Hausdorff measures and dimensions in non equiregular sub-Riemannian manifolds

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Abstract This paper is a starting point towards computing the Hausdorff dimension of submanifolds and the Hausdorff volume of small balls in a sub-Riemannian manifold with singular points. We first consider the case of a strongly equiregular submanifold, i.e., a smooth submanifold N for which the growth vector of the distribution $\mathscr{D}$ and the growth vector of the intersection of $\mathscr{D}$ with TN are constant on N. In this case, we generalize the result in [12], which relates the Hausdorff dimension to the growth vector of the distribution. We then consider analytic sub-Riemannian manifolds and, under the assumption that the singular point p is typical, we state a theorem which characterizes the Hausdorff dimension of the manifold and the finiteness of the Hausdorff volume of small balls $B(p,\rho)$ in terms of the growth vector of both the distribution and the intersection of the distribution with the singular locus, and of the nonholonomic order at p of the volume form on M evaluated along some families of vector fields.

1 Introduction

The main motivation of this paper arises from the study of sub-Riemannian manifolds as particular metric spaces. Recall that a sub-Riemannian manifold is a triplet $(M,\mathscr{D},g)$, where M is a smooth manifold, $\mathscr{D}$ a Lie-bracket generating subbundle of TM and g a Riemannian metric on $\mathscr{D}$. The absolutely continuous paths which are almost everywhere tangent to $\mathscr{D}$ are called horizontal and their length is obtained as in Riemannian geometry integrating the norm of their tangent vectors. The sub-Riemannian distance d is defined as the infimum of length of horizontal paths between two given points.

Hausdorff measures and spherical Hausdorff measures can be defined on sub-Riemannian manifolds using the sub-Riemannian distance. It is well-known that for these metric spaces the Hausdorff dimension is strictly greater than the topological one. Although the presence of an extra structure, i.e., the differential one, constitute a considerable help, computing Hausdorff measures and dimensions of sets is a difficult problem. In [5] we study Hausdorff measures of continuous curves, whereas in [1] the authors analyze the regularity of the top-dimensional Hausdorff measure in the equiregular case (see the definition below). In the case of Carnot groups, Hausdorff measures of regular hypersurfaces have been studied in [4] and in a more general context, a representation formula for the perimeter measure in terms of Hausdorff measure has been proved in [2].

In this paper we consider three questions: given a sub-Riemannian manifold $(M,\mathscr{D},g)$, $p \in M$ and a small $\rho > 0$,

1. what is the Hausdorff dimension $\dim_H(M)$?
2. under which condition is the Hausdorff volume $\mathscr{H}^{\dim_H(M)}(B(p,\rho))$ finite?

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3. the two preceding questions when M is replaced by a submanifold N, i.e., what is $\dim_H(N)$ and when is $\mathscr{H}^{\dim_H(N)}(N \cap B(p, \rho))$ finite?

A key feature to be taken into account is whether p is regular or singular for the sub-Riemannian manifold. Given $i \geq 1$, define recursively the submodule $\mathscr{D}^i$ of $\mathrm{Vec}(M)$ by $\mathscr{D}^1 = \mathscr{D}$, $\mathscr{D}^{i+1} = \mathscr{D}^i + [\mathscr{D}, \mathscr{D}^i]$. Denote by $\mathscr{D}_p^i = \{X(p) \mid X \in \mathscr{D}^i\}$. Since $\mathscr{D}$ is Lie-bracket generating, there exists $r(p) \in \mathbb{N}$ such that

$$\{0\} = \mathscr{D}_p^0 \subset \mathscr{D}_p^1 \subset \cdots \subset \mathscr{D}_p^{r(p)} = T_p M.$$

A point p is regular if, for every i, the dimensions $\dim \mathscr{D}_q^i$ are constant as q varies in a neighborhood of p. Otherwise, p is said to be singular. A set $S \subset M$ is equiregular if, for every i, $\dim \mathscr{D}_q^i$ is constant as q varies in S. For equiregular manifolds, questions 1 and 2 have been answered in [12] (but with an incorrect proof, see [13] for a correct one). In that paper, the author shows that the Hausdorff dimension of an equiregular manifold M is

$$\dim_H(M) = Q, \quad \text{where} \quad \sum_{i=1}^{r(p)} i(\dim \mathscr{D}_p^i - \dim \mathscr{D}_p^{i-1}), \tag{1}$$

and that the Hausdorff Q-dimensional measure near a regular point is absolutely continuous with respect to any Lebesgue measure on M. As a consequence, when p is regular, the Hausdorff dimension of a small ball $B(p, \rho)$ is Q, and the Hausdorff Q-dimensional measure of $B(p, \rho)$ is finite.

When there are singular points, these problems have been mentioned in [8, Section 1.3.A]. In this case, the idea is to compute the Hausdorff dimension using suitable stratifications of M where the discontinuities of the dimensions $q \mapsto \dim \mathscr{D}_q^i$ are somehow controlled. Namely, as suggested in [8], we consider stratifications made by submanifolds N which are *strongly equiregular*, i.e., for which both the dimensions $\dim \mathscr{D}_q^i$ and $\dim(\mathscr{D}_q^i \cap T_q N)$ are constant as q varies in N.

The first part of the paper provides an answer to question 3 when N is strongly equiregular. The first result of the paper (Theorem 1) computes the Hausdorff dimension of a strongly equiregular submanifold N in terms of the dimensions of $\dim(\mathscr{D}_q^i \cap T_q N)$, generalizing formula (1) which corresponds to the case $N = M$. More precisely, $\dim_H(N) = Q_N$ where

$$Q_N := \sum_{i=1}^{r(p)} i(\dim(\mathscr{D}_p^i \cap T_p N) - \dim(\mathscr{D}_p^{i-1} \cap T_p N)).$$

This actually follows from a stronger property: indeed, we show that the Q_N-dimensional spherical Hausdorff measure in N is absolutely continuous with respect to any smooth measure (i.e. any measure induced locally by a volume form) on N. The Radon–Nikodym derivative computed in Theorem 1 generalizes [1, Lemma 32], which corresponds to the case $N = M$. The main ingredient behind the proofs of such results is the fact that for a strongly equiregular submanifold N the metric tangent cone to $(N, d|_N)$ exists at every $p \in N$ and can be identified to $T_p N$ via suitable systems of privileged coordinates (see Lemma 1).

