An introduction to BV functions in Wiener spaces

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Abstract.

We present the foundations of the theory of functions of bounded variation and sets of finite perimeter in abstract Wiener spaces.

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§1. Introduction

This paper is an extended version of two talks given by the second and third author during the summer school *Variational methods for evolving objects*. As both talks were concerned with some infinite dimensional analysis, we took the opportunity of this report to present the whole research area in a quite self-contained way, as it arises today. Indeed, even though geometric analysis on infinite dimensional spaces and the theory of BV functions is presently an active research field and there are still many important open problems (some are presented in Section

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8), the foundations of the theory and some methods that have proved to be useful are rape enough as to be presented in an introductory paper. In particular, we think that our purpose fits into the general aim of a collection of lecture notes – that of being useful to students and young researchers who attended the summer school and could be interested in having an active part in further developments of the theory.

Malliavin calculus is essentially a differential calculus in Wiener spaces and was initiated by P. Malliavin [44] in the seventies with the aim, among the others, of obtaining a probabilistic proof of Hörmander hypoellipticity theorem. This quickly led to study connections to stochastic differential equations and applications in various fields in Mathematics and Physics, such as mathematical finance, statistical mechanics and hydrodynamics and the path approach to quantum theory or stationary phase estimation in stochastic oscillatory integrals with quadratic phase function. In general, solutions of SDEs are not continuous (and sometimes not even everywhere defined) functionals, hence the notion of weak derivative and Sobolev functional comes into play. Notice that there is no Sobolev embedding in the context of Malliavin calculus, which requires very little regularity. Looking at weak differentiation and the study of the behaviour of stochastic processes in domains leads immediately to the need for a good comprehension of integration by parts formulae, something that in the Euclidean case has been completely understood in the frameworks of geometric measure theory, sets with finite perimeter and more generally functions of bounded variation. This approach has been considered by Fukushima in [32] and Fukushima-Hino in [33], where the first definition of BV functions in infinite dimensional spaces has been given, most likely inspired by a stochastic characterisation of finite perimeter sets in finite dimension given by Fukushima in [31], see Theorem 4 below. In this paper we follow the integralgeometric approach to BV functions developed in [9], [10], [5], [6]. Among the first applications of the theory, let us mention some results in a geometric vein in [18], [19] and in a probabilistic vein in [50]. On a more analytical perspective, some results are available on integral functionals, see [20], [21], and weak flows with Sobolev vector fields, see [4]. In this connection, the extension to BV vector fields seems to require the analysis of fine properties of BV functions and perimeters.

Added in proof. After the completion of the present paper, new contributions on the subject have appeared. We list the reference we know, whose results are not discussed in this paper.

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§2. Preliminaries

As explained in the Introduction, motivations and possible applications of the theory we are going to present come from different areas, as well as the possible audience of the present notes. Indeed, it sits in the intersection between Calculus of Variations, Geometric Measure Theory, Functional Analysis, Stochastics and Mathematical Physics. Therefore, we have collected several prerequisites, divided in subsections, also with the purpose of fixing notation and basic results. Our aim is to introduce basic ideas and connections between the different perspectives, rather than giving precise and general results (this would take too much room). At the end of each subsection some general references for the sketched arguments are indicated.

When dealing with finite dimensional spaces $\mathbb{R}^d$, we always use Euclidean inner product $x \cdot y$ and norm $|x|$. Balls of radius ϱ and centre x in a Banach space are denoted by $B_\varrho(x)$, omitting the centre if $x = 0$. The σ-algebra of Borel sets in X is denoted by $\mathcal{B}(X)$. Moreover, if X is a Banach space, we denote by $\|\cdot\|_X$ its norm and by X^* its topological dual, with duality $\langle \cdot, \cdot \rangle$.

2.1. Measure theory

In this subsection we briefly discuss a few properties of general measures with some details on *Gaussian measures* in finite and infinite dimensions.
A *measurable space* is a pair $(X, \mathcal{F})$, where X is a set and $\mathcal{F}$ a σ-algebra of subsets of X. By *measure* on $(X, \mathcal{F})$ we mean a countably additive function on $\mathcal{F}$ with values in a normed vector space; if a measure μ is given on $(X, \mathcal{F})$, we say that $(X, \mathcal{F}, \mu)$ is a *measure space* (a *probability space* if μ is positive and $\mu(X) = 1$) and omit $\mathcal{F}$ whenever it is clear from the context or $\mathcal{F} = \mathcal{B}(X)$. For a measure μ with values in a normed vector space V with norm $\|\cdot\|_V$ we define the total variation $|\mu|$ as the real valued positive measure

$$
\begin{aligned}
|\mu|(B) = \sup\Big\{ \sum_{j \in \mathbb{N}} \|\mu(B_j)\|_V : & B = \bigcup_{j \in \mathbb{N}} B_j,\ B_j \in \mathcal{F}, \\
& B_j \cap B_h = \emptyset \text{ for } j \neq h \Big\};
\end{aligned}
$$

the measure μ is said to be finite if $|\mu|(X) < +\infty$. Given two measurable spaces $(X, \mathcal{F})$ and $(Y, \mathcal{G})$, a measure μ on X and a measurable function $f : X \to Y$ (i.e., such that $f^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{G}$), the *push-forward* measure $\nu = f_{\#}\mu$ on Y is defined by setting $\nu(B) = \mu(f^{-1}(B))$ for every $B \in \mathcal{G}$. Let us also recall that, given two measure spaces $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$, the *product measure* $\mu_1 \otimes \mu_2$ is defined on $X_1 \times X_2$ by first defining the product σ-algebra $\mathcal{B}$ as that generated by $\{B_1 \times B_2,\ B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$ and then defining $\mu_1 \otimes \mu_2$ as the unique measure on $\mathcal{B}$ such that $\mu_1 \otimes \mu_2(B_1 \times B_2) = \mu_1(B_1)\mu_2(B_2)$ for all pairs $B_j \in \mathcal{B}_j$. The construction generalises to the product of several spaces.

In $\mathbb{R}^d$ we consider as reference measure either the Lebesgue measure $\mathcal{L}^d$ or some absolutely continuous measure $\lambda = \rho \mathcal{L}^d$ with nonnegative density ρ. The main examples among these are *Gaussian measures*. For $d = 1$, these measures have densities G given by

$$
G(x) = \frac{1}{\sqrt{2\pi q}} \exp\{-|x - a|^2/2q\} \tag{1}
$$

for some $a \in \mathbb{R}$ (*centre* or *mean*) and $q > 0$ (*variance*). For $d > 1$, a measure λ on $\mathbb{R}^d$ is Gaussian if $f_\# \lambda$ is Gaussian on $\mathbb{R}$ for every linear function $f : \mathbb{R}^d \to \mathbb{R}$. Generalising (1), a Gaussian measure γ on $\mathbb{R}^d$ is characterised by its centre $a = \int_{\mathbb{R}^d} x \, d\gamma$ and its *covariance matrix* $Q = (q_{hk})$ with

$$q_{hk} = \int_{\mathbb{R}^d} (x_h - a_h)\,(x_k - a_k)\; d\gamma(x), \quad h, k \in \{1, \dots, d\} \tag{2}$$

and is denoted $\mathcal{N}(a, Q)$. A Gaussian measures γ is *nondegenerate* if $\gamma = G\mathcal{L}^d$ with $G(x) > 0$ (equivalently, Q positive definite) for all $x \in \mathbb{R}^d$, and is *standard* if

$$G(x) = G_d(x) = (2\pi)^{-d/2} \exp\{-|x|^2/2\}, \tag{3}$$

i.e., $\gamma = \mathcal{N}(0, \mathrm{Id})$. According to the preceding discussion on products, a standard Gaussian measure γ_d on $\mathbb{R}^d = \mathbb{R}^k \times \mathbb{R}^m$ factors in the product of standard Gaussian measures $\gamma_d = \gamma_k \otimes \gamma_m$ for $k + m = d$. A measure γ on a Banach space $(X, \mathcal{B}(X))$ is said Gaussian if $x^*_\# \gamma$ is Gaussian in $\mathbb{R}$ for every $x^* \in X^*$. In this case, the centre is defined as above by (*Bochner integral*, see [13])

$$a = \int_X x \, d\gamma \tag{4}$$

and the covariance operator $Q \in \mathscr{L}(X^*, X)$ is a symmetric and positive operator uniquely determined by the relation, cf (2),

$$\langle Qx^*, y^* \rangle = \int_X \langle x - a, x^* \rangle \langle x - a, y^* \rangle d\gamma(x), \ \forall x^*, y^* \in X^*. \tag{5}$$

The fact that the operator Q defined by (5) is bounded is a consequence of Fernique's theorem (see e.g. [13, Theorem 2.8.5]), asserting the existence of a positive $\beta > 0$ such that

$$\int_X \exp\{\beta \|x\|_X^2\} d\gamma(x) < \infty; \tag{6}$$

indeed, Q belongs to a special ideal of compact operators called γ-*Radonifying*. As above, we write $\gamma = \mathcal{N}(a, Q)$ and we say that γ is nondegenerate if $\operatorname{Ker} Q = \{0\}$. Notice that the Dirac measure at x_0 is considered as the (fully degenerate) Gaussian measure with centre x_0 and covariance $Q = 0$. For the arguments of the present subsection we refer to [13], [14].

2.2. Geometric measure theory

A general class of (non absolutely continuous) measures of interest in the sequel is that of *Hausdorff measures*, which we briefly discuss here, together with the related notions of *rectifiable set* and *approximate tangent space*.
The measure $\mathcal{H}^s$, $0 < s < \infty$, is defined in a general metric space by

$$\mathcal{H}^s(B) = \frac{\omega_s}{2^s} \sup_{\delta>0} \inf \Big\{ \sum_{j=1}^{\infty} (\operatorname{diam} B_j)^s,\ B \subset \bigcup_{j=1}^{\infty} B_j,\ \operatorname{diam} B_j < \delta \Big\}, \tag{7}$$

where, using Euler's Γ function, $\omega_s = \Gamma(1/2)^s / \Gamma(s/2+1)$ ($= \mathcal{L}^d(B_1)$ if $s = d \in \mathbb{N}$) is a normalising constant and the infimum runs along all the countable coverings. Beside the Hausdorff measures, it is useful to introduce the *Minkowski content*, which provides a more elementary, though less efficient, way of measuring "thin" sets. Given a closed set $C \subset \mathbb{R}^d$ and an integer s between 0 and d, the idea is to look at the rate of convergence to 0 of $\varrho \mapsto \mathcal{L}^d\left(I_\varrho(C)\right)$ as $\varrho \downarrow 0$, where $I_\varrho(C)$ denotes the open ϱ-neighbourhood of C. In general, given a closed set $C \subset \mathbb{R}^d$, the upper and lower s-dimensional Minkowski contents $\mathcal{M}^{*s}(C)$, $\mathcal{M}^s_*(C)$ are defined by

$$\begin{aligned} \mathcal{M}^{*s}(C) &= \limsup_{\varrho\downarrow 0} \frac{\mathcal{L}^d(I_\varrho(C))}{\omega_{N-s}\varrho^{N-s}}, \\ \mathcal{M}^s_*(C) &= \liminf_{\varrho\downarrow 0} \frac{\mathcal{L}^d(I_\varrho(C))}{\omega_{N-s}\varrho^{N-s}}, \end{aligned} \tag{8}$$

respectively. If $\mathcal{M}^{*s}(S) = \mathcal{M}^s_*(C)$, their common value is denoted by $\mathcal{M}^s(C)$ (Minkowski content of C) and we say that C admits Minkowski content. Unlike the Hausdorff measures, the Minkowski content is not subadditive. Nevertheless, in some important cases the two procedures give the same result. We compare later the Hausdorff measures and the Minkowski contents.

The natural regularity category in geometric measure theory is that of *Lipschitz continuous functions*. Let us recall (Rademacher theorem) that a Lipschitz function defined on $\mathbb{R}^d$ with values in a finite dimensional vector space is differentiable $\mathcal{L}^d$-a.e. (the differentiability properties of Lipschitz functions defined on infinite dimensional vector spaces is a much more delicate issue, see [13], [46]). For s integer between 0 and d, we say that a $\mathcal{H}^s$ measurable set $B \subset \mathbb{R}^d$ is *countably s-rectifiable* if there are countably many Lipschitz functions $f_j : \mathbb{R}^s \to \mathbb{R}^d$ such that

$$B \subset \bigcup_{j=1}^{\infty} f_j(\mathbb{R}^s). \tag{9}$$

We say that B is *countably $\mathcal{H}^s$-rectifiable* if there are countably many Lipschitz functions $f_j : \mathbb{R}^s \to \mathbb{R}^d$ such that

$$\mathcal{H}^s\Big(B \setminus \bigcup_{j=0}^{\infty} f_j(\mathbb{R}^s)\Big) = 0. \tag{10}$$

Finally, we say that B is *$\mathcal{H}^s$-rectifiable* if B is countably $\mathcal{H}^s$-rectifiable and $\mathcal{H}^s(B) < \infty$. All these classes of sets are stable under Lipschitz mapping. Notice that countable $\mathcal{H}^s$-rectifiability is equivalent to the seemingly stronger requirement that $\mathcal{H}^s$-almost all of the set can be covered by a sequence of Lipschitz s-graphs. Notice that if the admissible coverings in (7) are made only by balls we get the *spherical* Hausdorff measure $\mathcal{S}^s$. The measures $\mathcal{H}^s$ and $\mathcal{S}^s$ are comparable in the sense that

$$\mathcal{H}^s \le \mathcal{S}^s \le 2^s \mathcal{H}^s$$

and coincide on $\mathcal{H}^s$-rectifiable sets. However, an important difference between $\mathcal{H}^s$ and $\mathcal{S}^s$ measures is relevant in Subsection 4.3, where Hausdorff measures are discussed in the infinite dimensional setting, see Lemma 6. Analogously, the Hausdorff measure coincide with the Minkowski content on rectifiable sets. Even though rectifiable sets can be very irregular from the point of view of classical analysis, nevertheless they enjoy useful properties from the point of view of geometric measure theory. Indeed, for $\mathcal{H}^s$-a.e point x of a countably $\mathcal{H}^s$-rectifiable set B there exists an s-dimensional subspace S (*approximate tangent space*) such that

$$\lim_{\varrho \to 0} \int_{\frac{B-x}{\varrho}} \varphi \, d\mathcal{H}^s = \int_S \varphi \, d\mathcal{H}^s \qquad \forall\, \varphi \in C_c(\mathbb{R}^d). \tag{11}$$

If $s = d-1$ an *approximate unit normal vector* $\nu(x)$ to B at x is defined (up to the sign) as the unit vector normal to S. In the same vein, we say that a function $u \in L^1_{\rm loc}(\mathbb{R}^d, \mathbb{R}^k)$ admits an *approximate limit* at x_0, if there is $z \in \mathbb{R}^k$ such that

$$\lim_{\varrho \to 0} \frac{1}{\omega_d \varrho^d} \int_{B_\varrho(x_0)} |u(x) - z| \, dx = 0 \tag{12}$$

($z = \mathrm{ap}\lim_{x \to x_0} u(x)$ for short) and in this case we say that u is *approximately continuous at* x_0 if x_0 is a Lebesgue point of u and (12) holds with $z = u(x_0)$. Analogously, if u is approximately continuous at x_0 we say that u is *approximately differentiable at* x_0 if there is a linear map $L : \mathbb{R}^d \to \mathbb{R}^k$ such that

$$\operatorname*{ap\,lim}_{x\to x_0} \frac{u(x) - u(x_0) - L(x - x_0)}{|x - x_0|} = 0. \tag{13}$$

For the arguments of the present subsection we refer to [7], [28].

