SYMMETRY RESULTS FOR STABLE AND MONOTONE SOLUTIONS TO FIBERED SYSTEMS OF PDES

SERENA DIPIERRO, ANDREA PINAMONTI

Abstract. We study the symmetry properties for solutions of elliptic systems of the type

$$\begin{cases} -\mathrm{div}(a_1(x, |\nabla u^1|(X))\nabla u^1(X)) = F_1(x, u^1(X), \dots, u^n(X)), \\ \vdots \\ -\mathrm{div}(a_n(x, |\nabla u^n|(X))\nabla u^n(X)) = F_n(x, u^1(X), \dots, u^n(X)), \end{cases}$$

where $x \in \mathbb{R}^m$ with $1 \le m < N$, $X = (x, y) \in \mathbb{R}^m \times \mathbb{R}^{N-m}$, and $F_1, \dots, F_n$ are the derivatives with respect to $\xi^1, \dots, \xi^n$ of some $F = F(x, \xi^1, \dots, \xi^n)$ such that for any $i = 1, \dots, n$ and any fixed $(x, \xi^1, \dots, \xi^{i-1}, \xi^{i+1}, \dots, \xi^n) \in \mathbb{R}^m \times \mathbb{R}^{n-1}$ the map $\xi^i \to F(x, \xi^1, \dots, \xi^i, \dots, \xi^n)$ belongs to $C^2(\mathbb{R})$. We obtain a Poincaré-type formula for the solutions of the system and we use it to prove a symmetry result both for stable and for monotone solutions.

1. Introduction

In this paper we deal with symmetry results for solutions to the following system of partial differential equations defined in an open subset Ω of $\mathbb{R}^N$

$$\begin{cases} -\mathrm{div}(a_1(x, |\nabla u^1|(X))\nabla u^1(X)) = F_1(x, u^1(X), \dots, u^n(X)), \\ \quad\vdots \\ -\mathrm{div}(a_n(x, |\nabla u^n|(X))\nabla u^n(X)) = F_n(x, u^1(X), \dots, u^n(X)). \end{cases} \tag{1}$$

Here $x \in \mathbb{R}^m$ with $1 \le m < N$, $X = (x, y) \in \mathbb{R}^m \times \mathbb{R}^{N-m}$, and $F_1, \dots, F_n$ are the derivatives with respect to $\xi^1, \dots, \xi^n$ of some $F = F(x, \xi^1, \dots, \xi^n)$ such that, for any $i = 1, \dots, n$ and any fixed $(x, \xi^1, \dots, \xi^{i-1}, \xi^{i+1}, \dots, \xi^n) \in \mathbb{R}^m \times \mathbb{R}^{n-1}$, the map $\xi^i \to F(x, \xi^1, \dots, \xi^i, \dots, \xi^n)$ belongs to $C^2(\mathbb{R})$. We also assume that

- $a_i \in L^\infty(\mathbb{R}^m \times [\alpha_-, \alpha_+])$, for any $\alpha_+ > \alpha_- > 0$, $i = 1, \dots, n$;
- for any fixed $t \in (0, +\infty)$ and any $i = 1, \dots, n$, $\inf_{x \in \mathbb{R}^m} a_i(x, t) > 0$;
- for any fixed $x \in \mathbb{R}^m$ and any $i = 1, \dots, n$, the map $t \to a_i(x, t)$ is C^1 on $(0, +\infty)$.

The physical motivation for (1) comes from "fibered", or "stratified" media: namely, the medium, say $\Omega \subset \mathbb{R}^N$, is nonhomogeneous, but this nonhomogeneity only occurs in

Key words and phrases. Elliptic systems, fibered media, monotone solutions, stable solutions, phase separation, Poincaré-type inequality.

The first author has been supported by EPSRC grant EP/K024566/1 "Monotonicity formula methods for nonlinear PDE". During the preparation of this paper the second author was supported by MIUR and by Fondazione CaRiPaRo Project "Nonlinear Partial Differential Equations: models, analysis, and control-theoretic problems". The second author is now supported by ERC ADG GeMeThnES and GNAMPA .

lower dimensional slices (here, the medium is supposed to be homogeneous with respect to $y \in \mathbb{R}^{N-m}$ and nonhomogeneous with respect to $x \in \mathbb{R}^m$).

Systems similar to (1) have been studied in [5]. Precisely, the authors considered the following system

$$\begin{cases} \Delta u = uv^2, \\ \Delta v = vu^2, \\ u, v > 0, \end{cases} \tag{2}$$

which arises in phase separation for multiple states Bose-Einstein condensates. They proved that there exists a solution to (2) in $\mathbb{R}$, which is nondegenerate and reflectionally symmetric, namely that there exists $x_0 \in \mathbb{R}$ such that $u(x - x_0) = v(x - x_0)$. Moreover, they obtained a result that may be seen as the analogue of a famous conjecture of De Giorgi for problem (2) in dimension 2, that is they proved that monotone solutions of (2) in $\mathbb{R}^2$ (see Definition 1.2 below) have one-dimensional symmetry under the additional growth condition

$$u(x) + v(x) \leq C(1 + |x|). \tag{3}$$

On the other hand, in [29], it has been proved that the linear growth is the lowest possible for solutions to (2); in other words, if there exists $\alpha \in (0, 1)$ such that

$$u(x) + v(x) \leq C(1 + |x|)^{\alpha},$$

then $u = v \equiv 0$.

In [6] the authors replaced the monotonicity condition by the stability of the solutions (which is a weaker assumption), showing that the above mentioned one-dimensional symmetry still holds in $\mathbb{R}^2$. Moreover, they proved that there exist solutions to (2) which do not satisfy the growth condition (3), constructing solutions with polynomial growth.

We mention the paper [37], where the author showed that, for any $n \geq 2$, a solution to (2) which is a local minimizer and satisfies the growth condition (3) has one-dimensional symmetry.

In [23] it is proved that the symmetry result stated in [5] holds also for a more general class of nonlinearities.

Finally, in [16], the author considered a class of quasilinear (possibly degenerate) elliptic systems in $\mathbb{R}^n$ and proved that, under suitable assumptions, the solutions have one-dimensional symmetry, showing that the results obtained in [5, 6, 23] hold in a more general setting. We also refer the reader to [18], where symmetry results for systems driven by non local operators are studied.

Results similar to the ones described above are also well-understood in the case of one equation. In particular, in low dimensions, De Giorgi conjecture on the flatness of level sets of standard phase transition ([15]) has been proved, see [2, 3, 4, 24, 25]. Later, Savin in [31] showed that the conjecture is true up to dimension 8 under an additional hypothesis on the behaviour of the solution at infinity. Finally, in dimension $n \geq 9$ Del Pino, Kowalczyk and Wei constructed a solution to the Allen-Cahn equation which is monotone in one direction but not one-dimensional, see [14].

It is also worth noticing that an analogous of the De Giorgi conjecture has been studied for more general operators. In particular, we mention [20], where quasilinear operators of p-Laplacian and curvature type are considered, and [8, 33], where the authors proved a similar De Giorgi-type result for an equation involving the fractional Laplacian in dimension $n = 2$; see also [7, 9, 10] for further extensions.

First of all, we give the following definition:

Definition 1.1. *An $n-$tuple $(u^1, \dots, u^n)$ is said to be a weak solution of (1) if, for any $\psi = (\psi^1, \dots, \psi^n) \in C_c^\infty(\Omega, \mathbb{R}^n)$,*

$$\int_\Omega \big\langle a_i(x, |\nabla u^i|(X)) \nabla u^i(X), \nabla \psi^i(X) \big\rangle \, \mathrm{d}X = \int_\Omega F_i(x, u^1, \dots, u^n)\, \psi^i(X) \mathrm{d}X, \quad i = 1, \dots, n. \tag{4}$$

In order to state our results we start pointing out our assumptions. In particular, from now on we will always assume that every weak solution $(u^1, \dots, u^n)$ of (1) is such that[1]

$$\begin{aligned} & u^i \in C^1(\Omega) \cap C^2(\Omega \cap \{\nabla u^i \neq 0\}) \cap L^\infty(\Omega) \\ & \text{and } \nabla u^i \in L^\infty(\Omega, \mathbb{R}^N) \cap W^{1,2}_{loc}(\Omega, \mathbb{R}^N), \quad i = 1, \dots, n. \end{aligned} \tag{5}$$

Moreover, we will also assume that, for every $k = 1, \dots, n$, the map

$$\mathcal{A}^k : \mathbb{R}^m \times (\mathbb{R}^N \setminus \{0\}) \longrightarrow Mat(N \times N)$$

defined by[2]

$$\mathcal{A}^k_{ij} = \mathcal{A}^k_{ij}(x, \eta) := a_k(x, |\eta|)\delta_{ij} + \frac{\partial a_k}{\partial t}(x, |\eta|) \frac{\eta_i \eta_j}{|\eta|}, \qquad 1 \le i, j \le N$$

is such that[3]

$$\begin{aligned} & (x, y) = X \to \mathcal{A}^i(x, \nabla u^i(X)) \text{ belongs to } L^\infty(\{\nabla u^i \neq 0\} \cap B_R) \\ & \text{for any } R > 0 \text{ and any } i = 1, \dots, n. \end{aligned} \tag{6}$$

The following definition was introduced in [23].

