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Analisi matematica. — *Some properties of Carnot-Carathéodory balls in the Heisenberg group.* Nota di Roberto Monti, presentata (*) dal Socio M. Miranda.

Abstract. — Using the exact representation of Carnot-Carathéodory balls in the Heisenberg group, we prove that: 1. $|\nabla_{\mathbb{H}^n} d(z,t)| = 1$ in the classical sense for all $(z,t) \in \mathbb{H}^n$ with $z \neq 0$, where d is the distance from the origin; 2. Metric balls are not optimal isoperimetric sets in the Heisenberg group.

Key words: Isoperimetric inequality; Heisenberg group; Eikonal equation.

Riassunto. — *Alcune proprietá delle sfere di Carnot-Carathéodory nel gruppo di Heisenberg.* Usando la rappresentazione esatta per le sfere di Carnot-Carathéodory nel gruppo di Heisenberg, proviamo che: 1. $|\nabla_{\mathbb{H}^n} d(z,t)| = 1$ in senso classico per ogni $(z,t) \in \mathbb{H}^n$ con $z \neq 0$, dove d è la distanza dall'origine; 2. Le sfere metriche non sono insiemi isoperimetrici ottimali nel gruppo di Heisenberg.

1. Introduction

In this *Note* we consider the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ as a metric space endowed with its Carnot-Carathéodory distance d. We shall write an element of the group indifferently $(x, y, t) = (x + iy, t) = (z, t) \in \mathbb{H}^n$ with $z \in \mathbb{C}^n$, $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The group product in $\mathbb{H}^n$ is

$$(1.1) \qquad (\zeta, \tau) \cdot (z, t) = (\zeta + z, \tau + t + 2\,\mathrm{Im}\zeta\bar{z}),$$

and the Lie algebra of the group is generated by the vector fields

$$(1.2) \qquad X_j = \partial_{x_j} + 2y_j\partial_t \text{ and } Y_j = \partial_{y_j} - 2x_j\partial_t, \quad j = 1, \dots, n.$$

In $\mathbb{H}^n$ there are natural translations and dilations. Left translations $\tau_h : \mathbb{H}^n \to \mathbb{H}^n$, $h = (\zeta, \tau) \in \mathbb{H}^n$, are defined by

$$(1.3) \qquad \tau_h(z, t) = (\zeta, \tau) \cdot (z, t).$$

Homogeneous dilations $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$, $\lambda > 0$, are defined by

$$(1.4) \qquad \delta_\lambda(z, t) = (\lambda z, \lambda^2 t).$$

The Carnot-Carathéodory distance d beetween two points is defined as the minumum time necessary to connect them by curves with derivative in the sub-bundle spanned pointwise by the X_j's and Y_j's, and with somehow bounded coefficients. This metric is well behaved with respect to translations and dilations

$$(1.5) \quad d(\tau_h(z,t), \tau_h(\zeta,\tau)) = d((z,t), (\zeta,\tau)), \quad d(\delta_\lambda(z,t), \delta_\lambda(\zeta,\tau)) = \lambda d((z,t), (\zeta,\tau))$$

where $h \in \mathbb{H}^n$ and $\lambda > 0$. See Section 2 for Carnot-Carathéodory metrics and see [23] for an exhaustive introduction to the Heisenberg group.

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The exact equations for geodesics in $\mathbb{H}^n$ yield parametric equations for the surface of Carnot-Carathéodory metric balls. These equations show that the distance $d(z, t) = d((z, t), 0)$ is regular outside the center of the group. Now, the natural intrinsic gradient in the Heisenberg group is $\nabla_{\mathbb{H}^n} = (X_1, \dots, X_n, Y_1, \dots, Y_n)$, whose squared modulus, when applied to a differentiable function, is $|\nabla_{\mathbb{H}^n} u|^2 = \sum_{j=1}^n (X_j u)^2 + (Y_j u)^2$. The first result contained in this paper is that the distance has modulus of the gradient equal to one

$$|\nabla_{\mathbb{H}^n} d(z, t)| = 1\,, \ \text{ for all } (z, t) \in \mathbb{H}^n \text{ with } z \neq 0\,. \tag{1.6}$$

This is quite analogous to the euclidean case. In fact, if $d(x) = |x|$ with $x \in \mathbb{R}^n$ then one immediately sees that $|\nabla d(x)| = 1$ for all $x \neq 0$.

The second result is the negative answer to the question whether metric balls are optimal sets for the isoperimetric problem in $\mathbb{H}^n$. The isoperimetric inequality in $\mathbb{H}^n$ is well established (see [22] and the generalizations [9, 13] where the related references can be found). We show that, within the class of sets with given measure metric balls have *not* minimum perimeter. This will follow by a simple convexity observation.

Property (1.6) suggests that among all homogeneous distances on $\mathbb{H}^n$ the Carnot-Carathéodory metric is really the most «intrinsic» one. On the contrary, the lack of the isoperimetric property for the balls in $\mathbb{H}^n$ points out an interesting difference from the euclidean setting.

The plan of the paper is as follows. Section 2 contains the preliminar material: we state some properties of Carnot-Carathéodory spaces and we recall the Pontryagin Maximum Principle to derive the geodesics equations. In particular we show by a simple proof that geodesics in the Heisenberg group are regular. In Section 3 we study the Carnot-Carathéodory metric of $\mathbb{H}^1$ and prove (1.6). In Section 4 we discuss the isoperimetric inequality in $\mathbb{H}^1$ and prove that metric balls in the Heisenberg group are not isoperimetric.

It is now my pleasant task to acknowledge with gratitude Francesco Serra Cassano, Daniele Morbidelli and Ermanno Lanconelli, who first introduced me in the study of the Heisenberg group.

