The nonlinear multidomain model: a new formal asymptotic analysis

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Abstract

We study the asymptotic analysis of a singularly perturbed weakly parabolic system of m- equations of anisotropic reaction-diffusion type. Our main result formally shows that solutions to the system approximate a geometric motion of a hypersurface by anisotropic mean curvature. The anisotropy, supposed to be uniformly convex, is explicit and turns out to be the dual of the star-shaped combination of the m original anisotropies.

1 Introduction

The bidomain model, a simplified version of the FitzHugh-Nagumo system, was originally introduced in electrocardiology as an attempt to describe the electric potentials and current flows inside and outside the cardiac cells, see [12, 13, 1, 11] and references therein. In spite of the discrete cellular structure, at a macroscopic level the intra (i) and the extra (e) cellular regions can be thought of as two superimposed and interpenetrating continua, thus coinciding with the domain Ω (the physical region occupied by the heart). Denoting the intra and extra cellular electric potentials respectively with $u_{\rm i} = u_{{\rm i}\epsilon}$, $u_{\rm e} = u_{{\rm e}\epsilon} \colon [0, T] \times \Omega \to \mathbb{R}$, the bidomain model can be formulated using the following weakly parabolic system of two singularly perturbed linearly anisotropic reaction-diffusion equations, of variational nature[1]:

$$\begin{cases} \epsilon\partial_t(u_{\rm i} - u_{\rm e}) - \epsilon{\rm div}\left(M^{\rm i}(x)\nabla u_{\rm i}\right) + \dfrac{1}{\epsilon}f\left(u_{\rm i} - u_{\rm e}\right) = 0, \\ \\ \epsilon\partial_t(u_{\rm i} - u_{\rm e}) + \epsilon{\rm div}\left(M^{\rm e}(x)\nabla u_{\rm e}\right) + \dfrac{1}{\epsilon}f\left(u_{\rm i} - u_{\rm e}\right) = 0, \end{cases} \tag{1.1}$$

coupled with suitable initial and boundary conditions. Here $\epsilon \in (0, 1)$ is a small positive parameter, f is the derivative of a double-well potential with minima at $s_\pm$ (the standard choice is $f(s) = \frac{d}{ds}\left((1 - s^2)^2\right)$, so that $s_\pm = \pm 1$), and $M^{\rm i}(x), M^{\rm e}(x)$ are two symmetric uniformly positive definite matrices.

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[1]See Remark 3.2 below.

The whole process that determines $u_{\rm i}, u_{\rm e}$, and in particular the behaviour of the transmembrane potential

$$u = u_\epsilon := u_{\rm i} - u_{\rm e},$$

is quite complicated: we refer the reader to the already quoted references for a more accurate description of the physiological phenomenon and its mathematical modelization. For our purposes, here it suffices to recall that the transmembrane potential typically exhibits a thin transition region (of order ϵ) which separates the advancing depolarized region where $u_\epsilon \approx s_+$ from the one where $u_\epsilon \approx s_-$, see [4, 7] and references therein. Remarkably, a not negligible *nonlinear anisotropy* occurs in the limit $\epsilon \to 0^+$, because of the fibered structure of the myocardium. To explain the appearence of the anisotropy, let us introduce the riemannian norms $\phi_{\rm i}, \phi_{\rm e}$, defined as

$$(\phi_{\rm i}(x,\xi^*))^2 = \alpha_{\rm i}(x,\xi^*) := M^{\rm i}(x)\xi^* \cdot \xi^*, \qquad (\phi_{\rm e}(x,\xi^*))^2 = \alpha_{\rm e}(x,\xi^*) := M^{\rm e}(x)\xi^* \cdot \xi^*,$$

where ξ^* denotes a generic covector of the dual $(\mathbb{R}^N)^*$ of $\mathbb{R}^N$, $N \geq 2$, and $\cdot$ is the euclidean scalar product. The squared norms $\alpha_{\rm i}$ and $\alpha_{\rm e}$ describe the microscopic structure of the intra and extra cellular regions[2], and their hessians $\frac{1}{2}\nabla^2_{\xi^*}\alpha_{\rm i}$, $\frac{1}{2}\nabla^2_{\xi^*}\alpha_{\rm e}$ (with respect to ξ^*) give $M^{\rm i}$ and $M^{\rm e}$ respectively. Then the anisotropy manifests, for instance, recalling the following *formal* result [4]: as $\epsilon \to 0^+$, the zero level set of u_ϵ approximates a geometric motion of a front, evolving by Φ^o-*anisotropic mean curvature flow*, where Φ^o denotes the dual of Φ, and the anisotropy Φ turns out to be the star-shaped combination (see [7]) of $\phi_{\rm i}$ and $\phi_{\rm e}$, i.e. its square satisfies

$$\Phi^2 := \left(\frac{1}{\alpha_{\rm i}} + \frac{1}{\alpha_{\rm e}}\right)^{-1}, \tag{1.2}$$

supposing a priori that Φ^2 is smooth and uniformly *convex*. This convergence result is substantiated by a Γ-convergence theorem (at the level of the corresponding actions) to a geometric functional, the integrand of which is strictly related to (1.2), see [1] and Theorem 3.6 below.

Note that Φ is *not riemannian* anymore (i.e., a nonlinear anisotropy in the language of the present paper), and it may also *fail* to be convex (this latter property can be seen through an explicit example described in [7]). Lackness of an underlying scalar product for Φ suggests that it is natural to depart from the riemannian structure of (1.1) and to consider, more generally, the *nonlinear* bidomain model, described by

$$\begin{cases} \epsilon\partial_t(u_{\rm i} - u_{\rm e}) - \epsilon{\rm div}\left(T_{\phi_{\rm i}}(x,\, \nabla u_{\rm i})\right) + \dfrac{1}{\epsilon}f\left(u_{\rm i} - u_{\rm e}\right) = 0, \\ \\ \epsilon\partial_t(u_{\rm i} - u_{\rm e}) + \epsilon{\rm div}\left(T_{\phi_{\rm e}}(x,\, \nabla u_{\rm e})\right) + \dfrac{1}{\epsilon}f\left(u_{\rm i} - u_{\rm e}\right) = 0, \end{cases} \tag{1.3}$$

where now $\phi_{\rm i}$ and $\phi_{\rm e}$ are two smooth symmetric uniformly convex[3] Finsler metrics [2], and setting as before $\alpha_{\rm i} = \phi_{\rm i}^2$, $\alpha_{\rm e} = \phi_{\rm e}^2$, the maps

$$T_{\phi_{\rm i}} := \frac{1}{2}\nabla_{\xi^*}\alpha_{\rm i}, \qquad T_{\phi_{\rm e}} := \frac{1}{2}\nabla_{\xi^*}\alpha_{\rm e}$$

[2] $\phi_{\rm i}$ and $\phi_{\rm e}$ depend on the spatial variable x, since the fibers' orientation changes from point to point.

[3] Convexity of $\phi_{\rm i}$ and $\phi_{\rm e}$ is of course required in order to ensure well-posedness of (1.3).

are the so-called duality maps, taking $(\mathbb{R}^N)^*$ into $\mathbb{R}^N$. An analog of the above mentioned formal convergence result, to the Φ^o-anisotropic mean curvature flow, appears to hold also in this nonlinear setting, still assuming Φ^2 to be uniformly convex, see [7], where a starting analysis of the geometric meaning of the star-shaped combination of two anisotropies is also carried on.

It is interesting to remark that, generalizing system (1.3) to an arbitrary number m of Finsler symmetric metrics $\phi_1, \ldots, \phi_m$, leads to rewrite the problem, that we have called the nonlinear multidomain model, in a slightly different and more natural way[4], as follows: we seek functions $w^r = w^r_\epsilon$ satisfying the weakly parabolic system

$$\begin{cases} \epsilon\partial_t u - \epsilon \mathrm{div}\left(T_{\phi_r}\left(x, \nabla w^r\right)\right) + \dfrac{1}{\epsilon} f\left(u\right) = 0, \qquad r = 1, \ldots, m, \\ \\ u = \displaystyle\sum_{r=1}^{m} w^r, \end{cases} \tag{1.4}$$

where

$$T_{\phi_r} := \frac{1}{2}\nabla_{\xi^*}\alpha_r \quad \text{and} \quad \alpha_r := \phi_r^2, \qquad r = 1, \ldots, m.$$

It is the purpose of the present paper to provide an asymptotic analysis of the zero level set of $u = u_\epsilon$ in (1.4): we will show, in particular, that $\{u_\epsilon(t,\cdot) = 0\}$ converges to the Φ^o-anisotropic mean curvature flow (see (5.54) below), where Φ^2, supposed to be uniformly convex, reads as

$$\Phi^2 := \left(\sum_{r=1}^{m} \frac{1}{\alpha_r}\right)^{-1},$$

thus generalizing the above mentioned convergence result for the linear and nonlinear bidomain models. Our proof, which remains at a *formal level*, is based on a new asymptotic expansion for (1.4), rewritten equivalently as a system of one parabolic equation and $(m-1)$ elliptic equations[5]. The asymptotic expansion is simpler, and at the same time carried on at a higher order of accuracy, with respect to the one exhibited in [7] for the case $m = 2$.

Before passing to describe the content of the paper, two observations are in order. The first one concerns the case in which Φ^2 is known a priori to be uniformly convex: since we are dealing with systems, confirming rigorously the convergence result[6] for the sets $\{u_\epsilon(t,\cdot) = 0\}$ is still an open problem, even in the simplest case (1.1) (see Theorem 3.5 for a precise statement). The second remark concerns the case when Φ is nonconvex[7]: the question arises on what could be in this case the limit behaviour (if any), as $\epsilon \to 0^+$, of solutions to (1.3) (or also to (1.1)). Indeed, for a nonconvex Φ, the corresponding anisotropic mean curvature flow is ill-posed, and consequently highly unstable. The answer to this question seems, at the moment, out of reach, even at a formal level.

[4] When $m = 2$, (1.3) and (1.4) are equivalent, with the positions $u_{\mathrm{i}} = w^1$ and $u_{\mathrm{e}} = -w^2$.

[5] This shows, among other things, the nonlocality of solutions of (1.4).

[6] This, however, could be hopely less hard to prove than a convergence result of the Allen-Cahn's (2×2)-system, to curvature flow of networks, see [9] for a formal result in this direction.

[7] In this case Φ is not the dual of a convex anisotropy.

Let us now briefly describe the plan of the paper. In Section 2 we recall the definition of star bodies, and we introduce the star-shaped operation for an arbitrary number of star-shaped anisotropies, using the formalism of gauges and radial functions. In Section 3 we recall some known results on the linear and nonlinear bidomain models. The nonlinear multidomain model is introduced in Section 4. Section 5 contains the main result concerning the convergence of $\{u_\epsilon(t,\cdot)=0\}$ to the Φ^o-anisotropic mean curvature flow.

2 Star-shaped combination of star bodies and of anisotropies

We start with the following definitions. Let V denote either $\mathbb{R}^N$ or its dual $(\mathbb{R}^N)^*$, endowed with the euclidean norm $|\cdot|$.

Definition 2.1 (Star-shaped anisotropies). *A* star-shaped anisotropy *on V is a continuous function $\phi : V \to [0,+\infty)$, positive out of the origin, and positively one-homogeneous. ϕ is said to be* symmetric *if $\phi(-v)=\phi(v)$ for any $v \in V$.*

Definition 2.2 (Convex and linear anisotropies). *Let ϕ be a star-shaped anisotropy on V. If ϕ is convex, then it is called a* convex anisotropy*. A convex anisotropy which is the square root of a quadratic form*[8] *is called a* linear anisotropy*.*

Denote with $\mathcal{S}$ the family of *star bodies*:

$$\mathcal{S} := \Big\{K \subset V \;:\; K = \overline{\mathrm{int}(K)} \text{ is compact, star-shaped with respect to } 0 \in \mathrm{int}(K)\Big\}.$$

Associated with every $K \in \mathcal{S}$, we define the function

$$\phi_K(v) := \inf\{\lambda > 0 \;:\; v \in \lambda K\}, \qquad v \in V,$$

which is sometimes called *gauge* of K, and is a star-shaped anisotropy with $K = \{\phi_K \le 1\}$. A convex set $K \in \mathcal{S}$ is called a *convex body*, see [15, 16] and references therein. In this latter case, ϕ_K is usually called *Minkowski functional* of K, see for instance [14],[9] and it is obviously a convex anisotropy.
For $K \in \mathcal{S}$, the function

$$\phi_K^o(v^*) := \sup\{\langle v^*, v\rangle \;:\; v \in K\}, \qquad v^* \in V^*,$$

where $\langle\cdot,\cdot\rangle$ is the duality (identified with the euclidean scalar product $\cdot$) between V and its dual V^*, is called *support function* of K. It is often denoted by h_K and is also called the dual of ϕ_K (or also the anisotropy dual to ϕ_K). The corresponding set $K^o := \{\phi_K^o \le 1\}$ is called the *dual* of K and it is always a convex body, see again [14]; equivalently ϕ_K^o is always a convex anisotropy on V^*.
For $K \in \mathcal{S}$, let $\varrho_K : \mathbb{S}^{N-1} := \{v \in V : |v| = 1\} \to (0,+\infty)$ be the *radial function* of K (see for instance [16]) defined as

$$\varrho_K(\nu) := \sup\big\{\lambda \ge 0 \;:\; \lambda\nu \in K\big\}, \qquad \nu \in \mathbb{S}^{N-1}. \tag{2.1}$$

[8] A linear anisotropy is obviously symmetric.

[9] When K is symmetric with respect to the origin, K is said a symmetric convex body, and ϕ_K turns out to be a norm equivalent to the euclidean one.

