A Bernstein-type result for the minimal surface equation

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Abstract

We prove the following Bernstein-type theorem: if \( u \) is an entire solution to the minimal surface equation, such that \( N - 1 \) partial derivatives \( \frac{\partial u}{\partial x_j} \) are bounded on one side (not necessarily the same), then \( u \) is an affine function. Its proof relies only on the Harnack inequality on minimal surfaces proved in [4] thus, besides its novelty, our theorem also provides a new and self-contained proof of celebrated results of Moser and of Bombieri & Giusti.

MSC: 53A10, 58J05, 35J15

1 Introduction and main results

In this short article we are concerned with a Bernstein-type theorem for solutions to the minimal surface equation

\[
- \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \quad \text{in} \quad \mathbb{R}^N, \quad N \geq 2. \tag{1.1}
\]

The classical Bernstein Theorem ([2],[7]) asserts that the affine functions are the only solutions of (1.1) in \( \mathbb{R}^2 \). This result has been generalized to \( \mathbb{R}^3 \) by E. De Giorgi [5], to \( \mathbb{R}^4 \) by J.F. Almgren [1] and, up to dimension \( N = 7 \), by J. Simons [9]. On the other hand, E. Bombieri, E. De Giorgi and E. Giusti [3] proved the existence of a non-affine solution of the minimal surface equation (1.1) for any \( N \geq 8 \). Nevertheless, J. Moser [8] was able to prove that, if \( \nabla u \) is bounded on \( \mathbb{R}^N \), then \( u \) must be again an affine function, and this for every dimension \( N \geq 2 \). Later, E. Bombieri and E. Giusti [4] generalized Moser's result by assuming that only \( N - 1 \) partial derivatives of \( u \) are bounded on \( \mathbb{R}^N, \ N \geq 2 \). To prove
their result, the Authors of [4] demonstrate a Harnack inequality for uniformly elliptic equations on minimal surfaces (oriented boundary of least area) and then they use it to show that, if \( N - 1 \) partial derivatives of \( u \) are bounded on \( \mathbb{R}^N \), then \( u \) has bounded gradient on \( \mathbb{R}^N \), and they conclude by invoking the result of Moser. Our main theorem (see Theorem 1.1 below) provides a further extension of the above results. Its proof relies only on the Harnack inequality on minimal surfaces proved in [4] thus, besides its novelty, it also provides a new and self-contained proof of the celebrated results of Moser and of Bombieri & Giusti. We believe that this is another interesting feature of our work.

Our main result is stated in the following theorem.

**Theorem 1.1.** Assume \( N \geq 2 \). Let \( u \) be a solution of the minimal surface equation (1.1) such that \( N - 1 \) partial derivatives \( \frac{\partial u}{\partial x_j} \) are bounded on one side (not necessarily the same). Then \( u \) is an affine function.

### 2 Auxiliary results and proofs

To prove our results we briefly recall some standard notations and some well-known facts concerning the solutions of the minimal surface equation (1.1) (cfr. [4], [6]). For a given solution \( u \) of equation (1.1), we denote by \( S \) the minimal graph \( x_{N+1} = u(x) \) over \( \mathbb{R}^N \) (i.e., the complete smooth area minimizing hypersurface without boundary \( S \subset \mathbb{R}^{N+1} \), given by the graph of \( u \) over the entire \( \mathbb{R}^N \)). Then the (upward pointing) unit normal to \( S \) at a point \( (x,u(x)) \) is \( \nu = (\nu_1,\ldots,\nu_{N+1}) = \frac{(-\nabla u(x),1)}{\sqrt{1+|\nabla u(x)|^2}} \) and we can define the tangential derivatives \( \delta_k \) by

\[
\delta_k := \frac{\partial}{\partial x_k} - \nu_k \sum_{h=1}^{N+1} \nu_h \frac{\partial}{\partial x_h} \quad \forall \ k = 1,\ldots,N+1.
\]  

(2.1)

Moreover the functions \( \nu_h \) satisfy the equation

\[
\sum_{k=1}^{N+1} \delta_k \delta_k \nu_h + c^2 \nu_h = 0 \quad \text{on} \ S, \quad \forall \ h = 1,\ldots,N+1
\]  

