Pure and Random strategies in differential game with incomplete informations*

P. Cardaliaguet[1], C. Jimenez[2], M. Quincampoix[2]

[1] CEREMADE, Université Paris-Dauphine,

Place du Maréchal de Lattre de Tassigny, 75016 Paris, France

[2] Laboratoire de Mathématiques de Bretagne Atlantique, CNRS-UMR 6205, Université de Brest,

6, avenue Victor Le Gorgeu, CS 93837, 29238 Brest cedex 3, France.

cardaliaguet@ceremade.dauphine.fr; chloe.jimenez@univ-brest.fr; marc.quincampoix@univ-brest.fr.

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Abstract

We investigate a two players zero sum differential game with incomplete information on the initial state: The first player has a private information on the initial state while the second player knows only a probability distribution on the initial state. This could be view as a generalization to differential games of the famous Aumann-Maschler framework for repeated games. In an article of the first author, the existence of the value in random strategies was obtained for a finite number of initial conditions (the probability distribution is a finite combination of Dirac measures). The main novelty of the present work consists in : first extending the existence of a value result in random strategies for infinite number of initial conditions and second - and mainly - proving the existence of a value in pure strategies when the initial probability distribution is regular enough (without atoms).

1 Introduction

We consider a two-player, zero-sum differential game with dynamics

$$\begin{cases} x'(t) &= f(x(t), u(t), v(t)) \quad u(t) \in U, \, v(t) \in V \\ x(t_0) &= x_0 \end{cases}$$

and terminal cost $g : \mathbb{R}^N \to \mathbb{R}$, which is evaluated at a terminal time $T > 0$. The first player acts on the system through his control $u(\cdot)$ in order to minimize a final cost

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$g(X(T))$ while the second player wants to maximize $g(X(T))$ by choosing his control $v(\cdot)$.

Let us now describe how the game is played: fix an initial time $t_0 \in [0,T]$.
- before the game starts, the initial position x_0 is chosen randomly according to a probability measure μ_0,
- the initial state x_0 is communicated to Player I but not to Player II,
- the game is played on the time interval $[t_0,T]$,
- both players know the probability μ_0 and observe their opponents controls.

Such a game with incomplete information (the first player has a private information not available for the second player) was introduced in the 1960's in the framework of repeated games by Aumann and Maschler and are extensively studied since then (see for instance [3]).

For differential games, the similar problem were introduced by the first author in [6]: in this paper the existence of a value in mixed strategies was obtained when the unknown information for the players belongs to a finite set. Further investigations and generalizations on this topics can be found in [4, 8] and the references therein. In all these works the private information is given as a finite number of types. The case of a continuum of types for games in continuous times has been addressed only recently in [9], in a very particular situation where there is no dynamics and where the information issue lies on the payoff.

In the game we investigate here, the role of the information is crucial. Indeed the second Player does not know what the current state of the game is. However he can try to guess it—at least partially—by observing the actions of the first Player. For this reason the first Player's interest is to hide as much as possible his actions by playing randomly (choosing a random strategy), of course still trying to achieve his own goals. The second Player's interest is also to reveal at least his action by playing random strategies.

The main phenomenon that appears here lies in the fact that when the initial measure μ_0 has no atoms, one can built on it a "kind of randomness" which avoids the use of random strategies. This is precisely this phenomenon that is explained in our main result (Theorem 4.1) of the paper. Such a statement is reminiscent of the existence of pure strategies in noncooperative, nonatomic games: see in particular Schmeidler [13]. Note however that the frameworks are very different. This is also related to the notion of purification of strategies (e.g. for instance [11]).

The paper is organized as follows: the first section concerns basic fact on probability measure spaces, the complete description of the model, and a brief summary of results and methods of proof. Section 2 is devoted to the regularity of the values in random strategies. As a byproduct of this regularity, the existence of the value in random strategies is obtained for arbitrary probability measure μ_0. The last section contains our main result showing the existence of a value in pure strategy when μ_0 has no atoms.

2 Preliminaries

2.1 Probability Distribution on the initial condition

Notations: Throughout the paper $|\cdot|$ denotes the euclidian norm in the ambient space (in general $\mathbb{R}^N$). Given a Lipschitz continuous function ϕ, we let $lip(\phi)$ denote its Lipschitz constant. Finally, for $m \in \mathbb{N}^*$, $\mathcal{L}^m$ is the Lebesgue measure on $\mathbb{R}^m$.

Throughout the paper, we will restrict ourselves to Borel probability measures μ_0 on $\mathbb{R}^N$ with compact support (denoted by $supp(\mu_0)$). We denote by $\mathcal{P}(\mathbb{R}^N)$ the set of such probability measures. It is well-known that $\mathcal{P}(\mathbb{R}^N)$ can be endowed with the Wasserstein distance

$$W_2(\mu,\nu) := \inf_{\gamma\in\Pi(\mu,\nu)} \left\{ \left(\int_{\mathbb{R}^{2N}} |x-y|^2 d\gamma(x,y) \right)^{\frac{1}{2}} \right\}$$

where $\Pi(\mu,\nu)$ is the set of probability measures γ on $\mathbb{R}^{2N}$ which has μ as first marginal and ν as second one. It is known that the infimum is actually a minimum. Such optimal measures γ are then called optimal plan from μ to ν (see [14]). It is well known that the distance W_2 is compatible with the weak convergence of measures (cf. for instance [14]).