Definition 1.2. *A solution[4] $(u^1, \dots, u^n)$ of (1) is said to be $F-$monotone if*

i) *for every $i \in \{1, \dots, n\}$, $\partial_{y_{N-m}} u^i \neq 0$ in Ω,*
ii) *for $i < j$, we have $F_{ij}\, \partial_{y_{N-m}} u^i \partial_{y_{N-m}} u^j \geq 0$ in Ω.*

As it is customary in this setting, we recall the notion of stability:

Definition 1.3. *A solution $(u^1, \dots, u^n)$ of (1) is said to be stable if*

$$\begin{aligned} & \sum_{i=1}^n \int_\Omega \big\langle \mathcal{A}^i(x, \nabla u^i(X)) \nabla \psi^i(X), \nabla \psi^i(X) \big\rangle \, \mathrm{d}X \\ & \qquad - \sum_{i,j=1}^n \int_\Omega F_{ij}(x, u^1, \dots, u^n)\, \psi^i(X)\, \psi^j(X) \mathrm{d}X \geq 0, \end{aligned} \tag{7}$$

[1] At the end of this paper we present some explicit cases in which these assumptions are fulfilled.

[2] Here, as usual, $Mat(N \times N)$ denotes the vector space of real $N \times N$ matrices.

[3] Let us observe that condition (6) is implied if, for example, $\frac{\partial a_k}{\partial t} \in L^\infty(\mathbb{R}^m \times [\alpha-, \alpha+])$, for all $\alpha_+ > \alpha_- > 0$.

[4] We recall that condition (5) is assumed throughout the paper.

for any $\psi = (\psi^1, \ldots, \psi^n) \in C_c^\infty(\Omega, \mathbb{R}^n)$.

Let us note that (1) represents the Euler-Lagrange system associated to a suitable energy functional I (see the appendix). In particular, the notion of stability given in Definition 1.3 states that I has positive (formal) second variation (we refer to [2, 3, 20] for more details, see also Lemma A.3 in the Appendix).

According to [11, 20, 34], for every fixed $x \in \mathbb{R}^m$ and $c \in \mathbb{R}$, we define

$$H_u = H_{u,x,c} := \{y \in \mathbb{R}^{N-m} \mid u(x,y) = c\}$$

and

$$L_u = L_{u,x,c} := \{y \in H_u \mid \nabla_y u(x,y) \neq 0\}.$$

We also define

$$\mathcal{R}_u := \{(x,y) \in \Omega \mid \nabla_y u(x,y) \neq 0\}, \qquad \mathcal{S}_u := \sum_{i=1}^{m} \sum_{j=1}^{N-m} (u_{x_i y_j})^2 - |\nabla_x |\nabla_y u||^2,$$

$$\mathcal{T}_u := \sum_{j=1}^{N-m} \left\langle \nabla u, \nabla u_{y_j} \right\rangle^2 - \left\langle \nabla u, \nabla |\nabla_y u| \right\rangle^2, \qquad \mathcal{U}_u := \sum_{j=1}^{N-m} |\nabla u_{y_j}|^2 - |\nabla |\nabla_y u||^2.$$

We recall that the tangential gradient along L_u, ∇_L, is defined for every $\bar{y} \in L_u$ and any $G : \mathbb{R}^{N-m} \longrightarrow \mathbb{R}$ smooth in the vicinity of $\bar{y}$ as

$$\nabla_L G(\bar{y}) := \nabla_y G(\bar{y}) - \left\langle \nabla_y G(\bar{y}), \frac{\nabla_y u(x,\bar{y})}{|\nabla_y u(x,\bar{y})|} \right\rangle \frac{\nabla_y u(x,\bar{y})}{|\nabla_y u(x,\bar{y})|},$$

and since L_u is a smooth $(N-m-1)-$manifold we define, for every $y \in L_u$, the length of the second fundamental form by

$$\mathcal{K}_u(x,y) := \sqrt{\sum_{j=1}^{N-m-1} k_j^2(x,y)},$$

where $k_{1,u}(x,y), \ldots, k_{N-m-1,u}(x,y)$ are the principal curvatures of L_u.

We are now in position to state our main results. We establish first a geometric inequality, which involves the tangential gradients and the curvatures of the level sets of the solution. This inequality holds in every open set $\Omega \subset \mathbb{R}^N$.

Theorem 1.4. *Let $(u^1, \ldots, u^n)$ be a weak stable solution of (1) satisfying (5).*

Then, for each $\psi = (\psi^1, \ldots, \psi^n) \in C_c^\infty(\Omega, \mathbb{R}^n)$, we have

$$\begin{aligned}
(8) \qquad & \sum_{k=1}^{n} \int_{\mathcal{R}_{u^k}} \left(\sum_{j=1}^{N-m} \left\langle \mathcal{A}^k \nabla u^k_{y_j}, \nabla u^k_{y_j} \right\rangle - \left\langle \mathcal{A}^k \nabla |\nabla_y u^k|, \nabla |\nabla_y u^k| \right\rangle \right) (\psi^k)^2 \\
& - \sum_{k,j=1, j\neq k}^{n} \int_\Omega F_{kj} \left((\psi^k)^2 \left\langle \nabla_y u^j, \nabla_y u^k \right\rangle - \psi^j \psi^k |\nabla_y u^k| |\nabla_y u^j| \right) \\
& \leq \int_\Omega \sum_{k=1}^{n} \left\langle \mathcal{A}^k \nabla \psi^k, \nabla \psi^k \right\rangle |\nabla_y u^k|^2.
\end{aligned}$$

Moreover,

$$\sum_{k=1}^{n}\int_{R_{u^k}}\left[a_k(x,|\nabla u^k|)(\mathcal{S}_{u^k}+\mathcal{K}_{u^k}^2|\nabla_y u^k|^2+|\nabla_L|\nabla_y u^k||^2)+\frac{\frac{\partial a_k}{\partial t}(x,|\nabla u^k|)}{|\nabla u^k|}\mathcal{T}_{u^k}\right](\psi^k)^2$$
$$\leq\int_\Omega\sum_{k=1}^{n}\left\langle\mathcal{A}^k\nabla\psi^k,\nabla\psi^k\right\rangle|\nabla_y u^k|^2$$
$$-\sum_{k,j=1,j\neq k}^{n}\int_\Omega F_{kj}\left((\psi^k)^2|\nabla_y u^k||\nabla_y u^j|-\psi^j\psi^k\left\langle\nabla_y u^i,\nabla_y u^k\right\rangle\right).$$

Next, we state our symmetry results both for stable and for monotone solutions to (1). In the proof of the subsequent theorems, we will use the geometric inequality in (8) with $\Omega=\mathbb{R}^N$.

Theorem 1.5. *Let $(u^1,\dots,u^n)$ be a weak stable solution of (1) in the whole $\mathbb{R}^N$ satisfying (5). Let us also assume that there exist non-zero functions $\theta^i\in C^1(\mathbb{R}^N)$, $i=1,\dots,n$, which do not change sign, such that for all i,j with $1\leq i<j\leq n$ it holds*

$$F_{ij}(x,u^1(X),\dots,u^n(X))\,\theta^i(X)\,\theta^j(X)\geq 0,\qquad\forall X\in\mathbb{R}^N. \tag{9}$$

Moreover, we assume that, for any $i=1,\dots,n$, $\mathcal{A}^i(x,\nabla u^i(X))$ satisfies (6), it is positive definite at almost any $X\in\mathbb{R}^N$ and there exist $C_1,\dots,C_n\geq 1$ such that the largest eigenvalue $\overline{\mathcal{A}}^i(X)$ of $\mathcal{A}^i(x,\nabla u^i(X))$ satisfies

$$\int_{B_R}\overline{\mathcal{A}}^i(X)|\nabla u^i(X)|^2\mathrm{d}X\leq C_iR^2 \tag{10}$$

for any $R\geq\max\{C_1,\dots,C_n\}$.

Then, for each $i=1,\dots,n$, there exist $\bar{u}^i:\mathbb{R}^m\times\mathbb{R}\longrightarrow\mathbb{R}$ and $\omega_i:\mathbb{R}^m\longrightarrow\mathbb{S}^{N-m-1}$ such that

$$u^i(X)=u^i(x,y)=\bar{u}^i(x,\langle\omega_i(x),y\rangle)$$

for any $(x,y)\in\mathbb{R}^m\times\mathbb{R}^{N-m}$. Moreover, each $\omega_i(x)$ is constant in any connected component of $\{\nabla_y u^i\neq 0\}$.

Theorem 1.6. *Let $(u^1,\dots,u^n)$ be a weak $F-$monotone solution of (1) in the whole $\mathbb{R}^N$ satisfying (5). We assume that, for any $i=1,\dots,n$, $\mathcal{A}^i(x,\nabla u^i(X))$ satisfies (6), it is positive definite at almost any $X\in\mathbb{R}^N$ and $\overline{\mathcal{A}}^i(X)$ satisfies (10).*

Then, there exist $\bar{u}^i:\mathbb{R}^m\times\mathbb{R}\longrightarrow\mathbb{R}$ and $\omega_i\in\mathbb{S}^{N-m-1}$ such that

$$u^i(X)=u^i(x,y)=\bar{u}^i(x,\langle\omega_i,y\rangle)$$

for any $(x,y)\in\mathbb{R}^m\times\mathbb{R}^{N-m}$.

If, in addition, there exists $U\subseteq\mathbb{R}^m\times\mathbb{R}^n$ open such that, for every $j,k=1,\dots,n$, $F_{jk}>0$ (or $F_{jk}<0$) in U, then $\omega_j=\omega_k=\omega$.