(2.2)

where \( c^2 := \sum_{j,k=1}^{N+1} (\delta_j \nu_k)^2 \) denotes the sum of the squares of the principal curvatures of the hypersurface \( S \) at the point \( (x,u(x)) \). Therefore, for any vector \( a := (a_1,\ldots,a_{N+1}) \in \mathbb{R}^{N+1} \), the function \( (a \cdot \nu) = \sum_{j=1}^{N+1} a_j \nu_j \) also solves

\[
\sum_{k=1}^{N+1} \delta_k \delta_k (a \cdot \nu) + c^2 (a \cdot \nu) = 0 \quad \text{on} \ S.
\]  

(2.3)

**Lemma 2.1.** Assume \( N \geq 2 \) and let \( S \) be a minimal graph \( x_{N+1} = u(x) \) over \( \mathbb{R}^N \). If \( v > 0 \) and \( w \) are smooth solutions of the equation (2.3) on \( S \), then the smooth function \( \theta := \arctan \left( \frac{w}{v} \right) \in L^\infty(S) \) solves the equation

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\[
\sum_{k=1}^{N+1} \delta_k \left[(v^2 + w^2)\delta_k \theta\right] = 0 \quad \text{on } S. \tag{2.4}
\]

Proof. Consider the smooth complex-valued function \( z := v + iw \). Since \( v > 0 \) everywhere, we have that \( z = \rho e^{i\theta} \) on \( S \) and

\[
\sum_{k=1}^{N+1} \delta_k \delta_k z + c^2 z = 0 \quad \text{on } S, \tag{2.5}
\]

where \( \rho := \sqrt{v^2 + w^2} > 0 \) everywhere on \( S \). Hence, by definition of \( \delta_k \) we get

\[
0 = \sum_{k=1}^{N+1} \delta_k \delta_k (\rho e^{i\theta}) + c^2 \rho e^{i\theta} = \sum_{k=1}^{N+1} \delta_k \left( e^{i\theta} \delta_k \rho + i\rho e^{i\theta} \delta_k \theta \right) + c^2 \rho e^{i\theta} =
\]

\[
\sum_{k=1}^{N+1} e^{i\theta} \delta_k \delta_k \rho + i e^{i\theta} \delta_k \theta \delta_k \rho + i e^{i\theta} \delta_k \delta_k \theta + i \left( e^{i\theta} \delta_k \rho + i\rho e^{i\theta} \delta_k \theta \right) \delta_k \theta + c^2 \rho e^{i\theta} =
\]

\[
\sum_{k=1}^{N+1} e^{i\theta} \delta_k \delta_k \rho - \rho e^{i\theta} \delta_k \theta \delta_k \theta + i e^{i\theta} \rho \delta_k \delta_k \theta + i e^{i\theta} \rho \delta_k \rho \delta_k \theta + c^2 \rho e^{i\theta} \quad \text{on } S.
\]

Hence

\[
0 = \sum_{k=1}^{N+1} \delta_k \delta_k \rho - \rho \delta_k \theta \delta_k \theta + i \rho \delta_k \delta_k \theta + 2\delta_k \rho \delta_k \theta + c^2 \rho \quad \text{on } S
\]

and taking the imaginary part of the latter identity we obtain

\[
0 = \sum_{k=1}^{N+1} \rho \delta_k \delta_k \theta + 2\delta_k \rho \delta_k \theta = \frac{1}{\rho} \sum_{k=1}^{N+1} \delta_k \left[ \rho^2 \delta_k \theta \right] \quad \text{on } S
\]

which immediately implies (2.4). \( \Box \)

Now we are in position to prove our main result.

Proof of Theorem 1.1. We divide the proof into three steps.

Step 1: Every partial derivative of \( u \) is bounded on one side.