2.3. Stochastic analysis

Let a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be given. If $(X, \mathcal{B})$ is a measurable space, a measurable function $\xi : \Omega \to X$ is called an X*-valued random variable* (r.v. for short) and its *law* is the push-forward measure of $\mathbb{P}$ under ξ, i.e., $\xi_{\#}\mathbb{P}(B) = \mathbb{P}(\xi^{-1}(B))$, $B \in \mathcal{B}$. If $\xi \in L^1(\Omega, \mathbb{P})$ we define its *expectation* by $\mathbb{E}[\xi] = \int_\Omega \xi\, d\mathbb{P}$; if $\xi \in L^2(\Omega, \mathbb{P})$ we define its *variance* by $Var(\xi) = \mathbb{E}[\xi - \mathbb{E}[\xi]] = \mathbb{E}[\xi^2] - \mathbb{E}^2[\xi]$ and for $\xi, \eta \in L^2(\Omega, \mathbb{P})$ we define the *covariance* by $\mathrm{cov}(\xi, \eta) = \mathbb{E}[\xi\eta] - \mathbb{E}[\xi]\mathbb{E}[\eta]$. Given a sub σ-algebra $\mathcal{G} \subset \mathcal{F}$, the *conditional expectation of a summable* ξ *given* $\mathcal{G}$ is the unique $\mathcal{G}$-measurable random variable $\eta = \mathbb{E}(\xi|\mathcal{G})$ such that $\int_B \xi\, d\mathbb{P} = \int_B \eta\, d\mathbb{P}$ for all $B \in \mathcal{G}$. Given N random variables $\xi_j : \Omega \to X_j$, they are *independent* if for every $A_j \subset X_j$, setting $B_j = \{\omega \in \Omega : \ \xi_j(\omega) \in A_j\}$, $\mathbb{P}(B_1 \cap \cdots \cap B_N) = \mathbb{P}(B_1) \cdots \mathbb{P}(B_N)$, or, equivalently, if the law of the r.v. $\xi : \Omega \to X = X_1 \times \cdots \times X_N$ whose components are the ξ_j is the product measure of the laws of the ξ_j on X. A random variable is Gaussian if its law is a Gaussian measure.

An X-valued *continuous stochastic process* ξ *on* $[0, \infty)$ is the assignment, for $t \in [0, \infty)$, of a family of random variables $\xi_t : (\Omega, \mathcal{F}, \mathbb{P}) \to X$. An increasing family of sub σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ is called a *filtration*; a process ξ is said *adapted* to a given filtration $\mathcal{F}_t$ if ξ_t is $\mathcal{F}_t$-measurable for every t. If the filtration is not explicitly assigned, the *natural filtration* is understood, i.e., $\mathcal{F}_t$ is the smallest σ-algebra such that ξ_s is measurable for all $s \le t$, $s \in I$. If ξ_t is an adapted process, summable for every t and $\mathbb{E}(\xi_t|\mathcal{F}_s) = \xi_s$ for all $s \le t$, the process ξ is a *martingale*. Due to the dependence of $\xi_t(\omega)$ on two variables, we may think of $\omega \mapsto \xi_t(\omega)$, for fixed t, as a family of r.v. defined on Ω, or as $t \mapsto \xi_t(\omega)$, for ω fixed, as a set of *trajectories*. A real stochastic process on an interval I defines the *distribution functions*

$$F_{t_1 \cdots t_n}(x_1, \ldots, x_n) = \mathbb{P}[\xi_{t_1} < x_1, \ldots, \xi_{t_n} < x_n],\ 0 \le t_1 < \ldots < t_n < \infty,$$

called *finite-dimensional joint distributions*. In general, $F(x)$ is said to be a distribution function if it is increasing with respect to all the x_k variables, left-continuous, $F(x_1, \ldots, x_n) \to 0$ if some $x_k \to -\infty$,

$F(x_1, \dots, x_n) \to 1$ if all $x_k \to +\infty$ and for any intervals $I_k = [a_k, b_k)$, $1 \le k \le n$, the inequality

$$\Delta_{I_1} \cdots \Delta_{I_n} F(x_1, \dots, x_n) \ge 0$$

holds, where $\Delta_{I_k} F(x) = F(x_1, \dots, b_k, \dots, x_n) - F(x_1, \dots, a_k, \dots, x_n)$. A (remarkable) result of Kolmogorov's states that, given a sequence $F_n(x_1, \dots, x_n)$ of distribution functions, there is always a stochastic process whose distribution functions are the given ones, provided the (necessary) *consistency condition*

$$\lim_{x_n \to +\infty} F_n(x) = F_{n-1}(x_1, \dots, x_{n-1})$$

holds. A stochastic process on $[0, \infty)$ is *stationary* if its distribution function is invariant under translations on time, i.e.,

$$F_{t_1+h \cdots t_n+h}(x_1, \dots, x_n) = F_{t_1 \cdots t_n}(x_1, \dots, x_n) \qquad \forall\ h \ge 0.$$

Given a filtration $\mathcal{F}_t,\ t \in I$, a random variable $\tau : \Omega \to I = [0, +\infty]$ is a *stopping time* if $\{\tau \le t\} \in \mathcal{F}_t$ for all $t \in I$. Accordingly, a process ξ is a *local martingale* if there is an increasing sequence of stopping times $\tau_n \to +\infty$ such that $(\xi_{t \wedge \tau_n})$ is a martingale for every $n \in \mathbb{N}$.

A particular class of processes which is relevant for our purposes is that of *Markov processes*. Let us start from the notion of *time-homogeneous Markov transition function*, i.e., a function $p(t, x, B)$, $t \in [0, \infty)$, $x \in X$, $B \in \mathcal{B}$, which is measurable with respect to x, is a probability measure on $(X, \mathcal{B})$ with respect to B (we also write $p(t, x, dy)$ to stress the last property) and verifies the *Chapman-Kolmogorov* equation

$$p(t, x, B) = \int_X p(t - s, y, B) p(s, x, dy), \qquad \forall\ 0 \le s \le t. \tag{14}$$

Given a transition function p as above and a probability distribution μ on $(X, \mathcal{B})$, there is a stochastic process ξ such that the law of ξ_0 is μ and $\mathbb{P}(\xi_t \in B) = p(t, \xi_t, B)$ for all $t \ge 0$ and it is called *Markov process associated with p with initial law μ*. The initial law μ is *invariant* with respect to the process (see also next Subsection) if

$$\mu(B) = \int_X p(t, y, B)\, \mu(dy), \qquad \forall\ t \ge 0,\ B \in \mathcal{B}. \tag{15}$$

An $\mathbb{R}^d$ valued *Q-Brownian motion* starting from a or *Wiener process B_t* is a stochastic process such that $B_0 = a \in \mathbb{R}^d$ $\mathbb{P}$-a.s., for every $0 \le s < t$ the difference $B_t - B_s$ is a Gaussian random variable with centre 0 and covariance $(t - s)Q$, i.e., $\mathcal{N}(0, (t - s)Q)$ and for every $0 \le t_1 < \ldots < t_n$

the random variables $B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}$ are independent. This in partucular implies that the Brownian motion has a version whose trajectories are continuous and is a martingale, since the independence of $B_t - B_s$ from B_s implies that $B_t - B_s$ is independent from $\mathcal{F}_s$, that is

$$\mathbb{E}(B_t - B_s|\mathcal{F}_s) = 0.$$

According to the quoted Kolmogorov theorem, Brownian motions exist. Notice that a Brownian motion is a Markov process whose transition function is Gaussian, $p(t,x,dx) = \mathcal{N}(x,tQ)$. Moreover, as we have already observed, any Brownian motion has a continuous version and is a martingale; in the sequel we always assume that the continuous version has been selected. A Brownian motion is *standard* (or *normalised*) if $a = 0$ and $Q = \mathrm{Id}$.

The *Itô integral* with respect to a given real Brownian motion B_t, whose (completed) natural filtration we denote by $\mathcal{F}_t$, can be defined through suitable Riemannian sums, even though the usual Stiltjes approach cannot be pursued, due to the fact that B_t has not bounded variation in time. Let $\xi_t,\ t \in [0,T]$ be an adapted continuous *simple* process, i.e., such that there are a partition $0 = t_0 < t_1 < \cdots < t_N \le T$ and $\mathcal{F}_{t_{j-1}}$-measurable r.v. $\xi_j,\ j = 1,\dots,N$, for which

$$\xi_t(\omega) = \sum_{j=1}^{N} \xi_j(\omega)\chi_{[t_{j-1},t_j)}(t).$$

For such a process, define

$$\int_0^T \xi_t\,dB_t = \sum_{j=1}^{N} \xi_j(B_{t_j} - B_{t_{j-1}}).$$

As a consequence of the independence of the increments of the Brownian motion, we get the *Itô isometry*

$$\mathbb{P}\Big(\int_0^T \xi_t\,dB_t \cdot \int_0^T \eta_t\,dB_t\Big) = \int_0^T \xi_t\eta_t\,dt \tag{16}$$

for every ξ, η as above. The Itô isometry extends to $\mathbb{R}^d$ valued processes and Brownian motion in an obvious way. Thanks to the Itô isometry and the fact that every adapted process ξ such that $\mathbb{P}(\int_0^T |\xi_s|^2 ds < \infty) = 1$ can be approximated by elementary processes, it is possible to extend the stochastic integral to the described class of processes, or to processes defined for $0 \le t < \infty$ such that the finiteness condition holds for every $T > 0$. Notice that the stochastic integral is, in turn, a random variable.

It can be proved as well that the function $t \mapsto \int_0^t \xi_s \, dB_s$ is continuous $\mathbb{P}$-a.s.

The stochastic integral allows for a rigorous theory of *stochastic differential equations*, SDEs for short, which are intuitively dynamical systems perturbed by noise. We deal here only with *autonomous SDEs* on $\mathbb{R}^d$, assuming that the noise is given in terms of a Brownian motion. Something more in the Wiener space will be added in Subsection 4.5 in connection with the OrnsteinUhlenbeck process. In the present case the Cauchy problem can be written (at least formally) as

$$d\xi_t = A(\xi_t)dt + \sigma(\xi_t)dB_t, \qquad \xi_0 \text{ given r.v.,} \tag{17}$$

where B_t is a Brownian motion, σ and A are the *diffusion* and *drift* term, respectively. The meaning of (17) is that the process ξ is a solution if

$$\xi_t = \xi_0 + \int_0^t A(\xi_s) \, ds + \int_0^t \sigma(\xi_s) \, dB_s.$$

Under general hypotheses a unique solution exists and is a continuous Markov process. Presenting a general theory goes far from the aim of this short presentation; detailed results are discussed on concrete cases. For the arguments of the present subsection we refer to [13], [30], [45].

2.4. Semigroup theory

The theory of one-parameter semigroups of linear operators in Banach spaces was born as a general method to solve autonomous evolution equations, has been widely studied and is very rich of abstract results and applications. We need very few basic results, and the main point which is worth discussing here is the link between semigroups as a tool for solving linear parabolic partial differential equations and the related stochastic differential equations, as explained at the end of this subsection. First, we say that $(S_t)_{t\geq 0}$ is a *semigroup of linear operators* on a Banach space E if $S_t \in \mathscr{L}(E)$, i.e., S_t is a bounded linear operator on E for every $t \geq 0$, $S_0 = \mathrm{Id}$, $S_{t+s} = S_t \circ S_s$; if $t \mapsto S_t f$ is norm continuous for every $x \in E$ then S_t is said to be C_0 (or strongly continuous). If S_t is strongly continuous then, setting $\omega_0 = \inf_{t\geq 0} \frac{1}{t} \log \|S_t\|_{\mathscr{L}(E)}$, for any $\varepsilon > 0$ there is $M_\varepsilon \geq 1$ such that $\|S_t\|_{\mathscr{L}(E)} \leq M_\varepsilon e^{(\omega_0+\varepsilon)t}$ for all $t \geq 0$. A semigroup defined on $E = C_b(X)$ (the space of bounded continuous functions on a Banach space X) is *Feller* if $S_t f \in C_b(X)$ for all $f \in C_b(X)$ and is *strong Feller* if $S_t f \in C_b(X)$ for all $f \in B_b(X)$ (the space of bounded Borel functions). A *Markov semigroup* is a semigroup S_t on $C_b(X)$ such that $S_t\mathbf{1} = \mathbf{1}$, $\|S_t\|_{\mathscr{L}(E)} \leq 1$ for every $t \geq 0$, and $S_t f \geq 0$ for every $f \geq 0$ and $t > 0$ (here $\mathbf{1}$ is the constant function with value 1). Given a time homogeneous Markov transition function p and the associated process ξ_t^x starting at x (which means that the law of ξ_0 is δ_x), the family of operators

$$S_t f(x) = \int_X f(y) p(t,x,dy) = \mathbb{E}[f(\xi_t^x)], \qquad x \in X, \tag{18}$$

due to (14), is a *Markov semigroup*. Notice that S_t can be extended to $B_b(X)$. With each semigroup it is possible to associate a *generator*, i.e., a linear closed operator $(L, D(L))$ such that $Lf = \lim_{t\to 0}(S_t f - f)/t$, f in the *domain* $D(L) \subset E$. Here the limit is in the norm sense if S_t is strongly continuous or can be in weaker senses (uniform convergence on bounded or compact sets or even pointwise with bounds on the sup norm) in the case of Markov semigroups. We are mainly interested in the case where p comes from a process which solves a SDE (17) on a Banach space X. In this case, L is a linear elliptic operator given by $L = -\frac{1}{2}\mathrm{Tr}[\sigma\sigma^* D^2] + \langle Ax, \nabla\rangle$, at least on suitable smooth functions, giving rise to the *Kolmogorov backward parabolic operator* $\partial_t - L$. Under suitable conditions, the solution of the Cauchy problem $\partial_t u - Lu = 0$, $u(0) = f \in C_b(X)$ will be given by $u(t) = S_t f$. In this setting, the trajectories of the Markov process play a role analogous to that of the characteristic curves in a hyperbolic problem. Finally, we introduce the notion of *invariant measure* associated with the semigroup S_t, i.e., a probability measure μ on X such that

$$\int_X S_t f(x)\, d\mu(x) = \int_X f(x)\, d\mu(x), \qquad f \in C_b(X).$$

The meaning of the above equality is that the distribution μ is invariant under the flow described by equation (17), see (15). Typically, if μ_t is the law of ξ_t and the weak limit $\mu = \lim_{t\to\infty} \mu_t$ exists, then μ is invariant and the semigroup S_t extends to a C_0 semigroup in all the $L^p(X,\mu)$ spaces, $1 \le p < \infty$. For the arguments of the present subsection we refer to [13], [30].