For $\mu \in \mathcal{P}(\mathbb{R}^N)$ and $\phi : \mathbb{R}^N \to \mathbb{R}^N$ a Borel measurable with at most a linear growth, we denote by $\phi\sharp\mu$ the push-forward of μ by ϕ, i.e., the measure in $\mathcal{P}(\mathbb{R}^N)$ such that

$$\phi\sharp\mu(A) = \mu\left(\phi^{-1}(A)\right) \qquad \forall A \subset \mathbb{R}^N, \text{ Borel measurable.}$$

Let us recall the following result (cf. [1] and [12]) that we will use several times in the paper:

Proposition 2.1 *Let $m \in \mathbb{N}$ and P and Q be two Borel probability measures on $\mathbb{R}^m$ with a compact support. If P has no atom, there exists a sequence $(h_n)_n$ of Borel measurable maps from $\mathbb{R}^m$ to $\mathbb{R}^m$ such that:*

$$h_n\sharp P = Q \text{ and } \lim_{n\to+\infty} \int_{\mathbb{R}^m} |x-h_n(x)|^2 \, dP(x) = W_2^2(P,Q).$$

If, moreover, P is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^m$, there exists a unique Borel measurable map $h : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$h\sharp P = Q \text{ and } \int_{\mathbb{R}^m} |x-h(x)|^2 \, dP(x) = W_2^2(P,Q).$$

2.2 Model, values and Strategies

2.2.1 Dynamics

We consider a two-player zero-sum differential game with dynamics given by the controlled differential equation

$$(1) \qquad \left\{ \begin{array}{rcl} x'(t) &=& f(x(t),u(t),v(t)) \quad u(t)\in U,\, v(t)\in V \\ x(t_0) &=& x_0 \end{array} \right.$$

In the above equation, $t_0 \in [0,T]$ is the initial time—T being the finite horizon of the game—and $x_0 \in \mathbb{R}^N$ is the initial position. We denote by U and V are the sets of actions for each player, U for the first one and V for the second one; we assume that U and V are compact subsets of some finite dimensional spaces. The dynamics $f : \mathbb{R}^N \times U \times V \to \mathbb{R}^N$

is continuous in all variables, Lipschitz continuous in the state variable, and bounded. We will denote by $\mathcal{U}(t_0)$ (respectively $\mathcal{V}(t_0)$) the set of measurable controls $u(\cdot) : [t_0, T] \mapsto U$ (respectively $v(\cdot) : [t_0, T] \mapsto V$).

The sets of controls $\mathcal{U}(t_0)$ and $\mathcal{V}(t_0)$ are endowed with the $L^1_U[t_0, T]$ and $L^1_V[t_0, T]$ topology associated with the following distance : for u_1 and u_2 in $L^1_U[t_0, T]$, $d_{L^1_U}(u_1, u_2) := \int_{t_0}^T d_U(u_1(t), u_2(t))dt$ where d_U denotes the distance on the compact metric space U (the definition of $d_{L^1_V}$ is similar).

Under our assumptions, to any pair of controls $(u(\cdot), v(\cdot)) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ one can associate in a unique way a solution to (1) that will be denoted by $t \mapsto X_t^{t_0,x_0,u,v}$.

Throughout the paper we will assume that Isaacs' condition holds:

$$\forall x \in \mathbb{R}^N,\ \forall \xi \in \mathbb{R}^N,\ \inf_{u\in U} \sup_{v\in V} f(x,u,v).\xi = \sup_{v\in V} \inf_{u\in U} f(x,u,v).\xi. \tag{2}$$

Let us recall that this condition is generally used for differential games with perfect information in order to prove the existence of a value.

2.2.2 Pure strategies and values in pure strategies

The main novelty of the differential game studied in this paper lies in its information structure: given a measure $\mu_0 \in \mathcal{P}(\mathbb{R}^N)$, we suppose that
- before the game starts, the initial position x_0 is chosen randomly according to a probability measure μ_0,
- the initial state x_0 is communicated to Player I but not to Player II,
- the game is played on the time interval $[t_0, T]$,
- both players know the probability μ_0 and observe their opponents controls.

Because of this structure of information, the strategies of the players should be defined according only their available information. This leads to the following notions of strategies (compare with [4, 6]).

Definition 2.2 *A pure strategy for Player II is a Borel measurable map*[1] $\beta : \mathcal{U}(t_0) \mapsto \mathcal{V}(t_0)$ *which is nonanticipative with delay (NAD in short): namely there exists* $\tau_\beta > 0$ *such that for any* $u_1, u_2 \in \mathcal{U}(t_0)$*, for any* $t \in [t_0, T)$*, if* $u_1 = u_2$ *a.e. on* $[t_0, t]$*, then* $\beta(u_1) = \beta(u_2)$ *a.e. on* $[t_0, (t + \tau_\beta) \wedge T]$.

A pure strategy for Player I is a Borel measurable map:

$$\begin{array}{rccc} \alpha : & \mathbb{R}^N \times \mathcal{V}(t_0) & \to & \mathcal{U}(t_0) \\ & (x_0, v(\cdot)) & \mapsto & \alpha(x_0, v). \end{array}$$

for which there is a delay $\tau_\alpha > 0$ *such that, for any* $x_0 \in \mathbb{R}^N$*, the map* $\alpha(x_0, \cdot) : \mathcal{V}(t_0) \mapsto \mathcal{U}(t_0)$ *is nonanticipative with delay* τ_α .

The set of pure strategies for Player I (resp. Player II) is denoted by $A(t_0)$ *(resp.* $B(t_0)$*).*

[1] This means that the measurability property is considered when $\mathcal{U}(t_0)$ and $\mathcal{V}(t_0)$ are endowed with the Borel σ-field associated with $L^1_U[t_0, T]$ and $L^1_V[t_0, T]$.

Observe that, in order to formalize the information of the players, the definition of their strategies is not symmetric: Player I knows the initial state x_0, but this initial state is not known by Player II.

In order to write the game in a normal form we need the following:

Lemma 2.3 *For any pair of pure strategies $(\alpha, \beta) \in A(t_0) \times B(t_0)$, and any initial condition $x_0 \in \mathbb{R}^N$, there is a unique pair $(u_{x_0}, v_{x_0}) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$, such that*

$$\alpha(x_0, v_{x_0}) = u_{x_0} \text{ and } \beta(u_{x_0}) = v_{x_0} \ . \tag{3}$$

Furthermore the map $x_0 \mapsto (u_{x_0}, v_{x_0}) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ is Borel measurable.