By assumption there exists an integer \( n \in \{1, ..., N\} \) such that for every integer \( j \in \{1, ..., N\} \setminus \{n\} := J \), the partial derivative \( \frac{\partial u}{\partial x_j} \) is bounded on one side. We set \( A := \{ \alpha \in J : \frac{\partial u}{\partial x_\alpha} \text{ is bounded from below} \} \) and \( B := \{ \beta \in J : \frac{\partial u}{\partial x_\beta} \text{ is bounded from above} \} \). Hence
∀ α ∈ A  \exists c_α > 0 : \frac{\partial u}{\partial x_\alpha} + c_\alpha > 1 \text{ on } \mathbb{R}^N, \quad (2.6)

∀ β ∈ B  \exists c_β > 0 : c_β - \frac{\partial u}{\partial x_\beta} > 1 \text{ on } \mathbb{R}^N. \quad (2.7)

Now we observe that

\left| \nabla u \right|^2 = \left( \frac{\partial u}{\partial x_n} \right)^2 + \sum_{\alpha \in A} \left( \frac{\partial u}{\partial x_\alpha} + c_\alpha \right)^2 + \sum_{\beta \in B} \left( c_\beta - \frac{\partial u}{\partial x_\beta} \right)^2 = \quad (2.8)

\left( \frac{\partial u}{\partial x_n} \right)^2 + \sum_{\alpha \in A} \left( \frac{\partial u}{\partial x_\alpha} + c_\alpha - c_\alpha \right)^2 + \sum_{\beta \in B} \left( c_\beta - \frac{\partial u}{\partial x_\beta} - c_\beta \right)^2 = \quad (2.9)

\left( \frac{\partial u}{\partial x_n} \right)^2 + \sum_{\alpha \in A} \left( \frac{\partial u}{\partial x_\alpha} + c_\alpha \right)^2 + \sum_{\alpha \in A} c_\alpha^2 - 2 \sum_{\alpha \in A} c_\alpha \left( \frac{\partial u}{\partial x_\alpha} + c_\alpha \right) + \quad (2.10)

\sum_{\beta \in B} \left( c_\beta - \frac{\partial u}{\partial x_\beta} \right)^2 + \sum_{\beta \in B} c_\beta^2 - 2 \sum_{\beta \in B} c_\beta \left( c_\beta - \frac{\partial u}{\partial x_\beta} \right) \leq \quad (2.11)

\left( \frac{\partial u}{\partial x_n} \right)^2 + \sum_{\alpha \in A} \left( \frac{\partial u}{\partial x_\alpha} + c_\alpha \right)^2 + \sum_{\beta \in B} \left( c_\beta - \frac{\partial u}{\partial x_\beta} \right)^2 + \sum_{j \in J} c_j^2 \leq \quad (2.12)

\left( \frac{\partial u}{\partial x_n} \right)^2 + \left[ \sum_{\alpha \in A} \left( \frac{\partial u}{\partial x_\alpha} + c_\alpha \right) + \sum_{\beta \in B} \left( c_\beta - \frac{\partial u}{\partial x_\beta} \right) \right]^2 + \sum_{j \in J} c_j^2 \quad (2.13)

where in the latter we have used (2.6) and (2.7).

Now we set \( \xi := \sum_{\alpha \in A} e_\alpha - \sum_{\beta \in B} e_\beta \in \mathbb{R}^N \), \( k_1 := \sum_{j \in J} c_j^2 > 0 \), \( k_2 := \sum_{j \in J} c_j > 0 \), where \( \{e_1, ..., e_N\} \) denotes the canonical basis of \( \mathbb{R}^N \) and we rewrite (2.13) as

\left( \frac{\partial u}{\partial x_n} \right)^2 + (\nabla u \cdot \xi + k_2)^2 + k_1 \text{ on } \mathbb{R}^N \quad (2.14)

and observe that

\nabla u \cdot \xi + k_2 > 1 \text{ on } \mathbb{R}^N. \quad (2.15)

again by (2.6) and (2.7).