The function ϱ_K is extended (keeping the same symbol) in a one-homogeneous way on the whole of V, i.e., $\varrho_K(v) = |v|\varrho_K(\frac{v}{|v|})$ for any $v \in V \setminus \{0\}$. Notice that

$$\varrho_K(\nu) = \frac{1}{\phi_K(\nu)}, \qquad \nu \in \mathbb{S}^{N-1}, \tag{2.2}$$

and

$$K = \{\lambda\nu \,:\, 0 \le \lambda \le \varrho_K(\nu),\ \nu \in \mathbb{S}^{N-1}\}.$$

Remark 2.3. The previous definitions of ϕ_K, ϕ_K^o and ϱ_K can be generalized, by allowing a continuous dependence on the space variable x in some open subset Ω of $\mathbb{R}^N$. In this way we have that $\phi_K = \phi_K(x, v)$, as well as $\varrho_K = \varrho_K(x, v)$, are defined for $(x, v) \in \Omega \times V$ and $\phi_K^o(x, v^*)$ is defined for $(x, v^*) \in \Omega \times V^*$[10]. In the present paper, however, we will be mostly interested in space-independent anisotropies.

Assumption: in this paper we deal only with sets $K \in S$ having smooth boundary. In the case K is a convex body, we will always suppose that K is smooth and uniformly convex, so that K^o is also smooth and uniformly convex[11]. In this case, we say that ϕ_K^2 (or also that ϕ_K) is smooth and uniformly convex.
Also, for simplicity all anisotropies we consider will be assumed to be symmetric.

Now, consider $K_1, K_2 \in \mathcal{S}$. We let $\varrho_{K_1} \star \varrho_{K_2} : \mathbb{S}^{N-1} \to (0, +\infty)$ be defined as follows [7]:

$$\varrho_{K_1} \star \varrho_{K_2}(\nu) := \sqrt{\Big(\varrho_{K_1}(\nu)\Big)^2 + \Big(\varrho_{K_2}(\nu)\Big)^2}, \qquad \nu \in \mathbb{S}^{N-1}.$$

Again, $\varrho_{K_1} \star \varrho_{K_2}$ is extended (keeping the same symbol) in a one-homogeneous way on the whole of V.

Definition 2.4 (Star-shaped combination of two sets). *Given $K_1, K_2 \in \mathcal{S}$, we define the* star-shaped combination

$$K_1 \star K_2$$

of K_1 and K_2 as the set whose radial function coincides with $\varrho_{K_1} \star \varrho_{K_2}$:

$$\varrho_{K_1 \star K_2} := \varrho_{K_1} \star \varrho_{K_2}.$$

One checks that $K_1 \star K_2 \in \mathcal{S}$, and that the identity element for $\star$ does not belong to $\mathcal{S}$. Moreover

$$K_1 \star K_2 = K_2 \star K_1.$$

It is clear that the set $K_1 \star K_2$ depends on K_1 and K_2 and not only on $K_1 \cup K_2$. However,

[10] A continuous function $\phi : \Omega \times V \to [0, +\infty)$ is called an *inhomogeneous star-shaped anisotropy* on Ω, provided $\phi(x, \cdot)$ is positively one-homogeneous for any $x \in \Omega$, and there exist two constants c, C with $0 < c \le C < +\infty$ such that $c|v| \le \phi(x, v) \le C|v|$ for any $x \in \Omega$ and $v \in V$. If in addition $\phi(x, \cdot)$ is convex for every $x \in \Omega$, then ϕ is called a (inhomogeneous) convex anisotropy (or also a Finsler metric) on Ω. Eventually, if $\phi(x, \cdot)$ is the square root of a quadratic form, then ϕ is a Riemannian metric (an inhomogeneous linear anisotropy).

[11] Hence, our Finsler metrics will be smooth and uniformly convex, in the sense that for any $x \in \Omega$, the function $\phi(x, \cdot)$ is uniformly convex and smooth.

it cannot be viewed as the union of an enlargement of K_1 with an enlargement of K_2.

The next formula gives the concrete way to compute the star-shaped combination of two sets $K_1,\, K_2 \in \mathcal{S}$:

$$\partial\,(K_1 \star K_2) := \left\{ \sqrt{\lambda_1^2 + \lambda_2^2}\;\nu \,:\, \nu \in \mathbb{S}^{N-1},\ \lambda_j = \varrho_{K_j}(\nu),\ j = 1, 2 \right\}. \tag{2.3}$$

Remark 2.5. The reason for using star bodies, instead of convex sets, in Definition 2.4 is the following: if K_1 and K_2 are two convex bodies, then $K_1 \star K_2$ is not in general a convex body. An explicit counterexample for $N = 2$ is given in [7], and it involves the two ellipses:

$$K_1 := \left\{(x, y) \in \mathbb{R}^2 : x^2 + \rho y^2 = 1\right\}, \qquad K_2 := \left\{(x, y) \in \mathbb{R}^2 : \rho x^2 + y^2 = 1\right\},$$

defined for $\rho > 0$. Then

(i) $K_1 \star K_2$ is (smooth and) strictly convex, for $\rho \in (\frac{1}{3}, 3)$;

(ii) $K_1 \star K_2$ is (smooth and) convex, for $\rho = \frac{1}{3}$ or $\rho = 3$, with zero boundary curvature at the points of intersection with the lines $\{(x, y) \in \mathbb{R}^2 : x \pm y = 0\}$;

(iii) $K_1 \star K_2$ is (smooth and) not convex, for $\rho < \frac{1}{3}$ or $\rho > 3$.

Observe that for any $K_1,\, K_2,\, K_3 \in \mathcal{S}$ we have:

$$(\varrho_{K_1} \star \varrho_{K_2}) \star \varrho_{K_3} = \varrho_{K_1} \star (\varrho_{K_2} \star \varrho_{K_3}),$$

or equivalently:

$$\varrho_{K_1 \star K_2} \star \varrho_{K_3} = \varrho_{K_1} \star \varrho_{K_2 \star K_3}.$$

This observation leads to the following definition.

Definition 2.6 (Star-shaped combination of m sets). *Given $m \geq 2$ and $K_1, \ldots, K_m \in \mathcal{S}$, we let*

$$\mathop{\star}_{j=1}^{m} \varrho_{K_j}(\nu) := \sqrt{\sum_{j=1}^{m} \left(\varrho_{K_j}(\nu)\right)^2}, \qquad \nu \in \mathbb{S}^{N-1}, \tag{2.4}$$

extended (keeping the same symbol) in a one-homogeneous way on the whole of V, and

$$\mathop{\star}_{j=1}^{m} K_j$$

be the set in $\mathcal{S}$ whose radial function is given by $\mathop{\star}_{j=1}^{m} \varrho_{K_j}$.

Again, note that

$$\partial \left(\mathop{\star}_{j=1}^{m} K_j \right) = \left\{ \sqrt{\sum_{j=1}^{m} \lambda_j^2}\;\nu \;:\; \nu \in \mathbb{S}^{N-1},\ \lambda_j = \varrho_{K_j}(\nu),\ j = 1, \ldots, m \right\}.$$

Problem 2.7. An open problem is to characterize those sets in $\mathcal{S}$ obtained as star-shaped combination of m symmetric convex bodies[12], more precisely to characterize the class

$$\left\{ \mathop{\star}_{j=1}^{m} K_j \;:\; K_1, \cdots, K_m \text{ smooth symmetric uniformly convex bodies} \right\}. \tag{2.5}$$

[12] In [7] some necessary conditions are given in the case $m = 2$, such as the impossibility of cusps or re-entrant corners in $\partial(K_1 \star K_2)$.

Remark 2.8. From (2.2) and (2.4), it follows the formula

$$\left(\phi_{\underset{j=1}{\overset{m}{\star}} K_j}(\nu)\right)^2 = \left(\sum_{j=1}^{m} \frac{1}{\left(\phi_{K_j}(\nu)\right)^2}\right)^{-1}, \qquad \nu \in \mathbb{S}^{N-1}, \tag{2.6}$$

extended (keeping the same symbol) in a one-homogeneous way on the whole of V.

Definition 2.9 (Combined anisotropy). *The function*

$$\phi_{\underset{j=1}{\overset{m}{\star}} K_j}$$

will be called the star-shaped combination *of $\phi_{K_1}, \ldots, \phi_{K_m}$, or combined anisotropy for short.*

According to (2.6), the star-shaped combination of the star-shaped anisotropies $\phi_1, \ldots, \phi_m : V \to [0, +\infty)$ is defined as:

$$\underset{j=1}{\overset{m}{\star}} \phi_j := \left(\sum_{j=1}^{m} \frac{1}{\phi_j^2}\right)^{-1/2}. \tag{2.7}$$

2.1 On the hessian of the combined anisotropy

Let $\Phi : (\mathbb{R}^N)^* \to [0, +\infty)$ be the star-shaped combination of the star-shaped anisotropies $\phi_1, \ldots, \phi_m : (\mathbb{R}^N)^* \to [0, +\infty)$. Set for notational convenience

$$\alpha := \Phi^2, \qquad \alpha_j := \phi_j^2, \quad j = 1, \ldots, m.$$

Then formula (2.7) can be rewritten as

$$\alpha = \left(\sum_{j=1}^{m} \frac{1}{\alpha_j}\right)^{-1}. \tag{2.8}$$

The aim of this short section is to find an appropriate representation of the hessian

$$\frac{1}{2}\nabla^2 \alpha$$

of α, which will be useful in Section 5. From formula (2.8) it follows:

$$\nabla \alpha = \alpha^2 \sum_{j=1}^{m} \frac{1}{\alpha_j^2} \nabla \alpha_j. \tag{2.9}$$

Set

$$Q := \frac{1}{2}\alpha^2 \sum_{j=1}^{m} \frac{1}{\alpha_j^2} \nabla^2 \alpha_j, \tag{2.10}$$

and

$$Q_0 := \frac{1}{2}\nabla^2\alpha - Q.$$

From (2.9), we obtain

$$\begin{aligned} Q_0 &= \alpha^3 \left(\sum_{j=1}^m \frac{\nabla\alpha_j}{\alpha_j^2}\right) \otimes \left(\sum_{k=1}^m \frac{\nabla\alpha_k}{\alpha_k^2}\right) - \alpha^2 \sum_{k=1}^m \frac{\nabla\alpha_k \otimes \nabla\alpha_k}{\alpha_k^3} \\ &= \sum_{k=1}^m \left(\frac{\alpha^3}{\alpha_k^4} - \frac{\alpha^2}{\alpha_k^3}\right) \nabla\alpha_k \otimes \nabla\alpha_k + \sum_{j\neq k} \frac{\alpha^3}{\alpha_j^2\alpha_k^2} \nabla\alpha_j \otimes \nabla\alpha_k \qquad (2.11) \\ &= \alpha^2 \sum_{k=1}^m \frac{\alpha - \alpha_k}{\alpha_k^4} \nabla\alpha_k \otimes \nabla\alpha_k + \alpha^3 \sum_{j\neq k} \frac{1}{\alpha_j^2\alpha_k^2} \nabla\alpha_j \otimes \nabla\alpha_k. \end{aligned}$$

For $m = 2$, formulas (2.10) and (2.11) coincide with those given in [7]. Furthermore, we can observe that, as in the case $m = 2$, we have:

$$Q_0(\xi^*)\xi^* = 0, \qquad \xi^* \in (\mathbb{R}^N)^*. \qquad (2.12)$$

This relation will be used in the asymptotics, see Section 5.2.4. In order to show (2.12) we use Euler's formula $\nabla\alpha_j(\xi^*)\xi^* = 2\alpha_j(\xi^*)$. We have

$$\begin{aligned} \frac{1}{2}Q_0(\xi^*)\xi^* =& \alpha^2(\xi^*) \sum_{k=1}^m \frac{\alpha(\xi^*) - \alpha_k(\xi^*)}{(\alpha_k(\xi^*))^4} \alpha_k(\xi^*)\nabla\alpha_k(\xi^*) \\ &+ \alpha^3(\xi^*) \sum_{j\neq k} \frac{1}{(\alpha_j(\xi^*))^2(\alpha_k(\xi^*))^2} \alpha_j(\xi^*)\nabla\alpha_k(\xi^*) \\ =& \sum_{k=1}^m \left[\frac{\alpha^2(\xi^*)\Big(\alpha(\xi^*) - \alpha_k(\xi^*)\Big)}{(\alpha_k(\xi^*))^3} + \frac{\alpha^3(\xi^*)}{(\alpha_k(\xi^*))^2} \sum_{j\neq k} \frac{1}{\alpha_j(\xi^*)}\right] \nabla\alpha_k(\xi^*), \end{aligned}$$

and each terms in the summation leads (recalling (2.8) and omitting the symbol ξ^*) to:

$$\frac{\alpha^2}{\alpha_k^2} \left[\frac{\alpha - \alpha_k}{\alpha_k} + \alpha\Big(\frac{1}{\alpha} - \frac{1}{\alpha_k}\Big)\right] = 0.$$

Using (2.10) and (2.11) we have therefore obtained a representation for

$$\frac{1}{2}\nabla^2\alpha = Q + Q_0.$$

3 The bidomain model

Before starting our analysis on the multidomain model, we briefly summarize some known results on the bidomain model (1.3), i.e., $m = 2$.

Remark 3.1. System (1.3) is equivalent to the following parabolic/elliptic system:

$$\begin{cases} \epsilon\partial_t u - \epsilon \mathrm{div}(T_{\phi_{\mathrm{i}}}(x, \nabla w(x)) + \dfrac{1}{\epsilon} f(u) = 0, \\ \mathrm{div}\Big(T_{\phi_{\mathrm{i}}}(x, \nabla w) + T_{\phi_{\mathrm{e}}}(x, \nabla w - \nabla u)\Big) = 0, \end{cases} \tag{3.1}$$

obtained by taking the difference of the two equations in (1.3), and setting

$$u = u_{\mathrm{i}} - u_{\mathrm{e}}, \qquad w = u_{\mathrm{i}}.$$

Note that, in the linear case, the elliptic equation can be rewritten as

$$-\mathrm{div}\Big(T_{\phi_{\mathrm{e}}}(x, \nabla u)\Big) + \mathrm{div}\Big((T_{\phi_{\mathrm{i}}} + T_{\phi_{\mathrm{e}}})(x, \nabla w)\Big) = 0.$$

Remark 3.2 (Degenerate variational structure). Let F be the primitive of f vanishing in $s_{\pm}$. System (1.3) is the formal gradient flow of the functionals $\mathcal{F}_\epsilon \colon L^2(\Omega; \mathbb{R}^2) \to [0, +\infty]$ defined as:

$$\mathcal{F}_\epsilon\big(v, \omega\big) := \begin{cases} \displaystyle\int_\Omega \left\{ \frac{\epsilon}{2}\Big[\alpha_{\mathrm{i}}(\nabla v) + \alpha_{\mathrm{e}}(\nabla \omega)\Big] + \frac{1}{\epsilon} F(v - \omega) \right\} \, dx & \text{if } v, \omega \in H^1(\Omega), \\ +\infty \qquad \text{otherwise,} \end{cases} \tag{3.2}$$

with respect to the degenerate scalar product

$$b\Big((v, \omega), (\psi_1, \psi_2)\Big) := \int_\Omega (v - \omega)(\psi_1 - \psi_2) \, dx.$$

Thus, system (1.3) can be reformulated as:

$$\epsilon b\Big(\partial_t (u_{\mathrm{i}}, u_{\mathrm{e}}), (\psi_1, \psi_2)\Big) + \delta\mathcal{F}_\epsilon\Big((u_{\mathrm{i}}, u_{\mathrm{e}}), (\psi_1, \psi_2)\Big) = 0, \quad (\psi_1, \psi_2) \in H^1(\Omega; \mathbb{R}^2).$$

The following result is proven in [11, Theorem 2], to which we refer for more details.

