GRADIENT BOUNDS FOR ANISOTROPIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We consider solutions in the whole of the space of a partial differential equation driven by the anisotropic Laplacian. We prove a pointwise energy bound and we derive from that some rigidity results.

Introduction

Given a domain $\Omega \subseteq \mathbb{R}^n$, with $n \geqslant 2$, we consider critical points of the functional

$$\mathcal{W}_\Omega(u) := \int_\Omega \frac{1}{2} H^2(\nabla u(x)) - F(u(x))\, dx. \tag{1}$$

Here we take $H \in C^{3,\beta}_{\rm loc}(\mathbb{R}^N \setminus \{0\})$, with $\beta \in (0,1)$, and we suppose that H is a positive homogeneous function of degree 1, with $H(0) = 0$,

$$H(\xi) > 0 \text{ for any } \xi \in \mathbb{R}^n \setminus \{0\}, \tag{2}$$

and

$$\text{Hess}\,(H^2) \text{ positive definite in } \mathbb{R}^n \setminus \{0\} \tag{3}$$

(see the Appendix for the basic notions on positive homogeneous functions that are in used). We also suppose that $F \in C^{2,\beta}_{\rm loc}(\mathbb{R})$. In particular, critical points of $\mathcal{W}_\Omega$ satisfy (weakly) the equation

$$\frac{\partial}{\partial x_i}\Big(H(\nabla u) H_i(\nabla u)\Big) + f(u) = 0, \tag{4}$$

where $f := F'$ and $H_i(\xi) := \partial_{\xi_i} H(\xi)$. Here and in the sequel, the standard summation convention on repeated indices is understood. The differential operator in (4) is known in the literature with the name of anisotropic, or Finsler, Laplacian, and it has attracted the attention of several authors (see, for instance, [AFTL97, CS09] and references therein – of course, in the special case when $H(\xi) = |\xi|$, it reduces to the ordinary Laplacian).

The functional in (1), and therefore equation (4), is motivated by the study of the Wulff shape of anisotropic crystals (see, e.g., [T78, FM91, DKS92] and the references therein).

Given $u : \mathbb{R}^n \to \mathbb{R}$, we set

$$c_u := \sup \Big\{ F(r)\,,\ r \in \Big[\inf_{\mathbb{R}^n} u, \sup_{\mathbb{R}^n} u \Big] \Big\}.$$

In this paper, we provide three types of results, namely: a pointwise gradient bound, some rigidity and symmetry results, and some results relating the important quantity c_u with the extremals of the solution u. We start with the following pointwise gradient estimate:

Theorem 1. *Let $u \in L^\infty(\mathbb{R}^n) \cap H^1_{\rm loc}(\mathbb{R}^n)$ be a weak solution of equation (4) in $\mathbb{R}^n$. Then $u \in C^{1,\alpha}_{\rm loc}(\mathbb{R}^n)$, for some $\alpha \in (0,1)$ and, for any $x \in \mathbb{R}^n$,*

$$\frac{1}{2} H^2(\nabla u(x)) \leqslant c_u - F(u(x)). \tag{5}$$

Key words and phrases. Pointwise estimates, P-function, maximum principle.

EV is supported by the FIRB Project A.&B. *"Analysis and Beyond"*. Both the authors are supported by the ERC Project E.P.S.I.L.O.N. *"Elliptic Pde's and Symmetry of Interfaces and Layers for Odd Nonlinearities"*.

Also, if there exists $x_o \in \mathbb{R}^n$ such that

$$\nabla u(x_o) \neq 0 \tag{6}$$

and

$$\frac{1}{2} H^2(\nabla u(x_o)) = c_u - F(u(x_o)), \tag{7}$$

then

$$\frac{1}{2} H^2(\nabla u(x)) = c_u - F(u(x)) \tag{8}$$

for any x belonging to the connected component of $\{\nabla u \neq 0\}$ that contains x_o.

In the very special case in which $H(\xi) = |\xi|$ (that may be thought as the isotropic case), we have that (4) reduces to the Laplace equation and Theorem 1 in this case was proved in [Mod85]. The result of [Mod85] is indeed, up to now, classical, and it has been generalized in several directions (see, for instance, [CGS94, FV09, CFV12]). Nevertheless, as far as we know, the case of the Wulff shape that is treated in Theorem 1 here was still open.

In particular, though very general functionals have been recently studied in [DG02], results similar to Theorem 1 have been obtained there under an assumption of rotationally invariance (namely, (1.5) there), that excluded the types of anisotropy dealt with in this paper (see in particular Theorems 4.7 and 4.8, and Corollary 4.9 in [DG02] for the isotropic case of our Theorem 1).

We think that it is quite intriguing to observe that if (7) holds in a neighborhood of x_o (satisfying (6)), then the anisotropic mean curvature (or H-mean curvature) of the regular level set $S_{x_o} := \{u = u(x_o)\}$ vanishes at x_o.