Some comments are now in order. We start recalling that a result similar to the one in Theorem 1.4 has been proved in [11, 12, 19, 20, 32, 34, 35] in the case of one equation and in [23, 17] for particular systems of equations. Precisely, a geometric inequality like the one in (8) has been obtained in [23] in the case

$$a_i(x,t) = 1, \qquad F_i(x,u^1,\dots,u^n) = F_i(u^1,\dots,u^n),$$

and in [17], for $n = 2$, in the case

$$a_i(x,t) = a_i(t), \qquad F_i(x,u^1,u^2) = F_i(u^1,u^2).$$

We also mention the papers [21, 22, 30], where an inequality similar to the one obtained in Theorem 1.4 has been established for solutions to semilinear problems in Riemannian and sub-Riemannian manifolds.

We remark that Theorems 1.5 and 1.6 generalize to fibered media the results contained in [23, 17] and allow us to take into account more general systems than the ones considered in [23, 17] (see also the appendix for some explicit examples).

We want to stress on the fact that among the operators considered in this paper there is the $p(x)$-Laplacian, and therefore Theorems 1.5 and 1.6 apply to the regular solutions of a system involving the $p(x)$-Laplacian. This case was not considered in the previous works, and so in this setting our symmetry results are new.

The paper is organized as follows. In the next section we prove the geometric inequality in Theorem 1.4. Sections 3 and 4 are devoted to the proof of Theorems 1.5 and 1.6. Finally, there is an Appendix, which contains some comments on the assumptions made in our theorems.

2. A geometric inequality: proof of Theorem 1.4

Aim of this section is to prove Theorem 1.4. We recall first the following lemma, which has been proved in [11, 34, 35].

Lemma 2.1. *For any $u \in C^2(\Omega)$, the following equalities hold*

$$\sum_{j=1}^{N-m} \left\langle \mathcal{A}\nabla u_{y_j}, \nabla u_{y_j} \right\rangle - \langle \mathcal{A}\nabla|\nabla_y u|, \nabla|\nabla_y u|\rangle = a(x,|\nabla u|)\,\mathcal{U}_u + \frac{\frac{\partial a}{\partial t}(x,|\nabla u|)}{|\nabla u|}\mathcal{T}_u \quad \text{on } \mathcal{R}_u,$$

$$\mathcal{U}_u - \mathcal{S}_u = \sum_{i,j=1}^{N-m} (u_{y_i y_j})^2 - |\nabla_y|\nabla_y u||^2 = \mathcal{K}_u^2|\nabla_y u|^2 + |\nabla_L|\nabla_y u||^2 \quad \text{on } \mathcal{R}_u.$$

Moreover, $\mathcal{S}_u, \mathcal{T}_u \geq 0$ on $\mathcal{R}_u$.

In the sequel, we will need also the following result.

Proposition 2.2. *Let $(u^1,\dots,u^n)$ be a weak solution of (1) satisfying (5). Suppose that, for each $i = 1,\dots,n$, $\mathcal{A}^i$ verifies (6).*

Then, for every $j = 1, \dots, N-m$, the family $(u^1_{y_j}, \dots, u^n_{y_j})$ is a weak solution of the following system

$$\begin{cases} -\operatorname{div}(\mathcal{A}^1(x, \nabla u^1(X))\nabla u^1_{y_j}(X)) = \displaystyle\sum_{i=1}^n F_{1i}(x, u^1(X), \dots, u^n(X))\, u^i_{y_j}(X), \\ \vdots \\ -\operatorname{div}(\mathcal{A}^n(x, \nabla u^n(X))\nabla u^n_{y_j}(X)) = \displaystyle\sum_{i=1}^n F_{ni}(x, u^1(X), \dots, u^n(X))\, u^i_{y_j}(X). \end{cases}$$

Proof. We need to prove that, for every $\psi = (\psi^1, \dots, \psi^n) \in C^\infty_c(\Omega, \mathbb{R}^n)$, the following equalities hold (we drop for short the dependence of $\mathcal{A}^i$):

$$\begin{cases} \displaystyle\int_\Omega \left\langle \mathcal{A}^1 \nabla u^1_{y_j}(X), \nabla \psi^1(X) \right\rangle = \sum_{i=1}^n \int_\Omega F_{1i}(x, u^1(X), \dots, u^n(X))\, u^i_{y_j}(X)\, \psi^1(X), \\ \vdots \\ \displaystyle\int_\Omega \left\langle \mathcal{A}^n \nabla u^n_{y_j}(X) \nabla \psi^n(X) \right\rangle = \sum_{i=1}^n \int_\Omega F_{ni}(x, u^1(X), \dots, u^n(X))\, u^i_{y_j}(X)\, \psi^n(X). \end{cases} \tag{11}$$

To this end, we fix $i \in \{1, \dots, N-m\}$, and we use (4) with $(\psi^1_{y_i}, \dots, \psi^n_{y_i})$ as test functions. Hence (dropping for short the dependence of F_j and a_j), we have

$$\int_\Omega F_j\, \psi^j_{y_i} = \int_\Omega \langle a_j \nabla u^j, \nabla \psi^j_{y_i} \rangle = -\int_\Omega \langle (a_j \nabla u^j)_{y_i}, \nabla \psi^j \rangle = -\int_\Omega \langle \mathcal{A}^j \nabla u_{y_i}, \nabla \psi^j \rangle . \tag{12}$$

Moreover,

$$\int_\Omega F_j\, \psi^j_{y_i} = -\int_\Omega (F_j)_{y_i}\, \psi^j = -\sum_{k=1}^n \int_\Omega F_{jk}\, u^k_{y_i}\, \psi^j, \tag{13}$$

and putting together (12) and (13) we get the thesis. □

Remark 2.3. *By an easy density argument (see* [12] *for the details), we have that (11) holds for any $\psi = (\psi^1, \dots, \psi^n) \in W^{1,2}_0(\Omega, \mathbb{R}^n)$.*

Proof of Theorem 1.4: Let us fix $1 \le j \le N-m$ and $\psi = (\psi^1, \dots, \psi^n) \in C^\infty_c(\Omega, \mathbb{R}^n)$. Then, by (5), we have that $\varphi^i := u^i_{y_j}(\psi^i)^2 \in W^{1,2}_0(\Omega)$. Hence, by Remark 2.3, we can use $\varphi = (\varphi^1, \dots, \varphi^n)$ as a test function in (11). It follows that (dropping for short the dependence of $\mathcal{A}^i$ and F_{ij}), for any $k = 1, \dots n$,

$$\int_\Omega \left\langle \mathcal{A}^k \nabla u^k_{y_j}, \nabla (u^k_{y_j} (\psi^k)^2) \right\rangle = \sum_{i=1}^n \int_\Omega F_{ki}\, u^i_{y_j}\, u^k_{y_j}\, (\psi^k)^2.$$

Summing over j, we have

$$\sum_{j=1}^{N-m} \int_\Omega \left\langle \mathcal{A}^k \nabla u^k_{y_j}, \nabla (u^k_{y_j} (\psi^k)^2) \right\rangle = \sum_{i=1}^n \int_\Omega F_{ki}\, (\psi^k)^2 \left\langle \nabla_y u^i, \nabla_y u^k \right\rangle ,$$

which implies

$$\begin{aligned}\int_\Omega F_{kk}\,(\psi^k)^2|\nabla_y u^k|^2 =& \sum_{j=1}^{N-m}\int_\Omega \left\langle \mathcal{A}^k\nabla u^k_{y_j}, \nabla(u^k_{y_j}(\psi^k)^2)\right\rangle \\ &- \sum_{i=1,i\neq k}^{n}\int_\Omega F_{ki}\,(\psi^k)^2\left\langle \nabla_y u^i, \nabla_y u^k\right\rangle.\end{aligned} \tag{14}$$

Using Stampacchia's Theorem (see, for istance, [28, Theorem 6.19]), we get

$$\nabla|\nabla_y u^k| = 0 = \nabla u^k_{y_j} \quad \text{for a.e. } x\in\mathbb{R}^m, \text{ and a.e. } y\in\mathbb{R}^{N-m}\ s.t.\ \nabla_y u(x,y)=0. \tag{15}$$

Hence, summing over $k=1,\dots,n$ in (14), we obtain

$$\begin{aligned}\sum_{k=1}^n\int_\Omega F_{kk}\,(\psi^k)^2|\nabla_y u^k|^2 =&\sum_{k=1}^n\int_{\mathcal{R}_{u^k}}\sum_{j=1}^{N-m}\left\langle \mathcal{A}^k\nabla u^k_{y_j}, \nabla(u^k_{y_j}(\psi^k)^2)\right\rangle \\ &- \sum_{k,i=1,i\neq k}^{n}\int_\Omega F_{ki}\,(\psi^k)^2\left\langle \nabla_y u^i, \nabla_y u^k\right\rangle.\end{aligned} \tag{16}$$

