \mathbb{N}$ such that

$$d(E_\infty) = \frac{R}{N_\infty}.$$

Since $d(E_{n_k}) \to d(E_\infty)$ as $k \to \infty$, it must be holds that $N_{n_k} = N_\infty$ for sufficiently large $k \in \mathbb{N}$. This means that $d(E_{n_k}) = d(E_\infty)$ holds for sufficiently large $k \in \mathbb{N}$. The analyticity of $d(\cdot)$ implies that $E_{n_k} = E_\infty$ for sufficiently large $k \in \mathbb{N}$. The relation $E_{n_k} = E_\infty$ yields $\tilde{\gamma}_{n_k} = \tilde{\gamma}_\infty$. Since $\tilde{\gamma}_n \in \widetilde{\Sigma}_A$ is uniquely determined with respect to E_n by identifying $\tilde{\gamma}_n$ with $\mathcal{R}\tilde{\gamma}_n$, this contradicts the uniqueness. q.e.d.

Since Lemma 3.3 implies that Assumption 2.3 holds, we see that Theorem 2.1 yields the following:

Theorem 3.2. *Let $\gamma(x,t) : I \times [0,\infty) \to \mathbb{R}^2$ be a solution of* (3.23)*. Then there exists a solution $\tilde{\gamma}$ of* (3.25) *such that*

$$\gamma(\cdot, t) \to \tilde{\gamma}(\cdot) \quad \textit{as} \quad t \to \infty$$

in the C^∞-topology.

3.3. Symmetric Navier boundary condition. We now consider the following more general boundary condition for the curvature:

$$\kappa(0) = \kappa(1) = \alpha, \tag{3.26}$$

where $\alpha \in \mathbb{R}$ is a given constant. The boundary conditions (3.2)-(3.26) is sometimes called the *symmetric Navier boundary condition* (e.g., see [**1**, **3**]). In Section 5, we will show that the flow (3.1) with the symmetric Navier boundary condition is the L^2-gradient flow of the functional

$$\mathcal{E}_{\lambda,\alpha}(\gamma) := \mathcal{E}_\lambda(\gamma) - 2\alpha \int_\gamma \kappa \, ds. \tag{3.27}$$

We now show that the functional $\mathcal{E}_{\lambda,\alpha}$ is bounded from below whenever $|\alpha| < |\lambda|$.

Lemma 3.4. *Let $\alpha, \lambda \in \mathbb{R}$ be such that $\lambda \neq 0$ and*

$$|\alpha| < |\lambda| \,. \tag{3.28}$$

Then there exists a positive constant $C = C(\alpha, \lambda)$ such that

$$\mathcal{E}_{\lambda,\alpha}(\gamma) \geq C \max\{\|\kappa\|^2_{L^2_\gamma}, \mathcal{L}(\gamma)\} \qquad \text{for all } \gamma. \tag{3.29}$$

Proof. Using Hölder's and Young's inequalities, for all $\varepsilon \in (0, 1]$ we have

$$\begin{aligned} \mathcal{E}_{\lambda,\alpha}(\gamma) &= \int_\gamma \kappa^2 \, ds - 2\alpha \int_\gamma \kappa \, ds + \lambda^2 \mathcal{L}(\gamma) \\ &\geq \int_\gamma \kappa^2 \, ds - 2\,|\alpha| \left\{ \int_\gamma \kappa^2 \, ds \right\}^{\frac{1}{2}} \left\{ \int_\gamma ds \right\}^{\frac{1}{2}} + \lambda^2 \mathcal{L}(\gamma) \\ &\geq (1-\varepsilon) \int_\gamma \kappa^2 \, ds + \left(\lambda^2 - \frac{\alpha^2}{\varepsilon} \right) \mathcal{L}(\gamma). \end{aligned} \tag{3.30}$$

Taking $\alpha^2/\lambda^2 < \varepsilon < 1$, we obtain (3.29). q.e.d.

The purpose of this subsection is to prove a convergence of a solution of the following initial boundary value problem

$$\begin{cases} \partial_t \gamma = -2\partial_s^2 \kappa - \kappa^3 + \lambda^2 \kappa & \text{in } I \times [0,\infty), \\ \gamma(0,t) = (0,0), \ \gamma(1,t) = (R,0), & \\ \kappa(0,t) = \kappa(1,t) = \alpha & \text{in } [0,\infty), \\ \gamma(x,0) = \gamma_0(x) & \text{in } I, \end{cases} \tag{3.31}$$

to a solution of

$$\begin{cases} 2\partial_s^2\kappa + \kappa^3 - \lambda^2\kappa = 0 \quad \text{in} \quad I, \\ \gamma(0) = (0,0),\ \gamma(1) = (R,0),\ \kappa(0) = \kappa(1) = \alpha, \end{cases} \tag{3.32}$$

as $t \to \infty$.

In Section 5, we shall prove that there exists a unique smooth solution for all times, satisfying Assumption 2.2.

We turn to Assumption 2.3. Let γ be a planar open curve satisfying the stationary equation

$$\begin{cases} 2\partial_s^2\kappa + \kappa^3 - \lambda^2\kappa = 0 \quad \text{in} \quad I, \\ \kappa(0) = \kappa(1) = \alpha. \end{cases} \tag{3.33}$$

Then there exists a constant $E \in (-\lambda^4/4, \infty)$ such that the pair (κ, E) satisfies

$$\begin{cases} \left(\dfrac{d\kappa}{ds}\right)^2 + F(\kappa) = E \quad \text{in} \quad I, \\ \kappa(0) = \kappa(1) = \alpha. \end{cases} \tag{3.34}$$

Let

$$L_0(E) = 0 \tag{3.35}$$

$$L_1(E) = 2\int_{\kappa_m}^{\alpha} \frac{d\kappa}{\sqrt{E - F(\kappa)}}, \tag{3.36}$$

$$L_2(E) = 2\int_{\alpha}^{\kappa_M} \frac{d\kappa}{\sqrt{E - F(\kappa)}}, \tag{3.37}$$

so that

$$L(E) = L_1(E) + L_2(E). \tag{3.38}$$

It is easy to see that the length of γ can be written as

$$\mathcal{L}(\gamma) = \tilde{L}(E) + NL(E)$$

for some $N \in \mathbb{N}$, with

$$\tilde{L}(E) \in \{L_0(E), L_1(E), L_2(E)\} \quad \text{for any} \quad E \in (F(\alpha), +\infty).$$

Let $\mathcal{S}$ be a set of all solutions of (3.32). For each $A \in \mathbb{R}$, we define

$$\widetilde{\Sigma}_A = \{\gamma \in \mathcal{S} \mid \mathcal{E}_{\lambda,\alpha}(\gamma) \le A\}. \tag{3.39}$$

Lemma 3.5. *Let α, $\lambda \in \mathbb{R}$ be such that $\lambda \neq 0$ and* (3.28)*. Then the set $\widetilde{\Sigma}_A$ is finite for each $A \in \mathbb{R}$.*

Proof. Assume by contradiction that there exists a sequence $\{\gamma_n\}_{n=1}^{\infty} \subset \widetilde{\Sigma}_A$ with $\gamma_l \neq \gamma_m$ if $l \neq m$. Then there exists a constant E_n for each $n \in \mathbb{N}$ such that the pair (κ_n, E_n) satisfies

$$\begin{cases} \left(\dfrac{d\kappa_n}{ds}\right)^2 + F(\kappa_n) = E_n, \\ \kappa_n(0) = \kappa_n(\mathcal{L}(\gamma_n)) = \alpha, \end{cases} \tag{3.40}$$

where κ_n denotes the curvature of γ_n. By the discussion above, for each $E_n \in (-\lambda^4/4, \infty)$ there exists a unique solution κ_n of (3.40) such that $\mathcal{L}(\gamma_n) = L_i(E_n) + N_n L(E_n)$, where $i \in \{0, 1, 2\}$.