The proof of this Lemma is a particular case of the proof of Lemma 2.6 (stated later on for the random strategies). We denote by $t \mapsto X_t^{t_0,x_0,\alpha(x_0,\cdot),\beta(\cdot)}$ the solution $x(\cdot)$ to (1) with the controls (u_{x_0}, v_{x_0}).

It is now time to introduce the payoff: let $g : \mathbb{R}^N \to \mathbb{R}$ be a Lipschitz continuous and bounded terminal map. The first player acts on the system by choosing the control $u(\cdot)$, his goal being to minimize a final cost $g(X(T))$ while the second player wants to maximize $g(X(T))$ by choosing the control $v(\cdot)$.

The *upper value function in pure strategies* is:

$$V^+(\mu_0) := \inf_{\alpha \in A(t_0)} \sup_{\beta \in B(t_0)} \int_{\mathbb{R}^N} g(X_T^{t_0,x,\alpha(x,\cdot),\beta(\cdot)}) \, d\mu_0(x), \tag{4}$$

while the *lower value in pure strategies* can be defined in the same way by:

$$V^-(\mu_0) := \sup_{\beta \in B(t_0)} \inf_{\alpha \in A(t_0)} \int_{\mathbb{R}^N} g(X_T^{t_0,x,\alpha(x,\cdot),\beta(\cdot)}) \, d\mu_0(x). \tag{5}$$

We will use the following notation:

$$J(\mu_0, \alpha, \beta) = \int_{\mathbb{R}^N} g(X_T^{t_0,x,\alpha(x,\cdot),\beta(\cdot)}) \, d\mu_0(x).$$

2.2.3 Random strategies and values in random strategies

Now we will give the definition of random strategy:

Definition 2.4 *Let S be the set of triples $(\Omega, \mathcal{F}, P)$ such that $\Omega = [0,1]^m$ for some m, $\mathcal{F}$ is a σ-field contained in the class of Borel sets $B([0,1]^m)$ and P a probability measure on $(\Omega, \mathcal{F})$.*

A random strategy for Player 1 is a pair $((\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha), \alpha)$ where $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha) \in S$, such that there exists a delay $\tau_\alpha > 0$ with

1- the map $\alpha : \mathbb{R}^N \times \Omega_\alpha \times V(t_0) \to U(t_0)$ is Borel measurable,

2- for any $\omega_\alpha \in \Omega_\alpha$ and $x \in \mathbb{R}^N$, the strategy $v \in \mathcal{V}(t_0) \mapsto \alpha(x, \omega_\alpha, v)$ is non anticipative with delay τ_α.

Similarly a random strategy for Player II is a pair $((\Omega_\beta, \mathcal{F}_\beta, P_\beta), \beta)$ where $\beta : \Omega_\beta \times U(t_0) \to V(t_0)$ is a Borel measurable map and there exists a delay $\tau_\beta > 0$ such that for all $\omega_\beta \in \Omega_\beta$ $\beta(\omega_\beta, \cdot) : \mathcal{U}(t_0) \mapsto \mathcal{V}(t_0)$ is nonanticipative with delay τ_β.

The set of random strategies for Player I (resp. Player II) is denoted by $A_r(t_0)$ (resp. $B_r(t_0)$).

Remark 2.5 *Our random strategy concept—borrowed from [6]—is closely related with Aumann's notion of strategies [2]. It is worth pointing out that it slightly differs from the mixed strategies for differential games introduced in [5]. In contrast to repeated games, there is no well admitted definition of mixed strategies or behavior strategies for differential games, at least up to now. Finding a concept which would allow to dispense of Isaacs' condition* (2) *is a difficult task and is out of the scope of the paper.*

Lemma 2.6 *For any pair of random strategies* $(\alpha, \beta) \in A_r(t_0) \times B_r(t_0)$*, for any* $\omega := (\omega_\alpha, \omega_\beta) \in \Omega_\alpha \times \Omega_\beta$ *and for any initial condition* $x_0 \in \mathbb{R}^N$*, there is a unique pair* $(u_{\omega,x_0}, v_{\omega,x_0}) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$*, such that*

$$\alpha(x_0, \omega_\alpha, v_{\omega,x_0}) = u_{\omega,x_0} \text{ and } \beta(\omega_\beta, u_{\omega,x_0}) = v_{\omega,x_0} \ . \tag{6}$$

Furthermore the map $(\omega, x_0) \mapsto (u_{\omega,x_0}, v_{\omega,x_0}) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ *is Borel measurable.*

Proof : Existence and uniqueness of $(u_{\omega,x_0}, v_{\omega,x_0})$ is due to the delays and is proved in [7]. We show here the measurability of $(\omega, x_0) \in \Omega_\alpha \times \Omega_\beta \times \mathbb{R}^N \mapsto (u_{\omega,x_0}, v_{\omega,x_0})$ when $\Omega_\alpha \times \Omega_\beta \times \mathbb{R}^N$ is equipped with the σ-field $\mathcal{F}_\alpha \otimes \mathcal{F}_\beta \otimes \mathcal{B}(\mathbb{R}^N)$ and $\mathcal{U}(t_0)$ and $\mathcal{V}(t_0)$ are endowed with the Borel σ-fields associated with $L^1_U[t_0, T]$ and $L^1_V[t_0, T]$.

Define $\tau := \min\{\tau_\alpha, \tau_\beta\}$. We will prove by induction on the integer n that the map $(\omega, x_0) \mapsto (u_{\omega,x_0}, v_{\omega,x_0})$ is measurable onto $L^1_U[t_0, t_0 + n\tau] \times L^1_V[t_0, t_0 + n\tau]$.

Consider the case $n = 1$. Fix $(\bar{u}, \bar{v}) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$. For any $\omega := (\omega_\alpha, \omega_\beta)$ and $x \in \mathbb{R}^n$ the maps $\alpha(x, \omega_\alpha, \cdot)$ and $\beta(\omega_\beta, \cdot)$ are nonanticipative with delay τ. So the restrictions of $\alpha(x, \omega_\alpha, \bar{v})$ and $\beta(\omega_\beta, \bar{u})$ on $[t_0, t_0 + \tau]$ are independent on $\bar{u}$ and $\bar{v}$. Thus the measurability of

$$(x, \omega) \mapsto (u_{\omega,x_0}, v_{\omega,x_0})_{|[t_0,t_0+\tau]} = (\alpha(x, \omega_\alpha, \bar{v}), \beta(\omega_\beta, \bar{u}))_{|[t_0,t_0+\tau]} \in L^1_U[t_0, t_0+\tau] \times L^1_V[t_0, t_0+\tau]$$

is derived from the fact that α and β are measurable.