2. Geodesics in Carnot-Carathéodory spaces

Consider a family $X = \{X_1, \dots, X_m\}$ of $C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ vector fields. One can define on $\mathbb{R}^n$ a new metric in the following way. A Lipschitz path $\gamma : [0, T] \longrightarrow \mathbb{R}^n$ is said to be subunit with respect to the fields if there exist measurable coefficients $h = (h_1, \dots, h_m)$ such that

$$\dot\gamma(s) = \sum_{j=1}^m h_j(s) X_j(\gamma(s))\,, \quad \text{and} \ \sum_{j=1}^m h_j^2(s) \le 1 \qquad \text{for a.e. } s \in [0, T]\,. \tag{2.7}$$

Then we define

$$d(x, y) = \inf\{T \ge 0 : \text{ there exists a subunit path } \gamma : [0, T] \to \mathbb{R}^n \text{ joining } x \text{ to } y\}.$$

It is not always possible to connect two points by a subunit path but if this happens d turns out to be a metric. In the sequel we shall always assume the X-connectivity. The metric space $(\mathbb{R}^n, d)$ is usually called Carnot-Carathéodory or subriemannian space (see for instance [2, 19, 24] for a general introduction to subriemannian geometry).

There are other different ways to define the metric d. Here we are interested in the characterization of d by means of Control Theory. A Lipschitz path $\gamma : [0, 1] \to \mathbb{R}^n$ is admissible if $\dot{\gamma}(s) = \sum_{j=1}^{m} h_j(s) X_j(\gamma(s))$ with measurable coefficients h_j. Define the «length» of γ as $l_2(\gamma) = \left(\int_0^1 |h(s)|^2 ds \right)^{\frac{1}{2}}$ and define

$$d_2(x, y) = \inf\{ l_2(\gamma) : \gamma \text{ is an admissible path joining } x \text{ to } y \}.$$

It is not immediately obvious that the definitions are actually equal, in fact

$$d = d_2. \tag{2.8}$$

For the proof see [18].

Definition 2.1. A Lipschitz-continuous subunit path $\gamma : [0, T] \longrightarrow \mathbb{R}^n$ is said to be a geodesic if $d(\gamma(0), \gamma(T)) = T$.

Theorem 2.2. *Assume that $\mathbb{R}^n$ is X-connected. Then every pair of points $x, y \in \mathbb{R}^n$ can be connected by a geodesic γ. Moreover $d(\gamma(0), \gamma(s)) = s$ for any $s \in [0, T]$.*

For a proof of this theorem see [17, Theorem 1.10].

Subriemannian geodesics are solutions of the following *control problem* (see [24, 19, 4, 8]). Taking into account (2.8), the cost functional to minimize is

$$J(h) = \frac{1}{2} \int_0^1 |h(s)|^2 ds = \int_0^1 L(h(s)) ds, \tag{2.9}$$

where L is the lagrangian, and $h \in L^2([0, 1]; \mathbb{R}^m)$ are the controls. The state equation is

$$\dot{x} = B(x) h = \sum_{j=1}^{m} h_j X_j(x) \tag{2.10}$$

where $x \in Lip([0, 1]; \mathbb{R}^n)$ and B is the $n \times m$-matrix which has the vector fields $X_1, \dots, X_m$ as columns. The constraints are $x(0) = x_0$ e $x(1) = x_1 \in \mathbb{R}^n$.

Assume that the problem is controllable. Then, in force of Theorem 2.2 the problem of minimizing J has a solution. A pair (x, h) is said to be *optimal* if the control h minimizes the functional (2.9) and x satisfies almost everywhere the corresponding state equation (2.10) and the constraints. The Pontryagin Maximum Principle gives necessary condition in order that a pair (x, h) be optimal.

Theorem 2.3 (Pontryagin Maximum Principle). *If the pair (x, h) is optimal then there exist a constant $\lambda \in \{0, 1\}$ and an absolutely continuous function $\xi \in AC([0, 1]; \mathbb{R}^n)$ such that:*

(*i*) $|\xi(s)| + \lambda \neq 0$ *for all* $s \in [0\,,1]$;
(*ii*) $\dot{\xi} = -\frac{\partial}{\partial x}\xi^T Bh$ *for a.e.* $s \in [0\,,1]$;
(*iii*) *finally*

$$\langle \xi(s)\,,Bh(s)\rangle - \lambda L(h(s)) = \max_{u\in\mathbb{R}^m}\langle \xi(s)\,,Bu\rangle - \lambda L(u)$$

for a.e. $s \in [0\,,1]$.

For the proof of this classical result we refer for example to the book of Barbu [1]. Before stating a regularity Lemma for geodesics, we draw some consequences from the Maximum Principle. If $(h\,,x)$ is an optimal pair which corresponds to the case $\lambda = 1$, then we deduce from (*iii*) the explicit expression for the optimal control

$$h = B^T\xi. \tag{2.11}$$

Thus, equations (*ii*) and (2.10), which hold for almost every $s \in [0\,,1]$, transform in the well known system of Hamilton equations (see for example [2])

$$\begin{cases} \dot{x} = \dfrac{1}{2}\dfrac{\partial H(x\,,\xi)}{\partial \xi} \\ \dot{\xi} = -\dfrac{1}{2}\dfrac{\partial H(x\,,\xi)}{\partial x}\,, \end{cases} \tag{2.12}$$

where the hamiltonian is $H(x\,,\xi) = \sum_{j=1}^m \langle X_j(x)\,,\xi\rangle^2$. We remark that $H(x\,,\xi)$ is the symbol of the laplacian («sum of squares») of the fields.