More precisely, given $x_o \in \{\nabla u \neq 0\}$, one considers the regular level set $S_{x_o} := \{u = u(x_o)\}$. Then, for any $x \in S_{x_o}$, we denote by $\mathcal{C}(x)$ the anisotropic mean curvature of S_{x_o} at the point x (see, e.g., pages 103–107 of [WX11] and references therein for definitions and basic properties). Then, the following result holds true :

Theorem 2. *Let U be an open subset of $\mathbb{R}^n$ and u be as in Theorem 1. Suppose that $U \subseteq \{\nabla u \neq 0\}$ and that*

$$\frac{1}{2} H^2(\nabla u(x)) = c_u - F(u(x)) \qquad \text{for any } x \in U. \tag{9}$$

Then,

$$\mathcal{C}(x) = 0 \text{ for any } x \in U. \tag{10}$$

As a consequence of (10), we obtain a rigidity result in the plane:

Theorem 3. *Let $n = 2$, U be an open subset of $\mathbb{R}^2$ and u be as in Theorem 1. Suppose that $U \subseteq \{\nabla u \neq 0\}$ and that*

$$\frac{1}{2} H^2(\nabla u(x)) = c_u - F(u(x)) \qquad \text{for any } x \in U.$$

Then the level sets of u are contained in straight lines.

When U is the whole of the plane, Theorem 3 may be precised, according to the following result:

Theorem 4. *Let $n = 2$ and u be as in Theorem 1. Suppose that*

$$\frac{1}{2} H^2(\nabla u(x)) = c_u - F(u(x)) \qquad \text{for any } x \in \mathbb{R}^2. \tag{11}$$

Then u possesses one-dimensional Euclidean symmetry, i.e. there exist $\varpi \in \mathrm{S}^1$ and $u_o : \mathbb{R} \to \mathbb{R}$ such that

$$u(x) = u_o(\varpi \cdot x) \qquad \text{for any } x \in \mathbb{R}^2.$$

We stress that it is *not necessary* to assume that $\{\nabla u = 0\} = \varnothing$ in Theorem 4.

Also, we think that it is an interesting *open problem* to decide whether or not Theorems 3 and 4 hold true in higher dimension.

As an example of positive homogeneous functions that satisfy our assumptions, one can take $p > 3$ and

$$H(\xi) := \left(\sum_{i=1}^{n} |\xi_i|^p \right)^{1/p} + \left(\sum_{i=1}^{n} (\xi_i)^2 \right)^{1/2}.$$

In particular, the corresponding anisotropic operator is not rotationally invariant. In fact, our framework is more general than this, and we do not even need to assume that H is a norm.

Now, we give a rigidity result for u in terms of c_u, which may be seen as the anisotropic version of Theorem 4.10 in [DG02] (see also Theorem 1 in [Mod85] and Theorem 1.8 in [CGS94]) :

Theorem 5. *Let u be as in Theorem 1. If there exists $p \in \mathbb{R}^n$ such that $F(u(p)) = c_u$ and $F'(u(p)) = 0$, then u is constant.*

Next result, which gives a precise characterization of c_u, may be seen as the extension of Theorem 2 of [FV09] to the anisotropic case:

Theorem 6. *Let u be as in Theorem 1. Then*

$$c_u = \max\left\{ F\left(\inf_{\mathbb{R}^n} u\right),\ F\left(\sup_{\mathbb{R}^n} u\right)\right\}.$$

Also, if there exists $x \in \mathbb{R}^n$ such that $F(u(x)) = c_u$, then either $u(x) = \inf_{\mathbb{R}^n} u$ or $u(x) = \sup_{\mathbb{R}^n} u$.

The paper is organized as follows. First, in Section 1, we introduce a suitable P-function, that we show to be a subsolution of a suitable (possibly degenerate) pde. This will be the core of the proof of Theorem 1, which is completed in Section 2. The subsequent sections take care of the proofs of Theorems 2, 3, 4, 5 and 6. To make the paper more self-contained, we enclose an Appendix collecting the basics of positive homogeneous functions that are in used in the main computations.

1. P-function computations

Now we introduce a suitable function P, related to (5) and we show that P is a subsolution of a (possibly degenerate) pde. This and the strong maximum principle will lead to the proof of Theorem 1. Though the technique of looking for such a P-function is, up to know, classical, several technical issues arise in the concrete cases of interest: for instance, a result similar to the following Proposition 1 was proved in Theorem 4.3 of [DG02] in the isotropic case – remarkably, the P function considered there (and the associated pde) seem to be slightly different from ours. The P-function introduced in Theorem 4 of [WX11] to deal with other anisotropic problems is also different. So, the choice of the appropriate P-function seems always to be a delicate point.