Consider now $n > 1$ and suppose that the map $(\omega, x_0) \mapsto (u_{\omega,x_0}, v_{\omega,x_0})$ is measurable onto $L^1_U[t_0, t_0 + n\tau] \times L^1_V[t_0, t_0 + n\tau]$. Let us prove that this still holds true when n is replaced by $n + 1$. Fix $(\bar{u}, \bar{v}) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$. For any $u \in \mathcal{U}(t_0)$ we denote by $u_{|[t_0,t_0+n\tau]} \odot \bar{u}_{|[t_0+n\tau,T]}$ the measurable control which restriction on $[t_0, t_0 + n\tau]$ is u and which restriction on $[t_0 + n\tau, T]$ is $\bar{u}$. Clearly the map

$$\Upsilon_U : u \in \mathcal{U}(t_0) \mapsto u_{|[t_0,t_0+n\tau]} \odot \bar{u}_{|[t_0+n\tau,T]} \in \mathcal{U}(t_0)$$

is continuous for the L^1_U norm. We clearly have the same property for a map Υ_V similarly defined in $\mathcal{V}(t_0)$. Because of the nonanticipativity property, the restrictions of $\alpha(x, \omega_\alpha, u_{|[t_0,t_0+n\tau]} \odot \bar{u}_{|[t_0+n\tau,T]})$ and $\beta(\omega_\beta, v_{|[t_0,t_0+n\tau]} \odot \bar{v}_{|[t_0+n\tau,T]})$ on $[t_0 + n\tau, t_0 + (n+1)\tau]$ do not depend on $\bar{u}$ and $\bar{v}$. Then because

$$(u_{\omega,x_0})_{|[t_0,t_0+(n+1)\tau]} = (\alpha(x, \omega_\alpha, (v_{\omega,x_0})_{|[t_0,t_0+n\tau]} \odot \bar{v}_{|[t_0+n\tau,T]}))_{|[t_0,t_0+(n+1)\tau]}$$

and

$$(v_{\omega,x_0})_{|[t_0,t_0+(n+1)\tau]} = (\beta(\omega_\beta, (u_{\omega,x_0})_{|[t_0,t_0+n\tau]} \odot \bar{u}_{|[t_0+n\tau,T]}))_{|[t_0,t_0+(n+1)\tau]}$$

do not depend on $\bar{u}$ and $\bar{v}$, the map

$$(x,\omega) \mapsto (u_{\omega,x_0}, v_{\omega,x_0})_{|[t_0,t_0+(n+1)\tau]} \in L^1_U[t_0,t_0+(n+1)\tau] \times L^1_V[t_0,t_0+(n+1)\tau]$$

is measurable as a composition of the maps α β, Υ_U, Υ_V and $(\omega,x_0) \mapsto (u_{\omega,x_0}, v_{\omega,x_0})_{|[t_0,t_0+n\tau]}$ which are measurable.

The result follows by induction.

QED

Lemma 2.6 allows to associate a trajectory $t \mapsto X_t^{t_0,x,\alpha(x,\omega,\cdot)\beta(\omega',\cdot)}$ to any pair of random strategies $(\alpha,\beta) \in A_r(t_0) \times B_r(t_0)$, any $\upsilon := (\omega,\omega') \in \Omega_\alpha \times \Omega_\beta$ and any initial condition $x \in \mathbb{R}^N$. So we may now define the *values of the game in random strategies*:

$$(7) \quad V_r^+(\mu_0) := \inf_{\alpha \in A_r(t_0)} \sup_{\beta \in B_r(t_0)} \int_{\Omega_\alpha} \int_{\Omega_\beta} \int_{\mathbb{R}^N} g(X_T^{t_0,x,\alpha(x,\omega,\cdot)\beta(\omega',\cdot)})\; d\mu_0(x) dP_\alpha(\omega) dP_\beta(\omega'),$$

$$(8) \quad V_r^-(\mu_0) := \sup_{\beta \in B_r(t_0)} \inf_{\alpha \in A_r(t_0)} \int_{\Omega_\alpha} \int_{\Omega_\beta} \int_{\mathbb{R}^N} g(X_T^{t_0,x,\alpha(x,\omega,\cdot)\beta(\omega',\cdot)})\; d\mu_0(x) dP_\alpha(\omega) dP_\beta(\omega').$$

We will use the notation:

$$J_r(\mu_0,\alpha,\beta) = \int_{\Omega_\alpha} \int_{\Omega_\beta} \int_{\mathbb{R}^N} g(X_T^{t_0,x,\alpha(x,\omega,\cdot)\beta(\omega',\cdot)})\; d\mu_0(x) dP_\alpha(\omega) dP_\beta(\omega').$$

Remark 2.7 *As usually for random strategies, one can show that the upper value does not change if the first player plays a random strategy against a pure strategy of the second player:*

$$(9) \qquad V_r^+(\mu_0) := \inf_{\alpha \in A_r(t_0)} \sup_{\beta \in B(t_0)} \int_{\Omega_\alpha} \int_{\mathbb{R}^N} g(X_T^{t_0,x,\alpha(x,\omega,\cdot)\beta(\cdot)})\; d\mu_0(x) dP_\alpha(\omega),$$

Similarly for the lower value we have

$$(10) \qquad V_r^-(\mu_0) := \sup_{\beta \in B_r(t_0)} \inf_{\alpha \in A(t_0)} \int_{\Omega_\beta} \int_{\mathbb{R}^N} g(X_T^{t_0,x,\alpha(x,\cdot)\beta(\omega',\cdot)})\; d\mu_0(x) dP_\beta(\omega').$$