Theorem 3.3 (Well-posedness in the linear case). *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Suppose that*

$$\phi_{\mathrm{i}}, \phi_{\mathrm{e}} : \Omega \times (\mathbb{R}^N)^* \to [0, +\infty) \text{ *are two convex* linear *anisotropies*.}$$

Let $T > 0$ and $\overline{u} \in L^2(\Omega)$. Then there exists a pair

$$(u_{\mathrm{i}}, u_{\mathrm{e}}) \in (L^2\big(0, T; H^1(\Omega)\big))^2,$$

uniquely determined up to a family of additive time-dependent constants, with

$$u := u_{\mathrm{i}} - u_{\mathrm{e}} \in C^0\big([0, T]; L^2(\Omega)\big) \cap L^2\big(0, T; H^1(\Omega)\big), \qquad \partial_t u \in L^2_{\mathrm{loc}}\big([0, T]; L^2(\Omega)\big),$$

such that $(u_{\mathrm{i}}, u_{\mathrm{e}})$ solves system (1.1) distributionally, with initial condition

$$u(0, \cdot) = \overline{u} \text{ in } \Omega, \tag{3.3}$$

and zero Neumann boundary condition

$$T_{\phi_{\mathrm{i}}}(x, \nabla u_{\mathrm{i}}(x)) \cdot \nu_\Omega(x) = T_{\phi_{\mathrm{e}}}(x, \nabla u_{\mathrm{e}}(x)) \cdot \nu_\Omega(x) = 0, \qquad (t, x) \in [0, T] \times \Omega, \tag{3.4}$$

where $\nu_\Omega(x)$ stands for the inward unit vector normal to $\partial\Omega$ at point $x \in \partial\Omega$.

The initial and boundary conditions (3.3), (3.4) are better understood remembering Remark 3.1.

Problem 3.4. To our best knowledge, a well-posedness result for the nonlinear bidomain model (1.3) (even for $\phi_{\rm i}, \phi_{\rm e}$ independent of x), coupled with (3.3) and (3.4), is an open problem, and it is under investigation.

The next *formal* result is obtained in [4], using an asymptotic expansion argument, developed up to the second order included.

Theorem 3.5 (Formal convergence in the linear case). *Suppose that*

$$\phi_{\rm i}, \phi_{\rm e} : \Omega \times (\mathbb{R}^N)^* \to [0, +\infty) \text{ are two convex linear anisotropies.}$$

Let $u_{\rm i}, u_{\rm e}$ and $u = u_\epsilon := u_{\rm i} - u_{\rm e}$ be given by Theorem 3.3, with initial condition $\overline{u} = \overline{u}_\epsilon = u_\epsilon(0, \cdot)$ well-prepared[13] *and possibly depending on ϵ, in particular so that*

$$\{x \in \Omega : \overline{u}_\epsilon(x) = 0\} = \partial E, \qquad \epsilon \in (0, 1),$$

where ∂E is smooth and compact in Ω.
Suppose furthermore that the combined anisotropy

$$\Phi = \phi_{\rm i} \star \phi_{\rm e} \text{ is uniformly convex.}$$

Then, for any $t \in [0, T]$ the sets $\{u_\epsilon(t, \cdot) = 0\}$ formally converge[14]*, as $\epsilon \to 0^+$, to a hypersurface $\partial E(t)$ evolving by anisotropic Φ^o-mean curvature for $T > 0$ sufficiently small, with $\partial E(0) = \partial E$.*

Theorem 3.5 is related to the following result, obtained in [1].

Theorem 3.6 (Γ-convergence in the linear case). *Suppose that*

$$\phi_{\rm i}, \phi_{\rm e} : \Omega \times (\mathbb{R}^N)^* \to [0, +\infty) \text{ are two convex linear anisotropies.}$$

Then

- *there exists the $\Gamma\big(L^2(\Omega; \mathbb{R}^2)\big) - \lim_{\epsilon \to 0^+} \mathcal{F}_\epsilon = \mathcal{F}$, and depends only on $\mathrm{u} = v - \omega$.*
- *$\mathcal{F}$ is finite if and only if $\mathrm{u} \in BV(\Omega; \{s_\pm\})$. Moreover*

$$\mathcal{F}(v, \omega) = \int_{S_{\rm u}} \sigma(x, \nu_{\rm u}) \, d\mathcal{H}^{N-1}, \tag{3.5}$$

  where $S_{\rm u}$ is the jump set of u, $\nu_{\rm u}(x)$ is a unit normal to $S_{\rm u}$ at $x \in S_{\rm u}$, and σ is a convex symmetric anisotropy[15]*.*

In addition (assuming for simplicity that $\phi_{\rm i}$ and $\phi_{\rm e}$, and hence σ, are independent of x)

- *$\{\sigma(\cdot) \le 1\}$ contains the convexified of $\{\phi_{\rm i} \star \phi_{\rm e} \le 1\}$,*
- *$\{\sigma(\cdot) \le 1\}$ is contained in the smallest ellipsoid circumscribing the convexified of $\{\phi_{\rm i} \star \phi_{\rm e} \le 1\}$ and tangent to it at the intersection with the coordinate axes. Moreover, the strict inclusion holds whenever the two anisotropies are not proportional.*

[13] See [4] for the details.
[14] With an expected speed rate of order ϵ, up to logarithmic corrections.
[15] It is also possible to explicitly characterize $\sigma(x, \cdot)$ as an infimum of an appropriate class of vector-valued functions, see [1] for the details.

The following problem has been pointed out in [1].

Problem 3.7. Is it true that the unit ball of σ coincides with the convexified of $\{\phi_{\rm i} \star \phi_{\rm e} \leq 1\}$?

Problem 3.8. To our best knowledge, in the nonlinear case a Γ-convergence result similar to the one in Theorem 3.6 is an open problem, which is under investigation.

The following *formal* result, generalizing Theorem 3.5, is obtained in [7], using an asymptotic expansion argument, developed up to the first order.

Theorem 3.9. *Theorem 3.5 holds when $\phi_{\rm i}$ and $\phi_{\rm e}$ are two smooth symmetric uniformly convex anisotropies, namely dropping the linearity assumption on $T_{\phi_{\rm i}}$ and $T_{\phi_{\rm e}}$.*

Remark 3.10. Set $w^1 = u_{\rm i}$, $w^2 = -u_{\rm e}$, so that $u := u_{\rm i} - u_{\rm e} = w^1 + w^2$ and $u_{\rm e} = -w^2$. Let also

$$T_{\phi_1} := T_{\phi_{\rm i}}, \qquad T_{\phi_2} := T_{\phi_{\rm e}}.$$

Then, observing that $T_{\phi_2}(x, -\xi^*) = -T_{\phi_2}(x, \xi^*)$, we can rewrite system (1.3) as

$$\begin{cases} \epsilon \partial_t u(t,\, x) - \epsilon \mathrm{div}\Big(T_{\phi_1}\big(x,\, \nabla w^1(t,\, x)\big)\Big) + \dfrac{1}{\epsilon} f\big(u(t,\, x)\big) = 0, \\ \\ \epsilon \partial_t u(t,\, x) - \epsilon \mathrm{div}\Big(T_{\phi_2}\big(x,\, \nabla w^2(t,\, x)\big)\Big) + \dfrac{1}{\epsilon} f\big(u(t,\, x)\big) = 0. \end{cases} \tag{3.6}$$

Note that (3.6), in turn, is equivalent to the parabolic/elliptic system

$$\begin{cases} \epsilon \partial_t u(t,\, x) - \epsilon \mathrm{div}\Big(T_{\phi_1}\big(x,\, \nabla w^1(t,\, x)\big)\Big) + \dfrac{1}{\epsilon} f\big(u(t,\, x)\big) = 0, \\ \\ \mathrm{div}\Big(T_{\phi_1}\big(x,\, \nabla w^1(t,\, x)\big)\Big) = \mathrm{div}\Big(T_{\phi_2}\big(x,\, \nabla w^2(t,\, x)\big)\Big). \end{cases} \tag{3.7}$$

This observation will be the starting point of the asymptotic analysis of Section 5.

4 The nonlinear multidomain model

We come now to the main topic of this paper. First of all, in order to treat an arbitrary number m of components, it seems convenient to rewrite the system in a slightly different way[16] (which is the generalization of (3.6)), showing also more clearly the parabolic character of the problem.

Accordingly, let $m \geq 2$, $\phi_1, \ldots, \phi_m : (\mathbb{R}^N)^* \to [0, +\infty)$ be smooth symmetric uniformly convex anisotropies, and consider the degenerate system of parabolic PDE's:

$$\begin{cases} \epsilon \partial_t u - \epsilon \mathrm{div}\left(T_{\phi_r}\big(\nabla w^r\big)\right) + \dfrac{1}{\epsilon} f\big(u\big) = 0, \qquad r = 1, \ldots, m, \\ \\ u := \displaystyle\sum_{r=1}^{m} w^r, \end{cases} \qquad \text{in } (0,T) \times \Omega, \tag{4.1}$$

[16]This, for $m = 2$, corresponds to write $w^1 = u_{\rm i}$, $w^2 = -u_{\rm e}$.

in the unknown $(w^1,\dots,w^m) \in (H^1([0,T];\Omega))^m$, where $T_{\phi_r} := \frac{1}{2}\nabla_{\xi^*}\phi_r^2$ is allowed to be nonlinear, $r = 1,\dots,m$ (no summation on the index r is obviously understood in (4.1)).
Our aim is to formally show that, in the limit $\epsilon \to 0^+$, solutions to (4.1) suitably approximate a Φ^o-anisotropic motion by mean curvature, where Φ is the star-shaped combination of the ϕ_r's, under the assumption that Φ is smooth and uniformly convex. We will assume existence of sufficiently smooth solutions to (4.1) (however, recall that even in the case $m = 2$, this is an open problem, see Problem 3.4).

Remark 4.1 (Simplest possible case). Assume that there exists a smooth symmetric uniformly convex anisotropy ϕ such that

$$\text{for any } r = 1,\dots,m \text{ there exists } \lambda_r > 0 \text{ so that } \phi_r = \lambda_r\phi.$$

If we put $T_\phi := \frac{1}{2}\nabla\phi^2$, system (4.1) can be rewritten as

$$\begin{cases} \epsilon\partial_t u - \epsilon\lambda_r^2 \mathrm{div}\Big(T_\phi\big(\nabla w^r\big)\Big) + \dfrac{1}{\epsilon}f\big(u\big) = 0, \quad r = 1,\dots,m, \\ \\ u = \displaystyle\sum_{r=1}^m w^r. \end{cases} \tag{4.2}$$

Suppose also that ϕ is a linear anisotropy, so that $\mathrm{div}\,(T_\phi(\nabla u)) = \sum_{r=1}^m \mathrm{div}\,(T_\phi(\nabla w^r))$. Dividing each parabolic equation in (4.1) by λ_r^2, summing over $r = 1,\dots, m$ and dividing by $\sum_{r=1}^m \frac{1}{\lambda_r^2}$, we obtain

$$\epsilon\partial_t u - \epsilon\left(\sum_{r=1}^m \frac{1}{\lambda_r^2}\right)^{-1} \mathrm{div}\Big(T_\phi\big(\nabla u\big)\Big) + \frac{1}{\epsilon}f(u) = 0. \tag{4.3}$$

Hence, by formula (2.7) it follows that u satisfies the scalar Allen-Cahn's equation where we take as anisotropy the star-shaped combination Φ of the original anisotropies, namely

$$\epsilon\partial_t u - \epsilon\mathrm{div}\,(T_\Phi(\nabla u)) + \frac{1}{\epsilon}f(u) = 0 \tag{4.4}$$

where as usual $T_\Phi := \frac{1}{2}\nabla\Phi^2$. Under the previous assumptions, we summarize this more precisely as follows. Let be given suitable functions $\overline{u}$ on $\{0\}\times\Omega$ and $d^1,\dots,d^m$ on $[0,T]\times\partial\Omega$, so that $\overline{u} = \sum_{r=1}^m d^r$ on $\{0\}\times\partial\Omega$. If $(w^1,\dots,w^m)$ solve (4.2) with an initial condition $\sum_{r=1}^m w^r = \overline{u}$ and m (Dirichlet) boundary conditions $w^r = d^r$, for $r = 1,\dots,m$, then $u := \sum_{r=1}^m w^r$ solves (4.4), with initial condition $u = \overline{u}$ and Dirichlet boundary condition $u = \sum_{r=1}^m d^r$.
Conversely, we can solve (4.2) with initial condition $u = \overline{u}$ and m (Dirichlet) boundary conditions $w^r = d^r$, for $r = 1,\dots,m$ by first solving the parabolic equation (4.4) (with boundary condition given by $u = \sum_{r=1}^m d^r$) and subsequently solving the first $m-1$ linear elliptic equations at each time t to recover the unknowns $w^1,\dots,w^{m-1}$, and hence also the last one $w^m := u - \sum_{r=1}^{m-1} w^r$. These elliptic equations are obtained by subtracting the first equation in (4.2) from (4.4) and read as

$$\lambda_r^2\mathrm{div}\Big(T_\phi(\nabla w^r)\Big) = \mathrm{div}\Big(T_\Phi(\nabla u)\Big), \tag{4.5}$$

with (Dirichlet) boundary condition $w^r = d^r$.