Combining (2.8)-(2.14) and (2.15) we find
\[ 1 + |\nabla u|^2 \leq \left( \frac{\partial u}{\partial x_n} \right)^2 + (2 + k_1) (\nabla u \cdot \xi + k_2)^2 \] (2.16)

\[ \leq (2 + k_1) \left[ \left( \frac{\partial u}{\partial x_n} \right)^2 + (\nabla u \cdot \xi + k_2)^2 \right] \] (2.17)

Set \( \chi := (-\epsilon_n,0) \in \mathbb{R}^{N+1} \), \( \tau := (-\xi, k_2) \in \mathbb{R}^{N+1} \) and consider the functions

\[ w := \frac{\partial u}{\sqrt{1 + |\nabla u|^2}} = (\chi \cdot \nu) \quad \text{and} \quad v := \frac{\nabla u \cdot \xi + k_2}{\sqrt{1 + |\nabla u|^2}} = (\tau \cdot \nu) > 0. \]

Since \( v > 0 \) and \( w \) are solutions of the equation (2.3), an application of Lemma 2.1 implies that

\[ \theta := \arctan \left( \frac{w}{v} \right) \in L^\infty(S) \] solves the equation

\[ \sum_{k=1}^{N+1} \delta_k \left[ (v^2 + w^2) \delta_k \theta \right] = 0 \quad \text{on} \quad S. \] (2.18)

Thanks to (2.16)-(2.17) we see that the above equation (2.18) is uniformly elliptic on \( S \). Indeed, from (2.16)-(2.17) we get

\[ \frac{1 + |\nabla u|^2}{2 + k_1} \leq \left[ \left( \frac{\partial u}{\partial x_n} \right)^2 + (\nabla u \cdot \xi + k_2)^2 \right] \leq 2(1 + k_2^2) \left[ 1 + |\nabla u|^2 \right] \] (2.19)

which implies

\[ \frac{1}{2 + k_1} \leq v^2 + w^2 \leq 2(1 + k_2^2) \quad \text{on} \quad S. \] (2.20)

Thus \( \theta \) must be constant, by an application of the Harnack inequality proved by Bombieri and Giusti (cfr. Theorem 5 of [4]), i.e., \( w = \lambda v \) on \( S \), for some \( \lambda \in \mathbb{R} \). The latter immediately implies that \( \frac{\partial u}{\partial x_n} \) has a sign. In particular, all the partial derivatives of \( u \) are bounded on one side.

**Step 2:** For every unit vector \( \eta \in \mathbb{R}^N \) the directional derivative \( \frac{\partial u}{\partial \eta} \) has sign, that is, one and only one of the following assertions holds: (i) \( \frac{\partial u}{\partial \eta}(x) = 0 \quad \forall x \in \mathbb{R}^N \), (ii) \( \frac{\partial u}{\partial \eta}(x) > 0 \quad \forall x \in \mathbb{R}^N \), (iii) \( \frac{\partial u}{\partial \eta}(x) < 0 \quad \forall x \in \mathbb{R}^N \).

Let \( \sigma \) be any unit vector of \( \mathbb{R}^N \) and set \( I := \{1, \ldots, N\} \), \( A := \{ \alpha \in I : \frac{\partial u}{\partial x_\alpha} \text{ is bounded from below} \} \) and \( B := \{ \beta \in I : \frac{\partial u}{\partial x_\beta} \text{ is bounded from above} \} \).

Hence

\[ \forall \alpha \in A \quad \exists c_\alpha > 0 : \frac{\partial u}{\partial x_\alpha} + c_\alpha > 1 \quad \text{on} \quad \mathbb{R}^N, \] (2.21)

\[ \forall \beta \in B \quad \exists c_\beta > 0 : c_\beta - \frac{\partial u}{\partial x_\beta} > 1 \quad \text{on} \quad \mathbb{R}^N. \] (2.22)

and proceeding as before we obtain
\[(\frac{\partial u}{\partial \sigma})^2 \leq |\nabla u|^2 \leq \left[ \sum_{\alpha \in A} \left( \frac{\partial u}{\partial x_{\alpha}} + c_{\alpha} \right) + \sum_{\beta \in B} \left( c_{\beta} - \frac{\partial u}{\partial x_{\beta}} \right) \right]^2 + \sum c_j^2 = (2.23)\]

\[= (\nabla u \cdot \xi + k_4)^2 + k_3 \quad \text{on } \mathbb{R}^N \quad (2.24)\]

and

\[\nabla u \cdot \xi + k_4 > 1 \quad \text{on } \mathbb{R}^N, \quad (2.25)\]

where \(\xi := \sum_{\alpha \in A} e_{\alpha} - \sum_{\beta \in B} e_{\beta} \in \mathbb{R}^N, k_3 := \sum_{j=1}^{N} c_j^2 > 0, k_4 := \sum_{j=1}^{N} c_j > 0.\)