Using $\varphi^k := |\nabla_y u^k|\psi^k$, with $\psi=(\psi^1,\dots,\psi^n)\in C_c^\infty(\Omega,\mathbb{R}^n)$, as a test function in (7) (we point out that by the regularity assumptions on u^i it follows that $\varphi^i\in W_0^{1,2}(\Omega)$, $i=1,\dots,n$), and recalling (15), we obtain

$$\begin{aligned}0 \le&\sum_{k=1}^n\int_{\mathcal{R}_{u^k}}\left\langle \mathcal{A}^k\nabla|\nabla_y u^k|, \nabla|\nabla_y u^k|\right\rangle(\psi^k)^2 + \left\langle \mathcal{A}^k\nabla\psi^k, \nabla\psi^k\right\rangle|\nabla_y u^k|^2 \\ &+2\left\langle \mathcal{A}^k\nabla|\nabla_y u^k|, \nabla\psi^k\right\rangle|\nabla_y u^k|\psi^k \\ &-\sum_{k,j=1,j\neq k}^{n}\int_\Omega F_{kj}|\nabla_y u^k||\nabla_y u^j|\psi^j\psi^k - \sum_{k=1}^n\int_\Omega F_{kk}|\nabla_y u^k|^2(\psi^k)^2,\end{aligned}$$

which together with (16) implies

$$\begin{aligned}0 \le&\sum_{k=1}^n\int_{\mathcal{R}_{u^k}}\left\langle \mathcal{A}^k\nabla|\nabla_y u^k|, \nabla|\nabla_y u^k|\right\rangle(\psi^k)^2 + \left\langle \mathcal{A}^k\nabla\psi^k, \nabla\psi^k\right\rangle|\nabla_y u^k|^2 \\ &+\frac{1}{2}\left\langle \mathcal{A}^k\nabla|\nabla_y u^k|^2, \nabla(\psi^k)^2\right\rangle \\ &+\sum_{k,j=1,j\neq k}^{n}\int_\Omega F_{kj}\left((\psi^k)^2\left\langle \nabla_y u^j, \nabla_y u^k\right\rangle - \psi^j\psi^k|\nabla_y u^k||\nabla_y u^j|\right) \\ &-\sum_{k=1}^n\int_{\mathcal{R}_{u^k}}\sum_{j=1}^{N-m}(\psi^k)^2\left\langle \mathcal{A}^k\nabla u^k_{y_j}, \nabla u^k_{y_j}\right\rangle \\ &-\sum_{k=1}^n\frac{1}{2}\int_{\mathcal{R}_{u^k}}\mathcal{A}^k\left\langle \nabla|\nabla_y u^k|^2, \nabla(\psi^k)^2\right\rangle.\end{aligned} \tag{17}$$

Rewriting the inequality in (17) in a more compact form, we obtain

$$\begin{aligned}0 \le \sum_{k=1}^{n} \int_{\mathcal{R}_{u^k}} &\Big[\Big(\Big\langle \mathcal{A}^k \nabla |\nabla_y u^k|, \nabla |\nabla_y u^k|\Big\rangle - \sum_{j=1}^{N-m} \Big\langle \mathcal{A}^k \nabla u^k_{y_j}, \nabla u^k_{y_j}\Big\rangle\Big)(\psi^k)^2 \\ &+ \Big\langle \mathcal{A}^k \nabla \psi^k, \nabla \psi^k\Big\rangle |\nabla_y u^k|^2\Big] \\ &+ \sum_{k,j=1, j\neq k}^{n} \int_{\Omega} F_{kj}\Big((\psi^k)^2 \Big\langle \nabla_y u^j, \nabla_y u^k\Big\rangle - \psi^j \psi^k |\nabla_y u^k||\nabla_y u^j|\Big),\end{aligned}$$

which is the first part of the statement. For the second part, we observe that, by Lemma 2.1, we get

$$\begin{aligned}&\Big\langle \mathcal{A}^k \nabla |\nabla_y u^k|, \nabla |\nabla_y u^k|\Big\rangle - \sum_{j=1}^{N-m} \Big\langle \mathcal{A}^k \nabla u^k_{y_j}, \nabla u^k_{y_j}\Big\rangle \\ &= - a_k(x, |\nabla u^k|)(\mathcal{S}_{u^k} + \mathcal{K}^2_{u^k}|\nabla_y u^k|^2 + |\nabla_L |\nabla_y u^k||^2) - \frac{\frac{\partial a_k}{\partial t}(x, |\nabla u^k|)}{|\nabla u^k|}\mathcal{T}_{u^k} \quad \text{on } \mathcal{R}_{u^k}.\end{aligned} \tag{18}$$

Therefore, plugging (18), into (17) we get the thesis. □

3. Stable solutions and proof of Theorem 1.5

Recalling the definition of stable solutions to (1) given in (7), in this section we will prove Theorem 1.5.

First, we recall the following lemma from [12].

Lemma 3.1. *Let $R > 0$ and $h : B_R \subset \mathbb{R}^N \longrightarrow \mathbb{R}$ be a nonnegative measurable function. For any $\rho \in (0, R)$, let*

$$\xi(\rho) := 2 \int_{B_\rho} h(X)\mathrm{d}X.$$

Then,

$$\int_{B_R \setminus B_{\sqrt{R}}} \frac{h(X)}{|X|^2}\mathrm{d}X \le \int_{\sqrt{R}}^{R} t^{-3}\xi(t)\mathrm{d}t + \frac{\xi(R)}{R^2}.$$

***Proof of Theorem 1.5**:* In order to prove Theorem 1.5, we use the geometric inequality in (8), with $\Omega = \mathbb{R}^N$. Since $\mathcal{A}^k$ is positive definite, inequality (8) becomes

$$\begin{aligned}&\sum_{k=1}^{n} \int_{\mathcal{R}_{u^k}} \left(\sum_{j=1}^{N-m} \Big\langle \mathcal{A}^k \nabla u^k_{y_j}, \nabla u^k_{y_j}\Big\rangle - \Big\langle \mathcal{A}^k \nabla |\nabla_y u^k|, \nabla |\nabla_y u^k|\Big\rangle\right)(\psi^k)^2 \\ &\quad - \sum_{k,j=1, j\neq k}^{n} \int_{\mathbb{R}^N} F_{kj}\Big((\psi^k)^2 \Big\langle \nabla_y u^j, \nabla_y u^k\Big\rangle - \psi^j \psi^k |\nabla_y u^k||\nabla_y u^j|\Big) \\ &\le \sum_{k=1}^{n} \int_{\mathbb{R}^N} \overline{\mathcal{A}}^k |\nabla \psi^k|^2 |\nabla_y u^k|^2.\end{aligned} \tag{19}$$

By Lemma 3.1 in [11], we have

$$\sum_{j=1}^{N-m} \left\langle \mathcal{A}^k \nabla u^k_{y_j}, \nabla u^k_{y_j} \right\rangle - \left\langle \mathcal{A}^k \nabla |\nabla_y u^k|, \nabla |\nabla_y u^k| \right\rangle \geq 0, \qquad k = 1, \dots, n.$$

Moreover, by (9), we have that there exist non-zero functions $\theta^1, \dots, \theta^n \in C^1(\mathbb{R}^N)$ with constant sign such that

$$F_{ij}\, \theta^i\, \theta^j \geq 0, \qquad \forall i, j \in \{1, \dots, n\}, i < j. \tag{20}$$

For any $R > 1$, we define $\eta_R : \mathbb{R}^N \longrightarrow \mathbb{R}$ as

$$\eta_R(X) := \begin{cases} 1 & \text{if } X \in B_{\sqrt{R}}, \\ 2\,\dfrac{\log R - \log |X|}{\log R} & \text{if } X \in B_R \setminus B_{\sqrt{R}}, \\ 0 & \text{if } X \in \mathbb{R}^N \setminus B_R, \end{cases}$$

and consider

$$\eta^i_R := sgn(\theta^i)\eta_R, \tag{21}$$

where $sgn(x)$ is the Sign function. It follows that, for each $i = 1, \dots, n$, $\eta^i_R \in C^\infty_c(B_R)$, $0 < |\eta^i_R(X)| < 1$ for any $X \in \mathbb{R}^N$, and

$$|\nabla \eta^i_R(X)| \leq \frac{\chi_R(X)}{2|X| \log R},$$

where

$$\chi_R(X) := \begin{cases} 1 & \text{if } X \in B_R \setminus B_{\sqrt{R}}, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, from (20), we have

$$F_{ij}\, sgn(\theta^i)\, sgn(\theta^j) \geq 0, \qquad \forall i, j \in \{1, \dots, n\}, i < j.$$

Using (21) as a test function in (19), and observing that

$$F_{kj}\, sgn(\theta^k)\, sgn(\theta^j) = sgn(F_{kj}) F_{kj},$$

we get

$$\begin{aligned} &\sum_{k=1}^{n} \int_{\mathcal{R}_{u^k}} \left[\sum_{j=1}^{N-m} \left\langle \mathcal{A}^k \nabla u^k_{y_j}, \nabla u^k_{y_j} \right\rangle - \left\langle \mathcal{A}^k \nabla |\nabla_y u^k|, \nabla |\nabla_y u^k| \right\rangle \right] \left(\eta^k_R \right)^2 \\ &\quad - \sum_{k,j=1, j\neq k}^{n} \int_{\mathbb{R}^N} \Big(sgn(F_{kj}) \left\langle \nabla_y u^j, \nabla_y u^k \right\rangle - |\nabla_y u^k| |\nabla_y u^j| \Big) F_{kj}\, sgn(F_{kj})\, \eta^2_R \\ &\leq \frac{1}{4 \log^2 R} \sum_{k=1}^{n} \int_{B_R \setminus B_{\sqrt{R}}} \frac{\overline{\mathcal{A}}^k |\nabla_y u^k|^2}{|X|^2} \\ &\leq C \frac{1}{\log R}, \end{aligned} \tag{22}$$

where in the last inequality we have used the fact that $|\nabla_y u|^2 \le |\nabla u|^2$, Lemma 3.1 with $h(X) := \overline{\mathcal{A}}^k |\nabla_y u^k|^2$ and the assumption (10).

