Stress constraints in simple bodies undergoing large strains: a variational approach

Mariano Giaquinta¹, Paolo Maria Mariano², Giuseppe Modica³

¹ Scuola Normale Superiore, piazza dei Cavalieri 7, I-50133 Pisa, Italy, e-mail: m.giaquinta@sns.it
² DICEA, Università di Firenze, via Santa Marta 3, I-50139 Firenze, Italy, e-mail: paolo.mariano@unifi.it
³ Università di Firenze, University of Florence, via Santa Marta 3, I-50139 Firenze, Italy, e-mail: giuseppe.modica@unifi.it

The date of receipt and acceptance will be inserted by the editor

Abstract We show how continuum mechanics of simple bodies undergoing large strain can be naturally expressed in terms of forms. The setting allows us to prove the existence of ground states in case we admit a constraint prescribing that the first Piola-Kirchhoff stress does not overcome a certain generic threshold – in other words it is constrained to stay in a subset of the stress space. We require minimality for a regular functional which involves the standard polyconvex elastic energy and its dual counterpart. The resulting stress appears naturally as a measure over the transplacement graph where it satisfies an appropriate integral balance. The parts of such a measure on the ‘vertical’ portions of the transplacement graph (if any in the constrained case considered) are concentrated on sets of zero volume measure in the reference place. The projection of the stress measure over the reference place gives rise, however, to the standard version of the balance of forces in Lagrangian representation. The general setting is applied to some specific cases.

Key words Simple materials, elastic-perfectly-plastic behavior, stress constraints, variational methods

1 Introduction

The choice of a function space as a natural ambient for solutions to boundary value problems in mechanics has per se constitutive nature. In short we can say just that to be fellow of a function space a map (a field if you want) has to enjoy some properties which allow us to describe some physical behaviors and exclude the representation of others.
Once read, such a statement could instinctively appear rather obvious. So, we could be pushed to consign it to an ancillary role, forgetting some way potentialities which emerge when we are clearly conscious of it, after specifying what we intend \textit{exactly} when we use the words ‘constitutive nature’.

A clear example emerges from non-linear elasticity of simple bodies. Let us focus for a while the attention on the determination of ground states for a non-linear hyperelastic body in isothermal environment. We are thinking of a body with mechanical behavior profitably represented by the standard scheme of simple materials, subjected to Dirichlet boundary condition on the transplacement, which is a bijective, orientation preserving, differentiable map from the reference place to the actual one. Forget even body forces, as we do here for the sake of simplicity.

In these conditions, ground states are described – it is well known – by minimizers of the elastic energy, expressed commonly by the integral over a reference place of an energy density \( e \), which is function of the spatial coordinate \( x \) (in case we foresee material inhomogeneity) and the spatial derivative of the transplacement map \( u \), i.e. \( Du(x) \). The assignment of an explicit structure of \( e(x,Du(x)) \) or bounds to it has, obviously, constitutive nature. Freedom in the selection of an expression for \( e \) is restricted by physics. A requirement of invariance with respect to isometric changes of atlas in the ambient (physical) space – it is invariance with respect to \( SO(3) \)-based changes in classical observers – appears in fact incompatible with convexity of \( e \) with respect to \( Du(x) \) (see [6]). So we are forced to evaluate alternatives. Physics helps us discriminating among possibilities. If we would impose to \( e \) quasiconvexity, for example, we would find minimizers in spaces where we would be not able to assure that they are orientation preserving maps (another restrictions imposed by physics). Polyconvexity appears in contrast a more appropriate attribute for \( e \). Precisely, the requirement is that the elastic energy density be a polyconvex function of \( Du(x) \), that is a convex function of the triple \( Du(x) \), \( \text{cof} Du(x) \), \( \det Du(x) \), and it is well known (see [4]). Under appropriate growth conditions for \( e \), minimizers exist in the Sobolev space \( W^{1,p} \). However, for \( p < 3 \) we realize the presence of possible minimizers describing deformations with holes (see [5]). From one side the conclusion could be that the theory of non-linear elasticity includes in itself its limits, which can be reached without foreseeing the presence of further ingredients describing the transition to non purely elastic phases. From another side, we could claim, rather, that \( W^{1,p} \) with \( p < 3 \) be a candidate for a state space describing in its whole a certain class of elastic-brittle solids rather than purely elastic ones. So, the point is the meaning that we attribute to the word \textit{elastic} when we use it. Intuitively, elastic is used imagining that we can deform a body at will, paying different amounts of energy, even an infinite one when we want to include extreme deformations, and being always able to come back to the initial state by following in reverse way the same path in the state space. Along the way we exclude the occurrence of holes, fractures, or strain localization altering
monotonicity between stress and strain. In this sense the appropriate state space excluding behaviors that can be referred to brittleness or plasticity appears to be the one of weak diffeomorphisms, indicated commonly by $\text{d}if^{1,1}$, or, more appropriately, its subset $\text{d}if^{p,1}$, with $p > 1$, a strict subset of $W^{1,1}$, as pointed out in [10]. It includes approximately differentiable Sobolev maps which are orientation preserving (at least almost everywhere) and do not display vertical parts in their graphs, meaning, in particular, that they do not have jumps representing disconnections of the matter or multivalued parts of the transplacement (the deformation), which can be attributed, for example, to elastic-plastic strain localization.