From this fact, because a pure strategy can be viewed as a particular case of a random strategy, one can derive the following inequalities for any $\mu_0 \in \mathcal{P}(\mathbb{R}^N)$

$$(11) \qquad V^-(\mu_0) \le V_r^-(\mu_0) \le V_r^+(\mu_0) \le V^+(\mu_0).$$

On can also show that the space of random strategies could be restricted to $\Omega = [0,1]$:

Lemma 2.8 *Let $((\Omega, B(\Omega), P), \alpha) \in S$ be such that :*

- $\Omega = [0,1]^m$

- $B(\Omega)$ *is the class of Borel subsets of* Ω*,*
- P *is a probability measure on* Ω
- $\alpha : \mathbb{R}^N \times \Omega \times V(t_0) \to U(t_0)$ *is a random strategy for the first player.*

Then there exists a random strategy $(([0,1], B([0,1]), \mathcal{L}^1), \bar{\alpha})$ *which is equivalent to* α *in the following sense:*

$$\forall x \in \mathbb{R}^N,\ \forall \beta \in B(t_0), \quad \int_\Omega g(X_T^{t_0,x,\alpha(x,\omega,\cdot),\beta(\cdot)})\ dP(\omega) = \int_{[0,1]} g(X_T^{t_0,x,\bar{\alpha}(x,z,\cdot),\beta(\cdot)})\ dz.$$

Remark 2.9 *Lemma 2.8 actually means that we could reduce the class* S *to the singleton* $([0,1], B([0,1]), \mathcal{L}^1)$ *(or extending it to any* $(\Omega, B(\Omega), P)$ *where* Ω *is a compact subset of* $\mathbb{R}^m$ *for some* $n \in \mathbb{N}^*$ *and* P *is any probability measure supported on* Ω*).*

Proof : Let $\bar{P} := 1_{[0,1]\times\{0\}^{m-1}}\ d\mathcal{L}^m$. It is a non-atomic probability measure. So, by Proposition 2.1, there exists $h : [0,1]^m \to [0,1]^m$ such that $h\sharp\bar{P} = P$. Set $\bar{\alpha}(x,z,\cdot) := \alpha(x, h(z,0,...,0), \cdot)$ for any $x \in \mathbb{R}^N$ and $z \in [0,1]$. Then, for any $x \in \mathbb{R}^N$ and $\beta \in B(t_0)$, we have:

$$\int_\Omega g(X_T^{t_0,x,\alpha(x,\omega,\cdot),\beta(\cdot)})\ dP(\omega) = \int_{[0,1]^m} g(X_T^{t_0,x,\alpha(x,h(\omega'),\cdot),\beta(\cdot)})\ d\bar{P}(\omega')$$

$$= \int_{[0,1]} g(X_T^{t_0,x,\alpha(x,h(z,0,..0),\cdot),\beta(\cdot)})\ dz = \int_{[0,1]} g(X_T^{t_0,x,\bar{\alpha}(x,z,\cdot),\beta(\cdot)})\ dz.$$

QED

Now we recall the result of the first author showing the existence of a value for a finite number of initial conditions. In our framework, it can be reformulated as follows:

Proposition 2.10 ([6], section 6) *Assume that Isaacs condition (2) holds and that the initial probability distribution is a finite combination of Dirac masses:* $\mu_0 = \sum_{i=1}^J a_i \delta_{x_0^i}$*. Then the differential game with incomplete information has a value in random strategies:*

$$V_r^+(t_0, \mu_0) = V_r^-(t_0, \mu_0).$$

2.3 Outline of the main results

Our paper contains two main results: the first one (Theorem 3.2) states that, under Isaacs' condition, the game has a value in mixed strategies: equality $V_r^+(t_0,\mu_0) = V_r^-(t_0,\mu_0)$ holds for any (t_0, μ_0). In other words, we can remove the "finite support condition" required in Proposition 2.10.

Our second main result (Theorem 4.1) is the existence of a value in pure strategies for measures without atoms: $V^+(t_0,\mu_0) = V^-(t_0,\mu_0)$ if μ_0 is non-atomic. We actually show a stronger statement: for any non-atomic measure μ_0, one always have $V^\pm(t_0,\mu_0) = V_r^\pm(t_0,\mu_0)$ (even without Isaacs' condition). Then the first Theorem gives the result under Isaacs' condition.

Both results are proved under the assumption that the measure μ_0 has compact support and for games with incomplete information on one side. Extensions to more general measures and to games with incomplete information on both sides are discussed in Remarks 4.2 and 4.3 below.

Comments on the proofs are now in order. To show our first result (existence of a value in mixed strategies), we have to overcome the issue that the method used in [6] is no longer available: indeed [6] heavily relies on techniques of partial differential equations which have—up to now—no counterpart for general measures. Our idea is to extend the existence of a value for measures with finite support to game to general measures. The main step for this is a Lipschitz continuity property of the value functions V_r^+ and V_r^- (Proposition 3.1), which is proved by optimal transport techniques: these technique appear to be extremely useful here because they allow to transport properties of one measure to properties for another one. With the regularity of $V_r^\pm$ we can conclude by using Proposition 2.10 and the density into $\mathcal{P}(\mathbb{R}^N)$ of measures with finite support. For the second statement (existence of a value in pure strategies), we start by approximating the measure μ_0 by discrete ones; then we transform $\epsilon-$optimal random strategies for these discrete measures into pure strategies for non-atomic ones by optimal transport techniques. The resulting pure strategies turn out to be also $\epsilon-$optimal for the initial measure μ_0.

3 Values in Random Strategies

This section is devoted to obtain a Lipschitz continuity property of the values in random strategies and then to deduce the existence of a value for arbitrary initial probability distribution μ_0.