2.5. Dirichlet forms

In this subsection we collect a few notions on Dirichlet forms, confining to what we need in Theorem 4, and to show some further connections between the various areas we are quickly touching.

Given a σ-finite measure space (X, μ) consider the Hilbert space $L^2(X,\mu)$ with the inner product $[u, v]$. A functional $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ is a *Dirichlet form* if it is

(1) *bilinear*: $\mathcal{E}(u+v, w) = \mathcal{E}(u,w) + \mathcal{E}(v,w)$, $\mathcal{E}(\alpha u, v) = \alpha\mathcal{E}(u,v)$ for all $u, v, w \in L^2(X,\mu)$, $\alpha \in \mathbb{R}$;

(2) *nonnegative*: $\mathcal{E}(u,u) \geq 0$ for all $u \in L^2(X,\mu)$;
(3) *closed*: $D(\mathcal{E})$ is complete with respect to the metric induced by the inner product $\mathcal{E}(u,v)+[u,v]$, $u,v \in D(\mathcal{E})$;
(4) *Markovian*: if $u \in D(\mathcal{E})$ then $v := (0 \vee u) \wedge 1 \in D(\mathcal{E})$ and $\mathcal{E}(v,v) \leq \mathcal{E}(u,u)$.

A Dirichlet form $\mathcal{E}$ is *symmetric* if $\mathcal{E}(u,v)=\mathcal{E}(v,u)$ for all $u,v \in L^2(X,\mu)$ and is *local* if $\mathcal{E}(u,v)=0$ whenever $u,v \in D(\mathcal{E})$ have disjoint compact supports. The subspace $D(\mathcal{E})$ of $L^2(X,\mu)$ is called the *domain* of the form $\mathcal{E}$.

Dirichlet forms are strictly connected with Markov semigroups and processes. First, notice that a nonnegative operator L can be associated with any Dirichlet form as shown in the following theorem of Kato's.

Theorem 1. *There is a one-to-one correspondence between closed symmetric forms and nonnegative self-adjoint operators given by*

$$u \in D(L) \;\Leftrightarrow\; \exists\, f \in L^2(X,\mu): \quad \mathcal{E}(u,v)=[f,v]\ \forall\, v \in D(\mathcal{E}),\ Lu := f.$$

Moreover, $D(\mathcal{E})=D(\sqrt{L})$ and the operator $(-L,D(L))$ is the generator of a strongly continuous Markov semigroup S_t of self-adjoint operators.

According to the discussion in the preceding Subsection and the above Theorem, it is possible to associate with a Markov process, beside a Markov semigroup, a Dirichlet form. Of course, not all the Markov processes give raise to a Dirichlet form. Moreover, the transition function must be *symmetric*, i.e., such that $p(x,y,B)=p(y,x,B)$ for all $x,y \in X$ and $B \in \mathcal{B}(X)$ in order to get a symmetric Dirichlet form and if the process has continuous trajectories then the associated form is local.

Viceversa, given a regular Dirichlet form, there is a unique (in a suitable sense) Markov process whose Dirichlet form is the given one. Let us now discuss two key examples that will play a relevant role in the sequel.

Example 1. Let $D \subset \mathbb{R}^d$ be open and bounded with Lipschitz continuous boundary, and define the Dirichlet form on $L^2(D)$ by

$$\mathcal{E}(u,v)=\int_D \nabla u \cdot \nabla v \, dx,$$

for $u,v \in D(\mathcal{E})=W^{1,2}(D)$. The operator L defined as in Theorem 1 is the Neumann Laplacean, i.e.,

$$L=-\Delta, \qquad D(L)=\{u \in H^{2,2}(D):\ \partial_\nu u = 0 \text{ on } \partial D\},$$

where ∂_ν denotes the differentiation with respect to the normal direction. Then, $(-L,D(L))$ is the generator of a strongly continuous Markov semigroup on $L^2(D)$ and the related Markov process is the *reflecting Brownian motion in* D.

Example 2. Let $\gamma = G_d\mathcal{L}^d$ be the standard Gaussian measure. Define the Dirichlet form $\mathcal{E}$ on $L^2(\mathbb{R}^d,\gamma)$ by

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \; d\gamma,$$

$u, v \in D(\mathcal{E}) = W^{1,2}(\mathbb{R}^d,\gamma) = \{u \in W^{1,2}_{\rm loc}(\mathbb{R}^d)\colon u, |\nabla u| \in L^2(\mathbb{R}^d,\gamma)\}$. The operator L defined as in Theorem 1 is the Ornstein-Uhlenbeck operator defined on smooth functions by $L = -\Delta + x\cdot\nabla$ and $D(L) = W^{2,2}(\mathbb{R}^d,\gamma)$, $(-L, D(L))$ is the generator of the strongly continuous Markov semigroup T_t on $L^2(\mathbb{R}^d,\gamma)$ defined in (21) and the related Markov process is the *Ornstein-Uhlenbeck process in* $\mathbb{R}^d$ given by (25) below. Moreover, γ is the invariant measure of T_t.

For the arguments of the present subsection we refer to [34], [42].

§3. BV functions in the finite-dimensional case

In this section we present the main properties of BV functions in $\mathbb{R}^d$. In order to pave the way to the generalisations to Wiener spaces, we discuss now at the same time the case when the reference measure is the Lebesgue one or the finite dimensional standard Gaussian measure. Of course, BV functions with general densities can be studied, but this is not of our concern here. Standard Gaussian measures have regular and non-degenerate densities, hence there is no basic difference at the level of *local* properties of BV functions, which are basically the same in the two cases. Instead, the *global* properties are different, due to the very different behaviour of the densities at infinity. Let us start from the classical case. There are various ways of defining BV functions on $\mathbb{R}^d$, which are useful in different contexts.

Theorem 2. *Let $u \in L^1(\mathbb{R}^d)$. The following are equivalent:*

1 *there exist real finite measures μ_j, $j = 1, \ldots, d$, on $\mathbb{R}^d$ such that*

$$\int_{\mathbb{R}^d} u D_j\phi dx = -\int_{\mathbb{R}^d} \phi d\mu_j, \quad \forall \phi \in C^1_c(\mathbb{R}^d), \tag{19}$$

i.e., the distributional gradient $Du = \mu$ is an $\mathbb{R}^d$-valued measure with finite total variation $|Du|(\mathbb{R}^d)$;

2 *the quantity*

$$V(u) = \sup\Big\{ \int_{\mathbb{R}^d} u \operatorname{div}\phi dx : \phi \in C^1_c(\mathbb{R}^d, \mathbb{R}^d), \|\phi\|_\infty \le 1\Big\}$$

is finite;

3 *the quantity*

$$L(u) = \inf\Big\{ \liminf_{h\to\infty} \int_{\mathbb{R}^d} |\nabla u_h|\, dx : u_h \in \mathrm{Lip}(\mathbb{R}^d),\ \ u_h \stackrel{L^1}{\to} u\Big\}$$

is finite;

4 *if* $(W_t)_{t\ge 0}$ *denotes the heat semigroup in* $\mathbb{R}^d$*, then*

$$\mathscr{W}[u] = \lim_{t\to 0} \int_{\mathbb{R}^d} |\nabla W_t u|\, dx < \infty.$$

Moreover, $|Du|(\mathbb{R}^d) = V(u) = L(u) = \mathscr{W}[u]$*.*

If one of (hence all) the conditions in Theorem 2 holds, we say that $u \in BV(\mathbb{R}^d)$. The statement above is well known, a sketch of its proof, with more references, can be found in [8]. We observe that in **3** we may replace Lipschitz functions with functions in $W^{1,1}(\mathbb{R}^d)$. The translation of the above result in the case of a standard Gaussian measure $\gamma = \mathcal{N}(0, \mathrm{Id}) = G_d\mathcal{L}^d$ is an easy matter, taking into account that the integration by parts formula has to be modified because the density of γ is not constant and reads

$$\int_{\mathbb{R}^d} u(x) D_j v(x)\, d\gamma(x) = -\int_{\mathbb{R}^d} \big[v(x)D_j u(x) - x_j u(x)v(x)\big]\, d\gamma(x). \tag{20}$$

Hence, $BV(\mathbb{R}^d, \gamma)$ functions and the weighted total variation measure $|D_\gamma u|$ can be defined, for $u \in L^1(\mathbb{R}^d, \gamma)$, as in the above Theorem, according to the following suggestions:

(1) replace the measure dx with $d\gamma$ everywhere;
(2) in **1**, replace $D_j\phi(x)$ with $D_j^*\phi(x) = D_j\phi(x) - x_j\phi(x)$;
(3) in **2**, replace $\operatorname{div}\phi$ with $\sum_{j=1}^d D_j^*\phi_j$;
(4) in **4**, replace the heat semigroup W_t with the *Ornstein-Uhlenbeck semigroup*

$$\begin{aligned} T_t u(x) &= \int_{\mathbb{R}^d} u(e^{-t}x + \sqrt{1-e^{-2t}}y)\, d\gamma(y) \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} u(e^{-t}x + \sqrt{1-e^{-2t}}y) e^{-|y|^2/2}\, dy \\ &= (2\pi(1-e^{-2t}))^{-d/2} \int_{\mathbb{R}^d} u(y) e^{-|y-e^{-t}x|^2/2\sqrt{1-e^{-2t}}} dy \end{aligned} \tag{21}$$

which plays a fundamental role in the infinite-dimensional analysis.

Using Dirichlet forms, a further characteriation of BV functions can be given in the Gaussian setting. Indeed, given $u \in L^1(\mathbb{R}^d,\gamma)$, for $j = 1,\dots,d$ the linear projections x_j^* belong to the domain of the form

$$\mathcal{E}^u(w,v) = \int_{\mathbb{R}^d} \nabla w \cdot \nabla v \; u \; d\gamma$$

and $u \in BV(\mathbb{R}^d,\gamma)$ if and only if there is $C > 0$ such that

$$|\mathcal{E}^u(x_j^*,v)| \le C\|v\|_\infty \qquad \forall\; v \in C_b^1(\mathbb{R}^d). \tag{22}$$

Notice that both the heat and the Ornstein-Uhlenbeck semigroups are Markov semigroups whose transition functions in $\mathbb{R}^d$ in the sense of (18) are given by

$$\begin{aligned} & p(t,x,dy) = G(t,x,y)dy, \\ & G(t,x,y) = \frac{1}{t^{d/2}} G_d\Big(\frac{x-y}{\sqrt{t}}\Big) = \frac{1}{(2\pi t)^{-d/2}} \exp\Big\{\frac{|x-y|^2}{2t}\Big\} \end{aligned} \tag{23}$$

$$\begin{aligned} & p(t,x,dy) = \psi(t,x,y)dy, \\ & \psi(t,x,y) = (2\pi(1-e^{-2t}))^{-d/2} \exp\Big\{-\frac{|y-e^{-t}x|^2}{2\sqrt{1-e^{-2t}}}\Big\}. \end{aligned} \tag{24}$$

The only non trivial point is (4), which is discussed in detail in the Wiener case. For the moment, as discussed also in Subsection 2.5 and in particular in Example 2, let us only point out that the infinitesimal generator of T_t is the operator defined on smooth functions by the expression

$$-Lu(x) = \Delta u(x) - x \cdot \nabla u(x)$$

and that γ turns out to be the invariant measure associated with T_t. The semigroup T_t is related to the Ornstein-Uhlenbeck process

$$\xi_t = e^{-t/2}\xi_0 + \int_0^t e^{(s-t)/2}\, dB_s, \tag{25}$$

solution of the *Langevin SDE*

$$d\xi_t = -\frac{1}{2}\xi_t\, dt + dB_t. \tag{26}$$

From this point of view, let us recall that the generator of W_t is the Laplace operator, and that the Lebesgue measure is invariant under the heat flow (this does not fit completely into the theory of invariant measures, as $\mathcal{L}^d$ is not finite).

Differently from the Sobolev case, BV functions are allowed to be discontinuous along hypersurfaces, and indeed *characteristic functions* χ_E may belong to BV. If $E \subset \mathbb{R}^d$ and $|D\chi_E|(\mathbb{R}^d)$ is finite, we say that E is a set with finite perimeter, use the notation $P(E)$ (perimeter of E) for the total variation of the measure $D\chi_E$ and write $P(E,\cdot)$ for $|D\chi_E|(\cdot)$. Analogously, we set $P_\gamma(E)$ and $P_\gamma(E,\cdot)$ in the Gaussian case. The study of structure of sets with finite perimeter is important on its own, but also because it gives information on general BV functions, through the *coarea formula*: if $u \in BV(\mathbb{R}^d)$, then $P(\{u > t\})$ is finite for a.e. $t \in \mathbb{R}$ and for every $B \in \mathcal{B}(\mathbb{R}^d)$ the following equality holds:

$$|Du|(B) = \int_{\mathbb{R}} P(\{u > t\}, B)\, dt, \tag{27}$$

with P_γ in place of P and $D_\gamma u$ in place of Du in the Gaussian case.

Let us come at a very short discussion of *fine properties* of BV functions. Observing that, as usual, $BV_{\rm loc}$ functions can be defined as those $L^1_{\rm loc}(\mathbb{R}^d)$ functions such that

$$V(u,A) = \sup\Big\{ \int_A u \operatorname{div}\phi dx : \phi \in C^1_c(A,\mathbb{R}^d), \|\phi\|_\infty \le 1\Big\} < \infty$$

for all bounded open sets $A \subset \mathbb{R}^d$, clearly $BV(\mathbb{R}^d,\gamma) \subset BV_{\rm loc}(\mathbb{R}^d)$, hence we may confine to $BV_{\rm loc}(\mathbb{R}^d)$ to treat both the Lebesgue and the Gaussian case. On the other hand , it is clear that $BV(\mathbb{R}^d) \subset BV(\mathbb{R}^d,\gamma)$ and that in this case $D_\gamma u = G_d\, Du$.

According to the general discussion on approximate limits, we may assume that all the functions are approximately continuous in their Lebesgue set, and we may call S_u the complement of the Lebesgue set of u. Let us list some properties of $BV_{\rm loc}$ functions.