We claim that there exists a positive number E^* such that $E_n \leq E^*$ for any $n \in \mathbb{N}$. Suppose that $E_n \to \infty$ as $n \to \infty$. Then Lemma 3.1 yields

$$\int_{\gamma_n} \kappa_n^2 \, ds \to \infty \quad \text{as} \quad n \to \infty.$$

By virtue of Lemma 3.4, this implies that $\mathcal{E}_{\lambda,\alpha}(\gamma_n) \to \infty$ as $n \to \infty$, which contradicts $\gamma_n \subset \widetilde{\Sigma}_A$. Thus we see that $\{E_n\}$ is a bounded sequence, and then, there exists a constant $E_\infty \leq E^*$ such that $E_n \to E_\infty$ up to extracting a suitable subsequence. Moreover, possibly passing to a further subsequence, there exists a curve $\gamma_\infty \in \widetilde{\Sigma}_A$ such that the curves γ_n smoothly converge to γ_∞ as $n \to \infty$. As $\mathcal{L}(\gamma_\infty) = L_i(E_\infty) + N_\infty L(E_\infty)$ for some $i \in \{0, 1, 2\}$, it follows that $\mathcal{L}(\gamma_n) = L_i(E_n) + N_\infty L(E_n)$ for sufficiently large n. We define

$$\begin{aligned} \boldsymbol{d}(E) &= \gamma(L(E)) - \gamma(0), \\ \tilde{\boldsymbol{d}}(E) &= \gamma(L_i(E)) - \gamma(0), \end{aligned}$$

where γ is a solution of $(\kappa_s)^2 + F(\kappa) = E$. Since $\gamma_n \in \widetilde{\Sigma}_A$ and $N_n = N_\infty$ for n big enough, we have

$$|\tilde{\boldsymbol{d}}(E_n) + N_n \boldsymbol{d}(E_n)| = |\tilde{\boldsymbol{d}}(E_n) + N_\infty \boldsymbol{d}(E_n)| = R \tag{3.41}$$

for n sufficiently large.

In the following we show that (3.41) leads to a contradiction. We may assume that $\alpha > 0$ without loss of generality. First we consider the case where $F(\alpha) \geq 0$. Since $F(\alpha) \geq 0$ implies $E_n > 0$ for any $n \in \mathbb{N}$, Lemmas 4.2-4.3 imply that the function $|\tilde{\boldsymbol{d}}(E) + N_\infty \boldsymbol{d}(E)|$ is analytic on $(F(\alpha), \infty)$. Then (3.41) yields

$$|\tilde{\boldsymbol{d}}(E) + N_\infty \boldsymbol{d}(E)| = R \quad \text{for any} \quad E \in (F(\alpha), \infty). \tag{3.42}$$

It follows from Lemma 4.3 that $L(E) \to 0$ as $E \to \infty$. Then (3.38) yields that $L_i(E) \to 0$ as $E \to \infty$ for any $i \in \{1, 2\}$. Thus we observe that

$$|\tilde{\boldsymbol{d}}(E) + N_\infty \boldsymbol{d}(E)| \le |\tilde{\boldsymbol{d}}(E)| + N_\infty |\boldsymbol{d}(E)| \to 0 \quad \text{as} \quad E \to \infty.$$

This contradicts (3.42).

Next we consider the case where $F(\alpha) < 0$ and $E_\infty \ge 0$. Since we may assume that $E_n \ge 0$ for sufficiently large $n \in \mathbb{N}$, we can obtain a contradiction along the same argument of the case where $F(\alpha) \ge 0$.

Finally we consider the case where $F(\alpha) < 0$ and $E_\infty < 0$. Since it holds that $E_n < 0$ for sufficiently large n, Lemmas 4.1–4.3 and (3.41) yield that

$$|\tilde{\boldsymbol{d}}(E) + N_\infty \boldsymbol{d}(E)| = R \quad \text{for any} \quad E \in (-\lambda^4/4, 0). \tag{3.43}$$

Suppose that $N_\infty \neq 0$. Since $N_\infty \ge 1$, we have

$$|\tilde{\boldsymbol{d}}(E) + N_\infty \boldsymbol{d}(E)| \ge N_\infty |\boldsymbol{d}(E)| - |\tilde{\boldsymbol{d}}(E)| \ge N_\infty \left(|\boldsymbol{d}(E)| - |\tilde{\boldsymbol{d}}(E)| \right).$$

Remark that Lemmas 4.1–4.3 and (3.38) imply that $L(E) \to \infty$, $L_1(E) \to \infty$, and $L(E) - L_2(E) \to \infty$ as $E \uparrow 0$. If $\tilde{L}(E) \in \{L_0(E), L_1(E)\}$, then it holds that

$$|\boldsymbol{d}(E)| - |\tilde{\boldsymbol{d}}(E)| \to \infty \quad \text{as} \quad E \uparrow 0,$$

and then

$$|\tilde{\boldsymbol{d}}(E) + N_\infty \boldsymbol{d}(E)| \to \infty \quad \text{as} \quad E \uparrow 0.$$

This contradicts (3.42). If $\tilde{L}(E) = L_2(E)$, since (3.38) gives us that

$$\frac{L(E)}{L_1(E)} = 1 - \frac{L_2}{L_1} \to 1 \quad \text{as} \quad E \uparrow 0,$$

we observe that

$$|d(E) - \tilde{d}(E)| \to 0 \quad \text{as} \quad E \uparrow 0. \tag{3.44}$$

Then it follows from (3.44) that

$$|\tilde{d}(E) + N_\infty d(E)| \ge (N_\infty + 1)|d(E)| - |\tilde{d}(E) - d(E)| \to \infty \quad \text{as} \quad E \uparrow 0.$$

This also contradicts (3.42). Thus it must hold that $N_\infty = 0$. Then (3.43) is reduced to

$$|\tilde{\boldsymbol{d}}(E)| = R \quad \text{for any} \quad E \in (-\lambda^4/4, 0). \tag{3.45}$$

Lemma 4.2 implies that $\tilde{L}(E) = L_2(E)$. With the aid of Lemma 4.3, we can replace (3.45) with

$$|\tilde{\boldsymbol{d}}(E)| = R \quad \text{for any} \quad E \in (F(\alpha), \infty). \tag{3.46}$$

Moreover, by virtue of Lemma 4.3, we see that $L_2(E) \to 0$ as $E \to \infty$, i.e., $|\tilde{\boldsymbol{d}}(E)| \to 0$ as $E \to \infty$. This contradicts (3.46). The proof of Lemma 3.5 is complete. q.e.d.

Applying Theorem 2.1 to the problem (3.31), we obtain the following:

Theorem 3.3. *Let α, $\lambda \in \mathbb{R}$ satisfy $\lambda \neq 0$ and* (3.28)*. Let $\gamma(x,t) : I \times [0,\infty) \to \mathbb{R}^2$ be a solution of* (3.31)*. Then there exists a solution $\tilde{\gamma}$ of* (3.32)

$$\gamma(\cdot,t) \to \tilde{\gamma}(\cdot) \quad \textit{as} \quad t \to \infty$$

in the C^∞-topology.

4. Appendix A

Lemma 4.1. *Let $\gamma(s) : [0,\infty) \to \mathbb{R}^2$ be a planar open curve with the curvature satisfying*

$$\left(\frac{d\kappa}{ds}\right)^2 + F(\kappa) = E.$$

Then the function

$$L(E) = 2\int_{\kappa_m(E)}^{\kappa_M(E)} \frac{d\kappa}{\sqrt{E - F(\kappa)}}.$$

is analytic on $(-\lambda^4/4, 0) \cup (0,\infty)$. Furthermore it holds that

$$L(E) \to \infty \quad \textit{as} \quad E \to 0. \tag{4.1}$$

Proof. To begin with, we show that $L(E)$ is analytic on $(0,\infty)$ and $L(E) \to \infty$ as $E \downarrow 0$. Recall that $L(E)$ is written as

$$L(E) = 4\int_{0}^{\kappa_M(E)} \frac{d\kappa}{\sqrt{E - F(\kappa)}}$$

for $E \in (0,\infty)$. Since F is analytic, it is clear that $\kappa_M(E)$ is analytic. Moreover the definition of $\kappa_M(E)$ implies that $F'(\kappa(E)) \neq 0$. The Taylor expansion of F at $\kappa = \kappa_M(E)$ is expressed as

$$\begin{aligned} F(\kappa) = F(\kappa_M) + F'(\kappa_M)(\kappa - \kappa_M) + \frac{F''(\kappa_M)}{2!}(\kappa - \kappa_M)^2 \\ + \frac{F^{(3)}(\kappa_M)}{3!}(\kappa - \kappa_M)^3 + \frac{F^{(4)}(\kappa_M)}{4!}(\kappa - \kappa_M)^4. \end{aligned}$$

It follows from $F'(\kappa_M) \neq 0$ that

$$\sqrt{E - F(\kappa)} = \sqrt{F'(\kappa_M)(\kappa_M - \kappa)} \sqrt{1 + \sum_{n=1}^{3} a_n(E)(\kappa_M - \kappa)^n}$$

for any $\kappa \in [0, \kappa_M]$, where $a_n(E)$ is given by

$$a_n(E) = \frac{(-1)^n F^{(n+1)}(\kappa_M(E))}{(n+1)! F'(\kappa_M(E))}$$

Since it holds that

$$|a_n(E)| \leq C\, |\lambda|^{-n}$$

for any $E > 0$, we see that

$$\frac{1}{\sqrt{E - F(\kappa)}} = \sum_{k=0}^{\infty} b_k (\kappa_M(E) - \kappa)^{k-1/2}$$

for any $\kappa \in (\kappa_M(E) - \varepsilon, \kappa_M(E))$, where ε is a positive constant satisfying

$$\max_{1 \leq n \leq 3} |a_n| \varepsilon (1 + \varepsilon + \varepsilon^2) < 1.$$