The results for strongly equiregular submanifolds provide a first step towards the answer of questions 1 and 2 in the general case, at least for analytic sub-Riemannian manifolds. This is the topic in the second part of the paper. Indeed, when $(M, \mathscr{D}, g)$ is analytic, M can be stratified as $M = \cup_{i \geq 0} M_i$ where each M_i is an analytic equiregular submanifold. Then, the Hausdorff dimension of a small ball B is the maximum of the Hausdorff dimensions of the intersections $B \cap M_i$. To compute the latter ones, we use that each strata M_i can further be decomposed as the disjointed union of strongly equiregular analytic submanifolds. In Lemma 3, using Theorem 1 we compute the Hausdorff dimension of an equiregular (but possibly not strongly equiregular) analytic submanifold and we estimate the density of the corresponding Hausdorff measure. Characterizing the finiteness of the corresponding Hausdorff measure of the intersection of a small ball with an equiregular analytic submanifold is rather involved. Yet this is the main issue in question 2, as whenever the Hausdorff measure of $B(p, \rho) \cap \{\text{regular points}\}$ is infinite at a singular point p then so is $\mathscr{H}^{\dim_H(M)}(B(p, \rho))$. To estimate $\mathscr{H}^{\dim_H(M)}(B(p, \rho) \cap \{\text{regular points}\})$, we assume that the singular point p is "typical", that is, it belongs to a strongly equiregular submanifold N of the singular set. In Theorem 2 we characterize the finiteness of the aforementioned measure at typical singular points through an algebraic relation involving the Hausdorff dimension Q_{reg}

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near a regular point, the Hausdorff dimension Q_N of N, and the nonholonomic order at p of the volume form on M evaluated along some families of vector fields, given by Lie brackets between generators of the distribution.

The proof of Theorem 2 (and of Proposition 1) will appear in a forthcoming paper.

The structure of the paper is the following. In Section 2 we recall shortly the definitions of Hausdorff measures and dimension and some basic notions in sub-Riemannian geometry. Section 3 is devoted to the the definition and the study of strongly equiregular submanifolds and contains the proof of Theorem 1 and the statement of Proposition 1. In Section 4 we treat analytic sub-Riemannian manifolds. First, we estimate the Hausdorff dimension $\bar{Q}_N$ of an analytic equiregular submanifold N in Section 4.1. Then, in Section 4.2, we prove that the $\bar{Q}_N$-dimensional Hausdorff measure of the intersection of a small ball $B(p,\rho)$ with N is finite if $p \in N$ and we state Theorem 2. Finally, we end by applying our results to some examples of sub-Riemannian manifolds in Section 4.3. In particular, the examples show that when the Hausdorff dimension of a ball centered at a singular point is equal to the Hausdorff dimension of the whole manifold, the corresponding Hausdorff measure can be both finite or infinite.

2 Basic notations

2.1 Hausdorff measures

Let (M,d) be a metric space. We denote by $\operatorname{diam} S$ the diameter of a set $S \subset M$, by $B(p,\rho)$ the open ball $\{q \in M \mid d(q,p) < \rho\}$, and by $\overline{B(p,\rho)}$ the closure of $B(p,\rho)$. Let $\alpha \geq 0$ be a real number. For every set $A \subset M$, the α-*dimensional Hausdorff measure* $\mathscr{H}^\alpha$ of A is defined as $\mathscr{H}^\alpha(A) = \lim_{\varepsilon \to 0^+} \mathscr{H}^\alpha_\varepsilon(A)$, where

$$\mathscr{H}^\alpha_\varepsilon(A) = \inf\left\{ \sum_{i=1}^\infty (\operatorname{diam} S_i)^\alpha : A \subset \bigcup_{i=1}^\infty S_i, \ S_i \text{ closed set, } \operatorname{diam} S_i \leq \varepsilon \right\},$$

and the α-*dimensional spherical Hausdorff measure* is defined as $\mathscr{S}^\alpha(A) = \lim_{\varepsilon \to 0^+} \mathscr{S}^\alpha_\varepsilon(A)$, where

$$\mathscr{S}^\alpha_\varepsilon(A) = \inf\left\{ \sum_{i=1}^\infty (\operatorname{diam} S_i)^\alpha : A \subset \bigcup_{i=1}^\infty S_i, \ S_i \text{ is a ball, } \operatorname{diam} S_i \leq \varepsilon \right\}.$$

For every set $A \subset M$, the non-negative number

$$D = \sup\{\alpha \geq 0 \mid \mathscr{H}^\alpha(A) = \infty\} = \inf\{\alpha \geq 0 \mid \mathscr{H}^\alpha(A) = 0\}$$

is called the *Hausdorff dimension of A*. The D-dimensional Hausdorff measure $\mathscr{H}^D(A)$ is called the Hausdorff volume of A. Notice that this volume may be 0, > 0, or ∞.

Given a subset $N \subset M$, we can consider the metric space $(N, d|_N)$. Denoting by $\mathscr{H}^\alpha_N$ and $\mathscr{S}^\alpha_N$ the Hausdorff and spherical Hausdorff measures in this space, by definition we have

$$\mathscr{H}^\alpha \llcorner N(A) := \mathscr{H}^\alpha(A \cap N) = \mathscr{H}^\alpha_N(A \cap N),$$
$$\mathscr{S}^\alpha \llcorner N(A) := \mathscr{S}^\alpha(A \cap N) \leq \mathscr{S}^\alpha_N(A \cap N). \tag{2}$$

These are a simple consequences of the fact that a set C is closed in N if and only if $C = C' \cap N$, with C' closed in M. Notice that the inequality (2) is strict in general, as coverings in the definition of $\mathscr{S}^\alpha_N$ are made with sets B which satisfy $B = \overline{B(p,\rho)} \cap N$ with $p \in N$, whereas coverings in the definition of $\mathscr{S}^\alpha \llcorner N$ include sets of the type $\overline{B(p,\rho)} \cap N$ with $p \notin N$. Moreover, by construction of Hausdorff measures, for every subset $S \subset N$, $\mathscr{H}^\alpha(S) \leq \mathscr{S}^\alpha(S) \leq 2^\alpha \mathscr{H}^\alpha(S)$ and $\mathscr{H}^\alpha_N(S) \leq \mathscr{S}^\alpha_N(S) \leq 2^\alpha \mathscr{H}^\alpha_N(S)$. Hence

$$\mathscr{H}^\alpha(S) \leq \mathscr{S}^\alpha_N(S) \leq 2^\alpha \mathscr{H}^\alpha(S),$$

and $\mathscr{S}_N^\alpha$ is absolutely continuous with respect to $\mathscr{H}^\alpha \llcorner N$.

2.2 Sub-Riemannian manifolds

A *sub-Riemannian manifold of class* $\mathscr{C}^k$ ($k = \infty$ or $k = \omega$ in the analytic case) is a triplet $(M, \mathscr{D}, g)$, where M is a $\mathscr{C}^k$-manifold, $\mathscr{D}$ is a Lie-bracket generating $\mathscr{C}^k$-subbundle of TM of rank $m < \dim M$ and g is a Riemannian metric of class $\mathscr{C}^k$ on $\mathscr{D}$. Using the Riemannian metric, the length of horizontal curves, i.e., absolutely continuous curves which are almost everywhere tangent to $\mathscr{D}$, is well-defined. The Lie-bracket generating assumption implies that the distance d defined as the infimum of length of horizontal curves between two given points is finite and continuous (Rashewski–Chow Theorem). We refer to d as the *sub-Riemannian distance*. The set M endowed with the sub-Riemannian distance d is a metric space (M, d) (often called *Carnot-Carathéodory space*) which has the same topology than the manifold M.

