Constant in two-dimensional p-compliance-network problem

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Abstract

We consider the problem of the minimization of the p-compliance functional where the control variables Σ are taking among closed connected one-dimensional sets. we prove some estimate from below of the p-compliance functional in terms of the one-dimensional Hausdorff measure of Σ and compute the value of the constant $\theta(p)$ appearing usually in Γ-limit functional of the rescaled p-compliance functional.

1 Introduction

Let $p > 1$ be fixed and $q = p/(p-1)$ the conjugate exponent of p. For an open set $\Omega \subset \mathbb{R}$ and l a positive given real number, we define

$$\mathcal{A}_l(\Omega) = \{\Sigma \subset \overline{\Omega}, \text{ closed and connected}, \ 0 < \mathcal{H}^1(\Sigma) \leq l\}.$$

For a nonnegative function $f \in L^q(\Omega)$ and Σ a compact set with positive p-capacity, we denote by $u_{f,\Sigma,\Omega}$ the weak solution of the equation

$$\begin{cases} -\Delta_p u &= f \ \text{ in } \Omega \setminus \Sigma \\ u &= 0 \ \text{ in } \Sigma \cup \partial\Omega, \end{cases} \tag{1.1}$$

that is $u \in W_0^{1,p}(\Omega \setminus \Sigma)$ and

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_\Omega f \varphi dx \quad \forall \varphi \in W_0^{1,p}(\Omega \setminus \Sigma).$$

It is well known that by by the maximum principle, the nonnegativity of the function f implies that of u. For $f \geq 0$, the p-compliance functional is defined by

$$\begin{aligned} C_p(\Sigma, f, \Omega) &= \int_\Omega f u_{f,\Sigma,\Omega} dx = \int_\Omega |\nabla u_{f,\Sigma,\Omega}|^p dx \\ &= q \max \left\{ \int_\Omega \left(fv - \frac{1}{p} |\nabla v|^p \right) dx : v \in W_0^{1,p}(\Omega \setminus \Sigma) \right\}, \end{aligned} \tag{1.2}$$

where q stands for the conjugate exponent of p. The minimization problem we are dealing with is the following

$$\min\{C_p(\Sigma, f, \Omega) : \Sigma \in \mathcal{A}_l(\Omega)\}. \tag{1.3}$$

The existence of a minimal p-compliance configuration is just a consequence of a generalized Šverák compactness-continuity result (see [1]). In [2], authors have studied the asymptotic behavior of the optimal set Σ_l of the p-compliance functional problem as $l \to +\infty$. To fix idea, let us recall their result. Let us denote by $\mathcal{P}(\overline{\Omega})$ the space of all probability measures defined on $\overline{\Omega}$. We endow the space $\mathcal{P}(\overline{\Omega})$ with the topology generated by the weak* convergence of measures. To every set $\Sigma \in \mathcal{A}_l(\Omega)$, we associate a probability measure on $\overline{\Omega}$, given by

$$\mu_\Sigma = \frac{\mathcal{H}^1 \llcorner \Sigma}{\mathcal{H}^1(\Sigma)}$$

and define a functional $F_l : \mathcal{P}(\overline{\Omega}) \to [0; +\infty]$ by

$$F_l(\mu) = \begin{cases} l^q C_p(\Sigma, f, \Omega) & \text{if} \quad \mu = \mu_\Sigma,\ \Sigma \in \mathcal{A}_l(\Omega) \\ \quad +\infty & \text{otherwise.} \end{cases} \tag{1.4}$$

We also define a functional F by setting, for $\mu \in \mathcal{P}(\overline{\Omega})$

$$F(\mu) := \theta(p) \int_\Omega \frac{f^q}{\mu_a}\, dx, \tag{1.5}$$

where μ_a stands for the density of the absolutely continuous part of the measure μ and f the right hand side of equation (1.1). The constant $\theta(p)$ is a positive and finite real number which is defined by

$$\theta(p) := \inf\{\liminf_{l\to+\infty} l^q C_p(\Sigma_l, 1, Y) : \Sigma_l \in \mathcal{A}_l(Y)\}, \tag{1.6}$$

being Y the unit square in $\mathbb{R}^2$.
The following theorem is the main result in [2].