Remark that $b_k = b_k(E)$ is analytic on $(0, \infty)$. In the following let us set $L(E)/4 = \mathfrak{L}_1(E) + \mathfrak{L}_2(E)$, which are written as

$$\mathfrak{L}_1(E) = \int_0^{\kappa_0(E)} \frac{d\kappa}{\sqrt{E - F(\kappa)}}, \quad \mathfrak{L}_2(E) = \int_{\kappa_0(E)}^{\kappa_M(E)} \frac{d\kappa}{\sqrt{E - F(\kappa)}},$$

where $\kappa_0(E) = \kappa_M(E) - \varepsilon/2$. First we check that $\mathfrak{L}_1(E)$ is analytic. Let us write $\mathfrak{L}_1(E)$ as

$$\mathfrak{L}_1(E) = \frac{1}{\sqrt{E}} \int_0^{\kappa_0} \left(1 - \frac{F(\kappa)}{E}\right)^{-\frac{1}{2}} d\kappa.$$

Notice that the function $(1 - y)^{-1/2}$ is analytic on $(-\infty, 1)$, and for any $y_0 < 1$ one can write

$$(1 - y)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} c_k (y - y_0) \quad \text{for all} \quad y \in (y_0, 1),$$

where the coefficients c_k depend on y_0. Letting $\bar{\kappa} = |\lambda|$ which is a minimum point of F and setting $y = F(\kappa)/E$, $y_0 = F(\bar{\kappa})/E$, we have

$$\begin{aligned} \mathfrak{L}_1(E) &= \frac{1}{\sqrt{E}} \sum_{k=0}^{\infty} c_k \int_0^{\kappa_0} \left(\frac{F(\kappa)}{E} - \frac{F(\bar{\kappa})}{E}\right)^k d\kappa \\ &= \sum_{k=0}^{\infty} c_k \frac{g_k(\kappa_0(E))}{E^{k+\frac{1}{2}}}, \end{aligned} \tag{4.2}$$

where

$$g_k(x) = \int_0^x (F(\kappa) - F(\bar{\kappa}))^k d\kappa.$$

Since it holds that

$$\left(\int_0^{\kappa_0} \left(\frac{F(\kappa)}{E} - \frac{F(\bar{\kappa})}{E} \right)^k d\kappa \right)^{\frac{1}{k}} \to \sup_{\kappa \in (0,\kappa_0)} \frac{F(\kappa) - F(\bar{\kappa})}{E} < 1 - y_0$$

as $k \to \infty$, we see that the series in (4.2) converges for each $E > 0$. Recalling $\kappa_M(E)$ is analytic, all the functions $g_k(\kappa_0(E))$ is also analytic. This implies that $\mathfrak{L}_1(E)$ is analytic for $E > 0$.

Regarding $\mathfrak{L}_2(E)$, we have

$$\mathfrak{L}_2(E) = \int_{\kappa_0}^{\kappa_M(E)} \frac{d\kappa}{\sqrt{E - F(\kappa)}} = -\sum_{k=0}^{\infty} \frac{b_k(E)}{(k+1/2)} \left(\frac{\varepsilon}{2} \right)^{k+1/2}.$$

Since $b_k(E)$ is analytic, this implies that $\mathfrak{L}_2(E)$ is also analytic for $E > 0$. Therefore we observe that $L(E)$ is analytic on $(0, \infty)$. On the other hand, it follows from (4.2) that

$$L(E) \to \infty \quad \text{as} \quad E \downarrow 0. \tag{4.3}$$

Next we prove that the function $L(E)$ is analytic on $(-\lambda^4/4, 0)$. Along the same line as above, we see that

$$\frac{1}{\sqrt{E - F(\kappa)}} = \sum_{k=0}^{\infty} b_k (\kappa_M(E) - \kappa)^{k-1/2}$$

for any $\kappa \in (\kappa_M - \varepsilon, \kappa_M)$, and

$$\frac{1}{\sqrt{E - F(\kappa)}} = \sum_{k=0}^{\infty} \tilde{b}_k (\kappa_m(E) - \kappa)^{k-1/2}$$

for any $\kappa \in (\kappa_m, \kappa_m + \varepsilon)$, where ε is an appropriate small number. Setting $L(E) = \tilde{\mathfrak{L}}_1(E) + \tilde{\mathfrak{L}}_2(E) + \tilde{\mathfrak{L}}_3(E) + \tilde{\mathfrak{L}}_4(E)$, where

$$\tilde{\mathfrak{L}}_1(E) = \int_{\kappa_m}^{\kappa_m + \varepsilon/2} \frac{d\kappa}{\sqrt{E - F(\kappa)}}, \qquad \tilde{\mathfrak{L}}_2(E) = \int_{\kappa_m + \varepsilon/2}^{\bar{\kappa}} \frac{d\kappa}{\sqrt{E - F(\kappa)}},$$

$$\tilde{\mathfrak{L}}_3(E) = \int_{\bar{\kappa}}^{\kappa_M - \varepsilon/2} \frac{d\kappa}{\sqrt{E - F(\kappa)}}, \qquad \tilde{\mathfrak{L}}_4(E) = \int_{\kappa_M - \varepsilon/2}^{\kappa_M} \frac{d\kappa}{\sqrt{E - F(\kappa)}}.$$

Regarding $\tilde{\mathfrak{L}}_1(E)$ and $\tilde{\mathfrak{L}}_4(E)$, we can verify that $\tilde{\mathfrak{L}}_1(E)$ and $\tilde{\mathfrak{L}}_4(E)$ are analytic on $(-\lambda^4/4, 0)$ along the same argument for $\mathfrak{L}_2(E)$. Next we turn to $\tilde{\mathfrak{L}}_2(E)$. Along the same line as the argument for $\mathfrak{L}_1(E)$, we have

$$\begin{aligned}\tilde{\mathfrak{L}}_2(E) &= \int_{\kappa_m+\varepsilon/2}^{\bar{\kappa}} \frac{1}{\sqrt{-F(\kappa)}} \frac{d\kappa}{\sqrt{1-E/F(\kappa)}} \\ &= \sum_{k=0}^{\infty} \tilde{c}_k \int_{\kappa_m+\varepsilon/2}^{\bar{\kappa}} \frac{1}{\sqrt{-F(\kappa)}} \left(\frac{E}{F(\kappa)} - \frac{E}{F(\bar{\kappa})}\right)^k d\kappa \\ &= \sum_{k=0}^{\infty} \tilde{c}_k \tilde{g}_k(\kappa_m(E)+\varepsilon/2) E^k,\end{aligned} \tag{4.4}$$

where

$$\tilde{g}_k(x) = \int_x^{\bar{\kappa}} \left(F(\kappa)^{-1} - F(\bar{\kappa})^{-1}\right) (-F(\kappa))^{-1/2}\, d\kappa.$$

Since it holds that

$$\tilde{\mathfrak{L}}_2(E) \leq \frac{1}{\sqrt{-E}} \sum_{k=0}^{\infty} \tilde{c}_k \int_{\kappa_m+\varepsilon/2}^{\bar{\kappa}} \left(\frac{E}{F(\kappa)} - \frac{E}{F(\bar{\kappa})}\right)^k d\kappa$$

and

$$\left(\int_{\kappa_m+\varepsilon/2}^{\bar{\kappa}} \left(\frac{E}{F(\kappa)} - \frac{E}{F(\bar{\kappa})}\right)^k d\kappa\right)^{\frac{1}{k}} \to \sup_{\kappa \in (\kappa_m+\varepsilon/2, \bar{\kappa})} \frac{E}{F(\kappa)} - \frac{E}{F(\bar{\kappa})} < 1 - y_0$$

as $k \to \infty$, we observe that the series in (4.4) converges for each $E \in (-\lambda^4/4, 0)$. Recalling $\kappa_m(E)$ is analytic, all the functions $\tilde{g}_k(\kappa_m(E) + \varepsilon/2)$ is also analytic. This implies that $\tilde{\mathfrak{L}}_2(E)$ is analytic for $E \in (-\lambda^4/4, 0)$. Since similar argument gives us that $\tilde{\mathfrak{L}}_3(E)$ is also analytic for $E \in (-\lambda^4/4, 0)$.