Proposition 1. *Let u be as in Theorem 1 and*

$$G(r) := c_u - F(r), \tag{12}$$

$a_{ij} := H_i H_j + H H_{ij}$, $d_{ij} := a_{ij}/H$, and

$$P(u; x) := H^2(\nabla u) - 2G(u). \tag{13}$$

Then, $u \in C^3(\{\nabla u \neq 0\})$ and, at points where $\nabla u \neq 0$,

$$(d_{ij} P_i)_j - H^{-2} G' H_\ell P_\ell \geqslant 0. \tag{14}$$

Proof. First of all, we remark that u is $C^3(\{\nabla u \neq 0\})$, see [LU68, T84, DiB83, GT83]. Then, the proof uses some ideas of [Mod85, CGS94] as developed in [FV09, CFV12]. In a sense, the calculations we perform are a modification of the classical Bernstein technique, as employed in [Pay76, Spe81], but several technical difficulties arise in this case, and this complicates the algebra. Though some related computations can be found in [WX11], the approach here is slightly different. We use the short notation $\partial_i = \partial_{x_i}$, $G = G(u)$, $G' = G'(u)$, $H = H(\nabla u)$, $H_i = H_i(\nabla u) = (\partial_i H)(\nabla u)$, etc.

By differentiating (13), for any i, we have

$$P_i = 2HH_k u_{ki} - 2G' u_i. \tag{15}$$

Hence

$$(d_{ij}P_i)_j = -2(G'd_{ij}u_i)_j + (2a_{ij}H_k u_{ki})_j = -2(G'd_{ij}u_i)_j + 2(a_{ij}u_{ki})_j H_k + 2a_{ij}u_{ki}H_{k\ell}u_{\ell j}. \tag{16}$$

Moreover, from (4),

$$a_{ij}u_{ij} = G'. \tag{17}$$

Now, since u is C^3,

$$\begin{aligned}
(a_{ij}u_{ki})_j - (a_{ij}u_{ij})_k &= (a_{ij})_j u_{ki} - (a_{ij})_k u_{ij}\\
&= (H_iH_j + HH_{ij})_j u_{ki} - (H_iH_j + HH_{ij})_k u_{ij}\\
&= (H_{i\ell}H_j + H_iH_{j\ell} + H_\ell H_{ij} + HH_{ij\ell})u_{ki}u_{j\ell}\\
&\quad -(H_{i\ell}H_j + H_iH_{j\ell} + H_\ell H_{ij} + HH_{ij\ell})u_{ij}u_{k\ell}.
\end{aligned}$$

Therefore, by exchanging the names of the indices i and ℓ in the last term,

$$\begin{aligned}
&(a_{ij}u_{ki})_j - (a_{ij}u_{ij})_k\\
&\quad = (H_{i\ell}H_j + H_iH_{j\ell} + H_\ell H_{ij} + HH_{ij\ell})u_{ki}u_{j\ell}\\
&\qquad -(H_{i\ell}H_j + H_\ell H_{ij} + H_iH_{\ell j} + HH_{ij\ell})u_{ki}u_{j\ell} = 0.
\end{aligned}$$

This and (17) give that

$$(a_{ij}u_{ki})_j = (a_{ij}u_{ij})_k = (G')_k = G''u_k. \tag{18}$$

By plugging (18) into (16), we obtain

$$\begin{aligned}
(d_{ij}P_i)_j &= -2(G'd_{ij}u_i)_j + 2G''u_kH_k + 2a_{ij}u_{ki}H_{k\ell}u_{\ell j}\\
&= -2G''d_{ij}u_iu_j - 2G'(d_{ij}u_i)_j + 2G''u_kH_k + 2a_{ij}u_{ki}H_{k\ell}u_{\ell j}.
\end{aligned} \tag{19}$$

On the other hand, using (49) and (50),

$$\begin{aligned}
&2G''u_kH_k - 2G''d_{ij}u_iu_j = 2G''H^{-1}(u_kH_kH - a_{ij}u_iu_j)\\
&\quad = 2G''H^{-1}(H^2 - H_iH_ju_iu_j - HH_{ij}u_iu_j) = 2G''H^{-1}(H^2 - H^2 - 0) = 0.
\end{aligned}$$

Consequently, (19) becomes

$$(d_{ij}P_i)_j = -2G'(d_{ij}u_i)_j + 2a_{ij}u_{ki}H_{k\ell}u_{\ell j}. \tag{20}$$

Now, since, from (17), we know that $d_{ij}u_{ij} = H^{-1}G'$, we can write (20) as

$$(d_{ij}P_i)_j = -2G'(d_{ij})_j u_i - 2H^{-1}(G')^2 + 2a_{ij}u_{ki}H_{k\ell}u_{\ell j}. \tag{21}$$

On the other hand,

$$\begin{aligned}
(d_{ij})_j u_i &= (H^{-1}a_{ij})_j u_i = (H^{-1}H_iH_j + H_{ij})_j u_i\\
&= (-H^{-2}H_\ell H_iH_j + H^{-1}H_{i\ell}H_j + H^{-1}H_iH_{j\ell} + H_{ij\ell})u_iu_{j\ell}.
\end{aligned}$$

Then, by using (49), (50) and (51),

$$\begin{aligned}
(d_{ij})_j u_i &= (-H^{-1}H_\ell H_j + 0 + H_{j\ell} - H_{j\ell})u_{j\ell}\\
&= -H^{-1}H_\ell H_j u_{j\ell} = -H^{-2}H_\ell(HH_ju_{j\ell}).
\end{aligned}$$