Proposition 3.1 *The values with random strategies V_r^- and $V_r^+ : \mathcal{P}(\mathbb{R}^N) \mapsto \mathbb{R}$ are Lipschitz continuous with respect to the Wasserstein distance W_2.*

Proof : We will only prove that V_r^+ is Lipschitz continuous, the proof for V_r^- being similar. Consider μ_0 and μ_1 in $\mathcal{P}(\mathbb{R}^N)$. Fix $\varepsilon > 0$. Let $((\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha), \alpha)$ be a random strategy which is ε-optimal for $V_r^+(\mu_0)$: namely

$$(12)\qquad \sup_{\beta\in B_r(t_0)} \int_{\Omega_\alpha}\int_{\Omega_\beta}\int_{\mathbb{R}^N} g(X_T^{t_0,x,\alpha(x,\omega,\cdot)\beta(\omega',\cdot)})\ d\mu_0(x)dP_\alpha(\omega)dP_\beta(\omega') \le V_r^+(\mu_0)+\varepsilon.$$

Fix $\beta \in B_r(t_0)$.

Let γ be an optimal plan for $W_2(\mu_1,\mu_0)$. Then we disintegrate the measure γ with respect to μ_1 as follows

$$d\gamma(x,y) = d\gamma_y(x)d\mu_1(y).$$

By Proposition 2.1, there exists[2] a map $\xi :\ (y,\omega'') \in supp(\mu_1)\times[0,1]^N \mapsto \xi(y,\omega'') \in \mathbb{R}^N$ such that

$$\xi(y,\cdot)\sharp\mathcal{L}^N = \gamma_y \text{ for } \mu_1\text{-almost all } y,\ \text{ and } W_2^2(\mathcal{L}^N,\gamma_y) = \int_{[0,1]^N} |\omega''-\xi(y,\omega'')|^2 d\omega''.$$

[2]Here we use the notation $supp(\mu_1)$ for the support of the probability measure μ_1.

We now prove the Borel measurability of ξ which is a technical but important claim for further considerations.

First, by the classical desintegration Theorem (cf. for instance [10]) we know that the map $y \in supp(\mu_1) \mapsto \gamma_y \in \mathcal{P}(\mathbb{R}^N)$ is Borel Measurable (when $\mathcal{P}(\mathbb{R}^N)$ is equipped with the Borel σ- fields associated with the distance W_2).

Second, the map $\nu \in \mathcal{P}(supp(\mu_0)) \mapsto h \in L^2([0,1]^N, \mathbb{R}^N, \mathcal{L}^N)$ is continuous (here $h : [0,1]^N \mapsto \mathbb{R}^N$ is the unique optimal transport such that $W_2^2(\mathcal{L}^N, \nu) = \int_{[0,1]^N} |\omega'' - h(\omega'')|^2 d\omega'')$). For this fact the reader can refer to [14] page 71. Since this map is continuous it is also Borel Measurable.

Last, it is well known that the map $(x, \phi) \in \mathbb{R}^N \times L^2(\mathbb{R}^N, \mathbb{R}^N, \mathcal{L}^N) \mapsto \phi(x) \in \mathbb{R}^N$ is Borel measurable.

The map ξ being the composition of the three above maps, it is Borel measurable. Our claim is proved.

This enables us to define the following random strategy for the first player

$$\tilde{\alpha} : (y, \omega, \omega'', v) \in \mathbb{R}^N \times \Omega_\alpha \times [0,1]^N \times \mathcal{V}(t_0) \mapsto \alpha(\xi(y, \omega''), \omega, v) \in \mathcal{U}(t_0).$$

Then for any $\beta \in B_r(t_0)$ we have

$$J(\mu_1, \tilde{\alpha}, \beta) = \int_{\Omega_\alpha \times [0,1]^N} \int_{\Omega_\beta} \int_{\mathbb{R}^N} g(X_T^{t_0, y, \tilde{\alpha}(y,\omega,\omega'',\cdot)\beta(\omega',\cdot)})\ d\mu_1(y) dP_\alpha(\omega) d(\omega'') dP_\beta(\omega')$$

$$= \int_{\Omega_\alpha \times \mathbb{R}^N \times \mathbb{R}^N \times \Omega_\beta} g(X_T^{t_0, y, \alpha(x,\omega,\cdot)\beta(\omega',\cdot)})\ dP_\alpha(\omega) d\gamma_y(x) d\mu_1(y) dP_\beta(\omega')$$

(Using Fubini Theorem and the definition of $\tilde{\alpha}$)

$$= \int_{\Omega_\alpha \times \mathbb{R}^N \times \mathbb{R}^N \times \Omega_\beta} g(X_T^{t_0, y, \alpha(x,\omega,\cdot)\beta(\omega',\cdot)})\ dP_\alpha(\omega) d\gamma(x,y) dP_\beta(\omega')$$

$$\leq \int_{\Omega_\alpha \times \mathbb{R}^N \times \mathbb{R}^N \times \Omega_\beta} g(X_T^{t_0, x, \alpha(x,\omega,\cdot)\beta(\omega',\cdot)})\ dP_\alpha(\omega) d\gamma(x,y) dP_\beta(\omega') + CLip(g) \int_{\mathbb{R}^N \times \mathbb{R}^N} |x-y| d\gamma(x,y)$$

$$= \int_{\Omega_\alpha \times \mathbb{R}^N \times \Omega_\beta} g(X_T^{t_0, x, \alpha(x,\omega,\cdot)\beta(\omega',\cdot)})\ dP_\alpha(\omega) d\mu_0(x) dP_\beta(\omega') + CLip(g) \int_{\mathbb{R}^N \times \mathbb{R}^N} |x-y| d\gamma(x,y)$$

because, by standard estimates on trajectories of (1), we have for some constant $C > 0$:

$$|X_T^{t_0, y, \alpha(x,\omega,\cdot)\beta(\omega',\cdot)} - X_T^{t_0, x, \alpha(x,\omega,\cdot)\beta(\omega',\cdot)}| \leq C|x-y|.$$

Using Cauchy-Schwarz inequality, we obtain

$$J(\mu_1, \tilde{\alpha}, \beta) \leq J(\mu_0, \alpha, \beta) + CLip(g) (\int_{\mathbb{R}^{2N}} |x-y|^2\ d\gamma(x,y))^{\frac{1}{2}} = J(\mu_0, \alpha, \beta) + CLip(g) W_2(\mu_0, \mu_1).$$

Hence by (12), by passing to the supremum over β we obtain

$$\sup_{\beta \in B_r(t_0)} J(\mu_1, \tilde{\alpha}, \beta) \leq V_r^+(\mu_0) + CLip(g) W_2(\mu_0, \mu_1) + \varepsilon.$$

Consequently since ε is arbitrary, this yields

$$V_r^+(\mu_1) \leq V_r^+(\mu_0) + CLip(g) W_2(\mu_0, \mu_1).$$

Interchanging μ_0 and μ_1, the proof is complete.