Finally we prove that $L(E) \to \infty$ as $E \uparrow 0$. Regarding $\tilde{\mathfrak{L}}_1(E)$, it holds that

$$\begin{aligned}\tilde{\mathfrak{L}}_1(E) &> \int_{\kappa_m}^{\kappa_m+\varepsilon/2} \frac{d\kappa}{\sqrt{-F'(\kappa_m)(\kappa-\kappa_m) - F''(\kappa_m)(\kappa-\kappa_m)^2}} \\ &= \frac{1}{2\sqrt{-F''(\kappa_m)}} \log \frac{1+\sqrt{\frac{\varepsilon}{\varepsilon+2a}}}{1-\sqrt{\frac{\varepsilon}{\varepsilon+2a}}},\end{aligned} \tag{4.5}$$

where $a = F'(\kappa_m)/F''(\kappa_m)$. Since $F'(\kappa_m(E)) \to 0$ and $F''(\kappa_m(E)) \to -\lambda^2$ as $E \uparrow 0$, it follows from (4.5) that

$$\tilde{\mathfrak{L}}_1(E) \to \infty \quad \text{as} \quad E \uparrow 0.$$

This clearly implies that $L(E) \to \infty$ as $E \uparrow 0$. q.e.d.

The arguments in the proof of Lemma 4.1 also implies an analyticity of $L_i(E)$ which are defined by (3.36)–(3.37).

Lemma 4.2. *Let $\alpha > 0$. If $F(\alpha) \geq 0$, then the function $L_1(E)$ is analytic on $(F(\alpha), \infty)$. If $F(\alpha) < 0$, then the function $L_1(E)$ is analytic on $(F(\alpha), 0) \cup (0, \infty)$. Moreover, as $E \to 0$, it holds that*

$$L_1(E) \to \infty \quad \text{as} \quad E \to 0. \tag{4.6}$$

Proof. The proof of Lemma 4.1 gives us the conclusion. q.e.d.

Lemma 4.3. *For each $\alpha > 0$, the function $L_2(E)$ is analytic on $(F(\alpha), \infty)$. Moreover, for each $\alpha \in (0, 2\sqrt{|\lambda|})$, it holds that*

$$L_2(E) \to 0 \quad \text{as} \quad E \to \infty. \tag{4.7}$$

Proof. An analyticity of $L_2(E)$ is followed from the same argument of the proof of Lemma 4.1. We shall prove (4.7). Since $0 < \alpha < \sqrt{2}\,|\lambda|$, we divide $L_2(E)$ into two part as follows:

$$\int_\alpha^{\kappa_M} \frac{d\kappa}{\sqrt{E - F(\kappa)}} = \int_\alpha^{\sqrt{2}|\lambda|} \frac{d\kappa}{\sqrt{E - F(\kappa)}} + \int_{\sqrt{2}|\lambda|}^{\kappa_M} \frac{d\kappa}{\sqrt{E - F(\kappa)}}. \tag{4.8}$$

Recalling $F(\sqrt{2}\,|\lambda|) = 0$, we have

$$\int_\alpha^{\sqrt{2}|\lambda|} \frac{d\kappa}{\sqrt{E - F(\kappa)}} \leq \frac{1}{\sqrt{E}} \int_\alpha^{\sqrt{2}|\lambda|} d\kappa \to 0 \quad \text{as} \quad E \to \infty.$$

Thus it is sufficient to estimate the second term of the right-hand side of (4.8). By changing the variable $\kappa/(4E)^{1/4} = x$, we have

$$\begin{aligned} \int_{\sqrt{2}|\lambda|}^{\kappa_M} \frac{d\kappa}{\sqrt{E - F(\kappa)}} &\leq \frac{1}{\sqrt{E}} \int_{\sqrt{2}|\lambda|}^{\kappa_M} \frac{d\kappa}{\sqrt{1 - \frac{\kappa^4}{4E}}} \\ &= \frac{\sqrt{2}}{E^{1/4}} \int_{\sqrt{2}|\lambda|/(4E)^{1/4}}^{\kappa_M/(4E)^{1/4}} \frac{dx}{\sqrt{1 - x^4}}. \end{aligned}$$

And then, the conclusion is obtained from the following calculation:

$$\begin{aligned} \frac{\sqrt{2}}{E^{1/4}} \int_{\sqrt{2}|\lambda|/(4E)^{1/4}}^{\kappa_M/(4E)^{1/4}} \frac{dx}{\sqrt{1 - x^4}} &\leq \frac{\sqrt{2}}{E^{1/4}} \int_{\sqrt{2}|\lambda|/(4E)^{1/4}}^{\kappa_M/(4E)^{1/4}} \frac{dx}{\sqrt{1 - x^2}} \\ &= \frac{\sqrt{2}}{E^{1/4}} \left\{ \sin^{-1} \frac{\kappa_M}{(4E)^{1/4}} - \sin^{-1} \frac{\sqrt{2}\,|\lambda|}{(4E)^{1/4}} \right\} \\ &\to 0 \quad \text{as} \quad E \to \infty. \end{aligned}$$

q.e.d.

5. Appendix B

The scope of this appendix is to prove that (3.31) has a unique smooth solution defined for all times.

Let us first show that the L^2-gradient flow for the functional $\mathcal{E}_{\lambda,\alpha}$ under (3.2)-(3.26) can be written as (3.1). Indeed, let $\gamma : [0,1] \to \mathbb{R}^2$ be a smooth planar curve satisfying the symmetric Navier boundary condition

$$\gamma(0) = (0,0), \;\; \gamma(1) = (R,0), \;\; \kappa(0) = \kappa(1) = \alpha, \tag{5.1}$$

We consider a variation of γ defined as follows:

$$\gamma(x,\varepsilon) = \gamma(x) + \phi(x,\varepsilon)\boldsymbol{\nu}(x),$$

where $\boldsymbol{\nu}$ is the unit normal vector, pointing in the direction of the curvature, given by

$$\boldsymbol{\nu} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\gamma_x}{|\gamma_x|} := \mathcal{R}\frac{\gamma_x}{|\gamma_x|},$$

and $\phi(x,\varepsilon) \in C^\infty((-\varepsilon_0,\varepsilon_0); C^\infty(0,1))$ is an arbitral smooth function with

$$\phi(x,0) \equiv \phi(0,\varepsilon) \equiv \phi(1,\varepsilon) \equiv 0.$$

In the following we shall derive a first variational formula for the functional $\mathcal{E}_{\lambda,\alpha}(\gamma)$. Put

$$\tau = \frac{\gamma_x}{|\gamma_x|}.$$

Since the curvature of γ is expressed as

$$\kappa = \frac{\gamma_{xx} \cdot \mathcal{R}\gamma_x}{|\gamma_x|^3}, \tag{5.2}$$

we have

$$\kappa = \gamma_{xx} \cdot \boldsymbol{\nu} \, |\gamma_x|^{-2},$$

and then Frenet-Serret's formula $\partial_s \boldsymbol{\nu} \cdot \tau = -\kappa$ yields that

$$\boldsymbol{\nu}_x \cdot \tau = -\kappa \, |\gamma_x| \, .$$

To begin with, we derive useful variational formulae. First we find the first variational formula of the local length.

$$\frac{d}{d\varepsilon} |\gamma_x(x,\varepsilon)| \bigg|_{\varepsilon=0} = \frac{\gamma_x \cdot (\phi_\varepsilon \boldsymbol{\nu})_x}{|\gamma_x|} = \tau \cdot \phi_\varepsilon \boldsymbol{\nu}_x = -\kappa \, |\gamma_x| \, \phi_\varepsilon, \tag{5.3}$$

where $\phi_\varepsilon(\cdot) = (\partial\phi/\partial\varepsilon)(\cdot, 0)$. Next we find the first variation formula of the curvature. From (5.2) and

$$\begin{aligned}
\boldsymbol{\nu} \cdot \boldsymbol{\nu}_{xx} &= -\left|\boldsymbol{\nu}_x\right|^2 = -\kappa^2 \left|\gamma_x\right|^2, \\
\gamma_{xx} \cdot \mathcal{R}\boldsymbol{\nu}_x &= \gamma_{xx} \cdot \mathcal{R}(-\kappa\gamma_x) = -\kappa^2 \left|\gamma_x\right|^3, \\
\left(\left|\gamma_x\right|^{-1}\right)_x &= \frac{\gamma_{xx} \cdot \mathcal{R}\boldsymbol{\nu}}{\left|\gamma_x\right|^2},
\end{aligned}$$

it follows that

$$\frac{d}{d\varepsilon}\kappa(x,\varepsilon)\bigg|_{\varepsilon=0} = \frac{\phi_{\varepsilon xx}}{\left|\gamma_x\right|^2} + \kappa^2\phi_\varepsilon + (\left|\gamma_x\right|^{-1})_x \frac{\phi_{\varepsilon x}}{\left|\gamma_x\right|}. \tag{5.4}$$