In the special case

$$\lambda_r^2 d^r = \lambda_s^2 d^s, \qquad r, s = 1, \ldots, m \tag{4.6}$$

or equivalently

$$\lambda_r^2 \left(\sum_{s=1}^m \frac{1}{\lambda_s^2} \right) d^r = \sum_{s=1}^m d^s, \tag{4.7}$$

we can recover the unknowns w^r as

$$w^r := \frac{1}{\lambda_r^2} \left(\sum_{s=1}^m \frac{1}{\lambda_s^2} \right)^{-1} u, \qquad r = 1, \ldots, m.$$

Remark 4.2. Generalizing the previous cases ($m = 2$), one can transform (4.1) into a parabolic equation and $(m-1)$ elliptic equations. This suggests a way to assign initial/boundary conditions for (4.1), in the form of one initial condition, and m Neumann or Dirichlet boundary conditions.

5 Formal asymptotics of the multidomain model

In this section we perform a new formal asymptotic expansion for the nonlinear multidomain model, assuming for simplicity $f(s) = \frac{d}{ds}\left((1-s^2)^2\right)$, in particular $s_\pm = \pm 1$. The computations will be simpler, and at the same time more general[17], than those made in [7]. Due to the strong reaction term, we expect the sum $u_\epsilon := \sum_{r=1}^m w_\epsilon^r$ to assume values near to ± 1 in most of the domain with a thin, smooth, transition region where it transversally crosses the unstable zero of f. We will denote by $\Omega^\pm = \cup_{t=0}^T (\{t\} \times \Omega^\pm(t))$ the two phases. This motivates the use of matched asymptotics in the outer $\Omega^- \cup \Omega^+$ region (*outer expansion*) and in the transition layer (*inner expansion*).
As a formal consequence (see (5.54) below), the front generated by (4.1) propagates with the same law, up to an error of order $\mathcal{O}(\epsilon)$, as the front generated by a Φ^o-anisotropic mean curvature flow starting from a smooth hypersurface $\partial E \subset \Omega$, where Φ is the star-shaped combination of the m original smooth uniformly convex[18] anisotropies $\phi_1, \ldots, \phi_m$.
Remembering Remark 3.10, assuming independence of x of all ϕ_r, we write the system in the convenient form

$$\begin{cases} \epsilon \partial_t u_\epsilon - \epsilon \mathrm{div}\Big(T_{\phi_1}(\nabla w_\epsilon^1)\Big) + \dfrac{1}{\epsilon} f(u_\epsilon) = 0, \\[2ex] \mathrm{div}\Big(T_{\phi_r}(\nabla w_\epsilon^r)\Big) = \mathrm{div}\Big(T_{\phi_s}(\nabla w_\epsilon^s)\Big), \qquad 1 \le r,\, s \le m, \\[2ex] u_\epsilon = \displaystyle\sum_{r=1}^m w_\epsilon^r. \end{cases} \tag{5.1}$$

[17] This will be apparent particularly in the inner expansion of Section 5.2 below.
[18] Convexity of all ϕ_r is necessary in order to make the multidomain model well-posed.

This system consists of one parabolic equation and $(m-1)$ elliptic equations, to be coupled with an initial condition at $\{t=0\}$, which in particular is required to satisfy

$$\{u_\epsilon(0,\cdot)=0\}=\partial E, \qquad \epsilon\in(0,1), \tag{5.2}$$

and m either Neumann or Dirichlet boundary conditions at $\cup_{t=0}^{T}(\{t\}\times\partial\Omega)$. We restore in this section the notational dependence on ϵ for $u=u_\epsilon$ and all $w^r=w^r_\epsilon$.

5.1 Outer expansion

Given $r=1,\dots,m$, we expand formally u_ϵ and w^r_ϵ in terms of $\epsilon\in(0,1)$:

$$u_\epsilon=u_0+\epsilon u_1+\epsilon^2 u_2+\dots, \qquad w^r_\epsilon=w^r_0+\epsilon w^r_1+\epsilon^2 w^r_2+\dots$$

Substituting these expressions into the parabolic equation in (5.1) and using the expansion

$$f(u_\epsilon)=f(u_0)+\epsilon f'(u_0)u_1+\epsilon^2\left(\frac{u_1^2 f''(u_0)}{2}+f'(u_0)u_2\right)+\mathcal{O}(\epsilon^3),$$

we get

$$f(u_0)=0, \qquad u_1 f'(u_0)=0.$$

Hence, excluding $u_0=0$ (the unstable zero of f), we get in $(0,T)\times\Omega$,

$$u_0\in\{1,-1\}, \tag{5.3}$$

$$u_1\equiv 0. \tag{5.4}$$

We denote by

$$\Sigma_0(t), \qquad t\in(0,T), \tag{5.5}$$

the jump set of $u_0(t,\cdot)$.
Coming back to the elliptic equations in (5.1), we find

$$\begin{cases} \operatorname{div}\Big(T_{\phi_r}(\nabla w^r_0)\Big)=\operatorname{div}\Big(T_{\phi_s}(\nabla w^s_0)\Big) \qquad 1\le r,\ s\le m \\[2ex] \displaystyle\sum_{r=1}^m w^r_0=u_0 \implies \sum_{r=1}^m \nabla w^r_0=0, \end{cases} \tag{5.6}$$

where the last implication is a consequence of (5.3).
Note also that

$$u_2=\frac{1}{f'(u_0)}\operatorname{div}\Big(T_{\phi_r}(\nabla w^r_0)\Big), \qquad r=1,\dots,m. \tag{5.7}$$

Remark 5.1. (5.6) is a system of $(m-1)$ *nonlinear* elliptic equations in the $(m-1)$ unknown functions w^r_0 (for $r=2,\dots,m$), since we can solve the algebraic equation in (5.6) with respect to w^1_0.

Remark 5.2. It is important to notice that the boundary conditions across the limit interface $\Sigma_0(t)$, to be coupled with (5.6), will arise by matching the outer expansion with the inner expansion, see (5.65) and (5.68) (jump conditions and jump of the normal derivative). We assume the elliptic problem expressed by (5.6), (5.65), (5.68) (and augmented with Neumann or Dirichlet boundary conditions on $\partial\Omega$) to be solvable, thus providing w_0^r for every $r = 1, \ldots, m$, and therefore u_2 by (5.7).

If we now perform a Taylor expansion for T_{ϕ_r}, we obtain

$$T_{\phi_r}(\eta^* + \epsilon\zeta^*) = T_{\phi_r}(\eta^*) + \epsilon M^r(\eta^*)\zeta^* + \mathcal{O}(\epsilon^2),$$

where $M^r = \frac{1}{2}\nabla^2\alpha_r$, which can be used in the elliptic equations of (5.1) to get equations for w_1^r for any $r = 1, \ldots, m$, namely:

$$\mathrm{div}\Big(M^r(\nabla w_0^r)\nabla w_1^r\Big) = \mathrm{div}\Big(M^s(\nabla w_0^s)\nabla w_1^s\Big), \qquad 1 \le r,\, s \le m.$$

Moreover, from the relation $\sum_{r=1}^m w_\epsilon^r = u_\epsilon$, and recalling from (5.4) that $u_1 = 0$, we obtain

$$\sum_{r=1}^{m} w_1^r = 0. \tag{5.8}$$

By solving this latter algebraic equation with respect (for instance) to w_1^1, and substituting it into the previous equation we obtain a system of $(m-1)$ *linear* elliptic equations in the unknowns w_1^r, for $r = 2, \ldots, m$.

Remark 5.3. The outer expansion has been performed without assuming Φ to be convex.

5.2 Inner expansion

For any $\epsilon \in (0, 1)$ let us consider the set

$$E_\epsilon(t) := \{x \in \Omega \,:\, u_\epsilon(t, x) \ge 0\},$$

the boundary of which will be denoted by

$$\Sigma_\epsilon(t) = \{x \in \Omega \,:\, u_\epsilon(t, x) = 0\}. \tag{5.9}$$

Our aim is to formally identify the geometric evolution law of $\Sigma_\epsilon(t)$ as $\epsilon \to 0^+$.
For $r = 1, \ldots, m$ we seek the shape, in the transition layer, of functions w_ϵ^r satisfying

$$\epsilon^2\partial_t u_\epsilon - \epsilon^2\mathrm{div}\Big(T_{\phi_r}(\nabla w_\epsilon^r)\Big) + f(u_\epsilon) = 0, \qquad r = 1, \ldots, m, \tag{5.10}$$

with $u_\epsilon = \displaystyle\sum_{r=1}^{m} w_\epsilon^r$. We put, as usual,

$$\alpha_r := \phi_r^2, \qquad T_{\phi_r} := \frac{1}{2}\nabla\alpha_r, \qquad M^r := \frac{1}{2}\nabla^2\alpha_r, \qquad r = 1, \ldots, m,$$

so that, by Euler's identities for homogeneus functions, we have

$$\alpha_r(\xi^*) = T_{\phi_r}(\xi^*)\cdot\xi^* = M^r(\xi^*)\xi^*\cdot\xi^*, \qquad \xi^* \in (\mathbb{R}^N)^*. \tag{5.11}$$

Remember that the matrix M^r depends on the covector ξ^*, unless ϕ_r is a linear anisotropy (i.e., unless T_{ϕ_r} is linear).

5.2.1 Main assumptions and basic notation

We assume in this section that

the star shaped combination Φ^2 is smooth, symmetric and uniformly convex.

This allows to look at Φ as the dual of a smooth uniformly convex anisotropy φ defined in $\mathbb{R}^N$,

$$\Phi = \varphi^o, \qquad \text{namely} \qquad \varphi = \Phi^o. \tag{5.12}$$

Keeping the simpler symbol φ instead of Φ^o, we can accordingly introduce the φ-anisotropic distance d_φ (i.e., $d_\varphi(x,y) = \varphi(y-x)$), and the φ-signed distance function from $\Sigma_\epsilon(t)$ (positive in the interior of $E_\epsilon(t)$):

$$d_\epsilon^\varphi(t,x) := d_\varphi\big(x,\, \mathbb{R}^N \setminus E_\epsilon(t)\big) - d_\varphi\big(x,\, E_\epsilon(t)\big).$$

Following [4], it is convenient to introduce the stretched variable y defined as

$$y = y_\epsilon^\varphi(t,x) := \frac{d_\epsilon^\varphi(t,x)}{\epsilon}.$$

We parametrize $\Sigma_\epsilon(t)$ with a parameter

$$s \in \Sigma, \tag{5.13}$$

Σ a fixed reference $(N-1)$-dimensional smooth manifold, and the function $x(s,\, t;\, \epsilon)$ gives the position in Ω of the point s at time t.
We let, for x in a tubular neighbourhood of $\Sigma_\epsilon(t)$,

$$n_\epsilon^\varphi(t,x) := -T_\Phi(\nabla d_\epsilon^\varphi(t,x)) \tag{5.14}$$

be the (outward) Cahn-Hoffman's vector field (remember the notation in (5.12)), for which we suppose the expansion:

$$n_\epsilon^\varphi := n_0^\varphi + \epsilon n_1^\varphi + \dots$$

Points on the evolving manifold $\Sigma_\epsilon(t)$ are assumed to move in the direction of n_ϵ^φ, i.e.,

$$\partial_t x(s,\, t;\, \epsilon) = V_\epsilon^\varphi \; n_\epsilon^\varphi,$$

where V_ϵ^φ is positive for an expanding set, and where we assume the validity of the following expansion:

$$V_\epsilon^\varphi = V_0^\varphi + \epsilon V_1^\varphi + \epsilon^2 V_2^\varphi + \dots \tag{5.15}$$

The anisotropic projection of a point x on $\Sigma_\epsilon(t)$ will be denoted by $s_\epsilon^\varphi(t,x)$, which satisfies

$$\partial_t s_\epsilon^\varphi = 0. \tag{5.16}$$

Hence

$$\partial_t d_\epsilon^\varphi(t,x) = V_\epsilon^\varphi\big(s_\epsilon^\varphi(t,x),\, t\big). \tag{5.17}$$

We also recall (see [8, 6]) that $\mathrm{div}\,(T_\Phi(\nabla d_\epsilon^\varphi))$ gives the anisotropic mean curvature of the level hypersurface at that point and can be approximated by the anisotropic mean curvature κ_ϵ^φ of $\Sigma_\epsilon(t)$ (positive when $E_\epsilon(t)$ is uniformly convex) as follows

$$\mathrm{div}\,(T_\Phi(\nabla d_\epsilon^\varphi(t,x))) = -\kappa_\epsilon^\varphi(s_\epsilon^\varphi(t,x),t) - \epsilon y_\epsilon^\varphi h_\epsilon^\varphi(s_\epsilon^\varphi(t,x),t) + \mathcal{O}(\epsilon^2(y_\epsilon^\varphi)^2) \tag{5.18}$$

for a suitable h_ϵ^φ depending on the local shape of $\Sigma_\epsilon(t)$. We assume the expansions

$$\kappa_\epsilon^\varphi = \kappa_0^\varphi + \epsilon\kappa_1^\varphi + \mathcal{O}(\epsilon^2), \qquad h_\epsilon^\varphi = h_0^\varphi + \mathcal{O}(\epsilon). \tag{5.19}$$

With abuse of notation, for a given ϵ, we let $x(y;\, s,\, t)$ be the point of Ω having signed distance ϵy and projection s on $\Sigma_\epsilon(t)$. We have

$$x(y;\, s,\, t) = x(s,\, t) - \epsilon y n_\epsilon^\varphi + \mathcal{O}(\epsilon^2 y^2). \tag{5.20}$$

For a given ϵ, the triplet $(y;\, s,\, t)$ will parametrize a tubular neighbourhood of $\cup_{t\in(0,T)}(\{t\}\times\Sigma_\epsilon(t))$. We look for functions $U_\epsilon(y;\, s,\, t)$ and $W_\epsilon^r(y;\, s,\, t,\, x)$ $(r=1,\ldots,m)$ respectively so that

$$u_\epsilon(t,x) = U_\epsilon\left(\frac{d_\epsilon^\varphi(t,x)}{\epsilon}, s_\epsilon^\varphi(t,x), t\right), \tag{5.21}$$

$$w_\epsilon^r(t,x) = W_\epsilon^r\left(\frac{d_\epsilon^\varphi(t,x)}{\epsilon}, s_\epsilon^\varphi(t,x), t, x\right), \qquad r=1,\ldots,m, \tag{5.22}$$

with

$$\sum_{r=1}^m W_\epsilon^r = U_\epsilon. \tag{5.23}$$

Remark 5.4. Formula (5.21) defines uniquely the function U_ϵ, since to evey (t,x) there corresponds uniquely the triplet (y,s,t). This observation does not apply to (5.22), in view of the explicit dependence of the functions W_ϵ^r on x.