Theorem 1.1. *(Buttazzo-Santambrogio) Given any bounded open set $\Omega \subset \mathbb{R}^2$ and a nonnegative function $f \in L^q(\Omega)$, the functional defined in* (1.4) *Γ-converges to F as $l \to +\infty$ with respect to the weak* topology on $\mathcal{P}(\overline{\Omega})$.*

The constant used in [2] is equal to $q^{-1}\theta(p)$. For the notion of Γ-convergence, one may consults [3]. In order to have the explicit value of the functional F defined in (1.5), we need to compute the exact value of the constant $\theta(p)$. But in [2] this value was not available . However, authors proved that the constant is finite and bounded below by

$$\theta(p) \geq \frac{(2q)^{-q}}{q+1}. \tag{1.7}$$

Finding the value of the constant $\theta(p)$ is the main motivation of our paper. Let us point out that in the case where $p = 2$, the constant $\theta(2)$ is proved to be bounded above by $\frac{1}{12}$ (in [2], this value is equal to $\frac{1}{24}$ since our $\theta(2)$ is twice their own). Moreover authors conjectured that $\theta(2) = \frac{1}{12}$ and the comb configuration is asymptotically optimal. Recently, it has been proved in [4] that this conjecture holds true.

2 Estimate of $\theta(p)$ from below

In this section, we estimate from below the p-compliance functional $C_p(\Sigma, 1, \Omega)$ in terms of the one-dimensional Hausdorff measure of $\Sigma \cup \partial\Omega$ as made in the case $p = 2$ in [4]. By taking Ω as a unit square, we prove an estimate from below of the constant $\theta(p)$ (see (2.13) in the sequel) which is better than (1.7) obtained in [2].
From now on, if Σ is a nonempty closed set in $\mathbb{R}^2$, we denote by

$$d_\Sigma(x) = \min_{y\in\Sigma} |y - x|, \quad x \in \mathbb{R}^2$$

the distance function to Σ. We also denote by meas(A) the two-dimensional Lebesgue measure of the measurable set $A \subset \mathbb{R}^2$ and by $\mathcal{H}^1(A)$ the one-dimensional Hausdorff measure of a measurable set $A \subset \mathbb{R}^2$. Let us recall the following definition.

Definition 2.1. *Let $N \geq 1$ be an integer. We say that a set $\Sigma \subset \mathbb{R}^2$ is an N-continuum if the following holds true*

1. *Σ is decomposed as*

$$\Sigma = \bigcup_{j=1}^{N} \Sigma_j, \quad \Sigma_j \cap \Sigma_k = \emptyset \quad j \neq k,$$

where each Σ_j is nonempty, compact, connected set,

2. *$0 < \mathcal{H}^1(\Sigma) < +\infty$.*

The following result is proved in [4].

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain with Lipschitz boundary, let M denote the number of connected components of $\partial\Omega$, and let $\Sigma \subset \overline{\Omega}$ be N-continuum for some $N \geq 1$. For $t \geq 0$, we define*

$$A_t := \{x \in \Omega : d_{\Sigma\cup\Omega}(x) < t\}, \tag{2.1}$$

where $d_\sigma(x)$ stands for the distance function to σ. Then the following estimate of the measure of A_t holds

$$\mathrm{meas(A_t)} \leq \min\{\mathrm{meas}(\Omega), 2\mathcal{H}^1(\Sigma \cup \partial\Omega)\mathrm{t} + (\mathrm{N} + \mathrm{M})\pi \mathrm{t}^2\} \quad \mathrm{t} \geq 0.$$

For convenience of notation, we set $\Lambda := \Sigma \cup \partial\Omega$ and $L = \mathcal{H}^1(\Sigma \cup \partial\Omega)$. Let us introduce the following auxiliary function

$$B(t) := \min\{\mathrm{meas}(\Omega), 2\mathrm{Lt} + (\mathrm{N}+\mathrm{M})\pi \mathrm{t}^2\}. \tag{2.2}$$