Sending $R \to +\infty$ in (22), we conclude that

$$\sum_{j=1}^{N-m} \left\langle \mathcal{A}^k \nabla u^k_{y_j}, \nabla u^k_{y_j} \right\rangle - \left\langle \mathcal{A}^k \nabla |\nabla_y u^k|, \nabla |\nabla_y u^k| \right\rangle = 0, \qquad \text{a.e. in } \mathcal{R}_{u^k},\ k = 1, \dots, n, \tag{23}$$

and

$$\Big(sgn(F_{kj}) \left\langle \nabla_y u^j, \nabla_y u^k \right\rangle - |\nabla_y u^k||\nabla_y u^j| \Big) sgn(F_{kj}) F_{kj} = 0, \qquad \text{a.e. in } \mathbb{R}^N, \tag{24}$$

for any $k, j = 1, \dots, n$, with $j \neq k$.

By (23) and Corollary 3.2 in [11], we obtain that, for any level set L of u^k and any $X \in \mathcal{R}_{u^k} \cap L$,

$$\mathcal{K}_{u^k} = 0 = |\nabla_L |\nabla_y u^k||.$$

Therefore, using Lemma 2.11 in [20], this implies that, for each $k = 1, \dots, n$, there exist $\omega_k : \mathbb{R}^m \longrightarrow \mathbb{S}^{N-m-1}$ and $\bar{u}^k : \mathbb{R}^m \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$u^k(x, y) = \bar{u}^k(x, \langle \omega_k(x), y \rangle).$$

Moreover, by Lemma A.1 in [12] we have that each $\omega_k(x)$ is constant in any connected component of $\{\nabla_y u^k \neq 0\}$. This concludes the proof. $\square$

There are some cases in which the directions $\omega_1, \dots, \omega_n$ may be related or may coincide. In fact, as a corollary of Theorem 1.5, we prove that this happens under some additional assumptions on the functions F_{kj}. For this, we denote by $\Im(u^1, \dots, u^n)$ the image of the map $(u^1, \dots, u^n) : \mathbb{R}^N \to \mathbb{R}^n$, that is

$$\Im(u^1, \dots, u^n) := \left\{ (u^1(X), \dots, u^n(X)), X \in \mathbb{R}^N \right\}.$$

Then, the following symmetry result holds:

Corollary 3.2. *Under the assumptions of Theorem 1.5, we assume that, for every $x \in \mathbb{R}^m$ and for every $j, k = 1, \dots, n$, $j \neq k$,*

(25)

there exist open intervals $I_1^x, \dots, I_n^x \subset \mathbb{R}$ such that $(I_1^x \times \dots \times I_n^x) \cap \Im(u^1, \dots, u^n) \neq \varnothing$

and $F_{kj}(x, \overline{u^1}, \dots, \overline{u^n}) > 0$ (or $F_{kj}(x, \overline{u^1}, \dots, \overline{u^n}) < 0$)

for any $(\overline{u^1}, \dots, \overline{u^n}) \in I_1^x \times \dots \times I_n^x$.

Then there exist $C_{jk} : \mathbb{R}^N \longrightarrow \mathbb{R}$ and $D_{jk} : \mathbb{R}^m \longrightarrow \{-1, 1\}$ such that

$$\nabla_y u^j(X) = C_{jk}(X) \nabla_y u^k(X) \quad \textit{and} \quad \omega_j(x) = D_{jk}(x) \omega_k(x). \tag{26}$$

If, in addition, $\mathcal{I} := \bigcap_{i=1}^n \{\nabla_y u^i \neq 0\} \neq \varnothing$ is connected, then $\omega_1 \equiv \dots \equiv \omega_n$ in $\mathcal{I}$.

Proof. By Theorem 1.5, for each $j, k = 1, \dots, n$,

$$u^j(x, y) = \bar{u}^j(x, \langle \omega_j(x), y \rangle), \qquad u^k(x, y) = \bar{u}^k(x, \langle \omega_k(x), y \rangle), \tag{27}$$

for some $\bar{u}^j, \bar{u}^k : \mathbb{R}^m \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\omega_j, \omega_k : \mathbb{R}^m \longrightarrow \mathbb{S}^{N-m-1}$.

Now, for any fixed $x \in \mathbb{R}^m$, arguing as in the proof of the second part of Theorem 1.8 in [18], one can prove that there exists a non-empty open set $V \subset \mathbb{R}^{N-m}$ such that $u^i(x,y) \in I_i^x$ and $\nabla_y u^i(x,y) \neq 0$, for all $y \in V$ and $i = 1, \dots, n$.

Therefore, using (24) and (25), we obtain that there exists $y_* \in V$ such that

$$sgn(F_{kj}) \left\langle \nabla_y u^j(x,y_*), \nabla_y u^k(x,y_*) \right\rangle - |\nabla_y u^k(x,y_*)||\nabla_y u^j(x,y_*)| = 0, \tag{28}$$

for any $j,k = 1,\dots,n$, $j \neq k$. Moreover, from (27), we have that $\nabla_y u^j(x,y_*)$ is proportional to $\omega_j(x)$ and $\nabla_y u^k(x,y_*)$ is proportional to $\omega_k(x)$. Hence, (28) together with (27) implies (26).

If $\mathcal{I} \neq \varnothing$ is connected, we have that

$$\omega_i(x) = \omega_i, \qquad i = 1,\dots,n,$$

because, from Theorem 1.5, we know that each ω_i is constant in any connected component of $\left\{\nabla_y u^i \neq 0\right\}$.

Now, plugging the functions in (27) into (28), we have that

$$|\partial_z \overline{u}^j||\partial_z \overline{u}^k| \Big(\pm \langle \omega_j, \omega_k \rangle - 1 \Big) = 0, \qquad j,k = 1,\dots,n, j \neq k,$$

where $\partial_z \overline{u}^i$ denotes the derivative of the function $\overline{u}^i$ with respect to the last variable. Therefore, from the last equality, we deduce that, for every $j,k = 1,\dots,n$,

$$\langle \omega_j, \omega_k \rangle = \pm 1, \qquad \text{in } \mathcal{I}.$$

If $\omega_k = -\omega_j$, we have that $u^j(x,y) = \overline{u}^j(x, \langle \omega_j, y\rangle)$ and $u^k(x,y) = \overline{u}^k(x, \langle -\omega_j, y\rangle)$; then, we can define $\tilde{u}^k(x,y) := \overline{u}^k(x,-y)$, and obtain $u^k(x,y) = \tilde{u}^k(x, \langle \omega_j, y\rangle)$. This means that we can take $\omega_j = \omega_k$ up to renaming the function that describes u^k. Hence, we have that $\omega_j = \omega_k$ for every $j,k = 1,\dots,n$, and this concludes the proof. □

4. Monotone solutions and proof of Theorem 1.6

In this section we prove our symmetry result for monotone solutions to (1). First of all, we show that $F-$monotonicity implies stability (see Definitions 1.2 and 1.3).

Proposition 4.1. *If $(u^1,\dots,u^n)$ is a $F-$monotone solution of (1), then it is also stable.*

Proof. Choosing $\psi^i := \frac{\xi_i^2}{u^i_{y_{N-m}}} \in W_0^{1,2}(\Omega)$ in (11), where $\xi_i \in C_c^\infty(\Omega)$ [5], we obtain

$$
\begin{aligned}
\sum_{j=1}^n \int_\Omega F_{ij} \frac{u^j_{y_{N-m}}}{u^i_{y_{N-m}}} \xi_i^2 &= \int_\Omega \left\langle \mathcal{A}^i \nabla u^i_{y_{N-m}}, \frac{(\nabla \xi_i^2)\, u^i_{y_{N-m}} - \xi_i^2\, \nabla u^i_{y_{N-m}}}{(u^i_{y_{N-m}})^2} \right\rangle \\
&= -\int_\Omega \frac{\xi_i^2}{(u^i_{y_{N-m}})^2} \left\langle \mathcal{A}^i \nabla u^i_{y_{N-m}}, \nabla u^i_{y_{N-m}} \right\rangle \\
&\quad + 2 \int_\Omega \frac{\xi_i}{u^i_{y_{N-m}}} \left\langle \mathcal{A}^i \nabla u^i_{y_{N-m}}, \nabla \xi_i \right\rangle \\
&\le \int_\Omega \left\langle \mathcal{A}^i \nabla \xi_i, \nabla \xi_i \right\rangle,
\end{aligned} \tag{29}
$$

where in the last inequality we have used the fact that since $\mathcal{A}^i$ is positive definite then the following inequality holds for each $a \in \mathbb{R}$, $v, w \in \mathbb{R}^N$, $i \in \{1, \dots, n\}$:

$$
2a \left\langle \mathcal{A}^i v, w \right\rangle - a^2 \left\langle \mathcal{A}^i w, w \right\rangle - \left\langle \mathcal{A}^i v, v \right\rangle = \left\langle \mathcal{A}^i (v - aw), aw - v \right\rangle \le 0.
$$

Summing over $i \in \{1, \dots, n\}$ in (29), we get

$$
\sum_{i,j=1}^n \int_\Omega F_{ij} \frac{u^j_{y_{N-m}}}{u^i_{y_{N-m}}} \xi_i^2 \le \sum_{i=1}^n \int_\Omega \left\langle \mathcal{A}^i \nabla \xi_i, \nabla \xi_i \right\rangle, \qquad \forall \xi_i \in C_c^\infty(\Omega). \tag{30}
$$