If $\lambda = 0$ the situation is less nice. From (*i*) in the Maximum Principle we deduce $\xi(s) \neq 0$. Moreover (*iii*) becomes $\langle B^T\xi(s)\,,h\rangle = \max_{u\in\mathbb{R}^m}\langle B^T\xi(s)\,,u\rangle$. This can happen only if $B^T\xi(s) \equiv 0$, and this means that

$$\langle X_j(x(s))\,,\xi(s)\rangle \equiv 0\,, \quad j = 1\,,\ldots,m\,. \tag{2.13}$$

The dual function ξ is orthogonal to the fields. In this case one is not allowed to conclude that the Hamilton equations are satisfied. Geodesics corresponding to the case $\lambda = 0$ are usually called singular or abnormal (see [20]).

Lemma 2.4. *Suppose that the vector fields* $X_1\,,\ldots,X_m$ *are of class* $C^\infty(\mathbb{R}^n)$ *and that* $\mathbb{R}^n$ *is* X*-connected. If* $(x\,,h)$ *is an optimal pair which corresponds to the case* $\lambda = 1$*, then* $x \in$ $\in C^\infty([0\,,1])$.

Proof. Theorems 2.2 and 2.3 show that $x(t)$ a Lipschitz-continuous function which satisfies (with its dual function ξ) equations (2.12) almost everywhere. Since the dual variable ξ is absolutely continuous, the explicit expression (2.11) for h shows that the control h, which *a priori* is only L^2, actually is absolutely continuous. Thus from the state equation (2.10) we deduce that $\dot{x}$ is absolutely continuous. This proves that $x \in C^1([0\,,1])$. Moreover

$$\ddot{x} = (\frac{d}{dt}B(x(s)))h + B(x(s))\dot{h}\,, \quad \text{a.e. in } [0\,,1]\,.$$

Note that by (*ii*) $\dot{\xi}$ is absolutely continuous, and thus by (2.11) $\dot{h} \in AC([0,1])$. Then the expression for $\ddot{x}$ shows that $\ddot{x} \in AC([0,1])$, and thus $x \in C^2([0,1])$. Proceeding in the same way we find by induction that $x \in C^\infty([0,1])$. $\square$

It is possible to write down explicitly the subriemannian geodesics in the Heisenberg group. We have to check that the case $\lambda = 0$ in the Maximum Principle cannot occur.

LEMMA 2.5. *Geodesics in $\mathbb{H}^n$ are curves of class C^∞.*

PROOF. Let $h_1, h_2 \in L^2([0,1];\mathbb{R}^n)$ be the controls and write $h = (h_1, h_2)^T$. Write $(z,t) = (x,y,t)$ and $(\zeta,\tau) = (\xi,\eta,\tau)$. If $B = \mathrm{col}\,[X_1, \dots, X_n, Y_1, \dots, Y_n]$ is the matrix of the vector fields, then $Bh = \sum_{j=1}^m h_{1j}X_j + h_{2j}Y_j = (h_1, h_2, 2\langle y, h_1\rangle - 2\langle x, h_2\rangle)^T$. Equations (*ii*) in the Maximum Principle give

$$\begin{cases} \dot{\xi}(s) = 2\beta h_2 \\ \dot{\eta}(s) = -2\beta h_1 \\ \tau(s) = \beta \end{cases}$$

with $\beta \in \mathbb{R}$. Suppose by contradiction that the optimal pair $((z,t), h)$ corresponds to the case $\lambda = 0$. Using the orthogonality condition (2.13) we find

$$\begin{cases} \xi + 2\beta y = 0 \\ \eta - 2\beta x = 0 \end{cases}$$

and from this we see that it must be $\beta \neq 0$. Indeed, if $\beta = 0$ then $\zeta(s) \equiv 0$ and this is not possible because of (*i*) in the Maximum Principle.

Differentiating the previous equations, and substituting above we get

$$\begin{cases} \dot{x}(s) = -h_1(s) \\ \dot{y}(s) = -h_2(s). \end{cases}$$

But, from the state equation (2.10) we find

$$\begin{cases} \dot{x}(s) = h_1(s) \\ \dot{y}(s) = h_2(s) \end{cases}$$

and thus $h(s) \equiv 0$. This is a contradiction and the case $\lambda = 0$ is excluded. $\square$

Geodesics in $\mathbb{H}^n$. Geodesics equations in the Heisenberg group can be found in the literature [15, 2], but without complete proofs. The regularity of Heisenberg geodesics is also a special case of [24]. Here we give a self contained exposition. In the case of Grushin vector fields in $\mathbb{R}^2$ the problem of geodesics was solved by Franchi in [8].

Lemma 2.5 proves that geodesics in $\mathbb{H}^n$ are solutions of the Hamiltonian system (2.12) with

$$H((z,t),(\zeta,\tau)) = \sum_{j=1}^n (\xi_j + 2y_j\tau)^2 + (\eta_j - 2x_j\tau)^2.$$

One finds the equations

$$\begin{cases} \dot{x}_j = \xi_j + 2\tau y_j \\ \dot{y}_j = \eta_j - 2\tau x_j \\ \dot{t} = \displaystyle\sum_{j=1}^{n} 2y_j\xi_j + 4\tau y_j^2 - 2x_j\eta_j + 4\tau x_j^2 \\ \dot{\xi}_j = 2\tau\eta_j - 4\tau^2 x_j \\ \dot{\eta}_j = -2\tau\xi_j - 4\tau^2 y_j \\ \dot{\tau} = 0\,. \end{cases}$$

Taking the initial data $(z(0)\,,\,t(0))=(0\,,0)$ and $(\zeta(0)\,,\tau(0))=(B_1,\dots,B_n\,,\,A_1,\dots,A_n\,,\phi/4)$, we find the solutions

$$\text{(2.14)}\qquad \begin{cases} x_j(s) = \dfrac{A_j(1-\cos\phi s) + B_j\sin\phi s}{\phi} \\ y_j(s) = \dfrac{-B_j(1-\cos\phi s) + A_j\sin\phi s}{\phi} \\ t(s) = 2\dfrac{\phi s - \sin\phi s}{\phi^2}\displaystyle\sum_{j=1}^{n}(A_j^2 + B_j^2)\,. \end{cases}$$

In the limit case $\phi = 0$ one gets the euclidean geodesics $(x(s)\,,\,y(s)\,,\,t(s)) = (Bs\,,\,As\,,\,0)$. The correct normalization is $\sum_{j=1}^{n}(A_j^2 + B_j^2) = 1$.