Theorem 3. *Let u belong to $BV_{\rm loc}(\mathbb{R}^d)$. Then, the following hold:*

(1) *S_u is an $\mathcal{L}^d$-negligible and countably $(d-1)$-rectifiable Borel set;*

(2) *there is $J_u \subset S_u$ such that for every $x \in J_u$ there are $u^+(x) \neq u^-(x) \in \mathbb{R}$ and $\nu_u(x) \in \mathbb{S}^{d-1}$ such that, setting*

$$B^+_\varrho(x) = B_\varrho(x) \cap \{(y-x)\cdot \nu_u(x) > 0\},$$
$$B^-_\varrho(x) = B_\varrho(x) \cap \{(y-x)\cdot \nu_u(x) < 0\},$$

the following equalities hold:

$$\lim_{\varrho\to 0}\frac{1}{\mathcal{L}^d(B_\varrho^+)}\int_{B_\varrho^+(x)}|u(y)-u^+(x)|\,dy=0,$$

(28)

$$\lim_{\varrho\to 0}\frac{1}{\mathcal{L}^d(B_\varrho^-)}\int_{B_\varrho^-(x)}|u(y)-u^-(x)|\,dy=0.$$

J_u *is called* approximate jump set*, the values* $u^\pm(x)$ approximate one-sided limits *and* $\nu_u(x)$ approximate normal to J_u at x*. Moreover, the triple* $(u^+(x),u^-(x),\nu_u(x))$ *is determined up to an exchange between* $u^+(x)$ *and* $u^-(x)$ *and a change of sign of* $\nu_u(x)$*;*

(3) $\mathcal{H}^{d-1}(S_u\setminus J_u)=0$*, the functions* $x\mapsto u^\pm(x)$*,* $x\in J_u$*, are Borel, if* B *is such that* $\mathcal{H}^{d-1}(B)=0$ *then* $|Du|(B)=0$ *and the measure* $Du\llcorner J_u$ *coincides with* $(u^+-u^-)\nu_u\mathcal{H}^{d-1}\llcorner J_u$*.*

If $u=\chi_E\in BV_{\rm loc}(\mathbb{R}^d)$ is a characteristic function, we say that the set E has *locally finite perimeter* and we can say more on the set where the measure $P(E)$ is concentrated. Simple examples show that the topological boundary ∂E is too large (it can be the whole space), hence some suitable relevant subsets should be identified. In this connection, the notion of *density*, which is slightly weaker than that of approximate limit but has a more direct geometric meaning, turns out to be useful. We say that $E\subset\mathbb{R}^d$ has density $\alpha\in[0,1]$ at $x\in\mathbb{R}^d$ if

$$\lim_{\varrho\to 0}\frac{\mathcal{L}^d(E\cap B_\varrho(x))}{\mathcal{L}^d(B_\varrho(x))}=\alpha \tag{29}$$

and in this case we write $x\in E^\alpha$. Of course, if $\operatorname{ap}\lim_{y\to x}\chi_E(y)=\alpha$ then $x\in E^\alpha$. We introduce the *essential boundary*

$$\partial^*E=\mathbb{R}^d\setminus(E^0\cup E^1)$$

and the *reduced boundary* $\mathcal{F}E$, defined as follows: $x\in\mathcal{F}E$ if the following conditions hold:

$$|D\chi_E|(B_\varrho(x))>0\ \forall\,\varrho>0 \text{ and } \exists\ \nu_E(x)=\lim_{\varrho\to 0}\frac{D\chi_E(B_\varrho(x))}{|D\chi_E|(B_\varrho(x))} \tag{30}$$

with $|\nu_E(x)|=1$. If $x\in\mathcal{F}E$, the hyperplane $T(x)=T_{\nu_E(x)}=\{y\in R^d: y\cdot\nu_E(x)=0\}$ is the approximate tangent space to $\mathcal{F}E$ as in (11). Indeed,

$$\lim_{\varrho\to 0}\frac{E-x}{\varrho}=\{y\in\mathbb{R}^d: y\cdot\nu_E(x)>0\} \tag{31}$$

locally in measure in $\mathbb{R}^d$. Looking at the properties of $u = \chi_E$, the following inclusions hold:

$$\mathcal{F}E = J_u \subset E^{1/2} \subset \partial^* E = S_u.$$

On the other hand, $\mathcal{H}^{d-1}(\mathbb{R}^d \setminus (E^0 \cup E^1 \cup E^{1/2})) = 0$ and in particular $\mathcal{H}^{d-1}(\partial^* E \setminus \mathcal{F}E) = 0$. For further reference, it is worth noticing that densities are related to the short-time behaviour of the heat semigroup, i.e.,

$$x \in E^\alpha \quad \Longrightarrow \quad \lim_{t\to 0} W_t \chi_E(x) = \alpha. \tag{32}$$

Let us point out now that there are still (at least) two relevant issues concerning the infinite dimensional setting, the *slicing* and the discussion of *embedding theorems*, both for Sobolev and BV spaces and the related *isoperimetric inequalities*. Of course, we are interested here in these arguments in the Gaussian case, and indeed they can be discussed directly in the Wiener case, because these results are dimension independent, hence there is not a big difference with respect to $(\mathbb{R}^d, \gamma)$ setting.

§4. The Wiener space

In this section we present the measure theoretic and the differential structure which characterise the Wiener spaces. After briefly describing the *classical Wiener space*, whose elements are stochastic processes, we introduce the abstract structure.

4.1. Classical Wiener space

For $a \in \mathbb{R}^d$, let $X = C_a([0,1], \mathbb{R}^d)$ be the Banach space of $\mathbb{R}^d$-valued continuous functions ω on $[0,1]$ such that $\omega(0) = a$, endowed with the sup norm and the Borel σ-algebra $\mathcal{B}(X)$. Looking at $(X, \mathcal{B}(X))$ as a measurable space, consider the *canonical process* $B_t(\omega) = \omega(t)$, $0 \le t \le 1$. Then, there is one probability measure $\mathbb{P}$ (called *Wiener measure*) such that B_t is a Brownian motion in $\mathbb{R}^d$ such that $B_0 = a$. If we want to identify the measure $\mathbb{P}$, we can exploit the fact that linear and bounded functionals on X, i.e., Radon measures, can be tought of as random variables. Using the fact that $B_t = \delta_t$ and that delta measures are dense in the dual of X, it is possible to conclude that $\mathbb{P} = \mathcal{N}(a, Q)$ is a Gaussian measure with covariance $Q = (q_{hk})$, $q_{hk} = q_h \delta_{hk}$ with

$$q_h(\mu, \nu) = \int_0^1 \int_0^1 s \wedge t \, \mu_h(ds) \, \nu_h(dt), \;\; \mu, \nu \in \mathcal{M}([0,1], \mathbb{R}^d), \;\; h = 1, \ldots, d.$$

Given Borel sets $B_j \in \mathcal{B}(\mathbb{R}^d)$, $j = 1, \dots, m$ and $0 = t_0 < t_1 < \dots < t_m \le 1$, define the *cylinder*

$$C = \{\omega \in X: \ \omega(t_j) \in B_j, \ j = 1, \dots, m\};$$

we have

$$\mathbb{P}(C) = \int_{B_1} G(t_1, a, x_1)\, dx_1 \int_{B_2} G(t_2 - t_1, x_1, x_2)\, dx_2 \cdots \int_{B_m} G(t_m - t_{m-1}, x_{m-1}, x_m)\, dx_m, \tag{33}$$

where G is defined in (23).

For what follows (see (39) below), it is important to know for which functions $h \in X$ the measure $\mathbb{P}_h(B) = \mathbb{P}(h+B)$ is absolutely continuous with respect to $\mathbb{P}$: this happens if and only if $h \in H = X \cap H^1(0,1)$ (Cameron-Martin Theorem [13]), i.e., if and only if $h \in X, \ h' \in L^2(0,1)$.

As a consequence of the above discussion, the space of the directions which give absolutely continuous measures under translation has a natural Hilbert space structure. As we are going to see, this is a general fact.

The same construction of the Wiener measure can be done in the (non separable) space of bounded Borel functions on $(0,1)$, but by Kolmogorov Theorem (see [53, Chapter 5]) the Wiener measure concentrates on $C_0([0,1], \mathbb{R}^d)$.

In this setting, we present a result due to Fukushima, see [31], which has been the starting point of the whole theory, as it highlights a strong connection between the theory of perimeters and the stochastic analysis. We use the notation of Section 2.5.

Theorem 4. *Given an open set $D \subset \mathbb{R}^d$, the following conditions are equivalent:*

- **i)** *D has finite perimeter;*
- **ii)** *the reflecting Brownian motion $(X_t, \mathbb{P}_x)$ on $\overline{D}$ is a* semimartingale*, in the sense that the decomposition*

  $$X_t = X_0 + B_t + N_t,$$

  holds, where B_t is the standard d-dimensional Brownian motion and each component N_t^i is of bounded variation and satisfies the property

  $$\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}\left[\int_0^t \chi_K(X_s) d|N_s^i|\right] < +\infty$$

  for any compact set $K \subset D$.

The idea is that if D is a set with finite perimeter, then in a weak sense the Brownian motion B_t is reflected when it reaches the boundary of D since an (approximate) tangent space is defined; using the language of processes, the reflecting Brownian motion admits an expression of the form

$$X_t = X_0 + B_t + \int_0^t \nu_D(X_s) dL_s,$$

where L_t describes the reflection on the boundary; it is the local time, i.e., it is an *additive functional* with *Revuz measure* given by $\mathcal{H}^{d-1} \llcorner \mathcal{F}D$, that is

$$\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E} \left[\int_0^t f(X_s) dL_s \right] = \int_{\mathcal{F}D} f d\mathcal{H}^{d-1}$$

for continuous f. We refer to [34] for the related notions. The idea expressed by this theorem is that, since the Brownian motion has trajectories that are not C^1 and the tangent space to ∂D exists only in an approximate sense, a reflection law is not properly defined in terms of classical calculus, but the reflection properties of the Brownian motion can be described only in a stochastic sense and are contained in the additive functional L_t, the local time.

Fukushima proves the result for a general BV function ρ, by considering the Dirichlet form

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \rho dx$$

with associated process $(X_t, \mathbb{P}_x)$. The idea of the proof is to show that the additive functional

$$A_t^{[u]} = u(X_t) - u(X_0)$$

admits a semimartingale decomposition

$$A_t^{[u]} = M_t^{[u]} + N_t^{[u]},$$

with $M_t^{[u]}$ a martingale and $N_t^{[u]}$ of bounded variation if and only if

$$|\mathcal{E}(u, v)| \leq c \|v\|_\infty,$$

for some positive constant $c > 0$. The particular choice $u(x) = x_i$, the projection onto the i-th coordinate gives the result.

4.2. Abstract Wiener spaces

Let us come to the notion of abstract *Wiener space*. Given a separable Banach space X, let $\gamma = \mathcal{N}(0, Q)$ be a nondegenerate centred Gaussian measure on $(X, \mathcal{B}(X))$. As a general comment, let us point out that a Gaussian measure can be defined in any Banach space, and it is always concentrated on a separable subspace, as briefly recalled in the preceding subsection. Moreover, a consequence of Fernique's theorem, see (6), is that any $x^* \in X^*$ defines a function $x \mapsto \langle x, x^* \rangle$ belonging to $L^p(X, \gamma)$ for all $p \geq 1$. In particular, we may think of any $x^* \in X^*$ as an element of $L^2(X, \gamma)$. Let us denote by $R^* : X^* \to L^2(X, \gamma)$ the embedding, $R^* x^*(x) = \langle x, x^* \rangle$. The closure of the image of X^* in $L^2(X, \gamma)$ under R^* is denoted $\mathscr{H}$ and is called the *reproducing kernel* of the Gaussian measure γ. The above definition is motivated by the fact that if we consider the operator $R : \mathscr{H} \to X$ whose adjoint is R^*, then

$$R\hat{h} = \int_X \hat{h}(x) x \, d\gamma(x), \qquad \hat{h} \in \mathscr{H} \tag{34}$$

(Bochner integral). In fact, denoting by $[\cdot, \cdot]_{\mathscr{H}}$ the inner product in $\mathscr{H}$ and by $|\cdot|_{\mathscr{H}}$ the norm, the equality

$$[\hat{h}, R^* x^*]_{\mathscr{H}} = \int_X \hat{h}(x) \langle x, x^* \rangle d\gamma(x) = \Big\langle \int_X \hat{h}(x) x d\gamma(x), x^* \Big\rangle,$$

that holds for all $x^* \in X^*$, implies (34). With the definition of R, R^* we obtain directly by (5) the decomposition $Q = RR^*$:

$$\langle RR^* x^*, y^* \rangle = [R^* x^*, R^* y^*]_{\mathscr{H}} = \int_X \langle x, x^* \rangle \langle x, y^* \rangle d\gamma(x) = \langle Q x^*, y^* \rangle.$$

The space $H = R\mathscr{H}$ is called the *Cameron-Martin space*; it is a Hilbert space, dense in X because γ is nondegenerate, with inner product defined by

$$[h_1, h_2]_H = [\hat{h}_1, \hat{h}_2]_{\mathscr{H}}$$

for all $h_1, h_2 \in H$, where $h_i = R\hat{h}_i$, $i = 1, 2$, and norm $|\cdot|_H$. As recalled in Subsection 2.1, Q is a compact operator. The same holds for R and R^*, hence the embeddings $X^* \hookrightarrow \mathscr{H}$, $H \hookrightarrow X$ are compact. Given the elements $x_1^*, \ldots, x_m^*$ in X^*, we denote by $\pi_{x_1^*, \ldots, x_m^*} : X \to \mathbb{R}^m$ the finite dimensional projection of X onto $\mathbb{R}^m$ induced by the elements $x_1^*, \ldots, x_m^*$, that is the map

$$\pi_{x_1^*, \ldots, x_m^*} x = (\langle x, x_1^* \rangle, \ldots, \langle x, x_m^* \rangle),$$

also denoted by $\pi_m : X \to \mathbb{R}^m$ if it is not necessary to specify the elements $x_1^*, \ldots, x_m^*$. The symbol $\mathscr{F}C_b^k(X)$ denotes the space of k times continuously differentiable *cylindrical* functions with bounded derivatives up to the order k, that is: $u \in \mathscr{F}C_b^k(X)$ if there are $m \in \mathbb{N}$, $x_1^*, \dots, x_m^* \in X^*$ and $v \in C_b^k(\mathbb{R}^m)$ such that $u(x) = v(\pi_m x)$. We denote by $\mathcal{E}(X)$ the cylindrical σ-algebra generated by X^*, that is the σ-algebra generated by the sets of the form $E = \pi_m^{-1}B$ with $B \in \mathcal{B}(\mathbb{R}^m)$. Since X is separable, $\mathcal{E}(X)$ and $\mathcal{B}(X)$ coincide, see [51, Theorem I.2.2], even if we fix a sequence $(x_j^*) \subset X^*$ which separates the points in X and use only elements from that sequence to generate π_m. We shall make later on some special choices of (x_j^*), induced by the Gaussian probability measure γ in X. Using the embedding $R^*X^* \subset \mathscr{H}$, we say that a family $\{x_j^*\}$ of elements of X^* is orthonormal if the corresponding family $\{R^*x_j^*\}$ is orthonormal in $\mathscr{H}$. It can be proved that $\gamma(H) = 0$, see [13, Theorem 2.4.7] Since X and X^* are separable, starting from a sequence in X^* dense in H, we may construct an orthonormal basis (h_j) in H with $h_j = Qx_j^*$. Set also $H_m = \operatorname{span}\{h_1, \dots, h_m\}$, and define $X^\perp = \operatorname{Ker} \pi_{x_1^*, \dots, x_m^*}$ and X_m the (m-dimensional) complementary space. Accordingly, we have the canonical decomposition $\gamma = \gamma_m \otimes \gamma^\perp$ of the measure γ; notice also that these Gaussian measures are *rotation invariant*, i.e., if $\varrho : X \times X \to X \times X$ is given by $\varrho(x, y) = (\cos\vartheta x + \sin\vartheta y, -\sin\vartheta x + \cos\vartheta y)$ for some $\vartheta \in \mathbb{R}$, then $\varrho_\#(\gamma \otimes \gamma) = \gamma \otimes \gamma$ and the following equality holds:

$$\int_X \int_X u(\cos\vartheta x + \sin\vartheta y) d\gamma(x) d\gamma(y) = \int_X u(x) d\gamma(x), \tag{35}$$

$u \in L^1(X, \gamma)$, which is obtained by the above relation by integrating the function $u \otimes \mathbf{1}$ on $X \times X$. Notice that if X is decomposed as $X = X_m \oplus X^\perp$, the same formula holds in X_m and $X^\perp$ separately, with measures γ_m and $\gamma^\perp$.