We shall write

$$W_\epsilon^r = W_0^r + \epsilon W_{1,\epsilon}^r = W_0^r + \epsilon W_1^r + \epsilon^2 W_{2,\epsilon}^r, \qquad r=1,\ldots,m, \tag{5.24}$$

where W_0^r and W_1^r are allowed to depend explicitly on x (and hence on ϵ). We suppose the remainders $W_{1,\epsilon}^r$, $W_{2,\epsilon}^r$ to be bounded as $\epsilon \to 0^+$.

We let also

$$S^r := \frac{1}{2}\nabla^3\alpha_r = \nabla M^r, \qquad r=1,\ldots,m,$$

be the 3-indices, (-1)-homogeneus completely symmetric tensor given by the third derivatives of $\frac{1}{2}\alpha_r$: in components we have

$$S_{ijk}^r := \nabla_k M_{ij}^r, \qquad r=1,\ldots,m,$$

where $\nabla_k = \frac{\partial}{\partial\xi_k^*}$. Finally, for any $k,j=1,\ldots,N$, we introduce the notation

$$M_{\cdot k}^r := \left(M_{1k}^r \ldots M_{Nk}^r\right), \qquad S_{\cdot jk}^r := \left(S_{1jk}^r \ldots S_{Njk}^r\right), \qquad r=1,\ldots,m.$$

Warning: We will adopt the convention of summation on repeated indices, *with the exception of the index* r, for which the explicit symbol $\sum_{r=1}^m$ will be always used. For instance, in formulas (5.28), (5.32), (5.33), (5.34) and (5.84) below, no summation on r is understood.

5.2.2 Preliminary expansions

Now we begin to Taylor–expand all terms in (5.10). We have, using the convention of summation on repeated indices,

$$\epsilon^2 \partial_t u_\epsilon = \epsilon^2 U_{\epsilon s_\beta}\ \partial_t s^\varphi_{\epsilon\beta} + \epsilon U'_\epsilon\ \partial_t d^\varphi_\epsilon + \epsilon^2 U_{\epsilon t} = \epsilon U'_\epsilon V^\varphi_\epsilon + \epsilon^2 U_{\epsilon t}, \tag{5.25}$$

where we used (5.16) and (5.17).
We write

$$U_\epsilon = U_0 + \epsilon U_{1,\epsilon} = U_0 + \epsilon U_1 + \epsilon^2 U_{2,\epsilon}, \tag{5.26}$$

where we require U_0 and U_1 not to depend on ϵ.
Using Taylor's expansion of the nonlinearity f, we get

$$f(U_\epsilon) = f(U_0) + \epsilon U_{1,\epsilon} f'(U_0) + \frac{1}{2}\epsilon^2 (U_{1,\epsilon})^2 f''(U_0) + \mathcal{O}(\epsilon^3). \tag{5.27}$$

To expand the divergence term, we need some additional work. First of all, by Taylor–expanding the operator T_{ϕ_r}, we get

$$T_{\phi_r}(\eta^* + \epsilon\zeta^*) = T_{\phi_r}(\eta^*) + \epsilon M^r(\eta^*)\zeta^* + \frac{1}{2}\epsilon^2 S^r_{\cdot jk}(\eta^*)\zeta^*_j\zeta^*_k + \mathcal{O}(\epsilon^3),$$

so that, for any $r = 1, \ldots, m$,

$$\begin{aligned}
&\epsilon^2 T_{\phi_r}(\nabla w^r_\epsilon) \\
&= T_{\phi_r}\Big(\epsilon W^{r\prime}_\epsilon\ \nabla d^\varphi_\epsilon + \epsilon^2 W^r_{\epsilon s_\beta}\nabla s^\varphi_{\epsilon\beta} + \epsilon^2 \nabla W^r_\epsilon\Big) \\
&= \epsilon W^{r\prime}_\epsilon\ T_{\phi_r}(\nabla d^\varphi_\epsilon) + \epsilon^2 W^r_{\epsilon s_\beta} M^r(\nabla d^\varphi_\epsilon)\nabla s^\varphi_{\epsilon\beta} + \epsilon^2 M^r(\nabla d^\varphi_\epsilon)\nabla W^r_\epsilon \\
&\quad + \frac{1}{2W^{r\prime}_\epsilon}\epsilon^3 S^r_{\cdot jk}(\nabla d^\varphi_\epsilon)\Big[W^r_{\epsilon s_\beta}\partial_{x_j} s^\varphi_{\epsilon\beta} + \partial_{x_j} W^r_\epsilon\Big]\Big[W^r_{\epsilon s_\beta}\partial_{x_k} s^\varphi_{\epsilon\beta} + \partial_{x_k} W^r_\epsilon\Big] \\
&\quad + \mathcal{O}(\epsilon^4).
\end{aligned} \tag{5.28}$$

Remark 5.5. Since we still have to apply the divergence operator (which produces an extra ϵ^{-1} factor), we need to go through the ϵ^3 term in (5.28). We also observe that the term $\mathcal{O}(\epsilon^4)$ in (5.28) is actually a term of order $\mathcal{O}\left(\epsilon^4 \frac{1}{W^{r\prime 2}_\epsilon}\right)$ which, a posteriori, turns out to be of order $\mathcal{O}(\epsilon^4)$: indeed, from (5.44) below it follows that $W^{r\prime}_\epsilon$ is nonvanishing in the transition layer.

We now recall that by Euler's identities for homogeneous functions we have

$$T_{\phi_r}(\xi^*) = \nabla_i T_{\phi_r}(\xi^*)\xi^*_i, \qquad \xi^* \in (\mathbb{R}^N)^*, \tag{5.29}$$

which implies

$$T_{\phi_r}(\nabla d^\varphi_\epsilon)\cdot\nabla s^\varphi_{\epsilon\beta} = M^r(\nabla d^\varphi_\epsilon)\nabla s^\varphi_{\epsilon\beta}\cdot\nabla d^\varphi_\epsilon, \qquad r = 1, \ldots, m. \tag{5.30}$$

Differentiating (5.29) with respect to ξ^*_k and using the notation $\nabla^2_{ik} = \frac{\partial^2}{\partial\xi^*_k\partial\xi^*_i}$, we also have

$$\nabla^2_{ik} T_{\phi_r}(\xi^*)\xi^*_i = S^r_{\cdot ik}\xi^*_i = 0 \in \mathbb{R}^N, \qquad \xi^* \in (\mathbb{R}^N)^*, \quad k = 1, \ldots, N,$$

which implies

$$S^r_{ijk}(\nabla d^\varphi_\epsilon)\nabla_i d^\varphi_\epsilon = 0, \qquad j,k = 1,\dots,N, \quad r = 1,\dots,m. \tag{5.31}$$

For any $r = 1,\dots,m$, we compute, using (5.30),

$$\begin{aligned}
&\epsilon^2 \mathrm{div}(M^r(\nabla d^\varphi_\epsilon)\nabla W^r_\epsilon) \\
&= \epsilon^2 \partial_{x_i}\left(M^r_{ij}(\nabla d^\varphi_\epsilon) W^r_{\epsilon x_j}\right) \\
&= \epsilon T_{\phi_r}(\nabla d^\varphi_\epsilon)\cdot \nabla W^{r\prime}_\epsilon \\
&\quad + \epsilon^2 W^r_{\epsilon x_j}\mathrm{div}\left(M^r_{\cdot j}(\nabla d^\varphi_\epsilon)\right) + \epsilon^2 M^r(\nabla d^\varphi_\epsilon)\nabla s^\varphi_{\epsilon\beta}\cdot \nabla W^r_{\epsilon s_\beta} \\
&\quad + \epsilon^2 W^r_{\epsilon x_i x_j} M^r_{ij}(\nabla d^\varphi_\epsilon).
\end{aligned} \tag{5.32}$$

By differentiating (5.28) we obtain, using also (5.11),

$$\begin{aligned}
&\epsilon^2 \mathrm{div}\left(T_{\phi_r}(\nabla w^r_\epsilon)\right) \\
&= \alpha_r(\nabla d^\varphi_\epsilon) W^{r\prime\prime}_\epsilon + 2\epsilon W^{r\ \prime}_{\epsilon s_\beta} T_{\phi_r}(\nabla d^\varphi_\epsilon)\cdot \nabla s^\varphi_{\epsilon\beta} \\
&\quad + 2\epsilon\ T_{\phi_r}(\nabla d^\varphi_\epsilon)\cdot \nabla W^{r\prime}_\epsilon + \epsilon W^{r\prime}_\epsilon\ \mathrm{div}(T_{\phi_r}(\nabla d^\varphi_\epsilon)) \\
&\quad + \epsilon^2 W^r_{\epsilon s_\beta s_\delta}\ M^r(\nabla d^\varphi_\epsilon)\nabla s^\varphi_{\epsilon\beta}\cdot \nabla s^\varphi_{\epsilon\delta} + \epsilon^2 M^r(\nabla d^\varphi_\epsilon)\nabla s^\varphi_{\epsilon\beta}\cdot \nabla W^r_{\epsilon s_\beta} \\
&\quad + \epsilon^2 W^r_{\epsilon s_\beta}\mathrm{div}\left(M^r(\nabla d^\varphi_\epsilon)\nabla s^\varphi_{\epsilon\beta}\right) \\
&\quad + \epsilon^2 W^r_{\epsilon x_j}\mathrm{div}\left(M^r_{\cdot j}(\nabla d^\varphi_\epsilon)\right) + \epsilon^2\ M^r(\nabla d^\varphi_\epsilon)\nabla s^\varphi_{\epsilon\beta}\cdot \nabla W^r_{\epsilon s_\beta} \\
&\quad + \epsilon^2 W^r_{\epsilon x_i x_j} M^r_{ij}(\nabla d^\varphi_\epsilon) \\
&\quad + \mathcal{O}(\epsilon^3),
\end{aligned} \tag{5.33}$$

where we notice that no contribution of order larger than $\mathcal{O}(\epsilon^3)$ can come from the $\mathcal{O}(\epsilon^3)$ term in (5.28) — because they can only be produced *via* differentiation with respect to y, which in turn gives rise to a scalar product between $\nabla d^\varphi_\epsilon$ and the tensor $S^r(\nabla d^\varphi_\epsilon)$ (which in the end vanishes, due to Euler's identities (5.31)).
Hence, in terms of U_ϵ and W^r_ϵ, the expansion of the r-th parabolic equation in (5.10), for $r = 1,\dots, m$, reads as, using also (5.25),

$$\begin{aligned}
0 = &-\alpha_r(\nabla d^\varphi_\epsilon) W^{r\prime\prime}_\epsilon + f(U_\epsilon) \\
&+ \epsilon\Big(V^\varphi_\epsilon U'_\epsilon - 2W^{r\prime}_{\epsilon s_\beta} T_{\phi_r}(\nabla d^\varphi_\epsilon)\cdot \nabla s^\varphi_{\epsilon\beta} - 2T_{\phi_r}(\nabla d^\varphi_\epsilon)\cdot \nabla W^{r\prime}_\epsilon - W^{r\prime}_\epsilon \mathrm{div}\left(T_{\phi_r}(\nabla d^\varphi_\epsilon)\right)\Big) \\
&+ \epsilon^2\Big(U_{\epsilon t} - W^r_{\epsilon s_\beta s_\delta} M^r(\nabla d^\varphi_\epsilon)\nabla s^\varphi_{\epsilon\beta}\cdot \nabla s^\varphi_{\epsilon\delta} - 2M^r(\nabla d^\varphi_\epsilon)\nabla s^\varphi_{\epsilon\beta}\cdot \nabla W^r_{\epsilon s_\beta} \\
&\qquad - W^r_{\epsilon s_\beta}\mathrm{div}\Big(M^r(\nabla d^\varphi_\epsilon)\nabla s^\varphi_{\epsilon\beta}\Big) - W^r_{\epsilon x_j}\mathrm{div}\left(M^r_{\cdot j}(\nabla d^\varphi_\epsilon)\right) - W^r_{\epsilon x_i x_j} M^r_{ij}(\nabla d^\varphi_\epsilon)\Big) \\
&+ \mathcal{O}(\epsilon^3).
\end{aligned} \tag{5.34}$$

5.2.3 Order 0

Recall [8] that $\nabla d^\varphi_\epsilon$ satisfies the anisotropic eikonal equation

$$(\Phi(\nabla d^\varphi_\epsilon))^2 = 1 \tag{5.35}$$

in the evolving transition layer.
Assuming the formal expansion

$$d_\epsilon^\varphi = d_0^\varphi + \epsilon d_1^\varphi + \epsilon^2 d_2^\varphi + \mathcal{O}(\epsilon^3), \tag{5.36}$$

where $d_0^\varphi(t,\cdot)$ is the φ-signed distance from $\Sigma_0(t)$ (positive in the interior of $\{u_0(t,\cdot) = 1\}$), equation (5.35) leads to

$$\begin{aligned} 1 &= \Phi^2(\nabla d_0^\varphi) + 2\epsilon T_\Phi(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi \\ &\quad + \epsilon^2 \Big(2T_\Phi(\nabla d_0^\varphi) \cdot \nabla d_2^\varphi + \nabla T_\Phi(\nabla d_0^\varphi) \nabla d_1^\varphi \cdot \nabla d_1^\varphi \Big) + \mathcal{O}(\epsilon^3), \end{aligned} \tag{5.37}$$

which in particular entails:

$$\Phi^2(\nabla d_0^\varphi) = 1, \tag{5.38}$$

$$T_\Phi(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi = 0, \tag{5.39}$$

$$2T_\Phi(\nabla d_0^\varphi) \cdot \nabla d_2^\varphi + \nabla T_\Phi(\nabla d_0^\varphi) \nabla d_1^\varphi \cdot \nabla d_1^\varphi = 0$$

(the latter equation will not be used in what follows).
Using formula (2.7), equation (5.38) reads as

$$\sum_{r=1}^{m} \frac{1}{\alpha_r(\nabla d_0^\varphi(t,x))} = 1, \tag{5.40}$$

again for all x in a suitable tubular neighbourhood of $\Sigma_\epsilon(t)$.