Using (5.3), we obtain

$$\begin{aligned}
&\frac{d}{d\varepsilon}\mathcal{E}_{\lambda,\alpha}(\gamma(\cdot,\varepsilon))\bigg|_{\varepsilon=0} \\
&\quad = \int_0^1 \left\{2(\kappa-\alpha)\frac{d}{d\varepsilon}\kappa\bigg|_{\varepsilon=0} - \left(\kappa^3 - 2\alpha\kappa^2 + \lambda^2\kappa\right)\phi_\varepsilon\right\}\left|\gamma_x\right|\,dx
\end{aligned}$$

Using (5.4) and integrating by parts, we get

$$\begin{aligned}
&\int_0^1 (\kappa-\alpha)\frac{d}{d\varepsilon}\kappa\bigg|_{\varepsilon=0}\left|\gamma_x\right|\,dx \\
&= \int_0^1 (\kappa-\alpha)\left\{\frac{\phi_{\varepsilon xx}}{\left|\gamma_x\right|^2} + \kappa^2\phi_\varepsilon + (\left|\gamma_x\right|^{-1})_x\frac{\phi_{\varepsilon x}}{\left|\gamma_x\right|}\right\}\left|\gamma_x\right|\,dx \\
&= \int_0^1 -\left(\frac{\kappa-\alpha}{\left|\gamma_x\right|}\right)_x \phi_{\varepsilon x} + (\kappa^3-\alpha\kappa^2)\phi_\varepsilon\left|\gamma_x\right| + (\left|\gamma_x\right|^{-1})_x(\kappa-\alpha)\phi_{\varepsilon x}\,dx \\
&\qquad + \left[\frac{\kappa-\alpha}{\left|\gamma_x\right|}\phi_{\varepsilon x}\right]_0^1 \\
&= \int_0^1 -\frac{\kappa_x}{\left|\gamma_x\right|}\phi_{\varepsilon x} + (\kappa^3-\alpha\kappa^2)\phi_\varepsilon\left|\gamma_x\right|\,dx \\
&= \int_0^1 \left(\frac{\kappa_x}{\left|\gamma_x\right|}\right)_x \phi_\varepsilon + (\kappa^3-\alpha\kappa^2)\phi_\varepsilon\left|\gamma_x\right|\,dx \\
&= \int_0^1 \left\{\left(\frac{\partial_x}{\left|\gamma_x\right|}\right)^2\kappa + (\kappa^3-\alpha\kappa^2)\right\}\phi_\varepsilon\left|\gamma_x\right|\,dx.
\end{aligned}$$

Here we use $\kappa(0) = \kappa(1) = \alpha$. Thus we find

$$\frac{d}{d\varepsilon}\mathcal{E}_{\lambda,\alpha}(\gamma(\cdot,\varepsilon))\bigg|_{\varepsilon=0} = \int_a^b \left\{2\left(\frac{\partial_x}{\left|\gamma_x\right|}\right)^2\kappa + \kappa^3 - \lambda^2\kappa\right\}\phi_\varepsilon\left|\gamma_x\right|\,dx. \tag{5.5}$$

Parameterizing by the arc length, the formula (5.5) is written as

$$\frac{d}{d\varepsilon}\mathcal{E}_{\lambda,\alpha}(\gamma(\cdot,\varepsilon))\bigg|_{\varepsilon=0}=\int_0^1\left\{2\kappa_{ss}+\kappa^3-\lambda^2\kappa\right\}\phi_\varepsilon\,ds.$$

Therefore we see that the flow (3.1) is the L^2-gradient flow for the functional $\mathcal{E}_{\lambda,\alpha}$ under the symmetric Navier boundary condition (5.1).

Since (3.31) is a nonlinear boundary value problem for a quasi-linear parabolic equation, a short time existence is a standard matter. In what follows we shall prove a long time existence of solutions to (3.31). Throughout the section, put

$$V^\lambda=2\partial_s^2\kappa+\kappa^3-\lambda^2\kappa.$$

Then the equation in (3.31) is written as

$$\partial_t\gamma=-V^\lambda\boldsymbol{\nu}. \tag{5.6}$$

Since s depends on t, remark that the following holds.

Lemma 5.1. *Under* (5.6), *the following commutation rule holds*:

$$\partial_t\partial_s=\partial_s\partial_t-\kappa V^\lambda\partial_s.$$

Lemma 5.1 gives us the following:

Lemma 5.2. *Let $\gamma(x,t)$ satisfy* (5.6). *Then it holds that*

$$\partial_t\kappa=-\partial_s^2V^\lambda-\kappa^2V^\lambda. \tag{5.7}$$

Furthermore, the line element ds of $\gamma(x,t)$ satisfies

$$\partial_t ds=\kappa V^\lambda ds. \tag{5.8}$$

The boundary conditions in (3.31) imply that several terms vanish on the boundary.

Lemma 5.3. *Suppose that γ satisfies* (3.31). *Then it holds that*

$$\partial_t\gamma=0 \quad \text{on} \quad \partial I\times[0,\infty), \tag{5.9}$$

$$\partial_t\kappa=0 \quad \text{on} \quad \partial I\times[0,\infty), \tag{5.10}$$

$$V^\lambda=0 \quad \text{on} \quad \partial I\times[0,\infty), \tag{5.11}$$

$$\partial_s^2V^\lambda=0 \quad \text{on} \quad \partial I\times[0,\infty), \tag{5.12}$$

$$\partial_tV^\lambda=0 \quad \text{on} \quad \partial I\times[0,\infty), \tag{5.13}$$

$$\partial_t\partial_s^2V^\lambda=0 \quad \text{on} \quad \partial I\times[0,\infty), \tag{5.14}$$

$$\partial_t\partial_s=\partial_s\partial_t \quad \text{on} \quad \partial I\times[0,\infty). \tag{5.15}$$

Proof. Since both $\gamma(t)$ and $\kappa(t)$ are fixed on ∂I, we observe (5.9)-(5.10). It follows from (5.6) and (5.9) that (5.11) holds. By virtue of (5.2), (5.10), and (5.11), we obtain (5.12). Then (5.11) and (5.12) implies (5.13) and (5.14), respectively. (5.15) is followed from Lemma 5.1 and (5.11) q.e.d.

Here we introduce interpolation inequalities for open curves, which has been inspired by [**4**] for closed curves and given in [**6**]. The interpolation inequalities are written in terms of the following the scale invariant Sobolev norms:

$$\|\kappa\|_{k,p} := \sum_{i=0}^{k} \left\|\partial_s^i \kappa\right\|_p , \quad \left\|\partial_s^i \kappa\right\|_p := \mathcal{L}(\gamma)^{i+1-1/p} \left(\int_I \left|\partial_s^i \kappa\right|^p \right)^{1/p} .$$

Lemma 5.4. ([**6**]) *Let $\gamma : I \to \mathbb{R}^2$ be a smooth curve. Then for any $k \in \mathbb{N} \cup \{0\}$, $p \geq 2$, and $0 \leq i < k$, we have*

$$\left\|\partial_s^i \kappa\right\|_p \leq c \, \|\kappa\|_2^{1-\alpha} \, \|\kappa\|_{k,2}^{\alpha} ,$$

where $\alpha = (i + \frac{1}{2} - \frac{1}{p})/k$ and $c = c(n, k, p)$.

In order to prove a long time existence of solutions to (3.31), we make use of the following Lemma, which is a modification of Lemma 2.2 in [**4**].

Lemma 5.5. *Let $\gamma : I \times [0, T) \to \mathbb{R}^2$ satisfy the equation* (5.6) *and $\phi : I \times [0, T) \to \mathbb{R}$ be a scalar function defined on γ satisfying*

$$\begin{cases} \partial_t \phi = -2\partial_s^4 \phi + Y & \text{in} \quad I \times [0, T), \\ \phi = 0, \quad \partial_s^2 \phi = 0 & \text{on} \quad \partial I \times [0, T). \end{cases} \tag{5.16}$$

Then it holds that

$$\frac{d}{dt} \frac{1}{4} \int_\gamma \phi^2 \, ds + \int_\gamma (\partial_s^2 \phi)^2 \, ds = \frac{1}{2} \int_\gamma \phi Y \, ds + \frac{1}{4} \int_\gamma \phi^2 \kappa V^\lambda \, ds. \tag{5.17}$$

Proof. It follows from the equation in (5.16) and Lemma 5.2 that

$$\begin{aligned} \frac{d}{dt} \frac{1}{4} \int_\gamma \phi^2 \, ds &= \frac{1}{2} \int_\gamma \phi \partial_t \phi \, ds + \frac{1}{4} \int_\gamma \phi^2 \partial_t (ds) \\ &= \frac{1}{2} \int_\gamma \phi(-2\partial_s^4 \phi + Y) \, ds + \frac{1}{4} \int_\gamma \phi^2 \kappa V^\lambda ds. \end{aligned}$$

With the aid of the boundary conditions in (5.16), we obtain

$$\int_\gamma \phi \partial_s^4 \phi \, ds = - \int_\gamma \partial_s \phi \partial_s^3 \phi \, ds = \int_\gamma (\partial_s^2 \phi)^2 \, ds.$$

Then we observe (5.17). q.e.d.