That is, making use of (15) and (49) once more,

$$\begin{aligned}(d_{ij})_j u_i &= -H^{-2}H_\ell\Big(\frac{P_\ell+2G'u_\ell}{2}\Big)\\ &= -\frac{1}{2}H^{-2}H_\ell P_\ell - H^{-1}G'.\end{aligned}$$

By plugging this into (21), we conclude that

$$\begin{aligned}(d_{ij}P_i)_j &= -2G'\Big(-\frac{1}{2}H^{-2}H_\ell P_\ell - H^{-1}G'\Big) - 2H^{-1}(G')^2 + 2a_{ij}u_{ki}H_{k\ell}u_{\ell j}\\ &= H^{-2}G'H_\ell P_\ell + 2a_{ij}u_{ki}H_{k\ell}u_{\ell j}.\end{aligned} \tag{22}$$

Now, we write $v_k := H_i u_{ki}$ and $\beta_{jk} := H_{ij}u_{ik}$.

We remark that

$$\beta_{jk}\beta_{kj} \geqslant 0. \tag{23}$$

To check this, we diagonalize Hess (H) by writing $H_{ij} = M_{pi}\lambda_p M_{pj}$, with $\lambda_p \geqslant 0$ (recall Lemma 5). We also set $\vartheta_{pr} := M_{pi}M_{rm}u_{mi}$. Then, for any fixed p and r, we have that

$$\begin{array}{rcl}0 &\leqslant& (\vartheta_{pr})^2\\ &=& (M_{pi}M_{rk}u_{ki})\,(M_{pj}M_{r\ell}u_{\ell j})\\ &=& M_{pi}M_{pj}M_{rk}M_{r\ell}u_{ki}u_{\ell j}.\end{array}$$

We multiply by $\lambda_p\lambda_r$ and we sum over p and r: we get

$$\begin{array}{rcl}0 &\leqslant& M_{pi}\lambda_p M_{pj}M_{rk}\lambda_r M_{r\ell}u_{ki}u_{\ell j}\\ &=& H_{ij}H_{k\ell}u_{ki}u_{\ell j}\\ &=& (H_{ij}u_{ik})\,(H_{\ell k}u_{\ell j})\\ &=& \beta_{jk}\beta_{kj},\end{array}$$

which proves (23).

Also, by means of (52), we see that $H_{k\ell}v_k v_\ell \geqslant 0$, therefore

$$\begin{aligned}a_{ij}H_{k\ell}u_{ki}u_{\ell j} &= (H_iH_j + HH_{ij})H_{k\ell}u_{ki}u_{\ell j}\\ &= H_{k\ell}v_kv_\ell + H\beta_{jk}\beta_{kj} \geqslant 0.\end{aligned}$$

By substituting this expression into (22), we obtain (14). □

2. Proof of Theorem 1

With the result of Proposition 1, we can now complete the proof of Theorem 1 by using the techniques of [Mod85, CGS94, FV09]. We provide the argument in detail for the facility of the reader.

By the classic results of [LU68, T84, DiB83] we have that $u \in C^{1,\alpha}_{loc}(\mathbb{R}^n)$ for some $\alpha \in (0,1)$. Then we define the set

$$\mathcal{E} \;:=\; \Big\{v \text{ is solution of (4), with} \inf_{\mathbb{R}^n} u \leqslant v(x) \leqslant \sup_{\mathbb{R}^n} u \text{ for any } x\in\mathbb{R}^n\Big\}$$

and we remark that, using once again [LU68, T84, DiB83] (cfr. for instance Theorem 4.1 of [DG02]) the above set $\mathcal{E}$ is compact in the topology of $C^{1,\alpha}_{loc}(\mathbb{R}^n)$.

We also note that

$$\text{if } v\in\mathcal{E}, \text{ then } G(v(x)) \geqslant 0 \text{ for any } x\in\mathbb{R}^n, \tag{24}$$

where G is defined in (12). Let also

$$P_o := \sup_{\substack{v\in\mathcal{E}\\ x\in\mathbb{R}^n}} P(v;x). \tag{25}$$

We claim that

$$P_o \leqslant 0. \tag{26}$$

To prove (26) we argue by contradiction and we suppose that

$$P_o > 0. \tag{27}$$

Let $v_k \in \mathcal{E}$ and $x_k \in \mathbb{R}^n$ such that

$$\lim_{k\to+\infty} P(v_k; x_k) = P_o. \tag{28}$$

Let us define $w_k(x) := v_k(x + x_k)$. Then,

$$w_k \in \mathcal{E} \tag{29}$$

and

$$P(w_k; 0) = P(v_k; x_k). \tag{30}$$

So, up to subsequences, we may suppose that w_k converges to some w in $C^{1,\alpha}_{\rm loc}(\mathbb{R}^n) \cap \mathcal{E}$. In particular,

$$\begin{aligned} P(w; 0) &= H^2(\nabla w(0)) - 2G(w(0)) \\ &= \lim_{k\to+\infty} H^2(\nabla w_k(0)) - 2G(w_k(0)) = \lim_{k\to+\infty} P(w_k; 0). \end{aligned}$$