From lemma 2.2 and the definition of $B(t)$, it holds

$$\mathrm{meas}(\mathrm{A_t}) \leq \mathrm{B(t)} \leq \mathrm{meas}(\Omega). \tag{2.3}$$

Now define α to be the positive root of the equation

$$2L\alpha + (M+N)\pi\alpha^2 = \mathrm{meas}(\Omega)$$

that is

$$\alpha = \frac{\mathrm{meas}(\Omega)}{L + \sqrt{L^2 + (M+N)\pi \mathrm{meas}(\Omega)}}. \tag{2.4}$$

From the definition of α, the function B may be written in the form

$$B(t) = \begin{cases} 2Lt + (N+M)\pi t^2 & \text{if } 0 \leq t \leq \alpha \\ \mathrm{meas}(\Omega) & \text{if } t > \alpha. \end{cases}$$

For the computation in next proposition, let us introduce the quantity

$$m := \max_{x \in \overline{\Omega}} d_\Lambda(x). \tag{2.5}$$

Clearly, $\mathrm{meas}(\mathrm{A_m}) = \mathrm{meas}(\Omega)$, so taking $t = m$ in (2.3) gives

$$\mathrm{meas}(\mathrm{A_m}) = \mathrm{B(m)} = \mathrm{meas}(\Omega). \tag{2.6}$$

As a consequence we have

$$0 < \alpha \leq m. \tag{2.7}$$

The function B is differentiable at t for any $t \neq \alpha$ and

$$B'(t) = \begin{cases} 2L + 2(N+M)\pi t & \text{if } 0 \leq t < \alpha \\ 0 & \text{if } \alpha < t \leq m. \end{cases} \tag{2.8}$$

We denote the perimeter of A_t in Ω by $B(A_t, \Omega)$(this notation of perimeter is not usual but we do not want to use p which is reserved for the "$p-Laplacian$" operator). Then, by the coarea formula (see [5]), we have

$$\mathrm{meas}(\mathrm{A_t}) = \int_0^{\mathrm{t}} \mathrm{B}(\mathrm{A_s}, \Omega)\ \mathrm{ds}, \quad \mathrm{t} \in (0, \mathrm{m}),$$

hence

$$B(A_t, \Omega) = \frac{d}{dt}\mathrm{meas}(\mathrm{A_t}) \quad \text{a.e. } \mathrm{t} \in (0, \mathrm{m}).$$

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain with Lipschitz boundary, let M denote the number of connected components of $\partial\Omega$, and let $\Sigma \subset \overline{\Omega}$ be N-continuum for some $N \geq 1$. For any $h : [0, \alpha] \mapsto \mathbb{R}$ $C^{1,1}$ function such that*

$$h(0) = 0, \quad h' \geq 0, \quad h'' \leq 0 \quad \text{on} \quad [0, \alpha], \tag{2.9}$$

we have the following estimate:

$$C_p(\Sigma, 1, \Omega) \geq q \int_0^\alpha \Big(h(t) - \frac{1}{p} h'(t)^p\Big) B'(t)\ dt. \tag{2.10}$$

where B' is given in (2.8) *and α in* (2.4).

Proof. To prove (2.10), we will construct a competitor u depending only on the distance function to $\mathit{\Lambda}$. Our proof is based on the one made in [4] for the case $p = 2$.
Let $h : [0, \alpha] \mapsto \mathbb{R}$ be as in our statement (2.9) and extended by $h'(\alpha)(t - \alpha)$ for $t \geq 0$. It is well known that the distance function is Lipschitzian and enjoys the property $|\nabla d_\mathit{\Lambda}| = 1$ almost everywhere. Noticing that $d_\mathit{\Lambda}$ vanishes along $\mathit{\Lambda}$, we consider the competitor u as the composition of h with the distance function namely

$$u(x) = h(d_\mathit{\Lambda}(x)), \quad x \in \overline{\Omega}.$$

One can check that $u \in W_0^{1,p}(\Omega \setminus \mathit{\Lambda}) = W_0^{1,p}(\Omega \setminus \Sigma)$ and

$$|\nabla d_\mathit{\Lambda}(x)| = |h'(d_\mathit{\Lambda}(x))|$$

for almost every $x \in \overline{\Omega}$. Using (1.2) with $f = 1$, we get

$$C_p(\Sigma, 1, \Omega) \geq q \int_\Omega \Big(u(x) - \frac{1}{p}|\nabla u(x)|^p\Big) dx = q \int_\Omega \Big(h(d_{\Sigma'}(x)) - \frac{1}{p}|h'(d_{\Sigma'}(x))|^p\Big) dx.$$