REMARK 2.6. If γ is a geodesic then the translated path $\tau_h\gamma$ is still a geodesic. This shows that the hypothesis $(z(0)\,,\,t(0)) = (0\,,\,0)$ is not restrictive. Notice also that, if $\gamma : [0\,,\,T] \to \mathbb{H}^n$ is a geodesic joining $(z_o\,,t_o)$ to $(z_1\,,t_1)$, then the path $\gamma_\lambda : [0\,,\,\lambda T] \to \mathbb{H}^n$ defined by $\gamma_\lambda(s) = \delta_\lambda\gamma(\frac{s}{\lambda})$ is a geodesic joining $\delta_\lambda(z_o\,,\,t_o)$ to $\delta_\lambda(z_1\,,\,t_1)$. If γ corresponds to the initial datum $(B\,,\,A\,,\,\frac{\phi}{4})$ then γ_λ solves Hamilton equations with initial datum $(B\,,\,A\,,\,\frac{\phi}{4\lambda})$. Now, by a general result in Control Theory, normal extremals are always locally optimal (see [19, Appendix C]). Since the renormalized dilation of a geodesic is still a geodesic, this automatically proves that equations (2.14) and their translated are all the geodesics in $\mathbb{H}^n$. Notice also that the points $(0\,,\,0)$ and $(z\,,\,t)$ can be connected only by one geodesic if $(z\,,\,t) \notin Z$ where $Z = \{(z\,,\,t) \in \mathbb{H}^n : z = 0\}$ is the center of the group.

3. Regularity of the Carnot-Carathéodory metric in the Heisenberg group

We now deduce from equations (2.14) a parametrization of the surface of the unitary metric ball centered at the origin. In order to simplify some calculations we fix $n = 1$. Let $S = \{(x\,,\,y\,,\,t) \in \mathbb{H}^1 : d((x\,,\,y\,,\,t)\,,\,0) = 1\}$. First note that the velocity of a geodesic γ is

$$\dot{\gamma}(s) = (A\sin\phi s + B\cos\phi s)X(\gamma(s)) + (A\cos\phi s - B\sin\phi s)\,Y(\gamma(s))\,,$$

with $A^2 + B^2 = 1$. The time s is exactly the Carnot-Carathéodory distance between 0 and $\gamma(s)$. If we choose $s=1$ and set $A=\cos\theta$ and $B=\sin\theta$ we obtain

$$\begin{cases} x(\theta\,,\phi) = \dfrac{\cos\theta(1-\cos\phi)+\sin\theta\sin\phi}{\phi} \\ y(\theta\,,\phi) = \dfrac{-\sin\theta(1-\cos\phi)+\cos\theta\sin\phi}{\phi} \\ t(\theta\,,\phi) = 2\dfrac{(\phi-\sin\phi)}{\phi^2}\,, \end{cases} \tag{3.15}$$

with $0\le\theta\le 2\pi$ and $-2\pi\le\phi\le 2\pi$. From equations (3.15) one sees that the surface S is of class C^1 where $z=(x\,,y)\ne 0$.

It is now easy to show that the function $(z\,,t)\to d(z\,,t)=d((z\,,t)\,,0)$ is of class C^1 for $z\ne 0$. Set $A=\{(\theta\,,\phi\,,\rho)\in\mathbb{R}^3 : -2\pi\le\phi\rho\le 2\pi\,,\ \rho\ge 0\}$ and define $\Phi : A\longrightarrow\mathbb{H}^1$ by $\Phi(\theta\,,\phi\,,\rho)=(x(\theta\,,\phi\,,\rho)\,,y(\theta\,,\phi\,,\rho)\,,t(\theta\,,\phi\,,\rho))$, where

$$\begin{cases} x(\theta\,,\phi\,,\rho) = \dfrac{\cos\theta(1-\cos\phi\rho)+\sin\theta\sin\phi\rho}{\phi} \\ y(\theta\,,\phi\,,\rho) = \dfrac{-\sin\theta(1-\cos\phi\rho)+\cos\theta\sin\phi\rho}{\phi} \\ t(\theta\,,\phi\,,\rho) = 2\dfrac{(\phi\rho-\sin\phi\rho)}{\phi^2}. \end{cases} \tag{3.16}$$

The range of Φ is $\mathbb{H}^1$. In fact, if $\rho>0$ is fixed, then equations (3.16) with $\theta\in[0\,,2\pi)$ and $-\frac{2\pi}{\rho}\le\phi\le\frac{2\pi}{\rho}$ parametrize $\partial B(0\,,\rho)$. One can compute the determinant of the jacobian

$$\det J\Phi(\theta\,,\phi\,,\rho) = 4\frac{\phi\rho\sin\phi\rho-2(1-\cos\phi\rho)}{\phi^4}.$$

It is easily seen that the equation $s\sin s+2\cos s=2$ has the solutions $s=0\,,\pm 2\pi$ for $|s|\le 2\pi$. This means that

$$\det J\Phi(\theta\,,\phi\,,\rho)=0$$

if and only if $\phi\rho=\pm 2\pi$ or $\rho=0$ (the case $\phi=0$ must be excluded). The set of the points $\Phi(\theta\,,\phi\,,\rho)$ with $\phi\rho=\pm 2\pi$ is exactly the center $Z=\{(z\,,t)\in\mathbb{H}^1 : z=0\}$. By the inverse function Theorem the function Φ is a local diffeomorphism in the open set $\{(\theta\,,\phi\,,\rho) : \rho>0 \text{ and } |\phi\rho|<2\pi\}$. By the definition of the distance d we have $\Psi(\theta\,,\phi\,,\rho):=d(\Phi(\theta\,,\phi\,,\rho))=\rho$. Thus the function Ψ is of class C^∞. Now write $d=\Psi\circ\Phi^{-1}$ to show that d is of class C^1 outside the center.