For every function $u \in L^1(X, \gamma)$, if $\{h_j\}$ is an orthonormal basis of H, its *canonical cylindrical approximations* u^m are defined as the conditional expectations relative to the σ-algebras $\mathcal{F}_m = \pi_m^{-1}(\mathcal{B}(\mathbb{R}^m))$,

$$u^m = \mathbb{E}(u|\mathcal{F}_m) = \mathbb{E}_m u \quad \text{s.t.} \quad \int_A u d\gamma = \int_A u^m d\gamma \tag{36}$$

for all $A \in \mathcal{F}_m$. Then, $u^m \to u$ in $L^1(X, \gamma)$ and γ-a.e. (see e.g. [13, Corollary 3.5.2]). More explicitly, we set

$$\mathbb{E}_m u(x) = \int_X u(P_m x + (I - P_m)y) d\gamma(y) = \int_{X^\perp} u(P_m x + y') d\gamma^\perp(y'),$$

where P_m is the projection onto X_m. Notice that the restriction of γ to $\mathcal{F}_m$ is invariant under translations along all the vectors in $X^\perp$, hence we may write $\mathbb{E}_m u(x) = v(P_m x)$ for some function $v \in L^1(X_m, \gamma_m)$, and, with an abuse of notation, $\mathbb{E}_m u(x_m)$ instead of $\mathbb{E}_m u(x)$.

The importance of the Cameron-Martin space relies mainly on the fact that the translated measure

$$\gamma_h(B) = \gamma(B - h), \qquad B \in \mathcal{B}(X), \quad h \in X$$

is absolutely continuous with respect to γ if and only if $h \in H$ and in this case, with the usual notation $h = R\hat{h}$, $\hat{h} \in \mathscr{H}$, we have, see e.g. [13, Corollary 2.4.3],

$$d\gamma_h(x) = \exp\Big\{\hat{h}(x) - \frac{1}{2}|h|_H^2\Big\} d\gamma(x). \tag{37}$$

Let us look for the basic integration by parts formula in the present context, that generalises (20) and allows to define weak derivatives and BV functions. For $h \in X$, define

$$\partial_h f(x) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}$$

(whenever the limit exists); we look for an operator ∂_h^* such that for every $f, g \in \mathscr{F}C_b^1(X)$ the equality

$$\int_X g(x) \partial_h f(x) d\gamma(x) = -\int_X f(x) \partial_h^* g(x) d\gamma(x) \tag{38}$$

holds. Starting from the incremental ratio, we get

$$\begin{aligned} \int_X \frac{f(x + th) - f(x)}{t} g(x) d\gamma(x) = &-\int_X f(y) \frac{g(y) - g(y - th)}{t} d\gamma_{th}(y) \\ &+ \int_X f(x) g(x) d\mu_t(x) \end{aligned} \tag{39}$$

where $\mu_t = \frac{1}{t}\Big(\mathcal{N}(th, Q) - \mathcal{N}(0, Q)\Big)$. From the Cameron-Martin formula (37) we know that $\mu_t \ll \gamma$ if and only if $h \in H$. In this case, we can use (37) and pass to the limit by dominated convergence as $t \to 0$, getting (38) with

$$\partial_h^* g(x) = \partial_h g(x) - g(x) \hat{h}(x),$$

where as usual $h = R\hat{h}$. Such notions can be extended to the more general class of *differentiable measures*, see [15]. Let us now define the gradient and the divergence operators. For $f \in \mathscr{F}C_b^1(X)$, the H-gradient of f, denoted by $\nabla_H f$, is the map from X into H defined by

$$[\nabla_H f(x), h]_H = \partial_h f(x), \quad h \in H,$$

where $\partial_h f(x)$ is defined as before. Notice that if $f(x) = f_m(\pi_m x)$ with $f_m \in C^1(\mathbb{R}^m)$, then

$$\partial_h f(x) = \nabla f_m(\pi_m x) \cdot \pi_m h.$$

If we fix an orthonormal basis $\{h_j\}_{j\in\mathbb{N}}$ of H, we can write

$$\nabla_H f(x) = \sum_{j\in\mathbb{N}} \partial_j f(x) h_j, \qquad \partial_j = \partial_{h_j},$$

where it is important to notice that the directional derivative ∂_h is computed by normalising h with respect to the norm in H. Considering the space $\mathscr{F}C_b^1(X,H)$, we may define $-\operatorname{div}_H$, the adjoint operator of ∇_H, as the linear map from $\mathscr{F}C_b^1(X,H)$ to $\mathscr{F}C_b(X)$ such that

$$\operatorname{div}_H \phi(x) = \sum_{j\in\mathbb{N}} \partial_j^* \phi_j(x) = \sum_{j\in\mathbb{N}} \partial_j \phi_j(x) - \phi_j(x)\hat{h}_j(x), \quad \phi_j = [\phi, h_j]_H.$$

4.3. Hausdorff measures

The definition of Hausdorff measures in Wiener spaces goes back to [29] and is based on a finite dimensional approximation. If $F \subset X$ is an m-dimensional subspace of H, $B \subset F$, recall that we are denoting by $\mathcal{S}^k(B)$ the *spherical k-dimensional Hausdorff measure* of B. We stress that the balls used in the minimisation above are understood with respect to the H distance and we do not emphasise the dependence on F. Occasionally we canonically identify F with $\mathbb{R}^m$, choosing a suitable orthonormal basis.

Let $F \subset QX^*$ be an m-dimensional subspace of H. We denote by $z = \pi_F(x)$ the canonical projection induced by an orthonormal basis $e_j = Qe_j^*$ of F, namely

$$\pi_F(x) = \sum_{j=1}^{m} \langle e_j^*, x\rangle e_j$$

and set $x = y + z$, so that $y = x - \pi_F(x)$ belongs to $\operatorname{Ker}(\pi_F)$, the kernel of π_F. This decomposition induces the factorisation $\gamma = \gamma^\perp \otimes \gamma_F$ with γ_F standard Gaussian in F and $\gamma^\perp$ Gaussian in $\operatorname{Ker}(\pi_F)$ (whose Cameron-Martin space is $F^\perp$).

Following [29], we can now define spherical $(\infty - 1)$-dimensional Hausdorff measures in X relative to F by

$$\mathcal{S}_F^{\infty-1}(B) = \int^*_{\operatorname{Ker}(\pi_F)} \int_{B_y} G_m(z)\, d\mathcal{S}^{m-1}(z)\, d\gamma^\perp(y) \qquad \forall B \subset X. \tag{40}$$

Here, for $y \in \mathrm{Ker}(\pi_F)$, by B_y we denote the *section* or *slice*

$$B_y = \{z \in F : \; y + z \in B\}. \tag{41}$$

The internal integral in (40) is understood in the Choquet sense, namely

$$\int_{B_y} G_m(z)\, d\mathcal{S}^{m-1}(z) = \int_0^\infty \mathcal{S}^{m-1}\big(\{z \in B_y : \; G_m(z) > \tau\}\big)\, d\tau.$$

If $B_y \in \mathcal{B}(F)$, as it happens in the case $B \in \mathcal{B}(X)$, the integral reduces to a standard one. Furthermore, we have used the outer integral in order to avoid the issue of the measurability of the map $y \mapsto \int_{B_y} G_m\, d\mathcal{S}^{m-1}$. The next basic additivity result is proved in [29].

Lemma 5. *$\mathcal{S}_F^{\infty-1}$ is a σ-additive Borel measure on $\mathcal{B}(X)$. In addition, for all Borel sets B the map $y \mapsto \int_{B_y} G_m\, d\mathcal{S}^{m-1}$ is $\gamma^\perp$-measurable in $\mathrm{Ker}(\pi_F)$.*

A remarkable fact is the monotonicity of $\mathcal{S}_F^{\infty-1}$ with respect to F, which crucially depends on the fact that we are considering spherical Hausdorff measures.

Lemma 6. *$\mathcal{S}_F^{\infty-1} \le \mathcal{S}_G^{\infty-1}$ on $\mathcal{B}(X)$ whenever $F \subset G$.*

The above property has been pointed out in [29], relying on [28, 2.10.27]. We refer to [10, Lemma 3.1] for details. It follows from Lemma 6 that the following definition of *spherical* $(\infty - 1)$-*Hausdorff measure* $\mathcal{S}^{\infty-1}$ in $\mathcal{B}(X)$ is well-posed; we set

$$\mathcal{S}^{\infty-1}(B) = \sup_F \mathcal{S}_F^{\infty-1}(B) = \lim_F \mathcal{S}_F^{\infty-1}(B), \tag{42}$$

the limits being understood in the directed set of finite-dimensional subspaces of QX^*. A direct consequence of Lemma 5 is that $\mathcal{S}^{\infty-1}$ is σ-additive on $\mathcal{B}(X)$. This measure does not coincide with the one of [29], since we are considering only subspaces in H generated by elements of QX^*. Our approach is a bit simpler because the corresponding projections are continuous, whereas general orthogonal decompositions of H give merely measurable projections, so that some technical points related to removing sets of small capacity has to be addressed.

4.4. Sobolev spaces and isoperimetric inequality

There are several possible definitions of Sobolev spaces on Wiener spaces. Since the operator ∇_H is a closable operator in $L^p(X,\gamma)$, one may define the Sobolev space $\mathbb{D}^{1,p}(X,\gamma)$ as the domain of the closure of ∇_H in $L^p(X,\gamma)$[1]. Another possible definition, which is closer to our point of view, is based on the integration by parts formula (38): $f \in L^p(X,\gamma)$ is in $W^{1,p}(X,\gamma)$ if there is $F \in L^p(X,\gamma;H)$ such that (38) holds with $[F,h]_H$ in place of $\partial_h f$ and any $g \in \mathscr{F}C^1_b(X,H)$. In this case, we denote F by $\nabla_H f$. Anyway, the spaces $W^{1,p}$ and $\mathbb{D}^{1,p}$ coincide, see [13, Section 5.2]. This approach requires some further explanations in the case $p=1$, as we shall see at the end of this subsection.

The *Gaussian isoperimetric inequality* says the following, see [41]. Let $E \subset X$, and set $B_r = \{x \in H : \|x\|_H < r\}$, $E_r = E + B_r$; then

$$\Phi^{-1}(\gamma(E_r)) \geq \Phi^{-1}(\gamma(E)) + r, \quad \text{where } \Phi(t) := \int_{-\infty}^{t} \frac{e^{-s^2/2}}{\sqrt{2\pi}} ds.$$

We sketch here why this inequality implies the isoperimetric inequality. We introduce the function

$$\mathscr{U}(t) := (\Phi' \circ \Phi^{-1})(t) \approx t\sqrt{2\log(1/t)}, \ t \to 0.$$

Since $\mathscr{U}(t) = \mathscr{U}(1-t)$, the function $\mathscr{U}$ has the same behaviour as $t \to 1$, $\mathscr{U}(t) \approx (1-t)\sqrt{2\log(1-t)}$. Notice that $\Phi(t)$ is the volume of the halfspace $\{\hat{h}(x) < t\}$ and that $\mathscr{U}(t)$ is the perimeter of a halfspace of volume t.

From the above estimate for $\Phi^{-1}(\gamma(E_r))$ we obtain that

$$\begin{aligned}\gamma(E_r) &\geq \Phi(\Phi^{-1}(\gamma(E)) + r) = \gamma(E) + r\Phi'(\Phi^{-1}(\gamma(E))) + o(r)\\ &= \gamma(E) + r\mathscr{U}(\gamma(E)) + o(r),\end{aligned}$$

and then

$$\liminf_{r \to 0} \frac{\gamma(E_r) - \gamma(E)}{r} \geq \mathscr{U}(\gamma(E)).$$

The quantity on the left hand side is related to the Minkowski content of the set E constructed using the Cameron-Martin balls, although negligible. For instance, if $X = \mathbb{R}^d$, $\gamma = G_d\mathcal{L}^d$ the standard centred Gaussian measure on $\mathbb{R}^d$ and E a set with smooth boundary, then

$$P_\gamma(E) = \lim_{r \to 0} \frac{\gamma(E_r) - \gamma(E)}{r} \geq \mathscr{U}(\gamma(E)).$$

It is also possible to prove in this case that equality holds if E is a hyperplane; this skecth of the isoperimetry property of hyperplanes is

[1]Notice that the space denoted by $\mathbb{D}^{1,p}(X,\gamma)$ by Fukushima is denoted by $W^{p,1}(X,\gamma)$ in [13].

essentially the proof contained in [41]. The original proof of the isoperimeric properties of hyperplanes in the finite dimensional Gaussian space has been estabilished first in [49]; since the isoperimetric function does not depend on the space dimension, the same proof can be extended to the infinite dimensional case. In [26], again in the finite dimensional case, it is proved that hyperplanes are isoperimetric by using a symmetrisation argument; also in this case, the proof implies that hyperplanes are isoperimetric in the infinite dimensional case. The proof that hyperplanes are the *unique* isoperimetric sets is rather recent and is contained in [17]. Let us also point out that the right Minkowski content uses enlargements E_r of the set E with respect to balls of H and not of X. The reason of this can be explained as follows: the Gaussian measure γ introduces an anisotropy on X due to the covariance operator Q. This anisotropy is compensated in the definition of total variation and perimeter by the gradient ∇_H, since it is defined using vectors that have unit H-norm. The corresponding compensation in the computation of the Minkowski content is achieved by using the balls of H.