Using the fact that $|\nabla d_\mathit{\Lambda}| = 1$ almost everywhere and a slicing along the level sets of the distance function, the coarea formula (see [5]) gives

$$C_p(\Sigma, 1, \Omega) \geq q \int_0^m \Big(h(t) - \frac{1}{p} h'(t)^p\Big) B(A_t, \Omega) dt,$$

where $B(A_t, \Omega)$ is the perimeter of the set A_t inside Ω (see (2.1) for the definition of A_t). Set $H_p(t) = h(t) - \frac{1}{p} h'(t)^p$ and integrate by part, we get

$$\begin{aligned}
C_p(\Sigma, 1, \Omega) &\geq q \int_0^m H_p(t) B(A_t, \Omega) dt \\
&= -q \int_0^m H_p'(t) \mathrm{meas}(\mathrm{A_t})\mathrm{dt} + \mathrm{meas}(\mathrm{A_m})\mathrm{H_p(m)q} - \mathrm{meas}(\mathrm{A_0})\mathrm{H(0)q} \\
&= -q \int_0^m H_p'(t) \mathrm{meas}(\mathrm{A_t})\mathrm{dt} + \mathrm{meas}(\mathrm{A_m})\mathrm{H_p(m)q}
\end{aligned}$$

from (2.9) and the way h is extended for all $t \geq \alpha$, we get $H_p'(t) \geq 0$ for all $t \in (0, m)$ so, (2.3) yields

$$-H_p'(t)\mathrm{meas}(\mathrm{A_t}) \geq -\mathrm{H_p'(t)B(t)}, \quad \mathrm{t \in (0, m)}.$$

Using this inequality, (2.8), (2.6) and (2.7), an integration by part gives

$$\begin{aligned}
C_p(\Sigma, 1, \Omega) &\geq -q \int_0^m H_p'(t)\mathrm{meas}(\mathrm{A_t})\mathrm{dt} + \mathrm{meas}(\mathrm{A_m})\mathrm{H_p(m)q} \\
&= q \int_0^m H_p(t)B'(t)\ dt + (\mathrm{meas}(\mathrm{A_m}) - \mathrm{B(m)})\mathrm{H_p(m)q} \\
&= q \int_0^\alpha H_p(t)B'(t)\ dt + (\mathrm{meas}(\mathrm{A_m}) - \mathrm{B(m)})\mathrm{H_p(m)q} \\
&= q \int_0^\alpha H_p(t)B'(t)\ dt,
\end{aligned}$$

and the proof is over. □

In the following result, we prove two estimates from below of the p-compliance functional in terms of the one-dimensional Hausdorff measure of the set Λ (made by Dirichlet regions and the boundary of Ω) and the number of its connected components. One of these estimates allows to get an estimate from below of the constant $\theta(p)$.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary, let M denote the number of connected components of $\partial\Omega$, and let $\Sigma \subset \overline{\Omega}$ be N-continuum for some $N \geq 1$. Then the following estimate of p-compliance holds*

$$C_p(\Sigma, 1, \Omega) \geq \int_0^\alpha \Big(\frac{B(\alpha) - B(t)}{B'(t)}\Big)^q B'(t)\ dt. \tag{2.11}$$

where α is given in (2.4) *and B' in* (2.8). *In particular an useful estimate can be written as follow*

$$C_p(\Sigma, 1, Y) \geq \frac{2L}{q+1}\alpha^{q+1} + \frac{(N+M)(q^2+q+2)\pi}{(q+1)(q+2)}\alpha^{q+2}. \tag{2.12}$$

As a consequence, if we choose Ω to be the unit square Y then

$$\theta(p) = \inf\{\liminf_{l \to +\infty} l^q C_p(\Sigma, 1, Y) : \Sigma \in \mathcal{A}_l(Y)\} \geq \frac{1}{(q+1)2^q} \tag{2.13}$$