Hence, recalling (28) and (30),

$$P(w; 0) = \lim_{k\to+\infty} P(v_k; x_k) = P_o. \tag{31}$$

In particular, from (27) and (24),

$$0 < P_o = P(w; 0) = H^2(\nabla w(0)) - 2G(w(0)) \leqslant H^2(\nabla w(0))$$

and so $\nabla w(0) \neq 0$. This implies that we can use Proposition 1 and conclude that

$$(d_{ij} P_i(w; x))_j - H^{-2}(\nabla w(x)) G'(w(x)) H_\ell(\nabla w(x)) P_\ell(w; x) \geqslant 0 \tag{32}$$

for any $x \in B_\rho$, for a suitably small $\rho > 0$.

On the other hand, by (25), we know that $P(w; x) \leqslant P_o$ and so, by (31), we obtain that 0 is a local maximum for $P(w; \cdot)$. Accordingly, (32) and the strong maximum principle (see, e.g., Theorem 8.19 in [GT83]), imply that $P(w; \cdot)$ is constant in B_ρ and, in fact, by connectedness, in the whole of $\mathbb{R}^n$: thus

$$H^2(\nabla w(x)) - 2G(w(x)) = P(w; x) = P_o \text{ for any } x \in \mathbb{R}^n. \tag{33}$$

But, since w is bounded, we have that there exists a sequence $q_k \in \mathbb{R}^n$ such that

$$\lim_{k\to+\infty} \nabla w(q_k) = 0.$$

Hence, from (24) and (33),

$$0 = H^2(0) = \lim_{k\to+\infty} H^2(\nabla w(q_k)) \geqslant \lim_{k\to+\infty} H^2(\nabla w(q_k)) - 2G(w(q_k)) = P_o.$$

This is in contradiction with (27) and therefore it proves (26). Since obviously $u \in \mathcal{E}$, we obtain from (26) that $P(u; x) \leqslant P_o \leqslant 0$ for any $x \in \mathbb{R}^n$, thus proving (5).

Now we show that once equality in (5) is attained at some non-critical point, then it is attained everywhere. For this let x_o be as in (6) and (7). By (7), (5) and (12), we have that

$$\begin{aligned} P(u; x_o) &= H^2(\nabla u(x_o)) - 2G(u(x_o)) = 0 \\ &\geqslant H^2(\nabla u(x)) - 2G(u(x)) = P(u; x) \end{aligned} \tag{34}$$

for any $x \in \mathbb{R}^n$, hence

$$x_o \text{ is a local maximum for } P(u; \cdot). \tag{35}$$

By (6), (35), Proposition 1 and the strong maximum principle, we obtain that $P(u;\cdot)$ is constant, and constantly equal to 0, in the whole of the connected component of $\{\nabla u \neq 0\}$ that contains x_o. This establishes (7). $\square$

3. Proof of Theorem 2

From formula (10) on page 107 of [WX11], we know that

$$\mathcal{C} = H_{ij} u_{ij}, \tag{36}$$

where the short hand notation $H = H(\nabla u)$ has been used. Also, by using (9) and the notation in (12), we have

$$\frac{1}{2} H^2 = G$$

in U, and so, by differentiating,

$$H H_j u_{ij} = G' u_i. \tag{37}$$

Moreover, from (4)

$$0 = f + (HH_i)_i = f + H_i H_j u_{ij} + H H_{ij} u_{ij}. \tag{38}$$

By plugging (37) into (38) and then recalling (49), we obtain

$$0 = f + \frac{G' H_i u_i}{H} + H H_{ij} u_{ij} = f + G' + H H_{ij} u_{ij} = H H_{ij} u_{ij}.$$

This and (36) imply that $\mathcal{C}$ is constantly equal to zero on any level set lying in U. This proves (10). $\square$

4. Proof of Theorem 3

First of all, we recall that, for any $\xi \in \mathrm{S}^1$, $\mathrm{Hess}\,(H)$ at ξ is positive definite on $\xi^\perp$: see Proposition 2 on page 102 of[1] [WX11]. In particular,

$$H_{11}(0,1) = \mathrm{Hess}\,(H)(0,1)\big[(1,0),(1,0)\big] > 0. \tag{39}$$

Now we take a point $x_\star \in U$ and we write, nearby, the level set $\{u = u(x_\star)\}$ as a graph. Up to rigid motions, we may suppose that $x_\star = 0$ and that $\{u = u(x_\star)\}$, near 0, may be written as the graph $\{x_2 = h(x_1)\}$, with $h \in C^2$, $h(0) = 0$ and

$$h'(0) = 0. \tag{40}$$

Then we can write the normal as

$$\nu = (\nu_1, \nu_2) = \frac{(-h', 1)}{\sqrt{1 + (h')^2}}$$

and use the anisotropic curvature in local representation (see formula (8) on page 106 of [WX11], here with $\alpha = 1$ since we are in dimension 2):

$$\begin{aligned}
\mathcal{C} &= H_{i1}(\nu)\partial_1 \nu_i \\
&= H_{11}(\nu)\partial_1 \nu_1 + H_{12}(\nu)\partial_1 \nu_2 \\
&= -H_{11}(\nu)\partial_1 \left(\frac{h'}{\sqrt{1 + (h')^2}} \right) + H_{12}(\nu)\partial_1 \left(\frac{1}{\sqrt{1 + (h')^2}} \right).
\end{aligned}$$