Moreover,

$$
\begin{aligned}
\sum_{i,j=1}^n F_{ij} \frac{u^j_{y_{N-m}}}{u^i_{y_{N-m}}} \xi_i^2 &= \sum_{i,j=1}^n F_{ij}\, u^i_{y_{N-m}} u^j_{y_{N-m}} \frac{\xi_i^2}{(u^i_{y_{N-m}})^2} \\
&= \sum_{i=1}^n F_{ii}\, \xi_i^2 + \sum_{i<j} F_{ij}\, u^i_{y_{N-m}}\, u^j_{y_{N-m}} \left(\frac{\xi_i^2}{(u^i_{y_{N-m}})^2} + \frac{\xi_j^2}{(u^j_{y_{N-m}})^2} \right) \\
&\ge \sum_{i,j=1}^n F_{ij}\, \xi_i\, \xi_j,
\end{aligned} \tag{31}
$$

where in the last inequality we have used ii) of the definition of $F-$monotone. Putting together (30) and (31) we get the thesis. □

Proof of Theorem 1.6: By Proposition 4.1, every monotone solution of (1) is also stable. Moreover, the assumption in (9) is verified (it is enough to take $\theta^i := u^i_{y_{N-m}}$, $\theta^j := u^j_{y_{N-m}}$ which belong to $C^1(\mathbb{R}^N)$ thanks to (5)). Then, the hypotheses of Theorem 1.5 are satisfied, and therefore we conclude that there exist $\bar{u}^i : \mathbb{R}^m \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\omega_i \in \mathbb{S}^{N-m-1}$ such that

$$
u^i(X) = u^i(x, y) = \bar{u}^i(x, \langle \omega_i, y \rangle) \tag{32}
$$

for any $(x, y) \in \mathbb{R}^m \times \mathbb{R}^{N-m}$, $i \in \{1, \dots, n\}$.

[5] We explicitly observe that this is possible thanks to Remark 2.3, moreover ψ^i is well defined by i) in the definition of $F-$monotone.

Let us now assume that there exists $U \subseteq \mathbb{R}^m \times \mathbb{R}^n$ open such that, for every $j, k = 1, \ldots, n$, $F_{jk} > 0$ (or $F_{jk} < 0$) in U. Using (24) and (32), we get

$$\pm|\partial_z \overline{u}^j||\partial_z \overline{u}^k| \langle \omega_j, \omega_k \rangle = |\partial_z \overline{u}^j||\partial_z \overline{u}^k| \qquad \text{in } U,$$

which implies $\langle \omega_j, \omega_k \rangle = \pm 1$, and hence $\omega_j = \omega_k = \omega$ (see the comments at the end of the proof of Corollary 3.2). This concludes the proof. $\square$

Appendix A

In this appendix we analyze the assumptions made in Section 1 in order to get our symmetry results.

A.1. **Optimality of the assumptions.** We start observing that the regularity assumptions (5) are fulfilled in a lot of interesting cases. Precisely, let $(u^1, \ldots, u^n)$ be a solution of (1) with $u^i \in W^{1,p_i}(\Omega) \cap L^\infty(\Omega)$, and define

$$b_i^j(x, \nabla u^i(X)) := a_i(x, |\nabla u^i|(X))\partial_j u^i(X), \qquad G_i(X) := F_i(x, u^1(X), \ldots, u^n(X)).$$

Let us assume that for every $i = 1, \ldots, n$

$$b_i^j \in C^0(\mathbb{R}^m \times \mathbb{R}^N) \cap C^1(\mathbb{R}^m \times \mathbb{R}^N \setminus \{0\}), \qquad j = 1, \ldots, N \tag{33}$$

$$\sum_{j,k=1}^N \frac{\partial b_i^j}{\partial \eta_k}(x, \eta)\xi_j \xi_k \geq \sigma(k + |\eta|)^{p_i - 1}|\xi|^2, \tag{34}$$

$$\sum_{j,k=1}^N \left| \frac{\partial b_i^j}{\partial \eta_k}(x, \eta) \right| \leq \Gamma(k + |\eta|)^{p_i - 2}, \tag{35}$$

$$\sum_{j,k=1}^N \left| \frac{\partial b_i^j}{\partial x_k}(x, \eta) \right| \leq \Gamma(k + |\eta|)^{p_i - 2}|\eta|, \tag{36}$$

$$|G_i(X)| \leq \Gamma, \tag{37}$$

for all $\eta \in \mathbb{R}^N \setminus \{0\}$, $\xi \in \mathbb{R}^N$, $X \in \mathbb{R}^N$, with $p_i \geq 2$, $k \in [0, 1]$, $\Gamma, \sigma > 0$.

Then, by [16, 27, 36, 26], we conclude that $u^i \in C^1(\mathbb{R}^N) \cap C^2(\{\nabla u^i \neq 0\})$ for each $i = 1, \ldots, n$.

Moreover, using (2.2.2) in [36] and Theorem 1.1 and Proposition 2.2 in [13], we conclude that also the assumption $u^i \in W^{1,2}_{loc}(\mathbb{R}^N)$ is always verified if either $\{\nabla u^i = 0\} = \varnothing$ for $i = 1, \ldots, n$ or $1 < p < 3$.

Therefore, the functions u^i satisfy the regularity assumptions in (5) provided the conditions in (33)-(37) hold.

It is interesting to note that, as in the scalar case, the assumption $|\nabla u| \in L^\infty(\Omega)$ cannot be removed. Indeed, without such an assumption, one can find a solution which is not one-dimensional, according to the following proposition (see Proposition 3.1 in [20]):

Proposition A.1. *Let $k > 0$ and $\psi \in C^1((k, +\infty))$ satisfying $\dot{\psi}(t) > 0$ in $(k, +\infty)$ and $\lim_{t \to +\infty} \psi(t) = +\infty$. Then, there exists $a \in C^1((0, +\infty))$ strictly positive, and $u \in C^2(\mathbb{R}^N)$ which is a stable solution of*

$$-\mathrm{div}\big(a(|\nabla u(X)|)\nabla u(X)\big) = N$$

and such that $|\nabla u(X)| = \psi(|X|)$ *for any* $|X|$ *suitably large.*

Moreover, u *does not possess one-dimensional symmetry.*

We also mention that, proceeding exactly as in [20, 17], the assumption on the regularity of F, i.e. for any $(x, \xi^1, \ldots, \xi^{i-1}, \xi^{i+1}, \ldots, \xi^n) \in \mathbb{R}^m \times \mathbb{R}^{n-1}$ the map $\xi^i \to F(x, \xi^1, \ldots, \xi^i, \ldots, \xi^n)$ belongs to $C^2(\mathbb{R})$, can be weakened requiring that the map $\xi^i \to F(x, \xi^1, \ldots, \xi^i, \ldots, \xi^n)$ is only $C^{1,1}_{loc}(\mathbb{R})$. Notice that the extension to locally Lipschitz nonlinearities could be very interesting from a physical viewpoint; indeed, very often, physical applications are run by locally Lipschitz forces.

A.2. **On the $F-$monotonicity condition.** Proceeding in our discussion about the consistency of assumptions made in Section 1, it is worth noticing that, as pointed out in [23], the notion of $F-$monotonicity (see Definition 1.2) seems to be crucial in order prove that a solution is one-dimensional. Indeed, let us consider the following system

$$-\Delta u + \nabla F(u) = 0 \quad \text{in } \mathbb{R}^2, \tag{38}$$

where $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is defined by:

$$F(x_1, x_2) := (x_1 - 1)^2 x_2^2 + (x_2^2 - 1)^2.$$

Then, F does not satisfy condition $ii)$ in the Definition 1.2, indeed

$$F_{12}(x_1, x_2) = 4x_2(x_1 - 1).$$

Moreover,

$$\begin{aligned}
&F \in C^2(\mathbb{R}^2), \qquad F((1,1)) = 0, \qquad F((1,-1)) = 0, \quad F(\xi) > 0 \quad \text{for } \xi \neq (1,1), (1,-1),\\
&\nabla^2 F((1,1)) \geq I, \qquad \nabla^2 F((1,-1)) \geq I\\
&\nabla F(\xi) \cdot \xi \geq 0 \quad \text{for } |\xi| \geq R_0, \quad \text{for some } R_0 > 1,
\end{aligned}$$

which, by [1, Theorem 1.1], imply that there exist entire solutions (u^1, u^2) of (38) which are not one-dimensional.

A.3. **Minimizers and stable solutions.** We point out some conditions which ensure the validity of (10). As mentioned in the introduction, the system in (1) is associated to a suitable energy functional. Precisely, let us define

$$\lambda_1^i(x,t) := \frac{\partial a_i}{\partial t}(x,t)t + a_i(x,t), \qquad \lambda_2^i(x,t) := a_i(x,t), \qquad i = 1, \ldots, n,$$

and

$$\Lambda_2^i(x,t) := \int_0^t \lambda_2^i(x, |\tau|)\tau \, \mathrm{d}\tau.$$

Then, it is a matter of computations that the energy functional related to (1) is

$$I_\Omega(u^1, \ldots, u^n) := \sum_{i=1}^n \int_\Omega \Lambda_2^i(x, |\nabla u^i|) - F(x, u^1, \ldots, u^n). \tag{39}$$

According to [12], we give the following definition:

Definition A.2. *A family* $(u^1, \dots, u^n)$ *is said to be a local minimizer for* I_Ω *if, for any bounded open set* $U \subset \Omega$, $I_U(u^1, \dots, u^n)$ *is well-defined and finite, and*

$$I_U(u^1 + \psi^1, \dots, u^n + \psi^n) \geq I_U(u^1, \dots, u^n)$$

for any $(\psi^1, \dots, \psi^n) \in C_c^\infty(U, \mathbb{R}^n)$.