Remark 5.6 (Weights). The quantities

$$\frac{1}{\alpha_r(\nabla d_0^\varphi)}, \qquad r = 1, \dots, m$$

can be used as "weights" to obtain a weighted mean of equations (5.34). This observation will be crucial in the sequel.

Collecting all terms of order zero in ϵ from each parabolic equation (5.34), dividing by $\alpha_r(\nabla d_0^\varphi)$, summing $r = 1, \dots, m$ and using (5.40), we obtain

$$-U_0'' + f(U_0) = 0, \tag{5.41}$$

where we used expansions (5.24), (5.26), (5.27), (5.36) for U_ϵ, W_ϵ^r, $f(U_\epsilon)$, d_ϵ^φ, and we have employed (5.23).
The only admissible solution of (5.41) (see for instance [5, 4]) is the standard standing wave

$$U_0(y,s,t) = \gamma(y), \qquad y \in \mathbb{R}, \tag{5.42}$$

where $\gamma(y) = \mathrm{tgh}(cy)$ (here c is a constant only depending on f); in particular, U_0 does not depend on $(s,\,t)$.
Now we can recover each of the m functions W_0^r, $r = 1, \dots, m$, by substituting $f(U_0) = U_0''$ into (5.34):

$$\alpha_r(\nabla d_0^\varphi) W_0^{r\prime\prime} = U_0'' = \gamma''.$$

Hence

$$W_0^{r\prime\prime} = \frac{1}{\alpha_r(\nabla d_0^\varphi)} U_0'' = \frac{1}{\alpha_r(\nabla d_0^\varphi)} \gamma'', \qquad r = 1, \dots, m. \tag{5.43}$$

We also get by integration[19]

$$W_0^{r\prime} = \frac{1}{\alpha_r(\nabla d_0^\varphi)} U_0' = \frac{1}{\alpha_r(\nabla d_0^\varphi)} \gamma', \qquad r = 1, \dots, m. \tag{5.44}$$

Remark 5.7. The functions $W_0^{r\prime}$ depend explicitly on x (and on t) through the coefficient $\frac{1}{\alpha_r(\nabla d_0^\varphi)}$. They are, on the other hand, independent of s.

5.2.4 Order 1

Let us consider the terms of order ϵ in equations (5.34). To this aim, we use the representation of $\frac{1}{2}\nabla^2\alpha = Q + Q_0$ given in section 2.1 for $\alpha = \Phi^2$, namely

$$Q = \alpha^2 \sum_{r=1}^m \frac{1}{\alpha_r^2} M^r,$$

where

$$Q_0(\xi^*)\xi^* = 0, \qquad \xi^* \in (\mathbb{R}^N)^*. \tag{5.45}$$

Remember that by Euler's identities for homogeneous functions we have

$$T_\Phi(\xi^*) = \frac{1}{2}\nabla^2\alpha(\xi^*)\xi^* = \left(Q(\xi^*) + Q_0(\xi^*)\right)\xi^*, \qquad \xi \in (\mathbb{R}^N)^*.$$

Hence, using (5.45),

$$\begin{aligned} T_\Phi(\nabla d_0^\varphi) =& \Big(Q(\nabla d_0^\varphi) + Q_0(\nabla d_0^\varphi)\Big)\nabla d_0^\varphi \\ =& Q(\nabla d_0^\varphi)\nabla d_0^\varphi = (\alpha(\nabla d_0^\varphi))^2 \sum_{r=1}^m \frac{1}{(\alpha_r(\nabla d_0^\varphi))^2} M^r(\nabla d_0^\varphi)\nabla d_0^\varphi \\ =& \sum_{r=1}^m \frac{1}{(\alpha_r(\nabla d_0^\varphi))^2} M^r(\nabla d_0^\varphi)\nabla d_0^\varphi, \end{aligned} \tag{5.46}$$

where the last equality follows from (5.35).
Therefore

$$\operatorname{div}\left(T_\Phi(\nabla d_0^\varphi)\right) = \sum_{r=1}^m \operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2} T_{\phi_r}(\nabla d_0^\varphi)\right). \tag{5.47}$$

For each $r = 1, \dots, m$, we now collect all terms of order one in (5.34).
Remembering once more that $U_0 = \gamma$ and $W_0^{r\prime}$ do not depend explicitly on s and t so that in particular $W_{0s_\beta}^{r\prime} = 0$, we obtain

$$\begin{aligned} &-\alpha_r(\nabla d_0^\varphi) W_1^{r\prime\prime} - 2W_0^{r\prime\prime} T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi + f'(\gamma)U_1 \\ &+ \gamma' V_0^\varphi - 2T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla W_0^{r\prime} - W_0^{r\prime}\operatorname{div}\left(T_{\phi_r}(\nabla d_0^\varphi)\right) = 0, \end{aligned} \tag{5.48}$$

[19]See Section 5.2.5 below, and in particular equation (5.63).

where we have taken into account that the term

$$-\alpha_r(\nabla d_0^\varphi)W_1^{r\,\prime\prime} - 2W_0^{r\,\prime\prime}T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi + f'(\gamma)U_1$$

arises from the expansion at the order ϵ of the first line on the right hand side of (5.34). Using formula (5.44), equation (5.48) can be rewritten as

$$\begin{aligned}&-\alpha_r(\nabla d_0^\varphi)W_1^{r\,\prime\prime} - 2\gamma''\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{\alpha_r(\nabla d_0^\varphi)} + f'(\gamma)U_1\\&+\gamma' V_0^\varphi - \alpha_r(\nabla d_0^\varphi)\gamma'\left[\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2}\operatorname{div}\left(T_{\phi_r}(\nabla d_0^\varphi)\right) + \frac{2}{\alpha_r(\nabla d_0^\varphi)}T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla\frac{1}{\alpha_r(\nabla d_0^\varphi)}\right] = 0.\end{aligned} \tag{5.49}$$

Since $\nabla\frac{1}{\alpha_r^2} = \frac{2}{\alpha_r}\nabla\frac{1}{\alpha_r}$, the expression in square brackets is simply

$$\operatorname{div}\Big(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2}T_{\phi_r}(\nabla d_0^\varphi)\Big), \qquad r = 1,\ldots,m. \tag{5.50}$$

Recalling (5.47), the sum over $r = 1,\ldots,m$ of the latter divergences gives $\operatorname{div}(T_\Phi(\nabla d_0^\varphi))$. The weighted sum of equations (5.49) finally produces

$$-\mathcal{L}(U_1) = \gamma'\Big[V_0^\varphi - \operatorname{div}\left(T_\Phi(\nabla d_0^\varphi)\right)\Big],$$

where

$$\mathcal{L}(g) := -g'' + f'(\gamma)g,$$

and we make use of (5.39).
Recall now that from (5.18) and the expansions of κ_ϵ^φ it follows

$$\operatorname{div}\left(T_\Phi(\nabla d_\epsilon^\varphi)\right) = -\kappa_0^\varphi - \epsilon\kappa_1^\varphi - \epsilon y h_0^\varphi + \mathcal{O}(\epsilon^2 y^2), \tag{5.51}$$

in particular

$$\operatorname{div}\left(T_\Phi(\nabla d_0^\varphi)\right) = -\kappa_0^\varphi.$$

We then obtain

$$-\mathcal{L}(U_1) = \gamma'\Big[V_0^\varphi + \kappa_0^\varphi\Big]. \tag{5.52}$$

We recall now from [5, 4, 3] that for equation $-\mathcal{L}(g) = v$ to be solvable, we must enforce the orthogonality condition

$$\int_{\mathbb{R}} \gamma' v\; dy = 0. \tag{5.53}$$

This and (5.52) imply the remarkable fact

$$V_0^\varphi = -\kappa_0^\varphi, \tag{5.54}$$

so that

$$U_1 = 0. \tag{5.55}$$

Remark 5.8 (Convergence to anisotropic mean curvature flow). Note carefully that (5.54) justifies the convergence of solutions of system (1.4) to Φ^o-anisotropic mean curvature flow.

Substituting (5.54) and (5.55) in (5.49), dividing by $\alpha_r(\nabla d_0^\varphi)$ and recalling that the square bracket in (5.49) equals (5.50), we end up with the equation for W_1^r, for any $r = 1, \dots, m$:

$$\begin{aligned} W_1^{r\prime\prime} &= \frac{1}{\alpha_r(\nabla d_0^\varphi)}\gamma' V_0^\varphi - \gamma' \mathrm{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2} T_{\phi_r}(\nabla d_0^\varphi)\right) \\ &\quad - 2\gamma'' \frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot \nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2} \\ &= \frac{1}{\alpha_r(\nabla d_0^\varphi)}\gamma' \mathrm{div}\,(T_\Phi(\nabla d_0^\varphi)) - \gamma' \mathrm{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2} T_{\phi_r}(\nabla d_0^\varphi)\right) \\ &\quad - 2\gamma'' \frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot \nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}, \end{aligned} \tag{5.56}$$

since, from (5.51) and (5.54),

$$\mathrm{div}\,(T_\Phi(\nabla d_0^\varphi)) = V_0^\varphi.$$

As a consequence, recalling (5.40), (5.47) and (5.39), we have

$$\sum_{r=1}^m W_1^{r\prime\prime} = U_1'' = 0, \tag{5.57}$$

where the last equality follows from (5.55).
Equation (5.56) can be written as[20]

$$\begin{aligned} W_1^{r\prime\prime} =& \gamma' \Bigg[\mathrm{div}\left(\frac{1}{\alpha_r(\nabla d_0^\varphi)} T_\Phi(\nabla d_0^\varphi)\right) - \mathrm{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2} T_{\phi_r}(\nabla d_0^\varphi)\right) \\ & - T_\Phi(\nabla d_0^\varphi)\cdot \nabla \frac{1}{\alpha_r(\nabla d_0^\varphi)}\Bigg] - 2\gamma'' \frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot \nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}. \end{aligned} \tag{5.58}$$

From (5.57) it follows that $\sum_{r=1}^m W_1^r$ minus a linear function vanishes, namely

$$\sum_{r=1}^m W_1^r - U_1 = C_1 y + C_0.$$

We now claim that $C_0 = C_1 = 0$, and hence

$$\sum_{r=1}^m W_1^r = U_1 (= 0). \tag{5.59}$$

[20] Although written in a somewhat different form, this result coincides with that of [4], where d_ϵ^φ has not been expanded (hence d_ϵ^φ appears in place of d_0^φ in (5.58), and accordingly the last addendum is not present).

The constant C_0 turns out to be zero for the following argument: as a consequence of (5.23) and (5.9),

$$0 = U_\epsilon(0,t,x) = \sum_{r=1}^{m} W_\epsilon^r(0,t,x), \qquad \epsilon \in (0,1),$$

which implies

$$\sum_{r=1}^{m} W_i^r(0,t,x) = 0, \qquad i \geq 0$$

and hence $C_0 = 0$.
For what concerns the constant C_1, we have, using (5.72) below and (5.39),

$$\begin{aligned} C_1 = & \sum_{r=1}^{m} W_1^{r\prime} = \sum_{r=1}^{m} \left\{ (\gamma-1)\Theta^r - 2\gamma' \, \frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot \nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2} + w_0^{r\prime} \right\} \\ = & \sum_{r=1}^{m} \left\{ (\gamma-1)\Theta^r + w_0^{r\prime} \right\}. \end{aligned}$$

On the other hand, from (5.70) below, it follows $\sum_{r=1}^m w_0^{r\prime} = 0$, so that $C_1 = (\gamma-1)\sum_{r=1}^{m}\Theta^r$.
In order to conclude the proof of claim (5.59) it is enough to observe that $\sum_{r=1}^{m}\Theta^r = 0$, as a consequence of the expression of Θ^r in (5.72), and of (5.40) and (5.47), and so $C_1 = 0$.