QED

We recall that any measure in $\mathcal{P}(\mathbb{R}^N)$ can be written as a limit—for the Wasserstein distance—of a sequences of probability measures which are finite combinations of Dirac masses. Then in view of Proposition 2.10, we can deduce from Proposition 3.1 the existence of a value for general probability measures:

Theorem 3.2 *Under Isaacs condition (2), the differential game with incomplete information has a value in random strategies:*

$$\forall \mu_0 \in \mathcal{P}(\mathbb{R}^N),\ V_r^-(\mu_0) = V_r^+(\mu_0).$$

4 Values in Pure Strategies

In this section, we prove that the game has a value in pure strategy when the initial probability measure μ_0 has no atoms.

Theorem 4.1 *Consider μ_0 is a compactly supported probability measure on $\mathbb{R}^N$ without atoms. Then*

$$V_r^+(\mu_0) = V^+(\mu_0) \text{ and } V_r^-(\mu_0) = V^-(\mu_0). \tag{13}$$

If moreover we suppose that Isaacs' condition (2) holds true then the value of the game exists in pure strategies for μ_0:

$$V^+(\mu_0) = V^-(\mu_0).$$

Proof : We only prove (13) for upper values, the proof being similar for lower values. Let us approximate μ_0 by a sequence of discrete probability measures μ_n:

$$\mu_n := \sum_{k=1}^n \theta_k(n)\delta_{x_k(n)} \text{ with } x_k(n) \text{ all different, } \lim_{n\to+\infty} W_2(\mu_0, \mu_n) = 0$$

Let $\varsigma \in]0, \frac{1}{n^2}[$ be small enough such that for all $k \neq k'$ the balls $B(x_k, \varsigma)$ and $B(x_{k'}, \varsigma)$ do not intersect. We introduce the following sequence of probability measures:

$$\nu_n := \sum_{k=1}^n \frac{\theta_k(n)}{\zeta_N \varsigma^N} 1_{B(x_k,\varsigma)}\ \mathcal{L}^N,$$

(where $\zeta_N = \mathcal{L}^N(B(0,1))$. We have that

$$W_2(\mu_n, \nu_n) \leq \left(\sum_{k=1}^n \varsigma^2 \mathcal{L}^N(B(x_k, \varsigma)) \frac{\theta_k(n)}{\zeta_N \varsigma^N} \right)^{1/2} = \varsigma \leq \frac{1}{n^2}. \tag{14}$$

Let $\varepsilon = \frac{1}{n}$ and $(([0,1]^m, B([0,1]^m), \mathcal{L}^m), \alpha)$ an ε-optimal mixed strategy for $V_r^+(\mu_n)$. Namely, in view of Remark 2.7:

$$\sup_{\beta\in B(t_0)} \int_{\Omega_\alpha} \int_{\mathbb{R}^N} g(X_T^{t_0,x,\alpha(x,\omega,\cdot)\beta(\cdot)})\ d\mu_n(x) dP_\alpha(\omega) \leq V_r^+(\mu_n) + \frac{1}{n}. \tag{15}$$

With the random strategy α we associate a pure strategy $\hat{\alpha}$ in the following way. There exists T_k a Borel measurable map such that $T_k\sharp\left(\frac{1}{\zeta_N\varsigma^N}1_{B(x_k,\varsigma)}\ \mathcal{L}^N\right) = 1_{[0,1]^m}\mathcal{L}^m$ (cf. [14] for instance). Then we set

$$\begin{array}{rlll}\hat{\alpha}: & \mathbb{R}^N\times\mathcal{V}(t_0) & \to & \mathcal{U}(t_0)\\ & (x,v) & \mapsto & \sum_{k=1}^n \alpha(x_k,T_k(x),v)1_{B(x_k,\varsigma)}(x).\end{array}$$

Then for all $\beta\in B(t_0)$, using the definition of $\hat{\alpha}$, we have:

$$\begin{array}{rcl} J(\nu_n,\hat{\alpha},\beta) & = & \displaystyle\int_{R^N} g(X_T^{t_0,x,\hat{\alpha}(x,\cdot),\beta})\ d\nu_n(x)\\ & = & \displaystyle\sum_{k=1}^n\int_{B(x_k,\varsigma)} g(X_T^{t_0,x,\alpha(x_k,T_k(x),\cdot),\beta})\frac{\theta_k(n)}{\zeta_N\varsigma^N}\ dx\\ & \leq & \displaystyle\sum_{k=1}^n\int_{B(x_k,\varsigma)} g(X_T^{t_0,x_k,\alpha(x_k,T_k(x),\cdot),\beta})\frac{\theta_k(n)}{\zeta_N\varsigma^N}\ dx + nCLip(g)\varsigma\\ & & \text{(thanks to standard estimates on trajectories of (1))}\\ & = & \displaystyle\sum_{k=1}^n\int_{[0,1]^m} g(X_T^{t_0,x_k,\alpha(x_k,\omega,\cdot),\beta})\theta_k(n)\ d\omega + nCLip(g)\varsigma\\ & \leq & \displaystyle J_r(\mu_n,\alpha,\beta) + nCLip(g)\varsigma \leq J_r(\mu_n,\alpha,\beta) + \frac{CLip(g)}{n}.\end{array}$$