We now state a general result recently proved, which will be needed. If d is a metric on $\mathbb{R}^n$ generated by a family of vector fields and $f:\mathbb{R}^n\to\mathbb{R}$ is a Lipschitz function with respect to d, then f is differentiable almost everywhere along the fields (for the proof see [11, 14]).

THEOREM 3.7. *Let* $(\mathbb{R}^n\,,d)$ *be a Carnot-Carathéodory space associated with a family of vector fields* $X=(X_1\,,\dots\,,X_m)$, *and suppose that*

1. *$d(x, y) < \infty$ for all $x, y \in \mathbb{R}^n$.*
2. *d is continuos with respect to the euclidean topology.*

Then, for every L-Lipschitz function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ there exists $X_j f(x)$ for almost all $x \in \mathbb{R}^n$, $j = 1, \dots, m$, and $|Xf(x)| = \left(\sum_{j=1}^m (X_j f(x))^2\right)^{\frac{1}{2}} \leq L$ a.e.

We are now in position to prove our result.

Theorem 3.8. *Consider $\mathbb{H}^n$ endowed with its Carnot-Carathéodory metric d. Then we have*

$$|\nabla_{\mathbb{H}^n} d(z, t)| = 1 \tag{3.17}$$

for all $(z, t) \in \mathbb{H}^n$ such that $z \neq 0$.

Proof. The Heisenberg group satisfies the hypotheses of Theorem 3.7, and thus, since d is clearly 1-Lipschitz

$$|\nabla_{\mathbb{H}^n} d(z, t)| \leq 1$$

almost everywhere on $\mathbb{H}^n$. Now fix a point $(z, t) \in \mathbb{H}^n \setminus Z$ where this inequality holds. Choose a geodesic $\gamma : [0, T] \to \mathbb{H}^n$ joining 0 to (z, t). In particular γ is subunit

$$\dot{\gamma}(s) = \sum_{j=1}^n h_{1j} X_j(\gamma(s)) + h_{2j} Y_j(\gamma(s)), \qquad \text{and} \qquad \sum_{j=1}^n \quad h_{1j}(s)^2 + h_{2j}(s)^2 \leq 1.$$

If $z \neq 0$ we may assume that $\gamma(s) \notin Z = \{(z, t) \in \mathbb{H}^n : z = 0\}$ for $s > 0$. If we differentiate the identity $s = d(\gamma(s))$ (Theorem 2.2, regularity of γ and regularity of d) we find

$$1 = \frac{d}{ds} d(\gamma(s)) = \langle Dd(\gamma(s)), \dot{\gamma}(s) \rangle = \sum_{j=1}^n h_{1j}(s) X_j d(\gamma(s)) + h_{2j}(s) Y_j d(\gamma(s)) \leq |\nabla_{\mathbb{H}^n} d(\gamma(s))|$$

for all $s \in (0, T]$. Choosing $s = T$ we get

$$|\nabla_{\mathbb{H}^n} d(z, t)| \geq 1.$$

Thus we find $|\nabla_{\mathbb{H}^n} d(z, t)| = 1$ almost everywhere in $\mathbb{H}^n$. But $\nabla_{\mathbb{H}^n} d(z, t)$ is continuous outside the center Z and thus $|\nabla_{\mathbb{H}^n} d(z, t)| = 1$ when $z \neq 0$. The proof is complete. □

Remark 3.9. The partial derivatives $X_j d$ and $Y_j d$, which are defined only in $\mathbb{H}^n \setminus Z$, cannot be separately extended with continuity to the whole $\mathbb{H}^n$. Thus one cannot hope to extend the eikonal equation (3.17) to $\mathbb{H}^n$.

Remark 3.10. If $K \subset \mathbb{H}^n$ is a closed set define $d_K(z, t) = \inf_{(\zeta, \tau) \in K} d((z, t), (\zeta, \tau))$. The function d_K needs not be differentiable in the classical sense. Nonetheless, the derivatives along the Heisenberg vector fields exist a.e. and the eikonal equation $|\nabla_{\mathbb{H}^n} d_K(z, t)| = 1$ holds for a.e. $(z, t) \in \mathbb{H}^n \setminus K$. This improvement of (3.17) will be proved in [21].

EXAMPLE 3.11. If $g \in L^1(\mathbb{H}^n)$ and $f : \mathbb{H}^n \to \mathbb{R}$ is Lipschitz with respect to d then one can prove the following intrinsic coarea formula

$$\int_{\mathbb{H}^n} g(z,t)|\nabla_{\mathbb{H}^n} f(z,t)|\, dzdt = \int_{-\infty}^{+\infty} \int_{\{f=s\}} g(z,t)\, dP(E_s)\, ds \tag{3.18}$$

where $E_s = \{(z,t) \in \mathbb{H}^n : f(z,t) > s\}$ and $P(E_s)$ is the perimeter-measure $P(E_s)(\Omega) = P(E_s;\Omega)$ (see [10, 13 and next section]). Actually formula (3.18) is a particular case of a more general coarea formula for vector fields (see [21]). If $g : [0,\infty) \to \overline{\mathbb{R}}$ is such that $g(d(z,t)) \in L^1(\mathbb{H}^n)$ formula (3.18) with $f = d$ reads (see also [7, Proposition 1.15])

$$\int_{\mathbb{H}^n} g(d(z,t))\, dzdt = P(B(0,1)) \int_0^\infty g(s) s^{Q-1}\, ds,$$

where $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$. In fact, if $P(B(0,s))$ is the Heisenberg perimeter of $B(0,s)$ then $P(B(0,s)) = s^{Q-1} P(B(0,1))$ for $s > 0$ (see next section).