5.2.5 Matching procedure

We are now in a position to recover the first term w_0^r of the outer expansion of w_ϵ^r, by adding to (5.6) a jump condition for w_0^r and a condition for $n_0^\varphi \cdot \nabla w_0^r$ across the interface $\Sigma_0(t)$, defined as the boundary of the external phase $\{u_0(t,\cdot) = 1\}$ (see (5.5)). We set

$$\Sigma_\epsilon(t) = \{x + \epsilon\sigma_1(s,t)n_0^\varphi + \mathcal{O}(\epsilon^2) : x \in \Sigma_0(t)\}, \tag{5.60}$$

for a suitable $\sigma_1 \colon \Sigma \times \mathbb{R} \to \mathbb{R}$, where Σ is the reference manifold in (5.13).
We will make use of the change of variables (5.20), and we will match the two expansions in the region of common validity $|y| \to +\infty$ and x approaching $\Sigma_\epsilon(t)$:

$$w_\epsilon^r \left(t, x(s,t) - \epsilon y n_\epsilon^\varphi + \mathcal{O}(\epsilon^2 y^2)\right) \approx W_\epsilon^r(y; s,t,x(s,t) - \epsilon y n_\epsilon^\varphi + \mathcal{O}(\epsilon^2 y^2)).$$

By expanding the left and right hand sides, understanding that w_ϵ^r is computed at points $x(s,t) \in \Sigma_\epsilon(t)$, we get

$$w_\epsilon^r - \epsilon y n_\epsilon^\varphi \cdot \nabla w_\epsilon^r + \mathcal{O}(\epsilon^2 y^2) \approx W_\epsilon^r - \epsilon y n_\epsilon^\varphi \cdot \nabla W_\epsilon^r + \mathcal{O}(\epsilon^2 y^2), \qquad r = 1,\ldots,m.$$

Expanding w_ϵ^r, W_ϵ^r in powers of ϵ, and matching the first two orders, we get in particular

$$\lim_{y\to\pm\infty} W_0^r(y, s(t,x), t, x) = w_0^r(t,x), \tag{5.61}$$

and

$$\lim_{y\to\pm\infty}\Big\{W_1^r(y,s(t,x),t,x)-w_1^r(t,x)-y\Big(n_0^\varphi\cdot\nabla W_0^r(y,s(t,x),t,x)-n_0^\varphi\cdot\nabla w_0^r(t,x)\Big)\Big\}=0, \tag{5.62}$$

where w_0^r and w_1^r are evaluated at each side of the interface according to when y goes to plus or minus infinity.
Equality (5.61) in particular suggests

$$\lim_{y\to\pm\infty} W_0^{r\prime}(y,s(t,x),t,x)=0, \qquad r=1,\dots,m, \tag{5.63}$$

and the jump $[\![w_0^r]\!]$ of w_0^r across the interface is given by

$$[\![w_0^r]\!](s(t,x),t)=\int_{\mathbb{R}} W_0^{r\prime}(y,s(t,x),t,x)\ dy, \qquad r=1,\dots,m. \tag{5.64}$$

From (5.44) we get

$$[\![w_0^r]\!]=\frac{c_0}{\alpha_r(\nabla d_0^\varphi)}, \qquad r=1,\dots,m, \tag{5.65}$$

where

$$c_0:=\int_{\mathbb{R}}\gamma'\ dy\in(0,+\infty).$$

To obtain the equation involving the conormal derivative, we formally differentiate equation (5.62):

$$\lim_{y\to\pm\infty}\left\{W_1^{r\prime}(y,s(t,x),t,x)-n_0^\varphi\cdot\nabla W_0^r(y,s(t,x),t,x)\right\}=-n_0^\varphi\cdot\nabla w_0^r(t,x), \tag{5.66}$$

where we used also the fact that

$$\lim_{y\to\pm\infty} y\ n_0^\varphi\cdot\nabla W_0^{r\prime}(y,s(t,x),t,x)=0,$$

since $\nabla W_0^{r\prime}=\gamma'\nabla\frac{1}{\alpha_r(\nabla d_0^\varphi)}$ by (5.44) and γ' decays exponentially to 0 as $y\to\pm\infty$. For the same reason, $W_1^{r\prime}$ is also bounded, thus

$$-[\![n_0^\varphi\cdot\nabla w_0^r]\!](t,x)=\int_{\mathbb{R}}\left(W_1^{r\prime\prime}(y,s(t,x),t,x)-n_0^\varphi\cdot\nabla W_0^{r\prime}(y,s(t,x),t,x)\right)\ dy. \tag{5.67}$$

Coupling (5.67) with (5.56) and (5.44), and recalling from (5.14) that $n_0^\varphi=-T_\Phi(\nabla d_0^\varphi)$ we end up with

$$-[\![n_0^\varphi\cdot\nabla w_0^r]\!]=c_0\ \mathrm{div}\left[\frac{1}{\alpha_r(\nabla d_0^\varphi)}T_\Phi(\nabla d_0^\varphi)-\frac{1}{\alpha_r^2(\nabla d_0^\varphi)}T_{\phi_r}(\nabla d_0^\varphi)\right], \qquad r=1,\dots,m. \tag{5.68}$$

The two jump conditions on w_0 across $\Sigma_0(t)$, together with the *far field* equation (5.6) and appropriate boundary conditions at $\partial\Omega$ allow to retrieve a unique solution w_0.
If we integrate (5.44), and use the matching condition for w_0^r in (5.61), we get for W_0^r the expression

$$W_0^r=\frac{1}{\alpha_r(\nabla d_0^\varphi)}(\gamma-1)+w_0^{r+}(s,t), \qquad r=1,\dots,m, \tag{5.69}$$

where w_0^{r+} is the trace on $\Sigma_0(t)$ of w_0^r from the external phase $\{u_0(t,\cdot)=1\}$. In particular

$$\sum_{r=1}^{m} w_0^{r+} = 1. \tag{5.70}$$

Thus

$$\begin{aligned} W^r_{0s_\beta} &= w^{r+}_{0s_\beta}, \qquad \nabla W_0^r = (\gamma-1)\nabla \frac{1}{\alpha_r}, \qquad \nabla W^r_{0s_\beta} = 0, \\ W^r_{0s_\beta s_\delta} &= w^{r+}_{0s_\beta s_\delta}, \qquad W^r_{0x_ix_j} = (\gamma-1)\partial_{x_ix_j}\frac{1}{\alpha_r}. \end{aligned} \tag{5.71}$$

In a similar fashion we can integrate (5.56), and use the matching condition (5.66), to get, for any $r=1,\dots,m$,

$$\begin{aligned} W_1^{r\prime} = &(\gamma-1)\left\{\frac{1}{\alpha_r(\nabla d_0^\varphi)}\mathrm{div}\,(T_\Phi(\nabla d_0^\varphi)) - \mathrm{div}\left(\frac{1}{\alpha_r^2(\nabla d_0^\varphi)}T_{\phi_r}(\nabla d_0^\varphi)\right)\right\} \\ &- 2\gamma' \,\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2} + w_0^{r\prime}(s,t) \\ =&(\gamma-1)\Theta^r(t,x) - 2\gamma' \,\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2} + w_0^{r\prime}(s,t), \end{aligned} \tag{5.72}$$

where

$$w_0^{r\prime} := T_\Phi(\nabla d_0^\varphi)\cdot\nabla w_0^{r+},$$

and Θ^r is a shorthand for the expression in braces. Observe that only the last term explicitly depends on s, while the other terms depend on y (by means of γ) and on x (by means of Θ^r). Thus

$$W^{r\prime}_{1s_\beta} = w^{r\prime}_{0s_\beta} \tag{5.73}$$

$$\nabla W_1^{r\prime} = (\gamma-1)\nabla\Theta^r - 2\gamma'\nabla\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\right). \tag{5.74}$$

Remark 5.9. Note that the jump in the *conormal* derivative $[\![n_0^\varphi\cdot\nabla w_0^r]\!]$ vanishes in the special case of equal anisotropic ratio, which, in our context, consists of choosing, for every $r=1,\dots,m$, $\alpha_r := \lambda_r\overline{\alpha}$ with some given smooth symmetric uniformly convex squared anisotropy $\overline{\alpha}$ and positive λ_r (indeed, in this case eikonal equation (5.40) leads to $\overline{\alpha}(\nabla d_0^\varphi) = \sum_{r=1}^m \lambda_r^{-1}$).

Remark 5.10. Given $r=1,\dots,m$, the function $W_1^r(\cdot,t,x)$ is expected to have linear growth at infinity (independent of ϵ)[21]; observe, however, that $\sum_{r=1}^m W_1^r(\cdot,t,x)=0$, see (5.59).

5.2.6 Order 2

We end our asymptotic analysis considering the $\mathcal{O}(\epsilon^2)$ terms in equation (5.34), which represents an improvement with respect to [7] (in which expansions are performed only up to the

[21] Differently with respect to $W_0^r(\cdot,t,x)$, which is expected to be bounded at infinity.

order $\mathcal{O}(\epsilon)$ and $m = 2$). Recall that $U_0' = \gamma'$ depends only on y and that $U_1 = 0$. Then the terms of order $\mathcal{O}(\epsilon^2)$ arising from the first line on the right hand side of (5.34) are:

$$\begin{aligned} &- \alpha_r(\nabla d_0^\varphi) W_2^{r\prime\prime} - 2W_1^{r\prime\prime} T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi \\ &- \Big[2T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_2^\varphi + M^r(\nabla d_0^\varphi) \nabla d_1^\varphi \cdot \nabla d_1^\varphi \Big] W_0^{r\prime\prime} + f'(\gamma) U_2. \end{aligned} \tag{5.75}$$

The terms of order $\mathcal{O}(\epsilon)$ arising from the terms in the round parentheses in the second line of (5.34) are:

$$\begin{aligned} &\gamma' V_1^\varphi - 2W_{1s_\beta}^{r\prime} T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla s_{0\beta}^\varphi - 2T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla W_1^{r\prime} \\ &- 2M^r(\nabla d_0^\varphi) \nabla d_1^\varphi \cdot \nabla W_0^{r\prime} - W_1^{r\prime} \mathrm{div}\big(T_{\phi_r}(\nabla d_0^\varphi)\big) - W_0^{r\prime} \mathrm{div}\big(M^r(\nabla d_0^\varphi) \nabla d_1^\varphi\big). \end{aligned} \tag{5.76}$$

Note that, using (5.73), if we set

$$\mathrm{A}^r := -2W_{1s_\beta}^{r\prime} T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla s_{0\beta}^\varphi, \tag{5.77}$$

then A^r is independent of y, hence

$$\int_{\mathbb{R}} \gamma' \mathrm{A}^r \, dy = \mathrm{A}^r \int_{\mathbb{R}} \gamma' \, dy = c_0 \mathrm{A}^r. \tag{5.78}$$

Remark 5.11. The term A^r is independent of d_1^φ.

The terms of order $\mathcal{O}(1)$ arising from the terms in the round parentheses in the third and fourth lines of (5.34) are:

$$\begin{aligned} &- W_{0s_\beta s_\delta}^r M^r(\nabla d_0^\varphi) \nabla s_{0\beta}^\varphi \cdot \nabla s_{0\delta}^\varphi - 2M^r(\nabla d_0^\varphi) \nabla s_{0\beta}^\varphi \cdot \nabla W_{0s_\beta}^r \\ &- W_{0s_\beta}^r \mathrm{div}\Big(M^r(\nabla d_0^\varphi) \nabla s_{0\beta}^\varphi \Big) - W_{0x_j}^r \mathrm{div}\left(M_{\cdot j}^r(\nabla d_0^\varphi) \right) - W_{0x_i x_j}^r M_{ij}^r(\nabla d_0^\varphi) =: \mathrm{B}^r. \end{aligned} \tag{5.79}$$

where B^r is independent of y. Observe that, from (5.69) and (5.71), it follows that the y-dependence of B^r is through γ only in the term $W_{0x_j}^r$, which is the only term that does not contribute when integrated on $\mathbb{R}$ against γ'. All the other terms contribute, so that

$$\int_{\mathbb{R}} \gamma' \, \mathrm{B}^r \, dy = c_0(\mathrm{B}^r - W_{0x_j}^r \mathrm{div}\left(M_{\cdot j}^r(\nabla d_0^\varphi) \right) - W_{0x_i x_j}^r M_{ij}^r(\nabla d_0^\varphi). \tag{5.80}$$

Remark 5.12. The term B^r is independent of d_1^φ.

Collecting together (5.75), (5.76) and (5.79) we get

$$\begin{aligned} &- \alpha_r(\nabla d_0^\varphi) W_2^{r\prime\prime} - 2W_1^{r\prime\prime} T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi \\ &- \Big[2T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_2^\varphi + M^r(\nabla d_0^\varphi) \nabla d_1^\varphi \cdot d_1^\varphi \Big] W_0^{r\prime\prime} + f'(\gamma) U_2 \\ &\gamma' V_1^\varphi - 2W_{1s_\beta}^{r\prime} T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla s_{0\beta}^\varphi - 2T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla W_1^{r\prime} \\ &- 2M^r(\nabla d_0^\varphi) \nabla d_1^\varphi \cdot \nabla W_0^{r\prime} - W_1^{r\prime} \mathrm{div}\big(T_{\phi_r}(\nabla d_0^\varphi)\big) - W_0^{r\prime} \mathrm{div}\big(M^r(\nabla d_0^\varphi) \nabla d_1^\varphi\big) + \mathrm{B}^r. \end{aligned} \tag{5.81}$$

Before continuing, let us write (5.72) in the form

$$W_1^{r\prime} = -2\gamma' \ \frac{T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2} + \mathrm{C}^r, \tag{5.82}$$

where

$$\mathrm{C}^r := (\gamma - 1)\Theta^r + w_0^{r\prime},$$

so that C^r depends on y only through the term $\gamma(y)\Theta^r(t,x)$, and therefore

$$\int_{\mathbb{R}} \gamma' \ \mathrm{C}^r \ dy = c_0 \left(-\Theta^r + w_0^{r\prime}\right). \tag{5.83}$$

Remark 5.13. The term C^r is independent of d_1^φ.