The following lemma is the exact counterpart for systems of the result proved in [12, Lemma B.1] for the case of one equation.

Lemma A.3. *Let* $\Omega \subset \mathbb{R}^N$ *be an open set. If* $(u^1, \dots, u^n)$ *is a local minimizer of* I_Ω, *then* $(u^1, \dots, u^n)$ *is a weak solution of* (1) *and is stable.*

Proof. We start proving that every local minimizer $u = (u^1, \dots, u^n)$ of I_Ω is a weak solution of (1). To this end, let $U \subset \Omega$ be open and bounded and consider $\psi \in C_c^\infty(U)$. Then, for every $i = 1, \dots, n$, we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} I_U(u^1, \dots, u^i + s\psi, \dots, u^n) = 0,$$

and, recalling the definition of I_U in (39),

$$\int_\Omega a_i(x, |\nabla u^i|(X)) \left\langle \nabla u^i(X), \nabla \psi(X) \right\rangle \mathrm{d}X = \int_\Omega F_i(x, u^1, \dots, u^n)\, \psi(X)\, \mathrm{d}X,$$

which is the first part of the thesis.

Finally, for every $\psi = (\psi^1, \dots, \psi^n) \in C_c^\infty(U, \mathbb{R}^n)$,

$$\begin{aligned}
0 &\leq \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=0} I_U(u^1 + s\psi^1, \dots, u^n + s\psi^n) \\
&= \sum_{i=1}^n \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \int_\Omega \Big(a_i(x, |\nabla u^i + s\nabla\psi^i|(X)) \left\langle \nabla u^i + s\nabla\psi^i, \nabla\psi^i \right\rangle \\
&\qquad\qquad - F_i(x, u^1 + s\psi^1, \dots, u^n + s\psi^n)\, \psi^i(X) \Big) \mathrm{d}X \\
&= \sum_{i=1}^n \int_\Omega \left\langle \mathcal{A}^i(x, \nabla u^i(X)) \nabla\psi^i(X), \nabla\psi^i(X) \right\rangle \mathrm{d}X \\
&\qquad - \sum_{i,j=1}^n \int_\Omega F_{ij}(x, u^1, \dots, u^n)\, \psi^i(X)\, \psi^j(X)\, \mathrm{d}X,
\end{aligned}$$

and the proof is accomplished. □

In the following proposition we give a sufficient condition for the assumption (10) to hold for local minimizers.

Proposition A.4. *Let* $N \leq 3$, *and assume that, for each* $i = 1, \dots, n$, *there exists* $C_i > 0$ *such that*

$$\lambda_1^i(x,t) > 0, \qquad \forall x \in \mathbb{R}^m, t \in (0, +\infty), \tag{40}$$

$$\lambda_1^i(x,t)t^2, \lambda_2^i(x,t)t^2 \leq C_i \Lambda_2^i(x,t), \qquad \forall x \in \mathbb{R}^m, t \in (0, C_i]. \tag{41}$$

Moreover, we assume that for all $x \in \mathbb{R}^m$*, and* $s, t \in [0, +\infty)$

$$\Lambda_2^i(x,s) \geq 0, \tag{42}$$

$$\Lambda_2^i(x,s+t) \leq \bar{C}_i \Big[\Lambda_2^i(x,s) + \Lambda_2^i(x,t)\Big], \tag{43}$$

$$\Lambda_2^i(x,s) \leq \alpha_i(x) g_i(s), \tag{44}$$

for some $\bar{C}_i > 0$*,* $\alpha_i \in L^\infty_{loc}(\mathbb{R}^m)$ *and* $g_i : [0, +\infty) \longrightarrow \mathbb{R}$ *monotone increasing. Finally, for all* $x \in \mathbb{R}^m$ *and* $\xi \in \mathbb{R}^n$*, we suppose that the following holds*

$$F(x, \xi) \leq 0, \tag{45}$$

$$F(x, \xi) = 0, \qquad \forall x \in \mathbb{R}^m, \ \forall \xi \in \mathbb{S}^{n-1}, \tag{46}$$

and

$$\sup_{x \in \mathbb{R}^m, |\xi| \leq 1} |F(x, \xi)| \leq +\infty. \tag{47}$$

Then, assumption (10) *is satisfied for every local minimizer* $(u^1, \ldots, u^n)$ *of* $I_{\mathbb{R}^N}$ *such that* $|u^i| + |\nabla u^i| \leq M$*,* $i = 1, \ldots, n$*.*

Proof. We start observing that, for each $i = 1, \ldots, n$, and $x \in \mathbb{R}^m$, if $\xi, v \in \mathbb{R}^N$ with $|v| \leq 1$ and $|\xi| \leq M$, then

$$|\xi|^2 \left\langle \mathcal{A}^i(x,\xi) v, v \right\rangle \leq C_M \Lambda_2^i(x, |\xi|). \tag{48}$$

Indeed, by a simple calculation we get

$$\begin{aligned}
|\xi|^2 \left\langle \mathcal{A}^i(x,\xi) v, v \right\rangle &= \frac{\partial a_i}{\partial t}(x, |\xi|)|\xi| \left\langle \xi, v \right\rangle^2 + a_i(x, |\xi|)|v|^2|\xi|^2 \\
&= \lambda_1^i(x, |\xi|) \left\langle \xi, v \right\rangle^2 + \lambda_2^i(x, |\xi|) \Big[|v|^2|\xi|^2 - \left\langle \xi, v \right\rangle^2\Big] \\
&\leq \Big(\lambda_1^i(x, |\xi|) + \lambda_2^i(x, |\xi|)\Big)|v|^2|\xi|^2 \\
&\leq C_M \Lambda_2^i(x, |\xi|),
\end{aligned}$$

where in the last inequality we have used (41) and the fact that $|v| \leq 1$.

Let $R > 1$ and take $\psi = (\psi^1, \ldots, \psi^n) \in (C_c^\infty(\mathbb{R}^N))^n$ with the property that, for each $i = 1, \ldots, n$, $\psi^i = -1$ in B_{R-1}, $\psi^i = 1$ on ∂B_R and $|\nabla \psi^i| \leq M$ in $B_R \setminus B_{R-1}$. Let us define

$$v^i(X) := \min\{u^i(X), \psi^i(X)\}, \qquad i = 1, \ldots, n,$$

and observe that, by the minimality of u and (45),

$$\sum_{i=1}^n \int_{B_R} \Lambda_2^i(x, |\nabla u^i|) \leq I_{B_R}(u^1, \ldots, u^n) \leq I_{B_R}(v^1, \ldots, v^n).$$

By (42), (46),(43) and (47) we get

$$\begin{aligned}\sum_{i=1}^n \int_{B_R} \Lambda_2^i(x,|\nabla u^i|) &\leq \int_{B_R\setminus B_{R-1}} \sum_{i=1}^n \Lambda_2^i(x,|\nabla v^i|) - F(x,v^1,\dots,v^n)\\ &\leq \max\{\bar{C}_i\} \int_{B_R\setminus B_{R-1}} \sum_{i=1}^n \Big(\Lambda_2^i(x,|\nabla u^i|)+\Lambda_2^i(x,|\nabla \psi^i|)\Big) + \sup_{\mathbb{R}^m\times Q}|F|,\end{aligned}$$

where $Q := [-1,1]\times\cdots\times[-1,1]$ is the cube in $\mathbb{R}^n$. Using (44), we obtain

(49)

$$\begin{aligned}\sum_{i=1}^n \int_{B_R} \Lambda_2^i(x,|\nabla u^i|) &\leq \max\{\bar{C}_i\} \int_{B_R\setminus B_{R-1}} \sum_{i=1}^n \alpha_i(x)\Big(g_i(|\nabla u^i|)+g_i(|\nabla \psi^i|)\Big) + \sup_{\mathbb{R}^m\times Q}|F|\\ &\leq 2\max\{\bar{C}_i\} \int_{B_R\setminus B_{R-1}} \sum_{i=1}^n g_i(M)\sup_x \alpha_i + \sup_{\mathbb{R}^m\times Q}|F|\\ &\leq CR^{N-1},\end{aligned}$$

for some $C>0$. Finally, thanks to (48), (49) and the fact that $\mathcal{A}^i$ is positive definite, we have

$$\int_{B_R} |\nabla u^i|^2 \left\langle \mathcal{A}^i(x,\nabla u^i)v,v\right\rangle \leq CR^{N-1},$$

and, taking as v the normalized eigenvector corresponding to $\overline{\mathcal{A}}^i$, we get the thesis. □

We conclude this Appendix providing an example of functional which satisfies the hypotheses in Proposition A.4, and hence the assumption (10), obtaining, from Theorem 1.5, the one-dimensional symmetry for local minimizers.