We denote by $\mathscr{D}_q \subset T_q M$ the fiber of $\mathscr{D}$ over q. The subbundle $\mathscr{D}$ can be identified with the module of sections

$$\{X \in \mathrm{Vec}(M) \mid X(q) \in \mathscr{D}_q, \forall q \in M\}.$$

Given $i \geq 1$, define recursively the submodule $\mathscr{D}^i$ of $\mathrm{Vec}(M)$ by

$$\mathscr{D}^1 = \mathscr{D}, \quad \mathscr{D}^{i+1} = \mathscr{D}^i + [\mathscr{D}, \mathscr{D}^i].$$

Set $\mathscr{D}_q^i = \{X(q) \mid X \in \mathscr{D}^i\}$. Notice that the identification between the submodule $\mathscr{D}^i$ and the distribution $q \mapsto \mathscr{D}_q^i$ is no more meaningful when the dimension of $\mathscr{D}_q^i$ varies as a function of q (see the discussion in [3, page 48]). The Lie-bracket generating assumption implies that for every $q \in M$ there exists an integer $r(q)$, the *non-holonomy degree at q*, such that

$$\{0\} \subset \mathscr{D}_q^1 \subset \cdots \subset \mathscr{D}_q^{r(q)} = T_q M. \tag{3}$$

The sequence of subspaces (3) is called the *flag of $\mathscr{D}$ at q*. Set $n_i(q) = \dim \mathscr{D}_q^i$ and

$$Q(q) = \sum_{i=1}^{r(q)} i(n_i(q) - n_{i-1}(q)), \tag{4}$$

where $n_0(q) = 0$.

We say that a point p is *regular* if, for every i, $n_i(q)$ is constant as q varies in a neighborhood of p. Otherwise, the point is said to be *singular*. A subset $A \subset M$ is called *equiregular* if, for every i, $n_i(q)$ is constant as q varies in A. When the whole manifold is equiregular, the integer $Q(q)$ defined in (4) does not depend on q and it is the Hausdorff dimension of (M, d) (see [12]).

Given $p \in M$, let $X_1, \ldots, X_m$ be a local orthonormal frame of $\mathscr{D}$. A multiindex I of length $|I| = j \geq 1$ is an element of $\{1, \ldots, m\}^j$. With any multiindex $I = (i_1, \ldots, i_j)$ is associated an iterated Lie bracket $X_I = [X_{i_1}, [X_{i_2}, \ldots, X_{i_j}] \ldots]$ (we set $X_I = X_{i_1}$ if $j = 1$). The set of vector fields X_I such that $|I| \leq j$ is a family of generators of the module $\mathscr{D}^j$. As a consequence, if the values of $X_{I_1}, \ldots, X_{I_n}$ at $q \in M$ are linearly independent, then $\sum_i |I_i| \geq Q(q)$.

Let Y be a vector field. We define the *length of Y* by

$$\ell(Y) = \min\{i \in \mathbb{N} \mid Y \in \mathscr{D}^i\}.$$

In particular, $\ell(X_I) \leq |I|$. Note that, in general, if a vector field Y satisfies $Y(q) \in \mathscr{D}_q^i$ for every $q \in M$, Y need not be in the submodule $\mathscr{D}^i$. By an *adapted basis* to the flag (3) at q, we mean n vector fields $Y_1, \ldots, Y_n$ such that their values at q satisfy

$$\mathscr{D}_q^i = \mathrm{span}\{Y_j(q) \mid \ell(Y_j) \leq i\}, \quad \forall i = 1, \ldots, r(q).$$

In particular, $\sum_{i=1}^n \ell(Y_i) = Q(q)$. As a consequence, a family of brackets $X_{I_1}, \ldots, X_{I_n}$ such that $X_{I_1}(q), \ldots, X_{I_n}(q)$ are linearly independent is an adapted basis to the flag (3) at q if and only if $\sum_i |I_i| = Q(q)$.

3 Hausdorff dimensions and volumes of strongly equiregular submanifolds

In this section, we answer question 3 when N is a particular kind of submanifold, namely a strongly equiregular one. These results include the case where M itself is equiregular.

3.1 Strongly equiregular submanifolds

Let $N \subset M$ be a smooth connected submanifold of dimension b. The *flag at $q \in N$ of $\mathscr{D}$ restricted to N* is the sequence of subspaces

$$\{0\} \subset (\mathscr{D}_q^1 \cap T_q N) \subset \cdots \subset (\mathscr{D}_q^{r(q)} \cap T_q N) = T_q N. \tag{5}$$

Set

$$n_i^N(q) = \dim(\mathscr{D}_q^i \cap T_q N) \qquad \text{and} \qquad Q_N(q) = \sum_{i=1}^{r(q)} i(n_i^N(q) - n_{i-1}^N(q)),$$

with $n_0^N(q) = 0$.

Definition 1. We say that N is *strongly equiregular* if

(i) N is equiregular, that is, for every i, the dimension $n_i(q)$ is constant as q varies in N.
(ii) for every i, the dimension $n_i^N(q)$ is constant as q varies in N.

In this case, we denote by Q_N the constant value of $Q_N(q)$, $q \in N$.

By an *adapted basis* to the flag (5) at $q \in N$, we mean b vector fields $Z_1, \ldots, Z_b$ such taht

$$\mathscr{D}_q^i \cap T_q N = \mathrm{span}\{Z_j(q) \mid \ell(Z_j) \le i\}, \quad \forall i = 1, \ldots, r(q).$$

In particular, when $Z_1, \ldots, Z_b$ is adapted to the flag (5), we have $T_q N = \mathrm{span}\{Z_1(q), \ldots, Z_b(q)\}$ and $Q_N = \sum_{i=1}^b \ell(Z_i)$.

Recall that the metric tangent cone[1] to (M, d) at any point p exists and it is isometric to $(T_p M, \widehat{d}_p)$, where $\widehat{d}_p$ denotes the sub-Riemannian distance associated with a nilpotent approximation at p (see [3]). The following lemma shows the relevance of strongly equiregular submanifolds as particular subsets of M for which a metric tangent cone exists. Such metric space is isometrically embedded in a metric tangent cone to the whole M at the point.

Lemma 1. *Let $N \subset M$ be a b-dimensional submanifold of M. Assume N is strongly equiregular. Then, for every $p \in N$:*

(i) *there exists a metric tangent cone to $(N, d|_N)$ at p and it is isometric to $(T_p N, \widehat{d}_p|_{T_p N})$;*
(ii) *the graded vector space*

$$\mathfrak{gr}_p^N(\mathscr{D}) := \oplus_{i=1}^{r(p)} (\mathscr{D}_p^i \cap T_p N)/(\mathscr{D}_p^{i-1} \cap T_p N)$$

is a nilpotent Lie algebra whose associated Lie group $\mathrm{Gr}_p^N(\mathscr{D})$ is diffeomorphic to $T_p N$;
(iii) *every b-form $\omega \in \bigwedge^b N$ on N induces canonically a left-invariant b-form $\widehat{\omega}^p$ on $\mathrm{Gr}_p^N(\mathscr{D})$. Moreover,*

$$\int_{N \cap B(p,\varepsilon)} \omega = \varepsilon^{Q_N} \int_{T_p N \cap \widehat{B}_p} \widehat{\omega}^p + o(\varepsilon^{Q_N}), \tag{6}$$

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[1] in Gromov's sense, see [7]

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where $o(\varepsilon^{Q_N})$ is uniform as p varies in N and $\widehat{B}_p$ is the ball centered at 0 of radius 1 in the nilpotent approximation at p of the sub-Riemannian manifold.