The isoperimetric inequality implies also the following:

$$\|\nabla_H f\|_{L^1} \geq \int_0^\infty \mathscr{U}\left(\gamma(\{|f| > s\})\right) ds,$$

and it follows that if $\nabla_H f \in L^1(X, \gamma, H)$ then f belongs to the *Orlicz space*

$$L\log^{1/2}L(X,\gamma) = \{u : X \to \mathbb{R} :\ A_{1/2}(|u|) \in L^1(X,\gamma)\}, \tag{43}$$

where $A_{1/2}(t) = \int_0^t \log^{1/2}(1+s)ds$. This is important in connection to the integration by parts formula (38), because for general $f \in L^1(X,\gamma)$ the product $\hat{h}fg$ is not summable. But, thanks to Fernique theorem, the linear function $\hat{h}$ belongs to the Orlicz space defined through the complementary N-function of $A_{1/2}$, $\psi(t) = \int_0^t (e^{s^2} - 1)ds$, i.e., $\psi(\lambda|h|_H) < \infty$ for some $\lambda > 0$. As a consequence, if $\nabla_H f \in L^1(X,\gamma)$ then $f \in L\log^{1/2}L(X,\gamma)$, the product $\hat{h}fg$ is summable, (38) does make sense and the embedding of $\mathbb{D}^{1,1}(X,\gamma)$ into $L\log^{1/2}L(X,\gamma)$ follows, see [33, Proposition 3.2].

4.5. The Ornstein-Uhlenbeck semigroup

Let us consider the Ornstein-Uhlenbeck semigroup $(T_t)_{t\geq 0}$, defined pointwise by Mehler's formula, which generalises (21):

$$T_t u(x) = \int_X u\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right) d\gamma(y) \tag{44}$$

for all $u \in L^1(X,\gamma)$, $t > 0$. Unlike the heat semigroup, the Ornstein-Uhlenbeck semigroup T_t does not map $L^1(X,\gamma)$ into $\mathbb{D}^{1,1}(X,\gamma)$. But, T_t is strongly continuous in $L\log^{1/2}L(X,\gamma)$ and it follows from (35) that $T_t u \in \mathbb{D}^{1,1}(X,\gamma)$ for any $u \in L\log^{1/2}L(X,\gamma)$, see [33, Proposition 3.6]. Moreover, it is a contraction semigroup in $L^p(X,\gamma)$ for every $p \in [1,+\infty]$ (and hence also in $L\log^{1/2}L(X,\gamma)$) and self-adjoint in $L^2(X,\gamma)$. Moreover, the following commutation relation holds for any $u \in \mathbb{D}^{1,1}(X,\gamma)$

$$\nabla_H T_t u = e^{-t} T_t \nabla_H u, \qquad t > 0. \tag{45}$$

Therefore, we get

$$\nabla_H T_{t+s} u = \nabla_H T_t(T_s u) = e^{-t} T_t \nabla_H T_s u,$$

for any $u \in L\log^{1/2}L(X,\gamma)$, see [13, Proposition 5.4.8]. It also follows from (45) that

$$\int_X T_t f \operatorname{div}_H \phi d\gamma = e^{-t} \int_X f \operatorname{div}_H(T_t \phi) d\gamma, \tag{46}$$

for all $f \in L^1(X,\gamma)$, $\phi \in \mathscr{F}C_b^1(X,H)$, see [8]. Another important consequence of (45) is that if $u \in \mathbb{D}^{1,1}(X,\gamma)$ then

$$\lim_{t\to 0} \|\nabla_H T_t u - \nabla_H u\|_{L^1(X,\gamma)} = 0. \tag{47}$$

Finally, notice that if u^m are the canonical cylindrical approximations of a function $u \in L\log^{1/2}L(X,\gamma)$ defined in (36) then the following inequality holds, see e.g. [8]

$$\int_X |\nabla_H T_t u^m|_H d\gamma \le \int_X |\nabla_H T_t u|_H d\gamma \quad \forall\, t > 0. \tag{48}$$

We end this brief discussion on the Ornstein-Uhlenbeck semigroup by presenting the related Ornstein-Uhlenbeck *process* in the Wiener space. Of course, this is close to the finite dimensional case, with important modifications. First, we define the *cylindrical Brownian motion* in X as an X-valued continuous process B_t^H such that for every $x^* \in X^*$ with $|Qx^*|_H = 1$ the one-dimensional process $\langle x^*, B_t^H\rangle$ is Wiener. After extending the notion of stochastic integral to the case of a cylindrical Brownian motion, we may deal with SDEs in X. The Ornstein-Uhlenbeck process is given by

$$\xi_t = e^{-t/2}\xi_0 + \int_0^t e^{(s-t)/2}\, dB_t^H$$

and, as in $\mathbb{R}^d$, it is the solution of the Cauchy problem for the Langevin equation

$$d\xi_t = -\frac{1}{2}\xi_t\, dt + dB_t^H, \qquad \xi_0 \text{ given r.v.},$$

where B_t^H is a cylindrical Brownian motion. If the law of ξ_0 is δ_x for $x \in X$, denoting by ξ_t^x the corresponding solution, we have the usual equality $T_t f(x) = \mathbb{E}[f(\xi_t^x)]$.

§5. BV functions in the Wiener space

A definition of BV functions in abstract Wiener spaces has been given by M. Fukushima in [32], M. Fukushima and M. Hino in [33], and is based upon Dirichlet form theory quoted in Subsection 2.5.

When X is a Hilbert space, different notions of BV functions can be given, one using the definitions of gradient and divergence that compensate the anysotropy of the measure γ, and a second one that uses the gradient and divergence of the ambient space X; we refer for instance to [2] where these two notions are compared. We also refer to the recent paper [48], where more general notions of BV functions are given in the *Gelfand triples* setting. In [8], [9] the main aim has been to compare the finite and infinite dimensional theory of BV functions from a purely analytical point of view, closer to the classical setting. After collecting, in the preceding section, the tools we need, we pass now to the definition of BV functions in the abstract Wiener space setting. We denote by $\mathcal{M}(X,H)$ the space of all H-valued finite measures μ on $\mathcal{B}(X)$.

Definition 5.1. *Let $u \in L\log^{1/2}L(X,\gamma)$. We say that u has bounded variation in X and we set $u \in BV(X,\gamma)$ if there exists $\mu \in \mathcal{M}(X,H)$ such that for any $\phi \in \mathscr{F}C_b^1(X)$ we have*

$$\int_X u(x)\partial_j^*\phi(x)d\gamma(x) = -\int_X \phi(x)d\mu_j(x) \qquad \forall j \in \mathbb{N}, \tag{49}$$

where $\mu_j = [h_j,\mu]_H$. In particular, if $u = \chi_E$ and $u \in BV(X,\gamma)$, then we say that E has finite perimeter.

Notice that, as in the Sobolev case $\mathbb{D}^{1,1}(X,\gamma)$, the assumption $u \in L\log^{1/2}L(X,\gamma)$ gives a meaning to (38), as discussed in Subsection 4.4. Moreover, in the previous definition we have required that the measure μ is defined on the whole of $\mathcal{B}(X)$ and is σ-additive there. Since cylindrical sets generate the Borel σ-algebra, the measure μ verifying (49) is unique, and will be denoted $D_\gamma u$ as in the finite dimensional Gaussian case. The total variation measure is denoted as usual by $|D_\gamma u|$. We also let

$P_\gamma(E) := |D_\gamma \chi_E|(X)$ be the (Gaussian) perimeter of a subset E of X and we set, as in the finite dimensional case, $P_\gamma(E, \cdot) = |D_\gamma \chi_E|(\cdot)$.

We state now a characteriation of $BV(X,\gamma)$ functions analogous to Theorem 2 and the discussion which follows.

Theorem 7. *Given $u \in L\log^{1/2}L(X,\gamma)$, the following are equivalent:*

(1) *u belongs to $BV(X,\gamma)$;*

(2) *the quantity*

$$V_\gamma(u) := \sup\Big\{\int_X u \operatorname{div}_H \Phi \, d\gamma;\ \Phi \in \mathscr{F}C_b^1(X,H), |\Phi(x)|_H \le 1 \ \forall\, x \in X\Big\}$$

is finite;

(3) *the quantity*

$$L_\gamma(u) = \inf\Big\{ \liminf_{n\to\infty} \int_X |\nabla_H u_n|_H d\gamma : u_n \in \mathbb{D}^{1,1}(X,\gamma), u_n \stackrel{L^1}{\to} u\Big\}$$

is finite;

(4) *the quantity*

$$\mathcal{T}[u] = \lim_{t\downarrow 0} \int_X |\nabla_H T_t u|_H d\gamma \tag{50}$$

is finite.

Moreover, $|D_\gamma u|(X) = V_\gamma(u) = L_\gamma(u) = \mathcal{T}[u]$.

As in the finite dimensional case, see (22), $u \in BV(X,\gamma)$ if and only if there is $C > 0$ such that

$$\Big|\int_X [\nabla_H \Phi, h]_H \ u \ d\gamma\Big| \le C\|\Phi\|_\infty$$

for all $\Phi \in \mathscr{F}C_b^1(X,H)$. The proof of Theorem 7 is contained in [32], [33], and also in [9]. The proof in the latter reference relies on a *slicing* argument, a technique that has proved to be very useful in the finite dimensional case and we shall use later. For $\nu \in \bigcup_m H_m$, denote by ∂_ν and ∂_ν^* the differentiation operator and its adjoint, respectively, and the *directional total variation along* ν as

$$V_\gamma^\nu(u) = \sup\Big\{ \int_X u\partial_\nu^*\phi d\gamma : \phi \in \mathscr{F}^\nu C_b^1(X),\ |\phi(x)| \le 1 \ \forall\, x \in X\Big\}, \tag{51}$$

where $\phi \in \mathscr{F}^\nu C_b^1(X)$ means that $\phi(x) = v(\langle x, x^*\rangle)$ with $v \in C_b^1(\mathbb{R})$ and $\nu = Qx^*$. Riesz theorem shows that $V_\gamma^\nu(u)$ is finite if and only if the integration by parts formula

$$\int_X u\partial_\nu^* \phi d\gamma = -\int_X \phi d\mu_\nu \qquad \forall \phi \in C_b^1(X) \tag{52}$$

holds for some real-valued measure μ_ν with finite total variation, that we denote by $D_\gamma^\nu u$; if this happens, $|\mu_\nu|(X)$ coincides with $V_\gamma^\nu(u)$. Finally,

$$V_\gamma^\nu(u) = \lim_{m\to\infty} V_\gamma^\nu(\mathbb{E}_m u). \tag{53}$$

Once a direction $\nu = Qx^* \in H$ is fixed, let $\pi_\nu(x) = \langle x, x^*\rangle$ be the induced projection and let us write $x \in X$ as $y + \pi_\nu(x)\nu$. Then, denoting by K the kernel of π_ν, γ admits a product decomposition $\gamma = \gamma^\perp \otimes \gamma_1$ with $\gamma^\perp$ Gaussian in K. For $u : X \to \mathbb{R}$ and $y \in K$ we define the function $u_y : \mathbb{R} \to \mathbb{R}$ by $u_y(t) = u(y + t\nu)$. The following slicing theorem holds

Theorem 8. *Let $u \in L\log^{1/2}L(X,\gamma)$ and let $\nu \in \bigcup_m H_m$; then*

$$V_\gamma^\nu(u) = \int_K V_{\gamma_1}(u_y)\, d\gamma^\perp(y).$$

In particular, the directional total variation of u is independent of the choice of the basis and makes sense for all $h \in H$.

The coarea formula (27) holds as well in Wiener spaces and can be proved by following *verbatim* the proof of [27, Section 5.5]: if $u \in BV(X,\gamma)$, then for a.e. $t \in \mathbb{R}$ the level set $\{u > t\}$ has finite perimeter and for every Borel set $B \subset X$ the following equality holds:

$$|D_\gamma u|(B) = \int_{\mathbb{R}} P_\gamma(\{u > t\}, B) dt. \tag{54}$$

We end this section with a recent example of application in the classical Wiener space, see [50].

Example 3. Let us fix a time $t \in [0,1]$ and consider the classical Wiener space $X = C_0([0,1],\mathbb{R})$, see Subsection 4.1. Define

$$M_t = \sup\{B_s,\ 0 \le s \le t\}.$$

It is well-known that $M_t \in \mathbb{D}^{1,p}(X,\mathbb{P})$, but $\nabla_H M_t$ is not differentiable. Nevertheless, $\nabla_H M_t$ belongs to $BV(X,\mathbb{P})$, i.e., there exists a $H \otimes H$-valued measure $D^2 M_t$ such that

$$\int_X [\Phi h_1 \otimes h_2, D^2 M_t]_{H\otimes H} = \int_X M_t \partial_{h_1}^* \partial_{h_2}^* \Phi\, d\gamma$$

for every $\Phi \in \mathscr{F}C_b^2(X)$, $h_1, h_2 \in H$. Moreover, the measure $|D_{\mathbb{P}} \nabla_H M_t|$ is concentrated on the trajectories that attain their maximum exactly twice, hence, in particular, all these measures are singular with respect to $\mathbb{P}$.

§6. Fine properties of sets with finite perimeter

We show in this section how is it possible to generalise in the infinite-dimensional setting the properties listed in Theorem 3; we restrict our attention to the case of sets with finite perimeter, so that we can use the geometric meaning of points of density stated by formula (29) to give a suitable notion of boundary of a set.

It is worth noticing that in the infinite-dimensional setting things do not work as well as for the Euclidean case; Preiss [47] gave an example of an infinite-dimensional Hilbert space X, a Gaussian measure γ and a set $E \subset X$ such that $0 < \gamma(E) < 1$ and

$$\lim_{\varrho \to 0} \frac{\gamma(E \cap B_\varrho(x))}{\gamma(B_\varrho(x))} = 1, \qquad \forall x \in X. \tag{55}$$

In the same work, it is also shown that if the eigenvalues of the covariance Q decay to zero sufficiently fast, then it is possible to talk about density points; in some sense, the requirement on the decay gives properties of X closer to the finite-dimensional case. For these reasons, in general the notion of point of density as given in (55) is not a good notion.

In the infinite-dimensional setting, the idea is to use the factorisation $\gamma = \gamma^\perp \otimes \gamma_F$, for $F \subset QX^*$ an m-dimensional space, described in Subsection 4.3.

Definition 6.1 (Essential boundary relative to F)**.** *If we write* $X = F \oplus \mathrm{Ker}(\pi_F)$*, we recall by* (41) *the definition of the slice of* E *in direction* F

$$E_y = \{z \in F : y + z \in E\} \subset F;$$

the essential boundary of E relative to F *is then defined as*

$$\partial^*_F E = \{x = y + z : z \in \partial^*(E_y)\}.$$

It is not difficult to show that $\partial^*_F E$ is a Borel set; moreover, in order to pass from the finite dimensional space F to the whole of the Cameron-Martin space H, we need the following property.

Lemma 9. *Let* $G \subset QX^*$ *be a* k*-dimensional Hilbert space, let* $F \subset G$ *be an* m*-dimensional subspace and let* E *be a set with finite* *perimeter in G. Then, with the orthogonal decomposition $G = F \oplus L$ and the notation*

$$E_w := \{z \in F : \ w + z \in E\} \qquad\qquad w \in L,$$

we have that $\mathcal{S}^{m-1}\left(\{z \in F : \ z \in \partial^* E_w, \, w + z \notin \partial^* E\}\right) = 0$ for $\mathcal{S}^{k-m}$-a.e. $w \in L$.