Corollary A.5. *Let $N\leq 3$, and let $\alpha\in L^\infty(\mathbb{R}^m)$ be strictly positive, and $F\in C^2(\mathbb{R}^2)$ such that $G:=\alpha F$ satisfies* (45)-(47)*. Suppose that F_{12} does not change sign.*

For every $p_1,p_2\in(1,3)$, let us define the functional

$$I_{\mathbb{R}^N} := \int_{\mathbb{R}^N} |\nabla u^1(X)|^{p_1} + |\nabla u^2(X)|^{p_2} - \alpha(x)F(u^1(X),u^2(X))\mathrm{d}X.$$

Then, for every local minimizer (u^1,u^2) of $I_{\mathbb{R}^N}$ such that $u^1\in W^{1,p_1}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $u^2\in W^{1,p_2}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, with $|\nabla u^1|,|\nabla u^2|\in L^\infty(\mathbb{R}^N)$, there exist $u_0^1,u_0^2:\mathbb{R}^m\times\mathbb{R}\longrightarrow\mathbb{R}$ and $\omega_1,\omega_2:\mathbb{R}^m\to\mathbb{S}^{N-m-1}$ such that

$$u^1(x,y)=u_0^1(x,\langle\omega_1(x),y\rangle), \qquad u^2(x,y)=u_0^2(x,\langle\omega_2(x),y\rangle),$$

for any $(x,y)\in\mathbb{R}^m\times\mathbb{R}^{N-m}$. Moreover, ω_i, $i=1,2$, is constant in any connected component of $\big\{\nabla_y u^i\neq 0\big\}$.

Proof. The proof easily follows from Theorem 1.5 and Proposition A.4 . Indeed,

$$\Lambda_2^1(x,t)=|t|^{p_1}, \qquad \Lambda_2^2(x,t)=|t|^{p_2}$$

satisfy conditions (42), (43), (44) and

$$\lambda_1^1(x,t) = (p_1-1)|t|^{p_1-2}, \qquad \lambda_1^2(x,t) = (p_2-1)|t|^{p_2-2},$$
$$\lambda_2^1(x,t) = |t|^{p_1-2}, \qquad \lambda_2^2(x,t) = |t|^{p_2-2}$$

satisfy (40) and (41). Moreover, as proved in [11], both $\mathcal{A}^1, \mathcal{A}^2$ are positive definite for every $p_1, p_2 > 1$ and satisfy (6) when $p_1, p_2 \geq 2$, and even for $p_1, p_2 > 1$ as long as $\{\nabla u^1 = 0\} = \{\nabla u^2 = 0\} = \varnothing$. $\square$

Acknowledgments

The authors would like to thank *Enrico Valdinoci* for his very deep and useful comments and suggestions.

References

[1] Alama, S., Bronsard, L., Gui, C.: *Stationary layered solutions in $\mathbb{R}^2$ for an Allen-Cahn system with multiple-well potential*, Calc. Var. Partial Differential Equations **5**, 359–390 (1997).

[2] Alberti, G., Ambrosio, L., Cabré, X.: *On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property*, Acta Appl. Math. **65** (1-3), 9–33 (2001).

[3] Ambrosio, L., Cabré, X.: *Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi*, J. Amer. Math. Soc. **13**, 725–739 (2000).

[4] Berestycki, H., Caffarelli, L., Nirenberg, L.: *Further qualitative properties for elliptic equations in unbounded domains*, Ann. Sc. Norm. Super. Pisa Cl. Sci.(4) **25**, no. 1–2 (1998), 69–94, 1997 dedicated to Ennio De Giorgi.

[5] Berestycki, H., Lin. T.C., Wei, J., Zhao, C.: *On phase-separation models: Asymptotics and qualitative properties*, Arch. Ration. Mech. Anal., **208**, no. 1, 163–200 (2013).

[6] Berestycki, H., Terracini, S., Wang, K., Wei, J.: *On entire solutions of an elliptic system modeling phase separations*, Adv. Math., **243**, 102–126 (2013).

[7] Cabré, X., Cinti, E.: *Sharp energy estimates for nonlinear fractional diffusion equations*,to appear in Calc. Var. Partial Differential Equations, available at http://arxiv.org/abs/1207.6194

[8] Cabré, X., Solà-Morales, J.: *Layer solutions in a half-space for boundary reactions*, Comm. Pure Appl. Math. **58**, no. 12, 1678–1732 (2005).

[9] Cabré, X., Sire, Y.: *Nonlinear equations for fractional Laplacians I: regularity, maximum principles, and Hamiltonian estimates*, preprint (2010), available at http://arxiv.org/abs/1012.0867

[10] Cabré, X., Sire, Y.: *Nonlinear equations for fractional Laplacians II: existence, uniqueness, and qualitative properties of solutions*, to appear in Trans. Amer. Math. Soc., available at http://arxiv.org/abs/1111.0796

[11] Chermisi, M., Valdinoci, E.: *A symmetry result for a general class of divergence form PDEs in fibered media.* Nonlinear Anal. **73**, 695–703 (2010).

[12] Chermisi, M., Valdinoci, E.: *Fibered nonlinearities for p(x)-Laplace equations*, Adv. Calc. Var. **2**, 185–205 (2009).

[13] Damascelli, L., Sciunzi, B.: *Regularity, monotonicity and symmetry of positive solutions of $m-$Laplace equations*, J. Differential Equations **206**, no. 1, 483–515 (2004).

[14] Del Pino, M., Kowalczyk, M., Wei, J.: *On De Giorgi's conjecture in dimension $N \geq 9$*, Ann. of Math. (2) **174**, no. 3, 1485–1569 (2011).

[15] De Giorgi, E.: *Convergence problems for functionals and operators*, in Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), pp. 131–188, Pitagora, Bologna (1979).

[16] DiBenedetto, E.: *$C^{1,\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. **7**, no. 8, 827–850 (1983).

[17] Dipierro, S.: *Geometric inequalities and symmetry results for elliptic systems*, Discrete Contin. Dyn. Syst. **33**, no. 8, 3473–3496 (2013).
[18] Dipierro, S., Pinamonti, A.: *A geometric inequality and a symmetry result for elliptic systems involving the fractional Laplacian*, J. Differential Equations **255**, no. 1, 30–66 (2013).
[19] Farina, A.: *Propriétés Qualitatives de Solutions d'équations et Systémes d'équations Non-linéaires*, Habilitation a Diriger des Recherches, Paris VI (2002).
[20] Farina, A., Sciunzi, B., Valdinoci, E.: *Bernstein and De Giorgi type problems: new results via a geometric approach*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **7**, 741–791 (2008).
[21] Farina, A., Sire, Y., Valdinoci, E.: *Stable solutions of elliptic equations on Riemannian manifolds*, J. Geom. Anal. **23**, no. 3, 1158–81172 (2013).
[22] Farina, A., Sire, Y., Valdinoci, E.: *Stable solutions of elliptic equations on Riemannian manifolds with Euclidean coverings*, Proc. Amer. Math. Soc. **140**, no. 3, 927–930 (2012).
[23] Fazly, M., Ghoussoub, N.: *De Giorgi type results for elliptic systems*, Calc. Var. Partial Differential Equations **47**, no. 3–4, 809–823 (2013).
[24] Ghoussoub, N., Gui, C.: *On a conjecture of De Giorgi and some related problems*, Math. Ann. **311**, no. 3, 481–491 (1998).
[25] Ghoussoub, N., Gui, C.: *On De Giorgi's conjecture in dimensions* 4 *and* 5, Ann. of Math. (2) **157**, no. 1, 313–334 (2003).
[26] Ladyzenskaja, O.A., Ural'tzeva N.N.: *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, (1968).
[27] Lewis, J.L.: *Regularity of the derivatives of solutions to certain degenerate elliptic equations*, Indiana Univ. Math. J. **32**, no. 6, 849–858 (1983).
[28] Lieb, E.H., Loss, M.: *Analysis*, in: Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI (1997).
[29] Noris, B., Tavares, H., Terracini, S., Verzini, G.: *Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition*, Comm. Pure Appl. Math. **63**, 267–302 (2012).
[30] Pinamonti, A., Valdinoci, E.: *A geometric inequality for stable solutions of semilinear elliptic problems in the Engel group*, Ann. Acad. Sci. Fenn. Math. **37**, 357–373 (2012).
[31] Savin, O.: *Regularity of flat level sets in phase transitions*, Ann. of Math. (2) **169** (1), 41–78 (2009).
[32] Savin, O., Valdinoci, E.: *Elliptic PDEs with fibered nonlinearities*, J. Geom. Anal. **19**, no. 2, 420–432 (2008).
[33] Sire, Y., Valdinoci, E.: *Fractional Laplacian phase transitions and boundary reactions: A geometric inequality and a symmetry result*, J. Funct. Anal. **256**, 1842–1864 (2009).
[34] Sternberg, P., Zumbrun, K.: *A Poincaré inequality with applications to volume-constrained area-minimizing surfaces*, J. Reine Angew. Math. **503**, 63–85 (1998).
[35] Sternberg, P., Zumbrun, K.: *Connectivity of phase boundaries in strictly convex domains*, Arch. Ration. Mech. Anal. **141**, 375-400 (1998).
[36] Tolksdorf, P.: *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51**, no. 1, 126–150 (1984).
[37] Wang, K.: *On the De Giorgi type conjecture for an elliptic system modeling phase separation*, preprint (2012), available at http://arxiv.org/abs/1207.5285

Maxwell Institute for Mathematical Sciences and School of Mathematics, University of Edinburgh, James Clerk Maxwell Building, Kings Buildings, Edinburgh EH9 3JZ, United Kingdom
E-mail address: serena.dipierro@ed.ac.uk

SNS Pisa, Piazza dei Cavalieri 7, 56126 Pisa, Italy
E-mail address: andrea.pinamonti@sns.it