Remark 1. When N is an open submanifold of M, assuming N strongly equiregular is equivalent to saying that N contains only regular points. In that case, Lemma 1 is well-known (point (i) follows by the fact that the nilpotent approximation is a metric tangent cone, point (ii) says that the tangent cone shares a group structure - which in this case satisfies the additional property $\mathfrak{gr}_p(\mathscr{D}) = \mathrm{span}_p\{\mathscr{D}^1\}$ - and (iii) has been remarked in [1] using the canonical isomorphism between $\bigwedge^n(\mathfrak{gr}_p(\mathscr{D})^*)$ and $\bigwedge^n(T_p^*M)$.

Proof. Note first that since the result is of local nature, it is sufficient that we prove it on a small neighbourhood $B(p_0,\rho)\cap N$ of a point $p_0 \in N$. For every p in a such a neighbourhood, there exists a coordinate system $\varphi_p : U_p \to \mathbb{R}^n$ on a neighborhood $U_p \subset M$ of p, such that φ_p are privileged coordinates at p, $p \mapsto \varphi_p$ is continuous, and N is rectified in coordinates φ_p, that is $\varphi_p(N\cap U_p) \subset \{x \in \mathbb{R}^n \mid x_{b+1} = \cdots = x_n = 0\}$. The construction is as follows.

Given $\rho > 0$ small enough, we can find b vector fields $Y_1,\ldots,Y_b$ defined on $B(p_0,\rho)$ which form a basis adapted to the flag (5) restricted to N at every $p \in B(p_0,\rho)\cap N$. Moreover, up to reducing ρ, we can find $Y_{b+1},\ldots,Y_n$ such that $Y_1,\ldots,Y_n$ is adapted to the flag (3) of the distribution at every point $p \in B(p_0,\rho)\cap N$. Using these bases, we define for $p \in N\cap B(p_0,\rho)$, a local diffeomorphism $\Phi_p : \mathbb{R}^n \to M$ by

$$\Phi_p(x) = \exp\left(\sum_{i=b+1}^n x_i Y_i\right) \circ \exp\left(\sum_{i=1}^b x_i Y_i\right)(p). \tag{7}$$

The inverse $\varphi_p = \Phi_p^{-1}$ of Φ_p provides a system of coordinates centered at p which are privileged (see [9]). Moreover, thanks to property (i) in Definition 1, the map from $B(p_0,\rho)\cap N$ to M which associates with p the point $\Phi_p(x)$ is smooth for every $x \in \mathbb{R}^n$. Finally, in coordinates φ_p, the submanifold $N\cap U$ coincides with the set

$$\left\{\exp\left(\sum_{i=1}^b x_i Y_i\right)(p) \mid (x_1,\ldots,x_b) \in \Omega\right\} \subset \{\Phi_p(x) \mid x_{b+1} = \cdots = x_n = 0\},$$

where Ω is an open subset of $\mathbb{R}^b$.

Using φ_p we identify M with $T_pM \simeq \mathbb{R}^n$. Since $Y_1(p),\ldots,Y_b(p)$ span T_pN, φ_p maps N in T_pN, where T_pN is identified with $\mathbb{R}^b \times \{0\} \subset \mathbb{R}^n \simeq T_pM$. Therefore, whenever $q_1,q_2 \in U\cap N$ we have

$$\widehat{d}_p(q_1,q_2) = \widehat{d}_p|_{T_pN}(q_1,q_2),$$

and obviously $d(q_1,q_2) = d|_N(q_1,q_2)$. Hence estimate (70) in [3, Theorem 7.32] holds when we restrict d to N and $\widehat{d}$ to T_qN. This allows to conclude that a metric tangent cone to $(N,d|_N)$ at p exists and it is isometric to $(T_pN, \widehat{d}_p|_{T_pN})$, where the inclusion of T_pN into T_pM is to be intended via φ_p.

The algebraic structure of $\mathfrak{gr}_p^N(\mathscr{D})$ and the fact that $Gr_p^N(\mathscr{D})$ is diffeomorphic to $\mathbb{R}^b$ are straightforward. As a consequence, there also exists a canonical isomorphism between $\bigwedge^b(\mathfrak{gr}_p^N(\mathscr{D})^*)$ and $\bigwedge^b(T_p^*N)$. Let $\tilde{\omega}_p$ be the image of ω_p under such isomorphism (see the construction in [13, Section 10.5]). Then $\widehat{\omega}^p$ is defined as the left-invariant b-form on T_pN which coincides with $\tilde{\omega}_p$ at the origin.

Finally, as a consequence of point (i), by definition of metric tangent cone $\varphi_p(B(p,\varepsilon)\cap N)$ converges to $\widehat{B}(0,\varepsilon)\cap T_pN$ in the Gromov–Hausdorff sense as ε goes to 0. By homogeneity of $\widehat{d}_p$ we have $\widehat{B}(0,\varepsilon)\cap T_pN = \varepsilon^{Q_N}(\widehat{B}_p\cap T_pN)$ and we get (6). Since $p \mapsto \varphi_p$ and $p \mapsto \widehat{B}_p$ are continuous [1, Section 4.1], the remainder $o(\varepsilon^{Q_N})$ in (6) is uniform with respect to p. $\qquad\square$

For the sake of completeness, let us give an explicit formula for $\widehat{\omega}^p$. Recall that the construction of the coordinates φ_p involves an adapted basis $Y_1,\ldots,Y_b$ to the flag (5) restricted to N at every $p \in B(p_0,\rho)\cap N$. In particular the vector fields $Y_1,\ldots,Y_b$ restricted to N form a local frame for the tangent bundle to N and

$$\omega = \omega(Y_1,\ldots,Y_b)d(Y_1|_N)\wedge\cdots\wedge d(Y_b|_N).$$

Let $X_1, \ldots, X_m$ be a local orthonormal frame for the sub-Riemannian structure in a neighborhood of p, and $X_{I_1}, \ldots, X_{I_n}$ be an adapted basis to the flag (3) at p, where X_{I_j} is the Lie bracket corresponding to the multi-index I_j. Since $X_{I_1}, \ldots, X_{I_n}$ is a local frame for the tangent bundle to M, for every $i = 1, \ldots, b$ we can write Y_i in this basis as

$$Y_i = \sum_{|I| \leq \ell(Y_i)} Y_i^I X_I,$$

where Y_i^I are smooth function (the fact that only multiindices with length smaller than $\ell(Y_i)$ appear in this sum is due to the definition of length of a vector field). Denote by $\widehat{X}_1^p, \ldots, \widehat{X}_m^p$ the nilpotent approximation of $X_1, \ldots, X_m$ at p obtained in coordinates φ_p, and by $\widehat{X}_{I_j}^p$ the Lie bracket between the $\widehat{X}_1^p, \ldots, \widehat{X}_m^p$ corresponding to the multiindex I_j. For every $i = 1, \ldots, b$ we define the vector field

$$\widehat{Y}_i^p = \sum_{|I| = \ell(Y_i)} Y_i^I(p) \widehat{X}_I.$$

This enables us to compute $\hat{\omega}^p$ as

$$\hat{\omega}^p = \omega_p(Y_1(p), \ldots, Y_b(p)) d(\widehat{Y}_1^p|_{T_pN}) \wedge \cdots \wedge d(\widehat{Y}_b^p|_{T_pN}). \tag{8}$$

The fact that the right-hand side of (8) does not depend on the X_I nor on the Y_i is a consequence of the intrinsic definition of $\hat{\omega}^p$.