We notice that \(\xi, k_3\) and \(k_4\) are independent of the unit vector \(\sigma\) and let \(\{\eta, \sigma_2, ..., \sigma_N\}\) be an orthonormal basis of \(\mathbb{R}^N\). From (2.23)-(2.24) we get

\[1+|\nabla u|^2 = 1 + \left( \frac{\partial u}{\partial \eta} \right)^2 + \sum_{j=2}^{N} \left( \frac{\partial u}{\partial \sigma_j} \right)^2 \leq 1 + \left( \frac{\partial u}{\partial \eta} \right)^2 + (N-1) \left( (\nabla u \cdot \xi + k_4)^2 + k_3 \right) \]

and using (2.25) in the latter we immediately infer that

\[1 + |\nabla u|^2 \leq (N + (N-1)k_3) \left[ \left( \frac{\partial u}{\partial \eta} \right)^2 + (\nabla u \cdot \xi + k_4)^2 \right] \leq (2.27)\]

\[3(N + (N-1)k_3)(1 + k_4^2) \left[ 1 + |\nabla u|^2 \right]. \quad (2.28)\]

Setting \(\chi := (-\eta, 0) \in \mathbb{R}^{N+1}, \tau := (-\xi, k_4) \in \mathbb{R}^{N+1}, w := \frac{\partial u}{\partial \eta} = (\chi \cdot \nu)\) and \(v := \frac{\sum_{j} c_j}{\sqrt{1+|\nabla u|^2}} = (\tau \cdot \nu) > 0,\) and applying Lemma 2.1 as before, we see that the function \(\theta := \arctan \left( \frac{x}{v} \right) \in L^\infty(S)\) solves the equation (2.4), which is again uniformly elliptic on \(S\) in view of the above (2.27)-(2.28). It follows that \(\theta\) is constant, which implies that the directional derivative \(\frac{\partial u}{\partial \eta}\) has a sign.

**Step 3: End of the proof.**

Either \(u\) is constant, and in this case we are done, or there exists \(x_0 \in \mathbb{R}^N\) such that \(\nabla u(x_0) \neq 0.\) In the latter case there are \(N-1\) unit vectors of \(\mathbb{R}^N,\) denoted by \(\sigma_1, ..., \sigma_{N-1},\) which are orthogonal to \(\nabla u(x_0),\) i.e., such that

\[0 = \nabla u(x_0) \cdot \sigma_j = \frac{\partial u}{\partial \sigma_j}(x_0) \quad \forall j = 1, ..., N-1. \quad (2.29)\]

By the previous step, we must have

\[\frac{\partial u}{\partial \sigma_j}(x) \equiv 0 \quad \text{on } \mathbb{R}^N, \quad \forall j = 1, ..., N-1, \quad (2.30)\]
thus \( u(x) = h(\tau \cdot x) \), where \( \tau = \frac{\nabla u(x_0)}{|\nabla u(x_0)|} \) and \( h = h(t) \) is a non constant solution of the ODE
\[
- \left( \frac{h'}{\sqrt{1+|h'|^2}} \right)' = 0 \text{ on } \mathbb{R}.
\]
A direct integration of the latter gives \( h(t) = at + b, \ a \neq 0 \). Thus \( u \) is an affine function. \( \square \)

Acknowledgements: The author thanks E. Valdinoci for a careful reading of a first version of this article. The author is supported by the ERC grant EPSILON (Elliptic Pde’s and Symmetry of Interfaces and Layers for Odd Non-linearities).

References