Then taking the suppremum on $\beta\in B(t_0)$, using the Lipschitz property of V_r^+ (Proposition 3.1) and the $\frac{1}{n}$-optimality of α (equation 15), we get

$$V_r^+(\nu_n)\leq V^+(\nu_n)\leq \sup_{\beta\in B(t_0)} J(\nu_n,\hat{\alpha},\beta)\leq \sup_{\beta\in B(t_0)} J_r(\mu_n,\alpha,\beta) + \frac{CLip(g)}{n}$$

$$\leq V_r^+(\mu_n) + \frac{CLip(g)}{n} + \frac{1}{n} \leq V_r^+(\nu_n) + CLip(g)(W_2(\mu_n,\nu_n) + \frac{1}{n}) + \frac{1}{n}.$$

In view of (14), this implies that there exists some constant $C'>0$ (independent of n) such that for n large enough

$$V_r^+(\nu_n)\leq \sup_{\beta\in B(t_0)} J(\nu_n,\hat{\alpha},\beta)\leq V_r^+(\nu_n) + \frac{C'}{n}. \tag{16}$$

This means that the *pure strategy* $\hat{\alpha}$ is $\frac{C'}{n}$-optimal for the value *in random strategies* $V_r^+(\nu_n)$.

From Proposition 2.1, the probability measure μ_0 being non-atomic, there exists a minimizing sequence of maps $(S_n)_n$ such that $S_n\sharp\mu_0 = \nu_n$ and :

$$\left(\int |x - S_n(x)|^2 d\mu_0(x)\right)^{1/2} \leq W_2(\mu_0,\nu_n) + \frac{1}{n}.$$

Let us define $\bar\alpha_n(x,\cdot) := \hat\alpha(S_n(x),\cdot)$. Then for any $\beta \in B(t_0)$, we obtain

$$\begin{aligned} J(\mu_0,\bar\alpha_n,\beta) &= \int_{\mathbb{R}^N} g(X_T^{t_0,x,\hat\alpha(S_n(x),\cdot),\beta})\ d\mu_0(x) \\ &\leq \int_{\mathbb{R}^N} g(X_T^{t_0,S_n(x),\hat\alpha(S_n(x),\cdot),\beta}) + CLip(g)|x-S_n(x)|\ d\mu_0(x) \\ &\leq \int_{\mathbb{R}^N} g(X_T^{t_0,x,\hat\alpha(x,\cdot),\beta})\ d\nu_n(x) + CLip(g)(W_2(\mu_0,\nu_n)+\frac{1}{n}) \end{aligned}$$

Passing to the supremum on $\beta \in B(t_0)$ in the above inequality yields in view of (16)

$$V^+(\mu_0) \leq \sup_{\beta\in B(t_0)} J(\mu_0,\bar\alpha_n,\beta) \leq \sup_{\beta\in B(t_0)} \int_{\mathbb{R}^N} g(X_T^{t_0,x,\hat\alpha(x,\cdot),\beta})\ d\nu_n(x) + CLip(g)(W_2(\mu_0,\nu_n)+\frac{1}{n})$$

$$\leq V_r^+(\nu_n) + \frac{C'}{n} + CLip(g)(W_2(\mu_0,\nu_n)+\frac{1}{n})$$

$$\leq V_r^+(\mu_0) + 2CLip(g)W_2(\mu_0,\nu_n) + \frac{1}{n}(C' + CLip(g)).$$

Since ν_n converge to μ_0 as $n \to +\infty$, passing to the limit on n of the above inequality gives

$$V^+(\mu_0) \leq V_r^+(\mu_0).$$

According to (11), the reverse inequality also holds true. So we obtain that $V^+(\mu_0) = V_r^+(\mu_0)$. This proves the first part (13) of the Theorem.

Assume now that the Isaacs condition holds true. The inequality $V^+(\mu_0) = V^-(\mu_0)$ is a direct consequence of (13) and of Theorem 3.2. The proof is complete.

QED

Remark 4.2 *Our results are also valid if we consider instead $\mathcal{P}(\mathbb{R}^N)$ (the set of Borel probability measures with compact support) by $\mathcal{P}_2(\mathbb{R}^N)$ (the set of Borel probability measures μ with finite second moment $\int_{\mathbb{R}^N}|x|^2 d\mu(x) < +\infty$) and if the assumption of Lipschitz continuity and boundedness of g is replaced by Lipschitz continuity and*

$$\exists a > 0,\ \forall x \in \mathbb{R}^N,\ g(x) \leq a(1+|x|^2).$$

We have chosen not to consider this more general case because this would require to make a new proof of measurability arguments used to obtain the measurability of the function ξ in the proof of Proposition 3.1 (in the compact support case, we can refer to [14]). This would be make much longer our paper which main aim is not optimal transport theory neither measurability.

Remark 4.3 *Our approach could be extended to differential games with a incomplete information for both player as follows. Suppose that the state space $\mathbb{R}^N$ is a product spaces $\mathbb{R}^N = Y \times Z$ (this is the case in particular for pursuit differential games where each player acts only on a component of the dynamics). The game is then played as follows: the game*

starts at time t_0, where the initial position $x_0 = (y_0, z_0)$ is chosen randomly according to a probability measure $\mu_0 \otimes \nu_0$,
- the component y_0 of the initial state is communicated to Player I but not to Player II, while the component z_0 of the initial state is communicated to Player II but not to Player I
- both players know the probability $\mu_0 \otimes \nu_0$ and they observe their opponents controls
- the game is played until the terminal time T.

The payoff is again of the form $g(x(T))$.

Both main results of the paper (i.e., the existence of a value in random strategy for $\mu_0 \otimes \nu_0 \in \mathcal{P}(\mathbb{R}^N)$ and the existence of value in pure strategy when $\mu_0 \otimes \nu_0$ has no atoms) are still valid in this context. Indeed our approach is based on Proposition 2.10 which is also valid for $\mu_0 \otimes \nu_0 = \sum_{i=1}^{J} a_i(\delta_{y_0^i} \otimes \delta_{z_0^i})$ (cf. [6]). The proofs of analogues of Proposition 3.1 and Theorem 4.1 can be generalized using the same arguments.

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