Proof. The inequality (2.10) holds for every $C^{1,1}$ function h satisfying (2.9), so

$$C_p(\Sigma, 1, \Omega) \geq q \max\Big\{\int_0^\alpha \Big(h(t) - \frac{1}{p}h'(t)^p\Big)B'(t)\ dt : h \quad \text{satisfies} \quad (2.9)\Big\}. \tag{2.14}$$

To find the maximizer of this problem, we first look for the maximizer of the variational problem

$$\max\Big\{\int_0^\alpha \Big(h(t) - \frac{1}{p}h'(t)^p\Big)B'(t)\ dt : \quad h(0) = 0\Big\} \tag{2.15}$$

and show that the solution of (2.15) satisfies (2.9). Therefore solution of (2.15) will turn out to be a solution of (2.14). The Euler equation associated to the variational problem (2.15) is given by

$$\begin{cases} \frac{d}{dt}\Big(h'(t)^{p-1}B'(t)\Big) = -B'(t), \quad t\in(0,\alpha) \\ h(0)=0, \quad h'(\alpha)=0 \end{cases}. \tag{2.16}$$

where $B'(t)=2(N+M)\pi t+2L$. Integrating (2.16) from t to α and taking into account the boundary condition (namely $h'(\alpha)=0$) we get

$$\begin{cases} h'(t) = \Big(\frac{B(\alpha)-B(t)}{B'(t)}\Big)^{\frac{1}{p-1}}, \quad t\in(0,\alpha) \\ h(0)=0. \end{cases} \tag{2.17}$$

The right hand side of (2.17) is a nice function and the solution of the equation is given by

$$h(t)=\int_0^t \Big(\frac{B(\alpha)-B(s)}{B'(s)}\Big)^{\frac{1}{p-1}}\, ds, \quad t\in(0,\alpha). \tag{2.18}$$

The interested reader may consult [4] for the explicit function h where $p=2$ replacing N therein by $N+M$. The function h, critical point of (2.15), is a maximizer. In fact from (2.18), we may notice that the function h is twice differentiable on $(0,\alpha)$ and satisfies $(h'(t))^{p-1}B'(t)=B(\alpha)-B(t), \quad t\in(0,\alpha)$. Therefore multiplying this relation by $h'(t)$, integrating by part from 0 to α and using the boundary condition $h(0)=0$, we get

$$\int_0^\alpha (h'(t))^p B'(t)\, dt = \int_0^\alpha h(t)B'(t)\, dt,$$

which shows the maximality of h. It remains to show that h satisfies the condition (2.9) but this is straightforward since $h(0)=0$, $h'>0$ on $(0,\alpha)$ by (2.17) and a direct computation shows that $h''\le 0$ on $(0,\alpha)$. From (2.14) and the maximality of h, it holds

$$C_p(\Sigma,1,\Omega)\ge q\Big(\frac{p-1}{p}\Big)\int_0^\alpha (h'(t))^p B'(t)\, dt = \int_0^\alpha \Big(\frac{B(\alpha)-B(t)}{B'(t)}\Big)^q B'(t)\, dt.$$

To prove (2.12) and (2.13), we will choose a particular function h which is not optimal (that is not maximizer of (2.14)) but satisfies the conditions (2.9). Let take h to be the function defined by

$$h(t) := \frac{1}{q}\Big(\alpha^q-(\alpha-t)^q\Big) \quad t\in(0,\alpha),$$

where α is defined in (2.4). Clearly, h satisfies (2.9). Let us point out that the function h is the solution of the variational problem

$$\max\Big\{\int_0^\alpha \Big(h(t)-\frac{1}{p}h'(t)^p\Big)\, dt : \quad h(0)=0\Big\}.$$