By virtue of Lemma 5.3, we observe that $\partial_t^m V^\lambda = 0$ and $\partial_s^2 \partial_t^m V^\lambda = 0$ hold on ∂I for any $m \in \mathbb{N} \cup \{0\}$. The fact implies that we can apply Lemma 5.5 to $\phi = \partial_t^m V^\lambda$. To do so, first we introduce the following notation for a convenience.

Definition 5.1. ([**2**]) *We use the symbol $\mathfrak{q}^r(\partial_s^l \kappa)$ for a polynomial with constant coefficients such that each of its monomials is of the form*

$$\prod_{i=1}^{N} \partial_s^{j_i} \kappa \quad \text{with} \quad 0 \le j_i \le l \quad \text{and} \quad N \ge 1$$

with

$$r = \sum_{i=1}^{N} (j_i + 1).$$

Making use of the notation, we obtain the following:

Lemma 5.6. *Suppose that $\gamma : I \times [0, \infty) \to \mathbb{R}^2$ satisfies* (3.31)*. Let ϕ be a scalar function defined on γ. Then the following formulae hold for any m, $l \in \mathbb{N}$*:

$$\partial_s^m V^\lambda = \mathfrak{q}^{3+m}(\partial_s^{2+m}\kappa) - \lambda^2 \partial_s^m \kappa, \tag{5.18}$$

$$\partial_t \partial_s^m \phi = \partial_s^m \partial_t \phi + \sum_{i=0}^{m-1} (\mathfrak{q}^{4+i}(\partial_s^{2+i}\kappa) + \mathfrak{q}^{2+i}(\partial_s^i \kappa)) \partial_s^{m-i} \phi, \tag{5.19}$$

$$\partial_t \partial_s^m \kappa = -2\partial_s^{m+4}\kappa + \mathfrak{q}^{m+5}(\partial_s^{m+2}\kappa) + \mathfrak{q}^{m+3}(\partial_s^{m+2}\kappa), \tag{5.20}$$

$$\partial_t \mathfrak{q}^l(\partial_s^m \kappa) = \mathfrak{q}^{l+4}(\partial_s^{m+4}\kappa) + \mathfrak{q}^{l+2}(\partial_s^{m+2}\kappa). \tag{5.21}$$

Proof. Since $V^\lambda = \mathfrak{q}^3(\partial_s^2\kappa) - \lambda^2\kappa$, the assertion (5.18) is followed from a simple calculation. Regarding (5.19), we proceed by induction on m. For $m = 1$, we have

$$\partial_t \partial_s \phi = \partial_s \partial_t \phi - \kappa V^\lambda \partial_s \phi = \partial_s \partial_t \phi - (\mathfrak{q}^4(\partial_s^2\kappa) + \mathfrak{q}^2(\kappa)) \partial_s \phi.$$

Assuming that (5.19) is true for some $m \ge 1$, we obtain

$$\begin{aligned}
\partial_t \partial_s^{m+1} \phi &= \partial_s \partial_t \partial_s^m \phi + (\mathfrak{q}^4(\partial_s^2\kappa) + \mathfrak{q}^2(\kappa)) \partial_s^{m+1} \phi \\
&= \partial_s \left\{ \partial_s^m \partial_t \phi + \sum_{i=0}^{m-1} (\mathfrak{q}^{4+i}(\partial_s^{2+i}\kappa) + \mathfrak{q}^{2+i}(\partial_s^i \kappa)) \partial_s^{m-i} \phi \right\} \\
&\quad + (\mathfrak{q}^4(\partial_s^2\kappa) + \mathfrak{q}^2(\kappa)) \partial_s^{m+1} \phi \\
&= \partial_s^{m+1} \partial_t \phi + \sum_{i=0}^{m} (\mathfrak{q}^{4+i}(\partial_s^{2+i}\kappa) + \mathfrak{q}^{2+i}(\partial_s^i \kappa)) \partial_s^{m+1-i} \phi.
\end{aligned}$$

(5.20) is followed from (5.2) and (5.19) directly. Finally we obtain (5.21) for m, $l \in \mathbb{N}$ fixed arbitrarily as follows:

$$\begin{aligned}
\partial_t \mathfrak{q}^l(\partial_s^m \kappa) &= \sum_{j=0}^{m} \mathfrak{q}^{l-j-1}(\partial_s^m \kappa) \cdot \partial_t \partial_s^j \kappa \\
&= \sum_{j=0}^{m} \mathfrak{q}^{l-j-1}(\partial_s^m \kappa) \cdot \{-2\partial_s^{j+4}\kappa + \mathfrak{q}^{j+5}(\partial_s^{j+2}\kappa) + \mathfrak{q}^{j+3}(\partial_s^{j+2}\kappa)\} \\
&= \sum_{j=0}^{m} \mathfrak{q}^{l+4}(\partial_s^{\max\{m,j+4\}}\kappa) + \sum_{j=0}^{m} \mathfrak{q}^{l+2}(\partial_s^{\max\{m,j+2\}}\kappa) \\
&= \sum_{j=0}^{4} \mathfrak{q}^{l+4}(\partial_s^{m+j}\kappa) + \sum_{j=0}^{2} \mathfrak{q}^{l+2}(\partial_s^{m+j}\kappa) \\
&= \mathfrak{q}^{l+4}(\partial_s^{m+4}\kappa) + \mathfrak{q}^{l+2}(\partial_s^{m+2}\kappa).
\end{aligned}$$

q.e.d.

With the aid of Lemma 5.6, we obtain a representation of $\partial_t^m V^\lambda$.

Lemma 5.7. *For each $m \in \mathbb{N}$, it holds that*

$$\begin{aligned}
\partial_t^m V^\lambda = (-1)^m 2^{m+1} \partial_s^{4m+2}\kappa + \mathfrak{q}^{4m+3}(\partial_s^{4m}\kappa) \\
+ \sum_{j=1}^{m} \mathfrak{q}^{4m+3-2j}(\partial_s^{4m+2-2j}\kappa).
\end{aligned} \tag{5.22}$$

Proof. We proceed by induction on m. For $m = 1$, we have

$$\begin{aligned}
\partial_t V^\lambda &= \partial_t(2\partial_s^2\kappa + \kappa^3 - \lambda^2\kappa) \\
&= 2(-2\partial_s^6\kappa + \mathfrak{q}^7(\partial_s^4\kappa) + \mathfrak{q}^5(\partial_s^4\kappa)) + 3\kappa^2\partial_t\kappa - \lambda^2\partial_t\kappa \\
&= -2^2\partial_s^6\kappa + \mathfrak{q}^7(\partial_s^4\kappa) + \mathfrak{q}^5(\partial_s^4\kappa).
\end{aligned}$$

Suppose that (5.22) holds for $m = k$. Then we have

(5.23)

$$\begin{aligned}
\partial_t^{k+1} V^\lambda &= \partial_t\{(-1)^k 2^{k+1}\partial_s^{4k+2}\kappa + \mathfrak{q}^{4k+3}(\partial_s^{4k}\kappa) \\
&\qquad + \sum_{j=1}^{k} \mathfrak{q}^{4k+3-2j}(\partial_s^{4k+2-2j}\kappa)\} \\
&= (-1)^k 2^{k+1}\{-2\partial_s^{4k+6}\kappa + \mathfrak{q}^{4k+7}(\partial_s^{4k+6}\kappa) + \mathfrak{q}^{4k+5}(\partial_s^{4k+6}\kappa)\} \\
&\qquad + \partial_t\{\mathfrak{q}^{4k+3}(\partial_s^{4k}\kappa) + \sum_{j=1}^{k} \mathfrak{q}^{4k+3-2j}(\partial_s^{4k+2-2j}\kappa)\}.
\end{aligned}$$

By virtue of (5.21), the last term in (5.23) is reduced to

$$\begin{aligned}
&\partial_t\{\mathfrak{q}^{4k+3}(\partial_s^{4k}\kappa)+\sum_{j=1}^{k}\mathfrak{q}^{4k+3-2j}(\partial_s^{4k+2-2j}\kappa)\}\\
&=\mathfrak{q}^{4k+7}(\partial_s^{4k+4}\kappa)+\mathfrak{q}^{4k+5}(\partial_s^{4k+2}\kappa)\\
&\qquad+\sum_{j=1}^{k}\{\mathfrak{q}^{4k+7-2j}(\partial_s^{4k+6-2j}\kappa)+\mathfrak{q}^{4k+5-2j}(\partial_s^{4k+4-2j}\kappa)\}\\
&=\mathfrak{q}^{4(k+1)+3}(\partial_s^{4(k+1)}\kappa)+\sum_{j=1}^{k+1}\mathfrak{q}^{4(k+1)+3-2j}(\partial_s^{4(k+1)+2-2j}\kappa).
\end{aligned}$$

This implies that (5.22) holds for any $m \in \mathbb{N}$. q.e.d.