3.2 Hausdorff volume

Assume now that N is an orientable submanifold. By a *smooth volume* on N we mean a measure μ associated with a never vanishing smooth form $\omega \in \bigwedge^b N$, i.e., for every Borel set $A \subset N$, $\mu(A) = \int_A \omega$. We will denote by $\hat{\mu}^p$ the smooth volume on T_pN associated with $\hat{\omega}^p$.

We are now in a position to prove the main result.

Theorem 1. *Let $N \subset M$ be a smooth orientable submanifold. Assume N is strongly equiregular. Then, for every smooth volume μ on N,*

$$\lim_{\varepsilon \to 0} \frac{\mathscr{S}_N^{Q_N}(B(q, \varepsilon))}{\mu(N \cap B(q, \varepsilon))} = \frac{\mathrm{diam}_{\widehat{d}_q}(T_qN \cap \widehat{B}_q)^{Q_N}}{\hat{\mu}^q(T_qN \cap \widehat{B}_q)}, \quad \forall q \in N, \tag{9}$$

where $\mathrm{diam}_{\widehat{d}_q}$ denotes the diameter with respect to the distance $\widehat{d}_q$. In particular, $\mathscr{S}_N^{Q_N}$ is absolutely continuous with respect to μ with Radon–Nikodym derivative equal to the right hand side of (9). As a consequence,

$$\dim_H N = Q_N, \tag{10}$$

and, for a small ball $B(p, \rho)$ centered at a point $p \in N$, the Hausdorff volume $\mathscr{H}^{Q_N}(N \cap B(p, \rho))$ is finite.

Remark 2. When N is an open submanifold of M, e.g., $N = \{p \in M \mid p \text{ is regular}\}$, the computation of Hausdorff dimension is well-known, see [12]. In particular, when p is a regular point the top-dimensional Hausdorff measure $\mathscr{H}^Q(B(p, r))$ is positive and finite. When $N = M$, equation (9) gives a new proof to [1, Theorem 1]. This is interesting since the latter was obtained as a consequence of [1, Lemma 32], whose proof is incorrect.

To prove Theorem 1 a fundamental step is the following lemma.

Lemma 2. *Let N and μ be as in Theorem 1. Let $p \in N$. Assume there exists positive constants ε_0 and $\mu_+ > \mu_-$ such that, for every $\varepsilon < \varepsilon_0$ and every point $q \in B(p, \varepsilon_0) \cap N$, there holds*

$$\mu_- \mathrm{diam}(B(q, \varepsilon) \cap N)^{Q_N} \leq \mu(B(q, \varepsilon) \cap N) \leq \mu_+ \mathrm{diam}(B(q, \varepsilon) \cap N)^{Q_N}. \tag{11}$$

Then, for every $\varepsilon < \varepsilon_0$,

$$\frac{\mu(B(p,\varepsilon)\cap N)}{\mu_+} \leq \mathscr{S}_N^{Q_N}(B(p,\varepsilon)) \leq \frac{\mu(B(p,\varepsilon)\cap N)}{\mu_-}.$$

Proof. Let $\bigcup_i B(q_i,r_i)$ be a covering of $B(p,\varepsilon)\cap N$ with balls of radius smaller than $\delta < \varepsilon_0$. If δ is small enough, every q_i belongs to $B(p,\varepsilon_0)\cap N$ and, using (11), there holds

$$\mu(B(p,\varepsilon)\cap N) \leq \sum_i \mu(B(q_i,r_i)\cap N) \leq \mu_+ \sum_i \mathrm{diam}(B(q_i,r_i)\cap N)^{Q_N}.$$

Hence, we have $\mathscr{S}_N^{Q_N}(B(p,\varepsilon)) \geq \frac{\mu(B(p,\varepsilon)\cap N)}{\mu_+}$.

For the other inequality, let $\eta > 0$, $0 < \delta < \varepsilon_0$ and let $\bigcup_i B(q_i,r_i)$ be a covering of $B(p,\varepsilon)\cap N$ such that $q_i \in B(p,\varepsilon)\cap N$ $r_i < \delta$ and $\sum_i \mu(B(q_i,r_i)\cap N) \leq \mu(B(p,\varepsilon))+\eta$. Such a covering exists due to the Vitali covering lemma. Using as above (11), we obtain

$$\mu(B(p,\varepsilon)\cap N)+\eta \geq \sum_i \mu(B(q_i,r_i)\cap N) \geq \mu_- \sum_i \mathrm{diam}(B(q_i,r_i)\cap N)^{Q_N}.$$

We then have $\mathscr{S}_{N,\delta}^{Q_N}(B(p,\varepsilon)) \leq \frac{\mu(B(p,\varepsilon)\cap N)}{\mu_-} + \frac{\eta}{\mu_-}$. Letting η and δ tend to 0, we get the conclusion. $\square$

Proof of Theorem 1. Fix $q \in N$. By point (ii) in Lemma 1 $(T_qN, \widehat{d}_q|_{T_qN})$ is a metric tangent cone to $(N, d|_N)$ at q, whence, from the definition of Gromov–Hausdorff convergence we get

$$\lim_{\varepsilon \to 0} \frac{\mathrm{diam}(N\cap B(q,\varepsilon))}{\varepsilon} = \mathrm{diam}_{\widehat{d}_q}(T_qN\cap \widehat{B}_q). \tag{12}$$

By (6) in Lemma 1, for every $q \in N$ there holds

$$\mu(N\cap B(q,\varepsilon)) = \varepsilon^{Q_N}\widehat{\mu}^q(T_qN\cap \widehat{B}_q)+o(\varepsilon^{Q_N}). \tag{13}$$

Since N is strongly equiregular, the limits in (12) and (13) hold uniformly as q varies in N.

Moreover, adapting the argument in [1, Section 4.1], we deduce that the map $q \mapsto \widehat{\mu}^q(\widehat{B}_q\cap T_qN)$ is continuous on N. As a consequence, for any $\eta > 0$ there exists $\varepsilon_1 > 0$ such that for every $q \in B(p,\varepsilon_1)$ and every $\varepsilon < \varepsilon_1$ we have

$$\mu_- \leq \frac{\mu(N\cap B(q,\varepsilon))}{\mathrm{diam}(N\cap B(q,\varepsilon))^{Q_N}} \leq \mu_+$$

with

$$\mu_\pm = \frac{\widehat{\mu}_q(T_qN\cap \widehat{B}_q)}{\mathrm{diam}_{\widehat{d}_q}(T_qN\cap \widehat{B}_q)^{Q_N}} \pm \eta.$$

Therefore, applying Lemma 2 and letting η tend to 0 we deduce (9).