Thanks to this fact, we have that if $F \subset G \subset QX^*$ are two finite dimensional spaces, then the relative essential boundary $\partial^*_F E$ of E is contained, up to negligible sets, into the essential boundary $\partial^*_G E$ of E relative to G, that is

$$\mathcal{S}^{\infty-1}_F(\partial^*_F E \setminus \partial^*_G E) = 0.$$

In [10] there is the proof of the following fact.

Proposition 1. *Let $\mathcal{F}$ be a countable family of finite-dimensional subspaces of QX^* stable under finite unions. For $F \in \mathcal{F}$, let $A_F \in \mathcal{B}(X)$ be such that*

(i) $\mathcal{S}^{\infty-1}_F(A_F \setminus A_G) = 0$ *whenever $F \subset G$;*
(ii) $\sup_F \mathcal{S}^{\infty-1}_F(A_F) < \infty$.

Then $\lim\limits_F(\mathcal{S}^{\infty-1}_F \llcorner A_F)$ exists, and it is representable as $(\lim\limits_F \mathcal{S}^{\infty-1}_F) \llcorner A$ with

$$A := \bigcup_{F \in \mathcal{F}} \ \bigcap_{G \in \mathcal{F}, \, G \supset F} A_G \in \mathcal{B}(X).$$

Such Proposition allows for the definition of the cylindrical essential boundary.

Definition 6.2 (Cylindrical essential boundary)**.** *Let $\mathcal{F}$ be a countable set of finite-dimensional subspaces of H stable under finite union, with $\cup_{F \in \mathcal{F}} F$ dense in H. Then, we define cylindrical essential boundary $\partial^*_{\mathcal{F}} E$ along $\mathcal{F}$ the set*

$$\partial^*_{\mathcal{F}} E := \bigcup_{F \in \mathcal{F}} \ \bigcap_{G \in \mathcal{F}, \, G \supset F} \partial^*_G E.$$

These definitions are used in [38] and with minor revisions in [10], to get a representation of the perimeter measure as follows.

Theorem 10. *Let $E \in \mathcal{B}(X)$ be a set with finite γ-perimeter in X, let $\mathcal{F}$ be as in Definition 6.2 and let $\partial^*_{\mathcal{F}} E$ be the corresponding cylindrical essential boundary. Then*

$$|D_\gamma \chi_E|(B) = \mathcal{S}^{\infty-1}_{\mathcal{F}}(B \cap \partial^*_{\mathcal{F}} E) \qquad \forall B \in \mathcal{B}(X). \tag{56}$$

In particular, $\partial^*_{\mathcal{F}} E$ is uniquely determined by (56) *up to $\mathcal{S}^{\infty-1}_{\mathcal{F}}$-negligible sets.*

In [10] also a weak rectifiability result of the cylindrical essential boundary is given; the term weak refers to the fact that rectifiability is done by using Sobolev functions instead of Lipschitz maps as in (10). This is not a minor difficulty, since in the infinite-dimensional setting no Lusin type properties are known; in particular, it is not known if any Sobolev function coincides with a Lipschitz map in a set of positive measure.

First, we recall the notion of H-graph.

Definition 6.3 (H-graph)**.** *A set $\Gamma \subset X$ is called an H-graph if there exist a unit vector $k \in QX^*$ and $u : D \subset \mathrm{Ker}(\pi_F) \to \mathbb{R}$ (here $F = \{sk,\ s \in \mathbb{R}\}$) such that*

$$\Gamma = \{y + u(y)k :\ y \in D\}.$$

We say that Γ is an entire Sobolev H-graph if moreover $D \in \mathcal{B}(\mathrm{Ker}(\pi_F))$, $\gamma^\perp\big(\mathrm{Ker}(\pi_F) \setminus D\big) = 0$ and $u \in W^{1,1}(\mathrm{Ker}(\pi_F), \gamma^\perp)$.

With this notion, in [10] the following theorem is proved.

Theorem 11. *For any set $E \subset X$ with finite perimeter the measure $|D_\gamma \chi_E|$ is concentrated on a countable union of entire Sobolev H-graphs.*

In [5], the Ornstein-Uhlenbeck semigroup is used to define points of density 1/2; the main result can be summarised in the following Theorem.

Theorem 12. *Let $E \subset X$ be a set with finite perimeter; then*

$$\lim_{t \downarrow 0} \int_X \left| T_t \chi_E - \frac{1}{2} \right|^2 d|D_\gamma \chi_E| = 0;$$

in particular, there exists a sequence $t_i \downarrow 0$ such that

$$\sum_i \int_X \left| T_{t_i} \chi_E - \frac{1}{2} \right| d|D_\gamma \chi_E| < +\infty, \tag{57}$$

which ensures that $T_{t_i}\chi_E \to \frac{1}{2}$ $|D_\gamma \chi_E|$-a.e. in X.

Thanks to the previous Theorem, a notion of points of density $\frac{1}{2}$ can be given. As explained in connection with the notion of essential boundary, the analogue (55) of the finite dimensional procedure (29) is not available in the present situation, hence it relies rather on an approach analogous to (32).

Definition 6.4 (Points of density 1/2)**.** *Let* $(t_i)_i$ *be a sequence such that*

$$\sum_i \sqrt{t_i} < +\infty \tag{58}$$

and (57) *holds. Then, we say that* x *is a point of density* $\frac{1}{2}$ *for* E *if it belongs to*

$$E^{1/2} := \left\{ x \in X : \exists \lim_{i\to+\infty} T_{t_i}\chi_E(x) = \frac{1}{2} \right\}. \tag{59}$$

The requirement in (58) is rather natural, since for a set with finite perimeter it is possible to prove (see [5, Lemma 2.3]) that

$$\int_X |T_t\chi_E - \chi_E| d\gamma \le c_t P_\gamma(E),$$

with

$$c_t = \sqrt{\frac{2}{\pi}} \int_0^t \frac{e^{-s}}{\sqrt{1-e^{-2s}}} ds \sim 2\sqrt{\frac{t}{\pi}}.$$

Theorem 13. *Let* $(t_i)_i$ *be a sequence such that* $\sum_i \sqrt{t_i} < +\infty$ *and* (57) *holds. Then* $|D_\gamma\chi_E|$ *is concentrated on* $E^{1/2}$ *defined in* (59)*; moreover* $E^{1/2}$ *has finite* $\mathcal{S}^{\infty-1}$ *measure and*

$$|D_\gamma\chi_E| = \mathcal{S}^{\infty-1} \llcorner E^{1/2}.$$

It is worth noticing that the sequence $(t_i)_i$ depends of the set E itself. In [6]it is also proved a part of the rectifiability result for the reduced boundary; with minor revision of the definition of cylindrical essential boundary, it is possible to define a *cylindrical reduced boundary* by setting

$$\mathcal{F}_F E = \{x \in X : x = y + z : z \in \mathcal{F}(E_y) \subset F\},$$

and

$$\mathcal{F}_H E = \liminf_{F\in\mathcal{F}} \mathcal{F}_F E = \bigcup_{F\in\mathcal{F}} \bigcap_{G\in\mathcal{F}, G\supset F} \mathcal{F}_G E, \tag{60}$$

where here $\mathcal{F}$ has two meanings, the first one to denote the reduced boundary, the second one when writing $F \in \mathcal{F}$ is meant as a countable collection of finite dimensional sets as in Proposition 1. The liminf of sets in (60) is also given in the sense of Proposition 1.

Given an element $h \in H$, the halfspace having h as its "inner normal" is defined as

$$S_h = \{x \in X : \hat{h}(x) > 0\}.$$

Notice that S_h is a closed halfspace if $\hat{h} = R^* x^*$ for some $x^* \in X^*$; otherwise, it is easily seen by approximation that $\hat{h}$ is linear on a subspace of X of full measure, hence the above definition does make sense. Since the convergence of sequences $h_n \in H$ to $h \in H$ in the norm of H implies the convergence of S_{h_n} to S_h in the sence of convergence of characteristic functions in $L^1(X, \gamma)$, then, denoting by

$$E_{x,t} := \frac{E - e^{-t}x}{\sqrt{1 - e^{-2t}}},$$

the following result holds true, see [6]. We notice that the idea underlying the following result is the last line in (21), which cannot be used directly in the infinite-dimensional framework.

Theorem 14. *Let $E \subset X$ be a set with finite perimeter in X, $x \in \mathcal{F}_H E$ and $S(x) = S_{\nu_E(x)}$ where ν_E is defined by the polar decomposition $D_\gamma \chi_E = \nu_E |D_\gamma \chi_E|$; then*

$$\lim_{t \downarrow 0} \int_X \int_X \left| \chi_E(e^{-t}x + \sqrt{1 - e^{-2t}}y) - \chi_{S(x)}(y) \right| d\gamma(y) d|D_\gamma \chi_E|(x) = 0.$$

In other terms, the previous results can be restated by saying that

$$\lim_{t \downarrow 0} \int_X \|\chi_{E_{x,t}} - \chi_{S(x)}\|_{L^1(X,\gamma)} d|D_\gamma \chi_E|(x) = 0,$$

that is, $E_{x,t}$ coverge to $S(x)$ in $L^1(X, \gamma)$, for $|D_\gamma \chi_E|$-a.e. $x \in X$. This result is in some sense the Wiener space formulation of (31).

6.1. Examples of sets with finite perimeter

We now provide some examples of sets with finite perimeter; in some cases the essential and reduced boundary are directly identifiable, in some other they are indicated as candidates, but a proof is not available so far.

6.1.1. *Cylindrical sets.* Let $\mathcal{F}$ be as in Definition 6.2. The easiest way to construct examples of sets with finite perimeter is to use the decomposition $X = F \oplus \mathrm{Ker}(\pi_F)$; if $B \subset F$ is a set with $\chi_B \in BV(F, \gamma_F)$, then $E = \pi_F^{-1}(B)$ has finite perimeter in X with

$$P_\gamma(E, X) = P_{\gamma_F}(B, F).$$

If $F \in \mathcal{F}$, then

$$\partial^*_{\mathcal{F}} E = \partial^*_F E = \partial^* B, \qquad \mathcal{F}_H E = \mathcal{F} B,$$

otherwise the previous equality holds up to $|D_\gamma \chi_E|$-negligible sets.

6.1.2. *Level sets of Lipschitz maps: comparison with the Airault-Malliavin surface measure.* By coarea formula (54), almost every level set of a BV function has finite perimeter; in particular, we can use almost every level set of Sobolev or Lipschitz functions. To prove that *every* level set, under some regularity assumption on the function, has finite perimeter is quite delicate in this framework. In [1], Airault and Malliavin constructed a surface measure on boundaries of regular level sets. More precisely, they considered functions f belonging to

$$W^\infty(X,\gamma) = \bigcap_{p>1, k\in\mathbb{N}} W^{k,p}(X,\gamma),$$

where $W^{k,p}(X,\gamma)$ is the Sobolev space of order k with p-integrability, such that

$$\frac{1}{|\nabla_H f|_H} \in \bigcap_{p\geq 1} L^p(X,\gamma);$$

what they proved is that the image measure $f_\# \gamma$ defined on $\mathcal{B}(\mathbb{R})$ by

$$f_\#\gamma(I) = \gamma(f^{-1}(I))$$

has smooth density ρ with respect to the Lebesgue measure and that, for each t such that $\rho(t) > 0$, there exists a Radon measure σ_t supported on $f^{-1}(t)$ such that

$$\int_{\{f<t\}} \operatorname{div}_H \Phi \, d\gamma = \int_{\{f=t\}} \frac{[\Phi, \nabla_H f]_H}{|\nabla_H f|_H} \, d\sigma_t.$$

The measure σ_t is constructed in terms of the Minkowski content as explained in Subsection 4.4. In [18], it is proved that, under the additional technical assumption that f is continuous, the set $\{f < t\}$ has finite perimeter whenever $\rho(t) > 0$ with the identity

$$P_\gamma(\{f<t\}) = \sigma_t(\{f=t\}) = \int_{\{f<t\}} \operatorname{div}_H \nu_H \, d\gamma,$$

where $\nu_H = \nabla_H f / |\nabla_H f|_H$. The set $\{f = t\}$ is expected to be the essential boundary of $\{f < t\}$, whereas the points in the reduced boundary are expected to be those x where $\nabla_H f(x) \neq 0$.

6.1.3. *Balls and convex sets.* If we fix a point $x_0 \in X$, the map

$$f(x) = \|x - x_0\|_X$$

is Lipschitz and then the sets

$$E_t = \{f < t\} = B_t(x_0)$$

have finite perimeter for almost every $t > 0$. The proof that *every* ball has finite perimeter is contained in [18]; if X is a Hilbert space, then the function $f(x)^2$ is continuous and satisfies all the condition imposed by Airault and Malliavin and then all balls in Hilbert spaces have finite perimeter. In addition, the normal vector in this case is given by

$$\nu(x) = \frac{Q(x - x_0)}{|Q(x - x_0)|_H}$$

(where Q is the covariance operator, $\gamma = \mathcal{N}(0, Q)$) and the function

$$g(t) = P_\gamma(B_t(x_0))$$

is continuous in $[0, +\infty)$ with

$$\lim_{t\to 0} P_\gamma(B_t(x_0)) = \lim_{t\to+\infty} P_\gamma(B_t(x_0)) = 0.$$

It is also possible to prove that there exist $t_1 < t_2$ such that g is increasing in $[0, t_1]$ and decreasing in $[t_2, +\infty)$.

The proof that any ball in an infinite-dimensional Banach space has finite perimeter is less explicit and is based on a Brunn-Minkowski argument stating that for every Borel sets $A, B \subset X$,

$$\gamma(\lambda A + (1 - \lambda)B) \geq \gamma(A)^\lambda \gamma(B)^{1-\lambda}, \qquad \lambda \in [0, 1].$$

In [18] it is proved that if C is an open convex set, then $\gamma(\partial C) = 0$ and C has finite perimeter. In this case, it is easily seen that $\partial^*_{\mathcal{F}} C \subset \partial C$ and

$$|D_\gamma \chi_C|(\partial C \setminus \partial^*_{\mathcal{F}} C) = 0; \tag{61}$$

indeed if $x \in C$ or $x \in X \setminus \overline{C}$, then for any $F \leq H$, if we write

$$x = y + z_x, \qquad y \in X^\perp, z_x \in X_m$$

then z_x is an interior point either of C_y or of $X_m \setminus \overline{C}_y$, so $\partial^*_{\mathcal{F}} C \subset \partial C$. Property (61) follows by the representation of the perimeter measure (56). The characterisation of the reduced cylindrical boundary is less clear.

The assumption that C is open is essential; indeed, it is also shown that, in the Hilbert space case, there exists a convex set with infinite perimeter. Such a set is constructed by fixing a sequence r_i such that

$$\sqrt{\frac{2}{\pi}}\frac{e^{-\frac{r_i^2}{2}}}{r_i} = \frac{1}{(i+1)(\log(i+1))^{\frac{3}{2}}},$$

defining

$$C_m = \pi_F^{-1}(Q_m), \quad Q_m = \prod_{i=1}^m [-r_i, r_i]$$

and letting $m \to +\infty$.