We are in the position to prove the main result of this section.

Theorem 5.1. *Let $\lambda \in \mathbb{R}$ be non-zero constant. Let $\gamma_0 : I \to \mathbb{R}^2$ be a smooth open curve satisfying*

$$\gamma_0(0)=(0,0),\ \gamma_0(1)=(R,0),\ \kappa_0(0)=\kappa_0(1)=\alpha,$$

where $\alpha \in \mathbb{R}$ is a given constant with $|\alpha| < |\lambda|$. Then there exists a unique family of smooth open planar curves $\gamma(x,t)$ satisfying (3.31) *for any finite time $t > 0$.*

Proof. Suppose not, there exists a time $t_1 > 0$ such that the smooth solution $\gamma(x,t)$ of (3.31) remains up to $t = t_1$. Setting $\phi = \partial_t^m V^\lambda$, Lemma 5.5 implies that

$$\begin{aligned}
\frac{d}{dt}\frac{1}{4}\int_\gamma(\partial_t^m V^\lambda)^2\,ds+\int_\gamma(\partial_s^2\partial_t^m V^\lambda)^2\,ds&\\
=\frac{1}{2}\int_\gamma\partial_t^m V^\lambda Y\,ds+\frac{1}{4}\int_\gamma(\partial_t^m V^\lambda)^2\kappa V^\lambda\,ds.&
\end{aligned}\tag{5.24}$$

Regarding the integral of $(\partial_s^2\partial_t^m V^\lambda)^2$, we have

$$\begin{aligned}
(\partial_s^2\partial_t^m V^\lambda)^2 &\ge (2^{2(m+1)}-\varepsilon)(\partial_s^{4m+4}\kappa)^2\\
&\qquad+\{\mathfrak{q}^{4m+5}(\partial_s^{4m+2}\kappa)+\sum_{j=1}^{m}\mathfrak{q}^{4m+5-2j}(\partial_s^{4m+4-2j}\kappa)\}^2\\
&=c_m(\partial_s^{4m+4}\kappa)^2+\sum_{j=0}^{m}\mathfrak{q}^{8m+10-2j}(\partial_s^{4m+2}\kappa)\\
&\qquad+\sum_{j,l=1}^{m}\mathfrak{q}^{8m+10-2(j+l)}(\partial_s^{4m+4-2\min\{j,l\}}\kappa).
\end{aligned}$$

Regarding the integral of $\partial_t^m V^\lambda Y$, setting

$$\partial_t^m V^\lambda Y = \partial_t^m V^\lambda \mathfrak{q}^{4m+7}(\partial_s^{4m+4}\kappa) + \partial_t^m V^\lambda \mathfrak{q}^{4m+5}(\partial_s^{4m+4}\kappa) + \partial_t^m V^\lambda \sum_{j=2}^{m+1} \mathfrak{q}^{4m+7-2j}(\partial_s^{4m+6-2j}\kappa) := I_1 + I_2 + I_3,$$

and integrating by part once the highest order term, we find

$$\begin{aligned}\int_\gamma I_1\, ds &= -\int_\gamma \partial_s \partial_t^m V^\lambda \{\mathfrak{q}^{4m+7}(\partial_s^{4m+3}\kappa) + \mathfrak{q}^{4m+6}(\partial_s^{4m+3}\kappa)\}\, ds \\ &= -\sum_{j=0}^{m} \int_\gamma \{\mathfrak{q}^{8m+11-2j}(\partial_s^{4m+3}\kappa) + \mathfrak{q}^{8m+10-2j}(\partial_s^{4m+3}\kappa)\}\, ds,\end{aligned}$$

and

$$\begin{aligned}\int_\gamma I_2\, ds &= -\int_\gamma \partial_s \partial_t^m V^\lambda \{\mathfrak{q}^{4m+5}(\partial_s^{4m+3}\kappa) + \mathfrak{q}^{4m+4}(\partial_s^{4m+3}\kappa)\}\, ds \\ &= -\sum_{j=0}^{m} \int_\gamma \{\mathfrak{q}^{8m+9-2j}(\partial_s^{4m+3}\kappa) + \mathfrak{q}^{8m+8-2j}(\partial_s^{4m+3}\kappa)\}\, ds.\end{aligned}$$

Hence we see that

$$\begin{aligned}\int_\gamma \partial_t^m V^\lambda Y\, ds = \int_\gamma \sum_{j=0}^{m+1} \{&\mathfrak{q}^{8m+11-2j}(\partial_s^{4m+3}\kappa) + \mathfrak{q}^{8m+10-2j}(\partial_s^{4m+3}\kappa) \\ &+ \mathfrak{q}^{8m+8-2j}(\partial_s^{4m+2}\kappa) + \mathfrak{q}^{8m+6-2j}(\partial_s^{4m}\kappa)\} \\ &+ \sum_{l=1,j=1}^{m} \mathfrak{q}^{8m+8-2(j+l)}(\partial_s^{4m+2-2\min\{j,l\}}\kappa)\, ds.\end{aligned}$$

Since it holds that

$$\begin{aligned}\int_\gamma (\partial_t^m V^\lambda)^2 \kappa V^\lambda\, ds = \int_\gamma \sum_{j=0}^{m+1} \{&\mathfrak{q}^{8m+10-2j}(\partial_s^{4m+2}\kappa) + \mathfrak{q}^{8m+10-2j}(\partial_s^{4m}\kappa)\} \\ &+ \sum_{l=1,j=1}^{m} \mathfrak{q}^{8m+10-2(j+l)}(\partial_s^{4m+2-2\min\{j,l\}}\kappa)\, ds,\end{aligned}$$

the equality (5.24) is reduced to

$$\text{(5.25)}\quad \frac{d}{dt}\frac{1}{4}\int_\gamma (\partial_t^m V^\lambda)^2\,ds + c_m(\varepsilon)\int_\gamma (\partial_s^{4m+4}\kappa)^2\,ds$$
$$= \int_\gamma \Bigg[\sum_{j=0}^{m+1} \Big\{ \mathfrak{q}^{8m+11-2j}(\partial_s^{4m+3}\kappa) + \mathfrak{q}^{8m+10-2j}(\partial_s^{4m+3}\kappa) + \mathfrak{q}^{8m+10-2j}(\partial_s^{4m+2}\kappa) + \mathfrak{q}^{8m+10-2j}(\partial_s^{4m}\kappa) \Big\} + \sum_{l=1,j=1}^{m} \mathfrak{q}^{8m+10-2(j+l)}(\partial_s^{4m+2-2\min\{j,l\}}\kappa) \Bigg]\,ds.$$

We estimate the integral of $\mathfrak{q}^{8m+11}(\partial_s^{4m+3}\kappa)$ which is the highest order term in the right-hand side of (5.25). By Definition 5.1, this term can be written as

$$\mathfrak{q}^{8m+11}(\partial_s^{4m+3}\kappa) = \sum_j \prod_{i=1}^{N_j} \partial_s^{c_{j_i}}\kappa$$

with all the c_{j_i} less that or equal to $4m+3$, and

$$\sum_{i=1}^{N_j}(c_{j_i}+1) = 8m+11$$

for every j. Hence we have

$$\left|\mathfrak{q}^{8m+11}(\partial_s^{4m+3}\kappa)\right| \le \sum_j \prod_{i=1}^{N_j}\left|\partial_s^{c_{j_i}}\kappa\right|.$$

Putting

$$Q_j = \prod_{i=1}^{N_j}\left|\partial_s^{c_{j_i}}\kappa\right|,$$

it holds that

$$\int_\gamma \left|\mathfrak{q}^{8m+11}(\partial_s^{4m+3}\kappa)\right|\,ds \le \sum_j \int_\gamma Q_j\,ds.$$

After collecting the derivatives of the same order in Q_j, we can write

$$Q_j = \prod_{l=0}^{4m+3}\left|\partial_s^l\kappa\right|^{\alpha_{j_l}} \quad \text{with} \quad \sum_{l=0}^{4m+3}\alpha_{j_l}(l+1) = 8m+11.$$