To show (10), notice that the right-hand side of (9) is continuous and positive as a function of q. Hence, for $\mathscr{S}_N^{Q_N}$-almost every $q \in N$ there exists $\rho > 0$ small enough such that

$$0 < \mathscr{S}^{Q_N}(N\cap B(p,\rho)) < \infty. \tag{14}$$

This is equivalent to (10). $\square$

We end this section by stating a result which gives a weak equivalent of the function $\widehat{\mu}^q(T_qN\cap \widehat{B}_q)$ appearing in Theorem 1. This will be useful in the following to determine whether the Hausdorff volume of a small ball is finite or not. This result stems from the uniform Ball-Box Theorem, [10] and [11, Th. 4.7].

Proposition 1. *Let M be orientable and ϖ be a volume form on M. Let N be an orientable submanifold of M of dimension b, and let ω be a volume form on N, with associated smooth volume μ. Assume N is strongly equiregular*

and set $Q[N]$ equal to the constant value of $Q(q)$, for $q \in N$. Then there exists a constant $C > 0$ such that, for every $q \in N$,

$$\frac{1}{C} v_q \leq \hat{\mu}^q(T_q N \cap \widehat{B}_q) \leq C v_q \quad (i.e.\ \hat{\mu}^q(T_q N \cap \widehat{B}_q) \asymp v_q\ uniformly\ w.r.t.\ q),$$

where $v_q = \max\{ \left(\omega \wedge dX_{I_{b+1}} \wedge \cdots \wedge dX_{I_n}\right)_q (X_{I_1}(q), \ldots, X_{I_n}(q))\}$, the maximum being taken among all n-tuples $(X_{I_1}, \ldots, X_{I_n})$ in $\arg\max\{ \varpi_q(X_{I_1'}(q), \ldots, X_{I_n'}(q)) \mid \sum_i |I_i'| = Q[N]\}$.

In particular, if N is an open equiregular subset of M, i.e., $b = n$, and if μ is the smooth measure on M associated with ϖ, we have

$$\hat{\mu}^q(\widehat{B}_q) \asymp \max\{\varpi_q(X_{I_1'}, \ldots, X_{I_n'}) \mid \sum_i |I_i'| = Q[M]\}, \quad uniformly\ w.r.t.\ q \in M.$$

This proposition, together with Theorem 1, allows to give an estimate of the Hausdorff volume of a subset of N. If $S \subset N$, then

$$\frac{1}{C'} \int_S \frac{1}{v_q} d\mu \leq \mathscr{H}^{Q_N}(S) \leq C' \int_S \frac{1}{v_q} d\mu, \tag{15}$$

where the constant $C' > 0$ does not depend on S.

4 Hausdorff dimensions and volumes of analytic sub-Riemannian manifolds

Let $(M, \mathscr{D}, g)$ be an analytic (C^ω) sub-Riemannian manifold. The set Σ of singular points is an analytic subset of M which admits a Whitney stratification $\Sigma = \bigcup_{i \geq 1} M_i$ by analytic and equiregular submanifolds M_i (see for instance [6]). Denoting $M_0 = M \setminus \Sigma$ the set of regular points, we obtain a Whitney stratification $M = \bigcup_{i \geq 0} M_i$ of M by analytic and equiregular submanifolds. Note that M_0 is an open and dense subset of M, but it may be disconnected. As a consequence, the Hausdorff dimension of M is

$$\dim_H(M) = \max_{i \geq 0} \dim_H(M_i),$$

and the α-dimensional Hausdorff measure of a ball $B(p, \rho)$, $p \in M$ and $\rho > 0$, is

$$\mathscr{H}^\alpha(B(p, \rho)) = \sum_i \mathscr{H}^\alpha(B(p, \rho) \cap M_i).$$

4.1 Hausdorff dimension

The first problem is then to determine the Hausdorff dimension of an equiregular - possibly not strongly equiregular - submanifold.

Lemma 3. *Let N be an analytic and equiregular submanifold of M. Set $\overline{Q}_N := \max_{q \in N} Q_N(q)$. Then*

$$\dim_H(N) = \overline{Q}_N,$$

and $Q_N(q) = \overline{Q}_N$ on an open and dense subset of N.

If moreover N is orientable, then for every smooth measure μ on N, $\mathscr{S}_N^{\overline{Q}_N}$ is absolutely continuous with respect to μ with Radon–Nikodym derivative

$$\frac{d\mathscr{S}_N^{\overline{Q}_N}}{d\mu}(q) = \frac{(\text{diam}_{\widehat{d}_q}(T_q N \cap \widehat{B}_q))^{\overline{Q}_N}}{\widehat{\mu}_q(T_q N \cap \widehat{B}_q)}, \quad for\ \mu\text{-}a.e.\ q \in N. \tag{16}$$

Proof. Since N is analytic and equiregular, it admits a stratification $N = \bigcup_i N_i$ by strongly equiregular submanifolds N_i of N. By Theorem 1, $\dim_H(N_i) = Q_{N_i}$ and thus $\dim_H(N) = \max_i Q_{N_i}$. In particular, $\dim_H(N) \leq \max_{q \in N} Q_N(q)$.

Now, recall that $Q_N(q) = \sum_{i=1}^{r_N} i(n_i^N(q) - n_{i-1}^N(q))$, where $r_N := r(q)$ is constant since N is equiregular, and $n_{r_N}^N(q) = \dim N$. This may be rewritten as

$$Q_N(q) = \sum_{i=0}^{r_N-1} \mathrm{codim}(\mathscr{D}_q^i \cap T_q N), \tag{17}$$

where $\mathrm{codim}(\mathscr{D}_q^i \cap T_q N) = n_{r_N}^N(q) - n_i^N(q)$ is the codimension of $\mathscr{D}_q^i \cap T_q N$ in $T_q N$. The submanifold N being equiregular, $Q_N(q)$ is a lower semi-continuous function on N with integer values. Hence $Q_N(q)$ takes its maximal value $\overline{Q}_N$ on the strata N_i which are open in N, and smaller values on non open strata. Since $Q_{N_i}(q) = Q_N(q)$ when N_i is an open subset of N and $Q_{N_i}(q) < Q_N(q)$ when N_i is a non open subset of N, the first part of the lemma follows.

As for the second part, notice that every non open stratum N_i is of μ-measure zero, since N_i is a subset of N of positive codimension, and of $\mathscr{S}_N^{\overline{Q}_N}$-measure zero, since $\dim_H(N_i) = Q_{N_i} < \overline{Q}_N$. A first consequence is that N is strongly equiregular near μ-a.e. point q. Therefore the measure $\widehat{\mu}_q$ on $T_q N$ is defined μ-a.e. – and so is the right-hand side of (16). Applying then Theorem 1 to every open stratum N_i, we get the conclusion. $\qquad\square$

Corollary 1. $\dim_H(M) = \max\{Q_{M_i}(q) \;:\; i \geq 0, \; q \in M_i\} = \max\{\overline{Q}_{M_i} \;:\; i \geq 0\}$.

4.2 Finiteness of the Hausdorff volume of balls

Let $p \in M$ and $\rho > 0$ (ρ is assumed to be arbitrarily small). The aim of this section is to determine under which conditions the small ball $B(p,\rho)$ has a finite Hausdorff volume $\mathscr{H}^{\dim_H(B(p,\rho))}(B(p,\rho))$. We make first two preliminary remarks.