4. ISOPERIMETRIC PROBLEM IN $\mathbb{H}^1$: A NEGATIVE ANSWER

In this section we show that, given a metric ball in $\mathbb{H}^1$ there exists a new set with the same measure, but with strictly smaller Heisenberg perimeter.

Let us introduce the notion of perimeter in $\mathbb{H}^n$. This is a particular case of a more general construction for vector fields originally proposed in [10, 13]. Fix an open set $\Omega \subset \mathbb{H}^n$, and if $\phi \in C^1_o(\Omega; \mathbb{R}^{2n})$ is a vector valued function define the Heisenberg divergence of ϕ

$$\mathrm{div}_{\mathbb{H}^n} \phi(z,t) = \sum_{j=1}^n X_j \phi_j(z,t) + Y_j \phi_{j+n}(z,t).$$

Now take $f \in L^1(\Omega)$ and set

$$\|\nabla_{\mathbb{H}^n} f\|(\Omega) = \sup\left\{ \int_\Omega f(z,t)\, \mathrm{div}_{\mathbb{H}^n}\phi(z,t)\, dzdt : \phi \in C^1_o(\Omega;\mathbb{R}^{2n}),\ |\phi| \le 1 \right\}.$$

If $\|\nabla_{\mathbb{H}^n} f\|(\Omega) < +\infty$ we say that $f \in BV_{\mathbb{H}^n}(\Omega)$. Now, let $E \subset \mathbb{H}^n$ be measurable with finite measure. Define the perimeter of E in Ω

$$P(E;\Omega) = \sup\left\{ \int_{\Omega\cap E} \mathrm{div}_{\mathbb{H}^n}\phi(z,t)\, dzdt : \phi \in C^1_o(\Omega;\mathbb{R}^{2n}),\ |\phi| \le 1 \right\}.$$

We write $P(E) = P(E;\mathbb{H}^n)$ and say that E has finite perimeter if $P(E) < \infty$. By means of Riesz Theorem $P(E;\cdot)$ is a Radon measure on $\mathbb{R}^{2n+1}$.

The variational perimeter measure is the correct way to define an intrinsic surface measure on $\mathbb{H}^n$. There are at least three reasons for that. First, this measure is, as in the euclidean setting, lower semicontinuous with respect to the $L^1(\mathbb{H}^n)$ convergence of sets. Together with a compactness theorem this is a crucial condition in order to have existence in minimum problems. Moreover, if $E \subset \mathbb{H}^n$ is a bounded open set with C^∞ boundary then $P(E)$ coincides with the Minkowski content of ∂E (see [21]). Finally, if E is a set with finite perimeter it can be proved that (up to a multiplicative geometric constant) $P(E;\cdot) = \mathcal{S}^{Q-1} \llcorner \partial^* E$, where $\mathcal{S}^{Q-1}$ is the $(Q-1)$-dimensional *spherical* Hausdorff measure in the CC metric d and $\partial^* E$ is the reduced boundary of E (to be understood in a suitable sense, see [12]). Thus all reasonable definitions of surface measure in $\mathbb{H}^n$ seem to coincide (on regular sets) with the perimeter.

LEMMA 4.12. *Fix $\lambda > 0$, $h \in \mathbb{H}^n$ and $E \subset \mathbb{H}^n$ with finite perimeter. Then $P(\tau_h(E)) = P(E)$ and $P(\delta_\lambda(E)) = \lambda^{Q-1} P(E)$.*

PROOF. Let $\phi \in C^1_o(\mathbb{H}^n; \mathbb{R}^{2n})$ with $|\phi| \le 1$. Since X_j and Y_j are left invariant

$$\int_{\tau_h(E)} \mathrm{div}_{\mathbb{H}^n} \phi(z, t)\, dzdt = \int_E \mathrm{div}_{\mathbb{H}^n} \phi(\tau_h(z, t))\, dzdt = \int_E \mathrm{div}_{\mathbb{H}^n} (\phi \circ \tau_h)(z, t)\, dzdt$$

with $\phi \circ \tau_h \in C^1_o(\mathbb{H}^n; \mathbb{R}^{2n})$ and $|\phi \circ \tau_h| \le 1$. Thus $P(\tau_h E) \le P(E)$. Beginning from E one gets the other inequality. This proves that $P(\tau_h(E)) = P(E)$.

As far as the dilation property is concerned, note first the homogeneous property of the fields

$$(X_j \psi(\delta_\lambda(z, t)) = \frac{1}{\lambda} X_j (\psi \circ \delta_\lambda)(z, t), \quad (Y_j \psi(\delta_\lambda(z, t)) = \frac{1}{\lambda} Y_j (\psi \circ \delta_\lambda)(z, t).$$

Thus

$$\int_{\delta_\lambda(E)} \mathrm{div}_{\mathbb{H}^n} \phi(z, t)\, dzdt = \int_E \mathrm{div}_{\mathbb{H}^n} \phi(\delta_\lambda(z, t)) \lambda^Q dzdt = \lambda^{Q-1} \int_E \mathrm{div}_{\mathbb{H}^n} (\phi \circ \delta_\lambda)(z, t))\, dzdt.$$

Since $\phi \circ \delta_\lambda \in C^1_o(\mathbb{H}^n; \mathbb{R}^{2n})$ and $|\phi \circ \delta_\lambda| \le 1$, we get $P(\delta_\lambda(E)) \le \lambda^{Q-1} P(E)$. The other inequality follows analogously. □

REMARK 4.13. Suppose that the open set $\Omega \subset \mathbb{H}^n$ satisfies $\tau_h(\Omega) = \Omega$ for some $h \in \mathbb{H}^n$. The same proof as in the Lemma shows that $P(E; \Omega) = P(\tau_h(E); \Omega)$.