Hence, recalling (40),

$$\mathcal{C}(0) = -H_{11}(0,1)\, h''(0).$$

From (10), we obtain that $-H_{11}(0,1)\, h''(0) = \mathcal{C}(0) = 0$, and so, by (39), that $h''(0) = 0$. This says, in general, that the curvature of the level sets of u vanishes in the whole of U. $\square$

[1] We remark that in [WX11] the function H is taken to be even, which is not assumed here: on the other hand, this parity assumption is used elsewhere in [WX11] but not in the proof of Proposition 2.

5. Proof of Theorem 4

First, we take $\mathcal{S}$ to be any connected component of $\{\nabla u \neq 0\}$. We claim that

$$\begin{aligned}&\mathcal{S} \text{ is foliated by level sets of } u \text{ which are union of parallel straight lines,}\\&\text{and so } \partial\mathcal{S} \text{ is the union of (at most two) parallel straight lines.}\end{aligned} \tag{41}$$

To establish this, we fix any $x_\star \in \mathcal{S}$ and look at the level set $S_{x_\star} = \{u = u(x_\star)\}$. By Theorem 3, we know that

$$\text{any connected component of } S_{x_\star} \text{ is contained in a straight line, say } r_{x_\star}. \tag{42}$$

We observe that

$$S_{x_\star} \subseteq \{\nabla u \neq 0\}. \tag{43}$$

Indeed, taking any $\tilde{x} \in S_{x_\star}$, we have from (11) that

$$\frac{1}{2}H^2(\nabla u(\tilde{x})) = c_u - F(u(\tilde{x})) = c_u - F(u(x_\star)) = \frac{1}{2}H^2(\nabla u(x_\star)) \neq 0,$$

hence $\tilde{x} \in \{\nabla u \neq 0\}$, thus proving (43).

Now we point out that

$$\text{the connected component of } S_{x_\star} \text{ that contains } x_\star \text{ must be equal to } r_{x_\star}. \tag{44}$$

Indeed, $S_{x_\star}$ is closed in the relative topology of $r_{x_\star}$, because u is continuous, and $S_{x_\star}$ is also open in that topology, thanks to (43) and Theorem 3.

By (43) and (44), we obtain (41).

Now, we denote by ϖ a vector normal to all the straight lines in (41). We claim that

$$u(x_o) = u(y_o) \text{ if } (x_o - y_o) \cdot \varpi = 0. \tag{45}$$

To check this, fix $x_o \in \mathbb{R}^2$. If $\nabla u = 0$ at all points of r_{x_o}, then (45) follows from the Fundamental Theorem of Calculus. Conversely, if there exists $x_\sharp \in r_{x_o} \cap \{\nabla u \neq 0\}$, by (44) (applied to $x_\sharp$), we have that u is constant on $r_{x_\sharp}$, which, in turn, is equal to r_{x_o}, thus proving (45).

Then, (45) gives the desired one-dimensional Euclidean symmetry. $\square$

6. Proof of Theorem 5

We need an appropriate modification of some arguments in [CGS94, FV09]. We take p as in the statement of Theorem 5 and $r := u(p)$. We fix $q \in \mathbb{R}^n \setminus \{p\}$ and we show that $u(q) = r$: with this, the thesis of Theorem 5 would be established.

To this aim, we consider the function

$$[0,1] \ni t \mapsto \varphi(t) := u\big(tq + (1-t)p\big) - r.$$

By the homogeneity of H and (2), we have that

$$H^2(\xi) = |\xi|^2 H^2\left(\frac{\xi}{|\xi|}\right) \geqslant \kappa|\xi|^2, \tag{46}$$

for any $\xi \in \mathbb{R}^n \setminus \{0\}$, with

$$\kappa := \inf_{\substack{\eta\in\mathbb{R}^n\\|\eta|=1}} H^2(\eta).$$

As a matter of fact, since $H(0) = 0$, we infer from (46) that

$$H^2(\nabla u) \geqslant \kappa|\nabla u|^2.$$

Hence

$$\begin{aligned}(\dot\varphi(t))^2 &\leqslant |p-q|^2\big|\nabla u\big(tq+(1-t)p\big)\big|^2\\&\leqslant \kappa^{-1}|p-q|^2 H^2\Big(\nabla u\big(tq+(1-t)p\big)\Big).\end{aligned}$$