- If p is a regular point, then there exists a neighbourhood of p in M which is strongly equiregular, and Theorem 1 implies that $\mathscr{H}^{\dim_H(B(p,\rho))}(B(p,\rho))$ is finite. We then assume in the following that p is a singular point.
- The results of this section are local. Up to reducing to a neighbourhood of p, we can assume that M is an oriented manifold with volume form ϖ.

Recall that, by definition, the stratification $M = \bigcup_{i \geq 0} M_i$ is locally finite. That is, there exists a finite set $\mathscr{I}$ of indices such that $p \in \overline{M_i}$ if and only if $i \in \mathscr{I}$, where $\overline{M_i}$ denotes the closure of the stratum M_i. Therefore, for ρ small enough, the ball $B(p,\rho)$ admits a finite stratification $B(p,\rho) = \bigcup_{i \in \mathscr{I}}(B(p,\rho) \cap M_i)$. Applying Corollary 1, the Hausdorff dimension D_p of $B(p,\rho)$ is

$$D_p = \max\{Q_{M_i}(q) \;:\; i \in \mathscr{I}, \; q \in M_i\}.$$

Let $\mathscr{J} \subset \mathscr{I}$ be the subset of indices i such that $\dim_H(M_i) = D_p$. We have

$$\mathscr{H}^{D_p}(B(p,\rho)) = \sum_{i \in \mathscr{J}} \mathscr{H}^{D_p}(B(p,\rho) \cap M_i).$$

Proposition 2. *Let N be an analytic and equiregular submanifold of M, $\dim_H(N) = \overline{Q}_N$. If $p \in N$ and if $\rho > 0$ is small enough, then the Hausdorff volume $\mathscr{H}^{\overline{Q}_N}(B(p,\rho) \cap N)$ is finite.*

Proof. Up to replacing N with a small neighbourhood of p in N, we assume that N is orientable. We then choose a smooth measure μ on N and we have, for ρ small enough, $\mu(B(p,\rho) \cap N) < +\infty$. From Lemma 3,

$$\mathscr{S}_N^{\overline{Q}_N}(B(p,\rho) \cap N) = \int_{B(p,\rho) \cap N} \frac{(\mathrm{diam}_{\widehat{d}_q}(T_q N \cap \widehat{B}_q))^{\overline{Q}_N}}{\widehat{\mu}_q(T_q N \cap \widehat{B}_q)} d\mu.$$

The submanifold N is strongly equiregular near μ-a.e. $q \in N$. We can then apply Proposition 1 near μ-a.e. $q \in N$ and we get

$$\mathscr{S}_N^{\overline{Q}_N}(B(p,\rho) \cap N) \leq C \int_{B(p,\rho) \cap N} \frac{(\mathrm{diam}_{\widehat{d}_q}(T_q N \cap \widehat{B}_q))^{\overline{Q}_N}}{v_q} d\mu.$$

The function $q \mapsto v_q$ is positive and continuous on N, so the integrand function in the previous formula is finite and continuous on N, and we have $\mathscr{S}_N^{\overline{Q}_N}(B(p,\rho) \cap N) \leq \mathrm{Cst}\ \mu(B(p,\rho) \cap N) < +\infty$. Since $\mathscr{H}^{\overline{Q}_N}$ is absolutely continuous with respect to $\mathscr{S}_N^{\overline{Q}_N}$, the conclusion follows. $\qquad\square$

As a consequence, the Hausdorff volume $\mathscr{H}^{D_p}(B(p,\rho))$ is finite if and only if $\mathscr{H}^{D_p}(B(p,\rho) \cap M_i)$ is finite for every stratum M_i such that $\dim_H(M_i) = D_p$ and $p \in \partial M_i$. To go further, we will assume that p is a *typical* singular point, that is, that p satisfies the following assumptions for ρ small enough:

(A1) p belongs to a strongly equiregular submanifold N of M, $N \subset \Sigma$, and $B(p,\rho) \cap \Sigma \subset N$;
(A2) for every $q \in N \cap B(p,\rho)$, there exists a family $X_{I_1}, \dots, X_{I_n}$ such that $\sum_i |I_i| = Q_{\mathrm{reg}}$ and $\mathrm{ord}_q \varpi(X_{I_1}, \dots, X_{I_n}) = \sigma$, where Q_{reg} is the constant value of $Q(q)$ for $q \in M \setminus \Sigma$, and

$$\sigma = \max\{s \in \mathbb{N} \ : \ q \in N \cap B(p,\rho) \text{ and } \sum_i |I_i| = Q_{\mathrm{reg}} \text{ imply } \mathrm{ord}_q \varpi(X_{I_1}, \dots, X_{I_n}) \geq s\}.$$

Let us recall the definition of ord_q (see [3] for details). Given $f \in \mathscr{C}^k(M)$, we say that f has *non-holonomic order at p greater than or equal to s*, and we write $\mathrm{ord}_p f \geq s$ if for every $j \leq s - 1$

$$(X_{i_1} \dots X_{i_j} f)(p) = 0 \quad \forall\ (i_1, \dots, i_j) \in \{1, \dots, m\}^j,$$

where $X_i f$ denotes the Lie derivative of f along X_i. Equivalently, $f(q) = O(d(p,q)^s)$. If moreover we do not have $\mathrm{ord}_p f \geq s + 1$, then we say that f has *non-holonomic order at p equal to s*, and we write $\mathrm{ord}_p f = s$.

Theorem 2. *Assume p satisfies (A1) and (A2). Let Q_N be the constant value of $Q_N(q)$ for $q \in N$, and r_N be the maximal integer i such that $n_i(p) - n_{i-1}(p) > n_i^N(p) - n_{i-1}^N(p)$. Then*

$$\mathscr{H}^{Q_{\mathrm{reg}}}(B(p,\rho) \setminus \Sigma) < \infty \quad \Leftrightarrow \quad \sigma \leq Q(p) - Q_N - r_N.$$

As a consequence,

- *if $Q_{\mathrm{reg}} < Q_N$, then $D_p = Q_N$ and $\mathscr{H}^{D_p}(B(p,\rho))$ is finite;*
- *if $Q_{\mathrm{reg}} \geq Q_N$, then $D_p = Q_{\mathrm{reg}}$ and $\mathscr{H}^{D_p}(B(p,\rho))$ is finite if and only if $\sigma \leq Q(p) - Q_N - r_N$.*

The proof of this theorem is postponed to a forthcoming paper. It relies on the use of Proposition 1.

Remark 3. Assumption (A2) is actually not necessary for the computations. If p satisfies only (A1), we introduce two integers $\sigma_- \leq \sigma_+$:

$$\sigma_+ = \min\{s \in \mathbb{N} \ : \ \forall q \in N \cap B(p,\rho), \exists X_{I_1}, \dots, X_{I_n} \text{ s.t. } \sum_i |I_i| = Q_{\mathrm{reg}} \text{ and } \mathrm{ord}_q \varpi(X_{I_1}, \dots, X_{I_n}) \leq s\},$$
$$\sigma_- = \max\{s \in \mathbb{N} \ : \ \exists \text{ an open subset } \Omega \text{ of } N \cap B(p,\rho) \text{ s.t. } q \in \Omega \text{ and } \sum_i |I_i| = Q_{\mathrm{reg}} \text{ imply } \mathrm{ord}_q \varpi(X_{I_1}, \dots, X_{I_n}) \geq s\}.$$

Assumption (A2) is equivalent to $\sigma_- = \sigma_+ = \sigma$. The generalization of the criterion of Theorem 2 to the case where p satisfies only (A1) is then:

- if $\sigma_+ \leq Q(p) - Q_N - r_N$, then $\mathscr{H}^{Q_{\mathrm{reg}}}(B(p,\rho) \setminus \Sigma) < \infty$;
- if $\sigma_- > Q(p) - Q_N - r_N$, then $\mathscr{H}^{Q_{\mathrm{reg}}}(B(p,\rho) \setminus \Sigma) = \infty$.