Using Hölder's inequality we get

$$\int_\gamma Q_j \, ds \leq \prod_{l=0}^{4m+3} \left(\int_\gamma \left| \partial_s^l \kappa \right|^{\alpha_{j_l} \lambda_l} \right)^{1/\lambda_l} = \prod_{l=0}^{4m+3} \left\| \partial_s^l \kappa \right\|_{\alpha_{j_l} \lambda_l}^{\alpha_{j_l}},$$

where the value λ_l are chosen as follows: $\lambda_l = 0$ if $\alpha_{j_l} = 0$ (in this case the corresponding term is not present in the product) and $\lambda_l = (8m+11)/\alpha_{j_l}(l+1)$ if $\alpha_{j_l} \neq 0$. Clearly $\alpha_{j_l}\lambda_l = \frac{8m+11}{l+1} \geq \frac{8m+11}{4m+4} > 2$ and

$$\sum_{l=0, \lambda_l \neq 0}^{4m+3} \frac{1}{\lambda_l} = \sum_{l=0, \lambda_l \neq 0}^{4m+3} \frac{\alpha_{j_l}(l+1)}{8m+11} = 1.$$

Let $k_l = \alpha_{j_l}\lambda_l - 2$. The fact $\alpha_{j_l}\lambda_l > 2$ implies that $k_l > 0$. Then we obtain

$$\left\| \partial_s^l \kappa \right\|_{\alpha_{j_l} \lambda_l} \leq c \left\| \kappa \right\|_2^{1-\sigma_{j_l}} \left\| \kappa \right\|_{4m+4,2}^{\sigma_{j_l}},$$

where $\sigma_{j_l} = (l + \frac{1}{2} - \frac{1}{\alpha_{j_l}\lambda_l})/(4m+4)$ and $c = c(j, l, m)$. Since

$$\left\| \kappa \right\|_{4m+4,2}^2 \leq C(m) \left(\left\| \partial_s^{4m+4} \kappa \right\|_2^2 + \left\| \kappa \right\|_2^2 \right),$$

we observe that

$$\left\| \partial_s^l \kappa \right\|_{\alpha_{j_l} \lambda_l} \leq C \left\| \kappa \right\|_2^{1-\sigma_{j_l}} \left(\left\| \partial_s^{4m+4} \kappa \right\|_2^2 + \left\| \kappa \right\|_2^2 \right)^{\sigma_{j_l}}.$$

Multiplying together all the estimates, we obtain

(5.26)

$$\begin{aligned} \int_\gamma Q_j \, ds &\leq C \prod_{l=0}^{4m+3} \left\| \kappa \right\|_2^{(1-\sigma_{j_l})\alpha_{j_l}} \left(\left\| \partial_s^{4m+4} \kappa \right\|_2 + \left\| \kappa \right\|_2 \right)^{\sigma_{j_l} \alpha_{j_l}} \\ &= C \left\| \kappa \right\|_2^{\sum_{l=0}^{4m+3} (1-\sigma_{j_l})\alpha_{j_l}} \left(\left\| \partial_s^{4m+4} \kappa \right\|_2 + \left\| \kappa \right\|_2 \right)^{\sum_{l=0}^{4m+3} \sigma_{j_l} \alpha_{j_l}}. \end{aligned}$$

Then the exponent in the last term of (5.26) is written as

$$\sum_{l=0}^{4m+3} \sigma_{j_l} \alpha_{j_l} = \sum_{l=0}^{4m+3} \frac{\alpha_{j_l}(l + \frac{1}{2} - \frac{1}{\alpha_{j_l}\lambda_l})}{4m+4} = \frac{\sum_{l=0}^{4m+3} \alpha_{j_l}(l+\frac{1}{2}) - 1}{4m+4},$$

and hence by using the rescaling condition we have

$$\begin{aligned} \sum_{l=0}^{4m+3} \sigma_{j_l} \alpha_{j_l} &= \frac{\sum_{l=0}^{4m+3} \alpha_{j_l}(l+1) - \frac{1}{2} \sum_{l=0}^{4m+3} \alpha_{j_l} - 1}{4m+4} \\ &= \frac{8m + 11 - \frac{1}{2} \sum_{l=0}^{4m+3} \alpha_{j_l} - 1}{4m+4} = \frac{16m + 20 - \sum_{l=0}^{4m+3} \alpha_{j_l}}{2(4m+4)}. \end{aligned}$$

Noting that

$$\sum_{l=0}^{4m+3} \alpha_{j_l} \geq \sum_{l=0}^{4m+3} \alpha_{j_l} \frac{l+1}{4m+4} = \frac{8m+11}{4m+4},$$

we see that

$$\sum_{l=0}^{4m+3} \sigma_{j_l} \alpha_{j_l} \leq \frac{16m+20-\frac{8m+10}{4m+4}}{2(4m+4)} = 2 - \frac{1}{(4m+4)^2} < 2.$$

Hence we can apply the Young inequality to the product in the last term of (5.26), in order to get the exponent 2 on the first quantity, that is,

$$\begin{aligned}\int_\gamma Q_j \, ds &\leq \frac{\delta_j}{2} \left(\left\| \partial_s^{4m+4} \kappa \right\|_2 + \|\kappa\|_2 \right)^2 + C_j \|\kappa\|_2^\beta \\ &\leq \delta_j \left\| \partial_s^{4m+4} \kappa \right\|_2^2 + \|\kappa\|_2^2 + +C_j \|\kappa\|_2^{\beta_j}\end{aligned}$$

for arbitrarily small $\delta_j > 0$ and some constant $C_j > 0$ and exponent $\beta_j > 0$. Hence we get

$$\begin{aligned}&\frac{d}{dt} \frac{1}{4} \int_\gamma (\partial_t^m V^\lambda)^2 \, ds + \frac{1}{2} \int_\gamma (\partial_s^2 \partial_t^m V^\lambda)^2 \, ds + \frac{c_m(\varepsilon)}{2} \int_\gamma (\partial_s^{4m+4} \kappa)^2 \, ds \\ &\leq \sum_{j=0}^{m+1} \delta_j \left\| \partial_s^{4m+4} \kappa \right\|_2^2 + C \sum_{j=0}^{m+1} \|\kappa\|_2^{\beta_j} .\end{aligned}$$

Letting $\delta_j > 0$ be sufficiently small, we obtain

$$\frac{d}{dt} \frac{1}{4} \int_\gamma (\partial_t^m V^\lambda)^2 \, ds \leq C \sum_{j=0}^{m+1} \|\kappa\|_2^{\beta_j} . \tag{5.27}$$

Since Lemma 3.4 gives us

$$\|\kappa\|_2^2 \leq C(\alpha, \lambda) \mathcal{E}_{\lambda,\alpha}(\gamma_0),$$

(5.27) implies that

$$\left\| \partial_t^m V^\lambda(t) \right\|_{L^2}^2 \leq C_1 t + \left\| \partial_t^m V^\lambda(0) \right\|_{L^2}^2 \tag{5.28}$$

for any time $t \in [0, t_1)$. Using (5.22) and the interpolation inequality, we reduce (5.28) to

$$\left\| \partial_s^{4m+2} \kappa \right\|_{L^2}^2 \leq C_1 t + \left\| \partial_t^m V^\lambda \right\|_{L^2}^2 (0) + C_2, \tag{5.29}$$

where C_2 depends on $\mathcal{E}_{\lambda,\alpha}(\gamma_0)$. Combining (3.30) and (5.29) with the interpolation inequality, we observe that there exists a positive constant depending only on $\mathcal{E}_{\lambda,\alpha}(\gamma_0)$ such that

$$\left\|\partial_s^l \kappa\right\|_{L^2} \leq C_1 t + \left\|\partial_t^m V^\lambda\right\|_{L^2}^2 (0) + C_3 \tag{5.30}$$

for any $0 \leq l < 4m+2$. For each $l \in \mathbb{N}$, it is easy to obtain that

$$\left\|\partial_s^{l-1} \kappa\right\|_{L^\infty} \leq C \left\|\partial_s^l \kappa\right\|_{L^1} + \mathcal{L}(\gamma)^{-1} \left\|\partial_s^{l-1} \kappa\right\|_{L^1}. \tag{5.31}$$

Applying Hölder's inequality to (5.31), we obtain

$$\left\|\partial_s^{l-1} \kappa\right\|_{L^\infty} \leq \mathcal{L}(\gamma)^{1/2} \left\|\partial_s^l \kappa\right\|_{L^2} + \mathcal{L}(\gamma)^{-1/2} \left\|\partial_s^{l-1} \kappa\right\|_{L^2}. \tag{5.32}$$

Then it follows from (5.30) and (5.32) that there exists a constant $C = C(\gamma_0, t_1, \alpha, \lambda)$ such that

$$\left\|\partial_s^{l-1} \kappa(t)\right\|_{L^\infty} \leq C \tag{5.33}$$

for each $l \in \mathbb{N}$ and any $t \in [0, t_1)$. This contradicts that the solution of (3.31) remains smooth to $t = t_1$. We thus complete the proof. q.e.d.

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