Consequently, by (5) and the assumptions of Theorem 5, we have that

$$\begin{aligned}
(\dot\varphi(t))^2 &\leqslant 2\kappa^{-1}|p-q|\Big[c_u - F\Big(u\big(tq+(1-t)p\big)\Big)\Big]\\
&= 2\kappa^{-1}|p-q|^2\Big[F(r) - F\Big(u\big(tq+(1-t)p\big)\Big)\Big]\\
&= -2\kappa^{-1}|p-q|^2\int_r^{u(tq+(1-t)p)} F'(\sigma)\,d\sigma\\
&= 2\kappa^{-1}|p-q|^2\int_r^{u(tq+(1-t)p)} F'(r)-F'(\sigma)\,d\sigma\\
&\leqslant 2\kappa^{-1}|p-q|^2\,\|F\|_{C^{1,1}(\mathcal{J})}\left|\int_r^{u(tq+(1-t)p)} |\sigma-r|\,d\sigma\right|\\
&\leqslant \kappa^{-1}|p-q|^2\,\|F\|_{C^{1,1}(\mathcal{J})}\,\big|u(tq+(1-t)p)-r\big|^2\\
&= \kappa^{-1}|p-q|^2\,\|F\|_{C^{1,1}(\mathcal{J})}\,(\varphi(t))^2,
\end{aligned}$$

where $\mathcal{J} := \Big[\inf\limits_{\mathbb{R}^n} u, \sup\limits_{\mathbb{R}^n} u\Big]$. Therefore, if $\varphi(t)\neq 0$,

$$\left|\frac{\dot\varphi(t)}{\varphi(t)}\right| \leqslant K := \kappa^{-1/2}|p-q|.$$

As a consequence, if $\psi(t) := (\varphi(t))^2 e^{-2Kt}$, we have that

$$\begin{aligned}
\dot\psi(t) &= 2\varphi(t)\dot\varphi(t)e^{-2Kt} - 2K(\varphi(t))^2e^{-2Kt}\\
&= \begin{cases} 0 & \text{if } \varphi(t)=0,\\ 2(\varphi(t))^2e^{-2Kt}\left[\frac{\dot\varphi(t)}{\varphi(t)} - K\right] & \text{if } \varphi(t)\neq 0\end{cases}\\
&\leqslant 0
\end{aligned}$$

and so ψ is non-increasing. Thus

$$\begin{aligned}
(u(q)-r)^2e^{-2K} &= (\varphi(1))^2e^{-2K} = \psi(1)\\
&\leqslant \psi(0) = (\varphi(0))^2 = (u(p)-r)^2 = 0.
\end{aligned}$$

This says that $u(q) = r = u(p)$ and so the proof of Theorem 5 is finished. □

7. Proof of Theorem 6

Without loss of generality, we may and do assume that

$$u \text{ is not constant,} \tag{47}$$

otherwise we are done. Then, the proof of Theorem 6 is by contradiction: if its thesis were false, there would exist $r_o \in \Big(\inf_{\mathbb{R}^n} u,\ \sup_{\mathbb{R}^n} u\Big)$ such that

$$\sup\Big\{F(r)\,,\ r\in\Big[\inf_{\mathbb{R}^n} u, \sup_{\mathbb{R}^n} u\Big]\Big\} = c_u = F(r_o),$$

and so, by the continuity of u, there would exist $x_o\in\mathbb{R}^n$ such that $u(x_o) = r_o$.

We deduce that r_o is a local maximum for F, and so $F'(r_o) = 0$. Therefore, $F(u(x_o)) = F(r_o) = c_u$ and $F'(u(x_o)) = F'(r_o) = 0$ and so, by Theorem 5, it follows that u is constant. This is in contradiction with (47) and so the proof of Theorem 6 is complete. □

Appendix

This part collects some elementary facts on positive homogeneous functions. Its purpose is to make the paper more self-contained, and it can be skipped by the expert reader. Here, H denotes a positive homogeneous function (not necessarily the one of the main results of this paper). As usual, a function $H : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is said to be positive homogeneous of degree $k \in \mathbb{Z}$ if $H(t\xi) = t^k H(\xi)$ for any $t > 0$ and $\xi \in \mathbb{R}^n \setminus \{0\}$.

Lemma 1 (Euler's formula)**.** *If $H \in C^1(\mathbb{R}^n \setminus \{0\})$ is positive homogeneous of degree k, then*

$$\frac{\partial H}{\partial \xi_i}(\xi)\xi_i = kH(\xi). \tag{48}$$

Proof. Differentiate in t the formula $H(t\xi) = t^k H(\xi)$ and then plug $t = 1$. □

Lemma 2. *If $H \in C^m(\mathbb{R}^n \setminus \{0\})$ is positive homogeneous of degree k and $\alpha \in \mathbb{N}^n$ with $\alpha_1 + \cdots + \alpha_n = m$, then $\partial^\alpha H$ is positive homogeneous of degree $k - m$.*

Proof. By induction over m. The inductive step goes like this: if $\alpha_1 + \cdots + \alpha_n = m$ and we know by inductive assumption that $\partial^\alpha H$ is positive homogeneous of degree $k - m$, we write

$$\partial^\alpha H(t\xi) = t^{k-m} \partial^\alpha H(\xi).$$

Now we take one derivative more in direction, say ξ_1: we obtain

$$t\partial^{\alpha+e_1} H(t\xi) = t^{k-m} \partial^{\alpha+e_1} H(\xi)$$

which gives that $\partial^{\alpha+e_1} H$ is positive homogeneous of degree $k - (m + 1)$. □

Following are the identities that we use in the course of the main proofs. We use the standard notation $H_i(\xi) = \partial_i H(\xi) = \partial_{\xi_i} H(\xi)$.