The isoperimetric inequality in $\mathbb{H}^1$ was originally established by Pansu [22]. It is well known that – in the euclidean framework – this inequality is equivalent to the Sobolev imbedding theorem. This approach works in the more general context of Carnot-Carathéodory spaces (see [9, 13]). It is an open problem the proof of the isoperimetric inequality in $\mathbb{H}^n$ by means of the direct method of the Calculus of Variations. Indeed, we have the following compactness theorem (see [13]).

THEOREM 4.14. *If $\Omega \subset \mathbb{H}^n$ is a bounded and sufficiently regular open set then the imbedding $BV_{\mathbb{H}^n}(\Omega) \subset L^1(\Omega)$ is compact.*

Now fix $k > 0$ and consider the following minimum problem for $r > 0$ large enough

$$\min\{P(E) : E \subset B(0, r) \text{ measurable and such that } |E| = k\}. \tag{4.19}$$

Using the compactness theorem and the lower semicontinuity of the perimeter it is easy to show that this problem has a solution. Now, try to solve the problem

$$\min\{P(E) : E \subset \mathbb{H}^n \text{ bounded, measurable and such that } |E| = k\}. \tag{4.20}$$

One can always take a minimizing sequence of sets $E_h \subset \mathbb{H}^n$, $h \in \mathbb{N}$. But, since we do not have the condition $E_h \subset B(0, r)$, the sets could loose their measure by shrinking at infinity. Within the euclidean setting De Giorgi proved in his celebrated paper [6] that the minimum is a sphere thus providing the value of the optimal constant in the isoperimetric inequality: given a set with finite measure one can always find a sphere with the same measure and less perimeter. The same question naturally arises in the Heisenberg group. Are metric balls solutions of problem (4.20)? Unfortunately the answer is negative. This is a remarkable difference between the Heisenberg group and the euclidean case. It is well known that balls are optimal isoperimetric sets also in some riemannian manifolds such as spherical and Lobachevsky spaces (see [5]). Furthermore, if one considers $\mathbb{R}^n$ with a suitable Finsler metric, the balls induced by this metric are isoperimetric (see [3]).

REMARK 4.15. One can easily check that $|\tau_h(E)| = |E|$ and $|\delta_\lambda(E)| = \lambda^Q |E|$ for every Lebesgue measurable set $E \subset \mathbb{H}^n$, $h \in \mathbb{H}^n$ and $\lambda > 0$. Here $|\cdot|$ stands for the Lebesgue measure. Moreover, if $\Omega \subset \mathbb{H}^n$ is an open set and E has finite Heisenberg perimeter then $P(E; \Omega) = P(\Omega \setminus E; \Omega)$.

PROPOSITION 4.16. *The Carnot-Carathéodory ball in* $\mathbb{H}^1$ *is not a solution of problem* (4.20).

PROOF. Let $B = B(0, 1)$ be the metric ball centered at the origin, and fix $k > 0$ in (4.20) such that $|B| = k$. We shall construct a new measurable set $A \subset \mathbb{H}^1$ such that $|A| = k$ and $P(A) < P(B)$.

The surface $S = \partial B$ of B has equations (3.15). These equations are invariant under the orthogonal transformations of $\mathbb{R}^3 = \mathbb{H}^1$ that fix the t-coordinate. Thus, B and S have the same invariance. Moreover, B is not convex. To see this fact put

$$\frac{\partial t(\theta, \phi)}{\partial \phi} = 0 \qquad \text{to find} \quad -\phi - \phi\cos\phi + 2\sin\phi = 0\,,$$

which has the solutions $\phi = \pi$ and $\phi = -\pi$. To these angular coordinates there correspond the points of the surface with maximum and minimum height. We define

$$\begin{aligned} S^+ &= \{(x(\theta, \phi), y(\theta, \phi), t(\theta, \phi)) \in S : \pi < \phi \leq 2\pi\}\,, \\ S^- &= \{(x(\theta, \phi), y(\theta, \phi), t(\theta, \phi)) \in S : -2\pi \leq \phi < -\pi\}. \end{aligned}$$

The basic idea is to replace S^+ with S^- by a translation and *vice versa*. The set with this new boundary has the same perimeter as the original one but its measure is greater. If we contract that set, we can find the desired set A.

Here are some of the details. Fix $h = (0, 0, \frac{4}{\pi})$ (note that $\frac{4}{\pi}$ is the t-size of B) and define a new surface Σ obtained from S in the following way. S^+ is replaced by $\tau_h(S^-)$ and S^- is replaced by $\tau_{-h}(S^+)$. The surface Σ is continuous and of class C^∞ outside a set of null euclidean 2-Hausdorff measure. Call D the open and bounded set whose boundary is Σ. It is straigthforward that $|D| > |B|$.