Using the relation between the conjugate exponents p and q, it holds

$$(h'(t))^p = (\alpha - t)^{p(q-1)} = (\alpha - t)^q$$

and

$$h(t) - \frac{1}{p}(h'(t))^p = \frac{1}{q}\alpha^q - (\alpha - t)^q.$$

So plugging this in (2.10), using the expression of $B'(t)$ and integrating by part, we get

$$\begin{aligned} C_p(\Sigma, 1, \Omega) &\geq \alpha^q \int_0^\alpha B'(t)\ dt - q\int_0^\alpha (\alpha - t)^q B'(t)\ dt \\ &= \alpha^q B(\alpha) - q\Big(2L\int_0^\alpha (\alpha - t)^q\ dt + 2(M+N)\pi \int_0^\alpha (\alpha - t)^q t\ dt\Big) \\ &= \alpha^q B(\alpha) - q\Big(\frac{2L}{q+1}\alpha^{q+1} + \frac{2(M+N)\pi}{(q+1)(q+2)}\alpha^{q+2}\Big). \end{aligned}$$

By observing that $B(\alpha) = 2L\alpha + (M+N)\pi\alpha^2$, we get

$$C_p(\Sigma, 1, \Omega) \geq \frac{2L}{q+1}\alpha^{q+1} + \frac{(N+M)(q^2+q+2)\pi}{(q+1)(q+2)}\alpha^{q+2}$$

which proves (2.12). For (2.13), if we choose Ω to be the unit square Y and $\Sigma \in \mathcal{A}_l(Y)$ then $\mathrm{meas}(\Omega) = 1$, $M = N = 1$ (since Σ and ∂Y are connected) and from (2.4), we have

$$\alpha = \frac{1}{L + \sqrt{L^2 + 2\pi}}.$$

The relation between L and l is given by $l \leq L \leq l + 4$ since $L = \mathcal{H}^1(\Sigma \cup \partial Y)$, $\Sigma \in \mathcal{A}_l(Y)$ and $\mathcal{H}^1(\partial Y) = 4$. So

$$\alpha \approx \frac{1}{2L} \approx \frac{1}{2l} \quad \text{as } l \to +\infty,$$

hence

$$\begin{aligned} \liminf_{l\to+\infty} l^q C_p(\Sigma, 1, Y) &\geq \liminf_{l\to+\infty} l^q\Big(\frac{2L}{q+1}\Big(\frac{1}{2l}\Big)^{q+1} + \frac{(N+M)(q^2+q+2)\pi}{(q+1)(q+2)}\Big(\frac{1}{2l}\Big)^{q+2}\Big) \\ &= \frac{1}{(q+1)2^q}. \end{aligned}$$

Taking the infimum over all sets $\Sigma \in \mathcal{A}_l(Y)$ yields (2.13). □

3 Estimate of $\theta(p)$ from above and optimal sequence

This section deals with the estimate of the constant $\theta(p)$ from above and optimal sequence. In fact we will prove that the reverse inequality of (2.13) holds true and the comb structure is asymptotically optimal.

Theorem 3.1. *We have $\theta(p) \leq \frac{1}{(q+1)2^q}$ and the comb configuration is asymptotically optimal.*

Proof. To prove the Theorem, we will construct a comb configuration Σ_n with a one-dimensional Hausdorff measure $\mathcal{H}^1(\Sigma_n) = l_n$ (with $l_n \to +\infty$ as $n \to +\infty$) and then show that

$$\liminf_{n\to+\infty} l_n^q C_p(\Sigma_n, 1, Y) \leq \frac{1}{(q+1)2^q}.$$

Let u be a function defined on $[0,1]$ by

$$u(t) := \begin{cases} \frac{1}{q}(\frac{1}{2})^q - \frac{1}{q}(\frac{1}{2} - t)^q, & t \in (0, \frac{1}{2}) \\ \frac{1}{q}(\frac{1}{2})^q - \frac{1}{q}(t - \frac{1}{2})^q, & t \in (\frac{1}{2}, 1) \end{cases}. \tag{3.1}$$

that we extend periodically on $\mathbb{R}$ with period 1. Notice that u is the explicit solution of the p-Laplacian equation with right side 1 that is $-\Delta_p u = 1$ on $(0,1)$ constraint to the homogeneous Dirichlet boundary conditions at 0 and 1 (that is $u(0) = u(1) = 0$). For a given integer $n \geq 1$, we consider the set γ_n to be the union of $n+1$ parallel vertical segments of unit length (including the two vertical sides of the unit square) uniformly distributed. Clearly this set is not connected. To make it connected, we add one horizontal side of the unit square which give it the comb structure. We denote this new set by Σ_n. The length of Σ_n is

$$l_n := \mathcal{H}^1(\Sigma_n) = n + 2 = \mathcal{H}^1(\gamma_n) + 1. \tag{3.2}$$