Lemma 3. *If $H \in C^3(\mathbb{R}^n \setminus \{0\})$ is positive homogeneous of degree 1, we have that*[2]

$$H_i(\xi)\xi_i = H(\xi), \tag{49}$$

$$H_{ij}(\xi)\xi_i = 0, \tag{50}$$

$$H_{ijk}(\xi)\xi_i = -H_{jk}(\xi). \tag{51}$$

Proof. We use (48) with $k := 1$ and we obtain (49). Then, by Lemma 2, we know that H_j is positive homogeneous of degree 0, so that (50) follows from (48) with $k := 0$. Similarly, by Lemma 2, we know that H_{jk} is positive homogeneous of degree -1, so that (51) follows from (48) with $k := -1$. □

Now, we justify the regularity on H needed to write (4):

Lemma 4. *If $H \in C^1(\mathbb{R}^n \setminus \{0\})$ is positive homogeneous of degree $k > 1/2$, than it can be extended by setting $H(0) := 0$ to a continuous function, such that $H^2 \in C^1(\mathbb{R}^n)$ and*

$$\partial_i(H^2)(0) = 0 = \lim_{x \to 0} H(x)H_i(x).$$

Proof. Of course $|H(\xi)| \leqslant |\xi|^k \sup_{\mathrm{S}^{n-1}} |H|$, showing the continuity of H at 0. Moreover, $H^2 \in C^1(\mathbb{R}^n \setminus \{0\})$, and

$$\partial_i(H^2)(0) = \lim_{t \to 0} \frac{H^2(te_i)}{t} = \lim_{t \to 0} \frac{H^2(\pm|t|e_i)}{h} = \lim_{t \to 0} \frac{|t|^{2k} H^2(\pm e_i)}{t} = 0.$$

On the other hand, by Lemma 2, $H_i(x) = |x|^{k-1} H(x)$ for any $x \in \mathbb{R}^n \setminus \{0\}$, and so

$$\lim_{x \to 0} |H(x)H_i(x)| \leqslant \lim_{x \to 0} |x|^{2k-1} \sup_{\mathrm{S}^{n-1}} |HH_i| = 0,$$

as desired. □

[2]As a matter of fact, $H \in C^1(\mathbb{R}^n \setminus \{0\})$ is enough for (49), and $H \in C^2(\mathbb{R}^n \setminus \{0\})$ is enough for (50) (and so for the subsequent Lemma 5).

We end this paper with a convexity remark:

Lemma 5. *If $H \in C^2(\mathbb{R}^n \setminus \{0\})$ is positive homogeneous of degree* 1 *and it satisfies* (2) *and* (3)*, then H is convex and*

$$H_{ij}(\xi)\,\eta_i\eta_j \geqslant 0 \qquad \text{for any } \xi \in \mathbb{R}^n \setminus \{0\} \text{ and } \eta \in \mathbb{R}^n. \tag{52}$$

Proof. To prove (52), we may suppose that $\eta \neq 0$ and so, up to dividing by $|\eta|$, that $\eta \in \mathrm{S}^{n-1}$. Moreover, by Lemma 2, we know that H_{ij} is positive homogeneous of degree -1, hence, to prove (52), we may also assume that $\xi \in \mathrm{S}^{n-1}$. Let us now decompose η along ξ and its orthogonal space, that is let $\lambda := \eta \cdot \xi$ and $\tau := \eta - \lambda\xi$. Notice that

$$\tau \in \xi^\perp. \tag{53}$$

Furthermore, $\eta = \lambda\xi + \tau$ and, by virtue of (50),

$$\begin{aligned} H_{ij}(\xi)\,\eta_i\eta_j &= \lambda^2 H_{ij}(\xi)\,\xi_i\xi_j + H_{ij}(\xi)\,\tau_i\tau_j + 2\lambda H_{ij}(\xi)\,\eta_i\tau_j \\ &= 0 + H_{ij}(\xi)\,\tau_i\tau_j + 0. \end{aligned} \tag{54}$$

On the other hand, by Proposition 2 on page 102 of [WX11], we have that Hess$\,(H)$ at ξ is positive definite on $\xi^\perp$, and so, by (53),

$$\text{either } \tau = 0 \text{ or } H_{ij}(\xi)\tau_i\tau_j > 0. \tag{55}$$

Then, the desired result follows from (54) and (55). □

It is worth pointing out that it is not possible to have the strict sign in (52): indeed, by (50),

$$H_{ij}(\xi)\,\xi_i\xi_j = 0.$$

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