Call G the projection of S^+ onto the plane xy, and consider the open set $\Omega = G \times \mathbb{R}$. If B^+ is the open set in Ω with «cap» S^+, then by Remark 4.15 we get

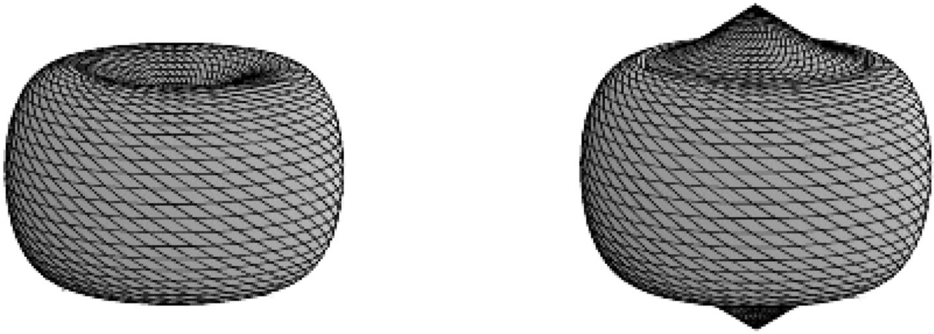

Fig. 1. – The ball B and the set D.

$P(B^+;\Omega) = P(\tau_h(B^+);\Omega) = P(\Omega \setminus \tau_h(B^+);\Omega)$. The same argument shows that $\tau_h(S^-)$ and S^- have the same perimeter-measure. This means that B and D can be split into parts with the same perimeter. Thus

$$P(B) = P(B;\mathbb{H}^n \setminus \Omega) + P(B;\Omega) = P(D;\mathbb{H}^n \setminus \Omega) + P(D;\Omega) = P(D).$$

Now fix $\lambda > 0$ such that $|\delta_\lambda(D)| = k = |B|$ which gives

$$\lambda = \left(\frac{k}{|D|}\right)^{\frac{1}{4}} < 1.$$

If we define $A = \delta_\lambda(D)$ then $|A| = |B|$ and using the homogeneous property of the perimeter we see that

$$P(A) = \lambda^3 P(D) = \lambda^3 P(B) < P(B). \quad \square$$

References

[1] V. Barbu, *Mathematical methods in optimization of differential systems*. Kluwer, Dordrecht 1994.

[2] A. Bellaïche, *The tangent space in subriemannian geometry*. In: A. Bellaïche - J. Risler (eds), *Sub-Riemannian Geometry*. Progress in Mathematics, 144, Birkhäuser, Basel 1996.

[3] G. Bellettini - M. Paolini - S. Venturini, *Some results in surface measure in Calculus of Variations*. Ann. Mat. Pura Appl., (4), 170, 1996, 329-357.

[4] R.W. Brockett, *Control theory and singular Riemannian geometry*. In: P.J. Hilton - G.S. Young, *New directions in applied mathematics*. Springer-Verlag, New York 1981.

[5] Yu.D. Burago - V.A. Zalgaller, *Geometric inequalities*. Springer-Verlag, Berlin 1988.

[6] E. De Giorgi, *Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita*. Atti Acc. Lincei Mem. fis., s. 8, v. 5, 1958, 33-44.

[7] G.B. Folland - E.M. Stein, *Hardy spaces on homogeneous groups*. Princeton University Press, 1982.

[8] B. Franchi, *Stime subellittiche e metriche riemanniane singolari II*. Seminario di Analisi Matematica, Università di Bologna, 1983, VIII-1 VIII-17.

[9] B. Franchi - S. Gallot - R.L. Wheeden, *Sobolev and isoperimetric inequalities for degenerate metrics*. Math. Ann., 300, 1994, 557-571.

[10] B. Franchi - R. Serapioni - F. Serra Cassano, *Meyers-Serrin Type Theorems and Relaxation of Variational Integrals Depending Vector Fields*. Houston Journal of Mathematics, 22, 4, 1996, 859-889.

[11] B. Franchi - R. Serapioni - F. Serra Cassano, *Approximation and Imbedding Theorems for Weighted Sobolev Spaces Associated with Lipschitz Continuous Vector Fields*. Bollettino U.M.I., 7, 11-B, 1997, 83-117.
[12] B. Franchi - R. Serapioni - F. Serra Cassano, *Rectifiability and perimeter in the Heisenberg group*. Preprint 2000.
[13] N. Garofalo - D.M. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces*. Comm. Pure Appl. Math., 49, 1996, 1081-1144.
[14] N. Garofalo - D.M. Nhieu, *Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot Carathéodory spaces*. Jour. Analyse Mathematique, 74, 1998, 67-97.
[15] B. Gaveau, *Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents*. Acta Math., 139, 1977, 1-2, 95-153.
[16] M. Gromov, *Carnot-Carathéodory spaces seen from within*. In: A. Bellaïche - J. Risler (eds.), *Sub-Riemannian Geometry*. Progress in Mathematics, 144, Birkhäuser, Basel 1996.
[17] M. Gromov, *Structures métriques pour les variétés riemanniennes*. CEDIC, Paris 1981.
[18] D. Jerison - A. Sanchez Calle, *Subelliptic, second order differential operators*. In: *Complex analysis III*. Springer, 1987, 46-77.
[19] W. Liu - H.J. Sussmann, *Shortest paths for sub-Riemannian metrics on rank-two distributions*. Mem. Amer. Math. Soc., 118, 1995.
[20] R. Montgomery, *Abnormal minimizers*. SIAM J. Control Optim., 32, 1994, 1605-1620.
[21] R. Monti - F. Serra Cassano, *Surface measures in Carnot-Carathéodory spaces*. Forthcoming.
[22] P. Pansu, *Une inégalité isopérimétrique sur le groupe de Heisenberg*. C.R. Acad. Sci. Paris, 295, I, 1982, 127-130.
[23] E.M. Stein, *Harmonic Analysis*. Princeton University Press, 1993.
[24] R.S. Strichartz, *Sub-Riemannian geometry*. J. Differential Geom., 24, 1986, 221-263. [*Corrections to* «*Sub-Riemannian geometry*». J. Differential Geom., 30, 1989, 595-596].

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