Let v_n be the weak solution of

$$\begin{cases} -\Delta_p v &= 1 \ \text{ in } \ Y \setminus \Sigma_n \\ \qquad v &= 0 \ \text{ in } \ \Sigma_n \cup \partial Y, \end{cases}$$

then

$$C_p(\Sigma_n, 1, Y) = \int_Y v_n(x,y)\, dxdy \tag{3.3}$$

To estimate the integral (3.3) from above, we will compare the function v_n with another solution of the p-Laplacian equation with mixed boundary conditions (namely Dirichlet and Neumann). Let us consider the function

$$u_n : \mathbb{R}^2 \mapsto \mathbb{R}, \quad u_n(x,y) := n^{-q} u(nx),$$

where u is the function defined in (3.1) and q the conjugate exponent of p. An easy computation shows that u_n satisfies $-\Delta_p u = 1$ in $Y \setminus \Sigma_n$ with homogeneous Dirichlet condition along γ_n (the set γ_n is made of $n+1$ parallel line segments of unit length) and homogeneous Neumann along the two horizontal sides of Y. Instead of homogeneous Neumann conditions along the two horizontal sides of Y, one may consider also the nonnegative inhomogeneous Dirichlet condition. Therefore by the maximum principle it holds $v_n \leq u_n$ in $Y \setminus \Sigma_n$. Integrating this inequality and taking into account the definition of u_n and the periodicity of u, we get

$$\int_Y v_n(x,y)\ dxdy \leq \int_Y u_n(x,y)\ dxdy = n^{1-q}\int_0^{1/n} u(nx)\ dx = n^{-q}\int_0^1 u(x)\ dx.$$

To compute the last integral, we use (3.1). From an elementary computation, we have

$$\int_0^{1/2} u(x)\ dx = \int_{1/2}^1 u(x)\ dx = \frac{1}{q+1}\Big(\frac{1}{2}\Big)^{q+1},$$

hence, recalling (3.3) we get

$$C_p(\Sigma_n, 1, Y) = \int_Y v_n(x,y)\ dxdy \leq \frac{n^{-q}}{(q+1)2^q}.$$

Since the length of Σ_n is $l_n = n+2$ (see (3.2)) it follows that

$$l_n^q C_p(\Sigma_n, 1, Y) \leq \frac{1}{(q+1)2^q}\Big(\frac{n+2}{n}\Big)^q.$$

Therefore, passing to liminf as $n \to +\infty$ in the inequality, and using the definition of $\theta(p)$, we get

$$\theta(p) \leq \liminf_{n\to+\infty} l_n^q C_p(\Sigma_n, 1, Y) \leq \frac{1}{(q+1)2^q}$$

which concludes the proof. □

Remark 3.2. *Combine the inequality of Theorem 3.1 and* (2.13)*, it follows that*

$$\theta(p) = \frac{1}{(q+1)2^q}.$$

For the proof of Theorem 3.1, we may choose Σ_n to be the union of γ_n with the two horizontal sides of the unit square Y instead of being the union of γ_n with one horizontal side of the unit square. In this case, we loose the comb configuration but the set Σ_n is still asymptotically optimal. the point is that the length of a side of Y we added is asymptotically irrelevant. More generally, adding to Σ_n some segments (or more generally connect set) of asymptotically irrelevant length will give another asymptotically optimal set.

Remark 3.3. *The case of higher dimension is still an open problem. For this setting, the scaling factor in the definition of $\theta(p)$ is $l^{\frac{q}{d-1}}$. To get an estimate from below, one has to find a right function h as we did in* (2.12)*. A big deal is to find an asymptotically optimal set which will lead to the value of $\theta(p)$.*

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