Notice that the order σ (and σ_- if p does not satisfies (A2)) always satisfies $\sigma \geq Q(p) - Q_{\mathrm{reg}}$. We thus obtain a simpler criterion for the non finiteness of the Hausdorff volume of a ball.

Corollary 2. *Assume p satisfies (A1). If $0 \leq Q_{\mathrm{reg}} - Q_N < r_N$, then $\mathscr{H}^{D_p}(B(p,\rho)) = \infty$.*

4.3 Examples

Example 1 (the Martinet space). Consider the sub-Riemannian manifold given by $M = \mathbb{R}^3$, $\mathscr{D} = \mathrm{span}\{X_1, X_2\}$,

$$X_1 = \partial_1, \quad X_2 = \partial_2 + \frac{x_1^2}{2} \partial_3,$$

and the metric $dx_1^2 + dx_2^2$. We choose $\varpi = dx_1 \wedge dx_2 \wedge dx_3$, that is, the canonical volume form on $\mathbb{R}^3$.

The growth vector is equal to $(2, 2, 3)$ on the plane $N = \{x_1 = 0\}$, and it is $(2, 3)$ elsewhere. As a consequence, N is the set of singular points. At a regular point, $Q_{\mathrm{reg}} = 4$. Every singular point $p = (0, x_2, x_3)$ satisfies (A1) and we have $Q(p) = 5$, $Q_N = 4$, and $r_N = 1$. Applying Corollaries 1 and 2, we obtain:

$$\dim_H(M) = 4, \quad \text{and} \quad \mathscr{H}^4(B(p, \rho)) < \infty \text{ if } p \text{ regular}, \quad \mathscr{H}^4(B(p, \rho)) = \infty \text{ otherwise.}$$

Thus small balls centered at singular points have infinite Hausdorff volume. This result can also be obtained by a direct computation based on the uniform Ball-Box Theorem, see [11].

Note that the only family $(X_{I_1}, X_{I_2}, X_{I_3})$ such that $\sum_i |I_i| = Q_{\mathrm{reg}}$ is $(X_1, X_2, [X_1, X_2])$. The volume form of this family equals x_1 and it is of order 1 at every point of N. Thus every singular point satisfies assumptions (A1) and (A2) with $\sigma = 1$ ($\sigma = Q(p) - Q_{\mathrm{reg}}$ here).

Example 2. Consider the sub-Riemannian manifold given by $M = \mathbb{R}^4$, $\mathscr{D} = \mathrm{span}\{X_1, X_2, X_3\}$, where

$$X_1 = \partial_1, \quad X_2 = \partial_2 + \frac{x_1^2}{2} \partial_4, \quad X_3 = \partial_3 + \frac{x_2^2}{2} \partial_4,$$

and $g = dx_1^2 + dx_2^2 + dx_4^2$. We choose ϖ as the canonical volume form on $\mathbb{R}^4$.

At a regular point, $Q_{\mathrm{reg}} = 5$. The set of singular points is $N = \{x_1 = x_2 = 0\}$. Every singular point satisfies (A1) and we have $Q(p) = 6$, $Q_N = 4$, and $r_N = 1$. Thus, by Corollary 1, $\dim_H(M) = 5$. However Corollary 2 does not allow to conclude on the finiteness of the Hausdorff volume.

The only families such that $\sum_i |I_i| = Q_{\mathrm{reg}}$ are $(X_1, X_2, X_3, [X_1, X_2])$ and $(X_1, X_2, X_3, [X_2, X_3])$. The volume form applied to these families is equal to x_1 and x_2 respectively, and both of them are of order 1 at every point of N. Thus every singular point satisfies assumptions (A1) and (A2) with $\sigma = 1$ ($\sigma = Q(p) - Q_{\mathrm{reg}}$ here). Applying Theorem 2, we obtain:

$$\dim_H(M) = 5, \quad \text{and} \quad \mathscr{H}^5(B(p, \rho)) < \infty \text{ for any } p \in M.$$

Example 3. Let $M = \mathbb{R}^5$, $\mathscr{D} = \mathrm{span}\{X_1, X_2, X_3\}$,

$$X_1 = \partial_1, \quad X_2 = \partial_2 + x_1 \partial_3 + x_1^2 \partial_5, \quad X_3 = \partial_4 + x_1^k \partial_5,$$

with $k > 2$, and $g = dx_1^2 + dx_2^2 + dx_3^2$. We choose ϖ as the canonical volume form on $\mathbb{R}^5$.

The singular set is $N = \{x_1 = 0\}$. A simple computation shows that every singular point p satisfies (A1) and (A2), and $Q_{\mathrm{reg}} = 7$, $Q(p) = 8$, $Q_N = 7$, $r_N = 1$, and $\sigma = k - 1$. Thus in this example $\sigma > Q(p) - Q_{\mathrm{reg}}$. Now Corollaries 1 and 2 apply and we obtain

$$\dim_H(M) = 7, \quad \text{and} \quad \mathscr{H}^7(B(p, \rho)) < \infty \text{ if } p \text{ regular}, \quad \mathscr{H}^7(B(p, \rho)) = \infty \text{ otherwise.}$$

Example 4. Let $M = \mathbb{R}^5$, $\mathscr{D} = \mathrm{span}\{X_1, X_2, X_3\}$,

$$X_1 = \partial_1, \quad X_2 = \partial_2 + x_1 \partial_3 + x_1^2 \partial_5, \quad X_3 = \partial_4 + (x_1^k + x_2^k) \partial_5,$$

with $k > 2$, and $g = dx_1^2 + dx_2^2 + dx_3^2$. We choose ϖ as the canonical volume form on $\mathbb{R}^5$.

The singular set is $N = \{x_1 = x_2 = 0\}$. Every singular point p satisfies (A1) and (A2) and we have $Q_{\mathrm{reg}} = 7$, $Q(0) = 8$, $Q_N = 6$, $r_N = 1$, and $\sigma = k - 1$. By Corollary 1 and Theorem 2, we obtain

$$\dim_H(M) = 7, \qquad \text{and} \qquad \mathcal{H}^7(B(p,\rho)) < \infty \text{ if } p \text{ regular}, \quad \mathcal{H}^7(B(p,\rho)) = \infty \text{ otherwise.}$$

Note that in this case we do not have $Q_{\text{reg}} - Q_N < r_{\hat{N}}$. This shows that the criterion in Corollary 2 does not provide a necessary condition for the Hausdorff volume to be infinite.

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