6.1.4. *An example in the classical Wiener space.* In [40] an example of a set with finite perimeter in the classical Wiener space is given, using the reflecting Brownian motion. The setting is given by a *pinned path space*, that is

$$X = \{\omega \in C([0,1], \mathbb{R}^d) : \omega(0) = a, \omega(1) = b\}$$

endowed with the pinned Wiener measure $\mathbb{P}_{a,b}$ defined in the same spirit as (33) by

$$\begin{aligned}\mathbb{P}_{a,b}(C) = & \frac{1}{G(1,a,b)} \int_{B_1} G(t_1 - t_0, a, x_1)\, dx_1 \int_{B_2} G(t_2 - t_1, x_1, x_2)\, dx_2 \\ & \cdots \int_{B_m} G(t_{m+1} - t_m, x_m, b)\, dx_m,\end{aligned}$$

where $B_j \in \mathcal{B}(\mathbb{R}^d)$, $j = 1, \ldots, m$, $0 = t_0 < t_1 <\leq t_m \ldots < t_{m+1} = 1$,

$$C = \{\omega \in X : \ \omega(t_j) \in B_j, \ j = 1, \ldots, m\}.$$

In such space, if $\Omega \subset \mathbb{R}^d$ is an open set containing the two points a and b, define the set

$$E^\Omega = \{\omega \in X : \omega(t) \in \Omega \ \forall t \in [0,1]\}.$$

Then E^Ω has finite perimeter in X under the assumption that Ω has positive reach, that is an uniform exterior ball condition: there exists $\delta > 0$ such that for every $y \in \partial\Omega$ there is $z \in \mathbb{R}^d \setminus \Omega$ such that $B_\delta(z) \cap \overline{\Omega} = \{y\}$. The proof of this fact is done constructing a sequence of Lipshitz functions ρ_n converging to $\chi_{\overline{E}^\Omega}$ in $L^1(X, \mathbb{P}_{a,b})$ and such that

$$\int_X |\nabla_H \rho_n|_H d\mathbb{P}_{a,b} \leq n \mathbb{P}_{a,b} \left(\left\{ \omega \in X : 0 \leq \inf_{t \in [0,1]} q(\omega(t)) \leq \frac{1}{n} \right\} \right);$$

the sequence is defined in terms of the signed distance function

$$q(x) = \inf_{\inf y \in \mathbb{R}^d \setminus \Omega} |x-y| - \inf_{y \in \Omega} |y-x|$$

as

$$\rho_n(\omega) = f_n(F(\omega)), \quad F(\omega) = \inf_{t \in [0,1]} q(\omega(t)),$$

where f_n is defined as

$$f_n(s) = \min\{\max\{0, ns\}, 1\}.$$

The keypoint in the proof where the positive reach condition is used is in estimating

$$\mathbb{P}_{a,b} \left(\left\{ \omega \in X : 0 \leq \inf_{t \in [0,1]} q(\omega(t)) \leq r \right\} \right) \leq cr,$$

since from that it comes that

$$\int_{X_{a,b}} |\nabla_H \rho_n|_H d\mathbb{P}_{a,b} \leq c.$$

In this case, Hino-Uchida prove also that the perimeter measure concentrates on the set

$$\partial' E^{\Omega} = \left\{ \omega \in X : \omega(t) \in \overline{\Omega} \text{ and } \exists \text{ an unique t } \in [0,1] \text{ s.t. } \omega(t) \in \partial\Omega \right\}.$$

The definition of the previous set has a meaning very close to the set of points of density 1/2 for E^{Ω}. Finally, it is worth noticing that the proof given by Hino and Uchida of the fact that E^{Ω} ha finite perimeter is close to the proof that a (sufficiently regular) set in the Euclidean setting has finite Minkowski content.

§7. Convex functionals on BV

Following [21], we now consider integral functionals on $BV(X, \gamma)$ of the form

$$u \mapsto \int_X F(D_\gamma u)$$

where $F : H \to \mathbb{R} \cup \{+\infty\}$ is a convex lower semicontinuous function. As $D_\gamma u$ is in general a measure, we have to give a precise meaning to the above expression.

Given a convex function $F : H \to \mathbb{R} \cup \{+\infty\}$ we denote by F^* its convex conjugate, defined as

$$F^*(\Phi) := \sup \{ [\Phi, h]_H - F(h) : h \in H \}, \qquad \Phi \in H,$$

and by F^∞ its recession function defined as

$$F^\infty(h) := \lim_{t\to+\infty} \frac{F(th)}{t} \qquad h \in H.$$

We shall consider functions $F : H \to \mathbb{R} \cup \{+\infty\}$ satisfying the following assumption:

(A) F is a proper (i.e., not identically $+\infty$), lower semi-continuous, convex function on H.

Notice that a convex function F with $p \geq 1$ growth, i.e., such that there are positive constants α_1, β_1, α_2, β_2 such that

$$\alpha_1 |h|_H^p - \beta_1 \leq F(h) \leq \alpha_2 |h|_H^p + \beta_2 \qquad \forall h \in H, \tag{62}$$

satisfies automatically assumption (A).

Given a function F satisfying (A) and $u \in L^2(X, \gamma)$, we define the functional

$$\int_X F(D_\gamma u) := \sup\Big\{ \int_X -u \operatorname{div}_H \Phi - F^*(\Phi)\, d\gamma,\ \Phi \in \mathscr{F}C_b^1(X, H) \Big\} \tag{63}$$

which is lower semicontinuous in $L^2(X, \gamma)$. Similarly, for $\mu \in \mathcal{M}(X, H)$ we set

$$\int_X F(\mu) := \sup\Big\{ \int_X [\Phi, d\mu]_H - \int_X F^*(\Phi) d\gamma,\ \Phi \in \mathscr{F}C_b^1(X, H) \Big\}.$$

The following result has been proved in [21, Theorem 3.2].

Theorem 15. *Let $F : H \to \mathbb{R} \cup \{+\infty\}$ satisfy (A) and let $\mu \in \mathcal{M}(X, H)$; then*

$$\int_X F(\mu) = \int_X F(\mu^a) d\gamma + \int_X F^\infty\left(\frac{d\mu^s}{d|\mu^s|} \right) d|\mu^s|$$

where $\mu = \mu^a \gamma + \mu^s$ is the Radon-Nikodym decomposition of μ w.r.t. γ.

From Theorem 15 we obtain a representation result for the functional in (63).

Theorem 16. *If $F : H \to \mathbb{R} \cup \{+\infty\}$ satisfies (A), then*

$$\int_X F(D_\gamma u) = \int_X F(\nabla_H u) d\gamma + \int_X F^\infty\left(\frac{dD_\gamma^s u}{d|D_\gamma^s u|} \right) d|D_\gamma^s u| \tag{64}$$

for all $u \in BV(X, \gamma)$, where $D_\gamma u = \nabla_H u \gamma + D_\gamma^s u$ is the Radon-Nikodym decomposition of $D_\gamma u$.

A natural question is whether the functional in (63) concides with the relaxation in $L^2(X,\gamma)$ of its restrictions to more regular functions. The following result has been proved in [21, Proposition 3.4].

Theorem 17. *If $F : H \to \mathbb{R} \cup \{+\infty\}$ satisfies (A), then the functional $\int_X F(D_\gamma u)$ is the relaxation in $L^2(X,\gamma)$ of the functional defined as $\int_X F(\nabla_H u) d\gamma$ for $u \in W^{1,1}(X,\gamma)$, and $+\infty$ for $u \notin W^{1,1}(X,\gamma)$.*

If F has $p \geq 1$ growth in the sense of (62)*, then the same relaxation result holds with the space $W^{1,1}(X,\gamma)$ replaced by $\mathscr{F}C_b^1(X)$.*

Condition (62) in the above statement is technical, and we expect that it is not necessary to obtain the relaxation result in $\mathscr{F}C_b^1(X)$.

7.1. Convexity of minimisers

The Direct Method of the Calculus of Variations is a well-known method to prove existence of minimisers of variational problems. The two conditions a functional has to satisfy in order to apply the method are the lower semicontinuity with respect to a given topology, and the compactness of a nonempty sublevel set in the same topology.

We now consider convex functionals of the form

$$\int_X F(D_\gamma u) + \frac{1}{2}\int_X (u-g)^2 d\gamma. \tag{65}$$

where $F : H \to \mathbb{R} \cup \{+\infty\}$ satisfies (A) and $g \in L^2(X,\gamma)$ is a convex function.

Notice that the functional in (65) is convex on $L^2(X,\gamma)$, hence it is also weakly lower semicontinuous. Moreover, its sublevel sets are (relatively) compact in the weak topology of $L^2(X,\gamma)$. By the Direct Method we then obtain the following existence result. The existence of a minimiser follows by the Direct Method of the Calculus of Variations, while the uniqueness follows from the strict convexity of the functional, due to the second term in (65).

Proposition 2. *There exists a unique minimiser $\bar{u} \in L^2(X,\gamma)$ of the functional* (65)*.*

We state a convexity result for minimisers of (65) which has been proved in [21, Theorem 5.1].

Theorem 18. *The minimiser $\bar{u}$ of* (65) *is convex.*

From Theorem 18 and the theory of maximal monotone operators (see [16]), one can easily get the following result:

Theorem 19. *Let $u_0 \in L^2(X,\gamma)$ be a convex initial datum. Then the solution $u(t)$ of the $L^2(X,\gamma)$-gradient flow of $\int_X F(D_\gamma u)$ with initial condition $u(0) = u_0$ is convex for every $t > 0$.*

Notice that, by taking $F(h) = |h|^p$ with $p \geq 1$, Theorem 18 applies to the functional

$$\int_X |D_\gamma u|_H^p + \frac{1}{2}\int_X (u-g)^2 d\gamma. \tag{66}$$

Recalling the coarea formula (54), when $p = 1$ the functional (66) can be written as

$$\int_X |D_\gamma u|_H^p + \frac{1}{2}\int_X (u-g)^2 d\gamma = \int_{\mathbb{R}} \Big(P_\gamma(\{u > t\}) - \int_{\{u>t\}} (g-t) d\gamma \Big) dt.$$

It then follows (see [19, 21]) that the level set $\{\bar{u} > t\}$ of the minimiser $\bar{u}$ minimises the geometric problem

$$P_\gamma(E) - \int_E (g-t)\, d\gamma \tag{67}$$

among the subsets $E \subset X$ of finite perimeter, for all $t \in \mathbb{R}$. Then, from Theorem 18 one can derive a convexity result for minimisers to (67) (see [21, Corollary 5.7]).

Theorem 20. *Let $g \in L^2(X, \gamma)$ be a convex function, and consider the functional*

$$F_g(E) = P_\gamma(E) - \int_E g\, d\gamma. \tag{68}$$

Then, two situations can occur:

- *If $\min F_g < 0$, there exists a unique nonempty minimiser of F_g, which is convex.*
- *If $\min F_g = 0$, there exists at most one nonempty minimiser of F_g, which is then convex.*

7.2. Relaxation of the perimeter in the weak topology

In view of the previous discussion, a natural problem which arises is the classification of the weakly lower semicontinuous functionals on $L^2(X, \gamma)$.

While convex functionals are lower semicontinuous with respect to both the weak and the strong topology, the perimeter functional

$$F(u) := \begin{cases} P_\gamma(E) & \text{if } u = \chi_E \\ +\infty & \text{otherwise} \end{cases}$$

is not weakly lower semicontinuous, as one can easily check by taking the sequence of halfspaces $E_n = \{\langle x, x_n^* \rangle < 0\}$, where x_n^* is a sequence in X^* such that $h_n = Qx_n^*$ is an orthonormal basis of H. Indeed, the characteristic functions of these sets weakly converge to the constant function $1/2$, which is not a characteristic function, while the perimeter of E_n is constantly equal to $1/\sqrt{2\pi}$.

In [35] the authors computed the relaxation $\overline{F}$ of F with respect to the weak $L^2(X,\gamma)$-topology, showing that

$$\overline{F}(u) = \begin{cases} \displaystyle\int_X \sqrt{\mathscr{U}^2(u) + |D_\gamma u|^2} & \text{if } u \in BV(X,\gamma) \text{ and } |u| \le 1 \\ +\infty & \text{otherwise} \end{cases}$$

where

$$\int_X \sqrt{\mathscr{U}^2(u) + |D_\gamma u|^2} = \int_X \sqrt{\mathscr{U}^2(u) + |\nabla_H u|_H^2} d\gamma + |D_\gamma^s u|(X) \tag{69}$$

with $D_\gamma u = \nabla_H u\, d\gamma + D_\gamma^s u$ as in Theorem 16. Observe that the functional $\overline{F}$ already appears in the seminal works by Bakry and Ledoux [11] and Bobkov [12], in the context of log-Sobolev inequalities. See also [9, Remark 4.3] where it appears in a setting closer to ours.

There is also a representation formula for $\overline{F}$, which is reminiscent of the definition of total variation:

$$\begin{aligned} \overline{F}(u) = \sup\Big\{ \int_X (u \operatorname{div}_H \Phi + \mathscr{U}(u)\xi) d\gamma \,:\, &\Phi \in \mathscr{F}C_b^1(X,H), \\ \xi \in \mathscr{F}C_b^1(X),\ |\Phi(x)|_H^2 + |\xi(x)|^2 \le 1\ \forall x \in X &\Big\}, \end{aligned}$$

for all $u \in BV(X,\gamma)$, with $|u| \le 1$.

§8. Open problems

We collect some open problems whose solution, in our opinion, would provide important information on the whole subject and would allow for a wide range of applications.

The first problems that should be solved and would have a great influence in the further developments concern the structure theory of reduced boundaries and general BV functions. For instance it would be important to check whether the well-known Euclidean decomposition result holds in Wiener spaces, i.e., whether the equality $X = E^1 \cup E^0 \cup E^{1/2}$ is true (up to negligible sets). Moreover, as we have seen, a *pointwise* characterisation of reduced boundary like that in (30) is missing, as well as suitable notions of *one-sided approximate limits*, see (28). In this respect, the Orstein-Uhlenbeck semigroup could come into play, but making density computations independent of the sequence (t_i), see (59), would certainly be useful, in connection with the coarea formula. Still on the side of the structure theory, it is important to improve the weak rectifiability Theorem 11, possibly getting Lipschitz rectifiability. All these problems are of course connected to the general problem of the traces of BV functions. Beside other instances, such as boundary value problems, closer to the arguments presented here are applications of the structure theory and fine properties to integral functionals. Indeed, it would be interesting to extend the results presented in Section 7 to integrands depending on u, see [7, Section 5.5] for the classical case. In this connection, it would be important to perform a deeper analysis of the singular part of the gradient, possibly distinguishing between the *jump part* and the *Cantor part*, and defining the one-sided approximate limits. This could probably give a representation formula more precise than (69). Finally, one could try to provide a complete characterisation of weakly lower semicontinuous integral functionals with integrands of linear growth.

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