Substituting (5.43), (5.44), (5.58), (5.82) into (5.81), and reordering terms we get, for any $r = 1, \ldots, m$,

$$\begin{aligned} 0 = \ & -\alpha_r(\nabla d_0^\varphi) W_2^{r\prime\prime} + U_2 f'(\gamma) + \gamma' V_1^\varphi \\ & + \gamma' \left\{ 2T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi \left[\frac{\kappa_0^\varphi}{\alpha_r(\nabla d_0^\varphi)} + \operatorname{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{(\alpha_r(\nabla d_0^\varphi))^2} \right) \right] \right\} \\ & + 4\gamma' T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla \left(\frac{T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2} \right) \\ & - 2\gamma' M^r(\nabla d_0) \nabla d_1^\varphi \cdot \nabla \left(\frac{1}{\alpha_r(\nabla d_0^\varphi)} \right) \\ & - \gamma' \frac{1}{\alpha_r(\nabla d_0^\varphi)} \operatorname{div}\big(M^r(\nabla d_0^\varphi) \nabla d_1^\varphi \big) \\ & + 2\gamma' \frac{T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2} \operatorname{div}\left(T_{\phi_r}(\nabla d_0^\varphi)\right) \\ & + \mathrm{A}^r + \mathrm{B}^r + \mathrm{C}^r + \gamma'' \mathrm{D}^r, \end{aligned} \tag{5.84}$$

where

$$\mathrm{D}^r := \left(\frac{2T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi}{\alpha_r(\nabla d_0^\varphi)} \right)^2 - \frac{\left[2T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_2^\varphi + M^r(\nabla d_0^\varphi) \nabla d_1^\varphi \cdot \nabla d_1^\varphi \right]}{\alpha_r(\nabla d_0^\varphi)}.$$

Remark 5.14. Note that D^r depends on d_1^φ, however

$$\int_{\mathbb{R}} \gamma' \gamma'' \ \mathrm{D}^r \ dy = \mathrm{D}^r \int_{\mathbb{R}} \gamma' \gamma'' \ dy = 0. \tag{5.85}$$

Let us now focus the attention to (5.84), where for the moment we neglect the first line and the term $\mathrm{A}^r + \gamma \mathrm{B}^r + \mathrm{C}^r + \mathrm{D}^r$: dividing by $\alpha_r(\nabla d_0^\varphi)$ we have

$$\begin{aligned} & \frac{2T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2} \kappa_0^\varphi + \frac{2T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi}{\alpha_r(\nabla d_0^\varphi)} \operatorname{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{(\alpha_r(\nabla d_0^\varphi))^2} \right) \\ & + 4 \frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)} \cdot \nabla \left(\frac{T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi)^2} \right) + 2 \frac{T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^3} \operatorname{div}\left(T_{\phi_r}(\nabla d_0^\varphi)\right) \\ & - \frac{2}{\alpha_r(\nabla d_0^\varphi)} M^r(\nabla d_0) \nabla d_1^\varphi \cdot \nabla \left(\frac{1}{\alpha_r(\nabla d_0^\varphi)} \right) - \frac{1}{(\alpha_r(\nabla d_0^\varphi))^2} \operatorname{div}\big(M^r(\nabla d_0^\varphi) \nabla d_1^\varphi \big) \end{aligned} \tag{5.86}$$

Observe now that the first term in (5.86) will disappear when summing up on $r = 1, \dots, m$, thanks again to (5.46) and (5.38). Moreover, the two terms in last line of (5.86) can be put together giving div $\left(\frac{M^r(\nabla d_0^\varphi)\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\right)$ so that, summing up on r, we get:

$$
\begin{aligned}
&\underbrace{2\sum_{r=1}^m \frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{\alpha_r(\nabla d_0^\varphi)}\operatorname{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{(\alpha_r(\nabla d_0^\varphi))^2}\right)}_{:=E} + \underbrace{4\sum_{r=1}^m \frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)}\cdot\nabla\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{\alpha_r^2(\nabla d_0^\varphi)}\right)}_{:=F} \\
&+\underbrace{2\sum_{r=1}^m \frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^3}\operatorname{div}T_{\phi_r}(\nabla d_0^\varphi)}_{:=G} \underbrace{-\sum_{r=1}^m \operatorname{div}\left(\frac{M^r(\nabla d_0^\varphi)\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\right)}_{:=H}.
\end{aligned}
\tag{5.87}
$$

Recall now that $-\kappa_1^\varphi - yh_0^\varphi = \operatorname{div}\big(\nabla T_\Phi(\nabla d_0^\varphi)\nabla d_1^\varphi\big)$. Using formulas (2.10), (2.11), and the relations $\nabla\alpha_r = 2T_{\phi_r}$, $\Phi^2(\nabla d_0^\varphi) = 1$, we get

$$
\begin{aligned}
-\kappa_1^\varphi - yh_0^\varphi =& \sum_{r=1}^m \operatorname{div}\left(\frac{M^r(\nabla d_0^\varphi)\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\right) + \sum_{r=1}^m \operatorname{div}\left(\frac{1-\alpha_r(\nabla d_0^\varphi)}{(\alpha_r(\nabla d_0^\varphi))^4}\nabla\alpha_r(\nabla d_0^\varphi)\otimes\nabla\alpha_r(\nabla d_0^\varphi)\nabla d_1^\varphi\right) \\
&+\sum_{j\neq r}\operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2(\alpha_j(\nabla d_0^\varphi))^2}\nabla\alpha_r(\nabla d_0^\varphi)\otimes\nabla\alpha_j(\nabla d_0^\varphi)\nabla d_1^\varphi\right).
\end{aligned}
$$

Adding and subtracting the term $4\sum_{r=1}^m \operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^4}T_{\phi_r}(\nabla d_0^\varphi)\otimes T_{\phi_r}(\nabla d_0^\varphi)\nabla d_1^\varphi\right)$ it follows

$$
\begin{aligned}
-\kappa_1^\varphi - yh_0^\varphi =& \sum_{r=1}^m \operatorname{div}\left(\frac{M^r(\nabla d_0^\varphi)\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\right) + 4\sum_{r=1}^m \operatorname{div}\left(\frac{1-\alpha_r(\nabla d_0^\varphi)}{(\alpha_r(\nabla d_0^\varphi))^4}(T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi)T_{\phi_r}(\nabla d_0^\varphi)\right) \\
&+4\sum_{j,r}\operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2(\alpha_j(\nabla d_0^\varphi))^2}T_{\phi_r}(\nabla d_0^\varphi)\otimes T_{\phi_j}(\nabla d_0^\varphi)\nabla d_1^\varphi\right) \\
&-4\sum_{r=1}^m\operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^4}T_{\phi_r}(\nabla d_0^\varphi)\otimes T_{\phi_r}(\nabla d_0^\varphi)\nabla d_1^\varphi\right).
\end{aligned}
$$

Fixing one of the two indices r, j, for instance r, and summing over the other one $j = 1, \dots, m$, we get

$$
\sum_{j,r}\operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2(\alpha_j(\nabla d_0^\varphi))^2}T_{\phi_r}(\nabla d_0^\varphi)\otimes T_{\phi_j}(\nabla d_0^\varphi)\nabla d_1^\varphi\right) = 0,
$$

thanks again to eikonal equation (5.38). We deduce

$$
\begin{aligned}
-\kappa_1^\varphi - yh_0^\varphi =& \sum_{r=1}^m \operatorname{div}\left(\frac{M^r(\nabla d_0^\varphi)\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\right) + 4\sum_{r=1}^m \operatorname{div}\left(\frac{1-\alpha_r(\nabla d_0^\varphi)}{(\alpha_r(\nabla d_0^\varphi))^4}(T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi)T_{\phi_r}(\nabla d_0^\varphi)\right) \\
&-4\sum_{r=1}^m\operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^4}T_{\phi_r}(\nabla d_0^\varphi)\otimes T_{\phi_r}(\nabla d_0^\varphi)\nabla d_1^\varphi\right) \\
=& \sum_{r=1}^m \operatorname{div}\left(\frac{M^r(\nabla d_0^\varphi)\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\right) - 4\sum_{r=1}^m \operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^3}(T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi)T_{\phi_r}(\nabla d_0^\varphi)\right) \\
:=& I,
\end{aligned}
$$

where we used

$$T_{\phi_r}(\nabla d_0^\varphi) \otimes T_{\phi_r}(\nabla d_0^\varphi)\nabla d_1^\varphi = (T_{\phi_r}(\nabla d_0^\varphi) \cdot \nabla d_1^\varphi) T_{\phi_r}(\nabla d_0^\varphi).$$

We claim now that $\kappa_1^\varphi + y h_0^\varphi$ is equal to (5.87) — namely:

$$E + F + G + H + I = 0. \tag{5.88}$$

We first observe that the first term appearing in I cancels with H, so that it is enough to show

$$E + F + G = 4\sum_{r=1}^m \operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^2}(T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi)T_{\phi_r}(\nabla d_0^\varphi)\right),$$

i.e.,

$$\begin{aligned}
&2\sum_{r=1}^m \frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{\alpha_r(\nabla d_0^\varphi)}\operatorname{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{(\alpha_r(\nabla d_0^\varphi))^2}\right) + 4\sum_{r=1}^m \frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)}\cdot\nabla\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\right)\\
&+2\sum_{r=1}^m \frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^3}\operatorname{div}T_{\phi_r}(\nabla d_0^\varphi) = 4\sum_{r=1}^m \operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^3}(T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi)T_{\phi_r}(\nabla d_0^\varphi)\right).
\end{aligned} \tag{5.89}$$

The right hand side of (5.89) can be rewritten as

$$\begin{aligned}
&4\sum_{r=1}^m \operatorname{div}\left(\frac{1}{(\alpha_r(\nabla d_0^\varphi))^3}(T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi)T_{\phi_r}(\nabla d_0^\varphi)\right)\\
=&4\sum_{r=1}^m \frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\operatorname{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)}\right) + 4\sum_{r=1}^m \frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)}\cdot\nabla\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{\alpha_r^2(\nabla d_0^\varphi)}\right),
\end{aligned}$$

so that its last addendum cancels with F. Thus, in order to show (5.88) it remains to prove that

$$\begin{aligned}
&2\sum_{r=1}^m \frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{\alpha_r(\nabla d_0^\varphi)}\operatorname{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{(\alpha_r(\nabla d_0^\varphi))^2}\right) + 2\sum_{r=1}^m \frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^3}\operatorname{div}T_{\phi_r}(\nabla d_0^\varphi)\\
=&4\sum_{r=1}^m \frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{(\alpha_r(\nabla d_0^\varphi))^2}\operatorname{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)}\right),
\end{aligned}$$

or equivalently

$$\begin{aligned}
\sum_{r=1}^m \frac{T_{\phi_r}(\nabla d_0^\varphi)\cdot\nabla d_1^\varphi}{\alpha_r(\nabla d_0^\varphi)}\Bigg\{&\operatorname{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{(\alpha_r(\nabla d_0^\varphi))^2}\right) + \frac{\operatorname{div}(T_{\phi_r}(\nabla d_0^\varphi))}{(\alpha_r(\nabla d_0^\varphi))^2}\\
&-\frac{2}{\alpha_r(\nabla d_0^\varphi)}\operatorname{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)}\right)\Bigg\} = 0.
\end{aligned} \tag{5.90}$$

Using the identity

$$\operatorname{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{(\alpha_r(\nabla d_0^\varphi))^2}\right) = \frac{1}{\alpha_r(\nabla d_0^\varphi)}\operatorname{div}\left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)}\right) + \frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)}\cdot\nabla\left(\frac{1}{\alpha_r(\nabla d_0^\varphi)}\right),$$

it follows that, for any $r = 1, \ldots, m$, the quantity in braces in (5.90) becomes

$$\frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)} \cdot \nabla \left(\frac{1}{\alpha_r(\nabla d_0^\varphi)} \right) + \frac{\operatorname{div}(T_{\phi_r}(\nabla d_0^\varphi))}{(\alpha_r(\nabla d_0^\varphi))^2} - \frac{1}{\alpha_r(\nabla d_0^\varphi)} \operatorname{div} \left(\frac{T_{\phi_r}(\nabla d_0^\varphi)}{\alpha_r(\nabla d_0^\varphi)} \right), \tag{5.91}$$

which is identically zero. This concludes the proof of our claim (5.88).

From (5.84), summing over $r = 1, \ldots, m$ and using (5.40) we deduce

$$0 = -U_2'' + U_2 f'(\gamma) + \gamma'(V_1^\varphi + \kappa_1^\varphi) + y\gamma' h_0^\varphi + \sum_{r=1}^m \frac{1}{\alpha_r(\nabla d_0^\varphi)} \left[\mathrm{A}^r + \mathrm{B}^r + \mathrm{C}^r + \gamma'' \mathrm{D}^r \right].$$

Note that we have used $U_2 = \sum_{r=1}^m W_2^r$: in general it may happen that $U_2 - \sum_{r=1}^m W_2^r = \mathcal{O}(\epsilon)$, but we have the freedom[22] to redefine the functions W_2^r up to discrepancies of order $\mathcal{O}(\epsilon)$, and put the subsequent errors in the terms U_3 and W_3^r, which we are not interested in.

Recalling (5.78), (5.80), (5.83) and (5.85), and observing also that

$$\int_{\mathbb{R}} y\gamma'\gamma' \, dy = 0,$$

(so that the orthogonality condition (5.53) leads to drop out the terms with h_0^φ), we end up with the following integrability condition:

$$0 = c_1(V_1^\varphi + \kappa_1^\varphi) + c_0 G,$$

where

$$c_1 = \int_{\mathbb{R}} (\gamma')^2 \, dy$$

and

$$G = \sum_{r=1}^m \frac{1}{\alpha(\nabla d_0^\varphi)} \int_{\mathbb{R}} \gamma' \left(\mathrm{A}^r + \mathrm{B}^r + \mathrm{C}^r \right) \, dy.$$

The term G is presumably nonzero, which shows that, in general, V_1^φ is nonzero. This is a difference with respect to the formal asymptotic analysis of the anisotropic Allen-Cahn's equation [5, 4, 3], and suggests an $\mathcal{O}(\epsilon)$-error estimate between the geometric front and $\Sigma_\epsilon(t)$ (while, in the Allen-Cahn's equation, the estimate can be improved to the order $\mathcal{O}(\epsilon^2)$).

Remark 5.15 (Approximate evolution law and forcing term). The integrability condition for function U_2 relates V_1^φ and κ_1^φ and together with the integrability condition for U_1 leads to the approximate evolution law

$$V_\epsilon = -\kappa_\epsilon^\varphi - \epsilon \frac{c_0}{c_1} G + \mathcal{O}(\epsilon^2)$$

for Σ_ϵ. By dropping the $\mathcal{O}(\epsilon^2)$ term we obtain a new approximation Σ_1 of Σ_ϵ which we assume to have an $\mathcal{O}(\epsilon^2)$ error. This allows in turn to recover the $\mathcal{O}(\epsilon)$ term for the signed distance d_1^φ by taking the difference between the signed distance from $\Sigma_1(t)$ and the signed

[22] This is because enforcing the relation between (t, x) and (y, s, t, x) introduces a dependence on ϵ.

distance from $\Sigma_0(t)$ and dividing by ϵ. Now we can recover the functions W_1^r (which indeed depend on ∇d_1^φ) and solve the differential equation for U_2 (which also depends on ∇d_1^φ) to get U_2. This argument works provided G does not depend on d_1^φ, since it is also through G that the function U_2 is determined. We see from Remarks 5.11, 5.12, 5.13 and the properties of D^r, that the function G is indeed independent of d_1^φ.

Problem 5.16. Investigate on the existence and regularity of solutions to the elliptic equation (5.6), coupled with (5.65), (5.68), leading to the function w_0^r for any $r = 1, \ldots, m$.

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