The constraint on the graph of these maps is imposed by resorting to typical tools of geometric measure theory: currents. They are functionals over the space of 3-forms. A 3-form is a map defined over the product of the reference place of the body and the actual configuration with values third-rank, skew-symmetric, covariant tensors. The values of the forms considered here are then elements of the space dual to the one of third-rank, skew-symmetric, contravariant tensors, say $M$. In this last space, we can select, in particular, fellows which contains as components the entries of $Du$, $\text{cof} Du$, and $\det Du$, which are the necessary ingredients for defining polyconvexity for the energy, by using in this case the notation $M(Du)$. For these reasons, we can consider the energy as a function of $M$. In particular, when $e$ is selected to be a convex function of $M$, the choice corresponds to consider it polyconvex with respect to $Du$ when $M$ is of the type $M(Du)$.

The extension of the energy density to the whole space of the $M$s is not merely arbitrary. In fact, when we consider $Du$, $\text{cof} Du$, and $\det Du$ as elements of three independent linear spaces, it is natural to define a convex function of them. In contrast, when we put $Du$, $\text{cof} Du$, and $\det Du$ all together in a unique geometric object $M(Du)$, which is what we call commonly 3-vector in the product space $\mathbb{R}^3 \times \mathbb{R}^3$, we are forced to consider at every point the energy density defined over the whole space of 3-vectors, and to realize, a posteriori, that the minimum is reached over a 3-vector of the type $M(Du)$.

Beyond formal aspects, however, if we want to interpret as a constitutive choice the selection of a function space to be the ambient of our analysis of some boundary value problem in mechanics, as we have claimed starting these notes, we have to clarify the physical meaning of the ingredients defining the ambient that we consider. So, in the case of non-linear elasticity mentioned here, some questions emerge naturally:

1. Has the notion of current a clear physical interpretation?
2. Does the extension of the energy to the whole space of 3-vectors give us advantages in describing some mechanical phenomena?

To answer the previous questions we first evaluate how traditional continuum mechanics of simple hyperelastic bodies can be expressed naturally in terms of forms.
First we notice that the $3-$vector $M(Du(x))$ can be constructed naturally by fixing a prism generated in the reference place by three linearly independent vectors, considering the maps assigning to every edge of the prism its counterpart in the current place, and then taking into account that they are linearly independent.

The $3-$vector $M(Du(x))$ has a geometrical rôle for it orients locally the graph of the map $x \mapsto Du(x)$. It characterizes then the oriented tangent hyperplane to the point $(x, u(x))$ of the graph of the transplacement map $x \mapsto u(x)$.

Then we realize that the first Piola-Kirchhoff stress admits a multiplicative decomposition into a third-rank, skew-symmetric, covariant tensor $\omega$, which is dual of $M(Du)$ and a fifth-rank tensor which is skew-symmetric in the first three indices. The second tensor is $\frac{dM}{dDu}(Du)$, a geometric projector not related with the energy and depending only on the specific deformation along which the first Piola-Kirchhoff stress is evaluated. In non-linear elasticity, $\omega$ is the energetic part of the stress, the rest being just a geometric factor.

A consequence of this picture is that the current associated with the map $x \mapsto u(x)$ has a natural interpretation in terms of virtual internal work of a simple body, evaluated by considering virtual the stress, rather than the strain.

The previous approach becomes even more useful when we try to determine the existence solutions to problems where, presumably, the transplacement displays ‘vertical’ parts, so it is not everywhere the graph of a one-to-one map, as in presence of constraints, the case treated here. In particular, in what follows the constraint that we consider is the prescription of an admissibility set for the first Piola-Kirchhoff stress, which is obtained through the projection generated by $\frac{dM}{dDu}(Du)$ (acting on the left on $\omega$) of an analogous admissibility convex set in the space of $3-$covectors, a constraint on $\omega$, indeed. This picture generalizes the standard view including admissibility criteria for the stress like Tresca’s, Beltrami’s, Huber-von Mises-Hecky’s ones, just to list a few examples. Admissibility criteria, however, define states which are considered critical with respect to some condition. Information on the post-critical behavior emerge, commonly, from flow rules which can be prescribed a priori or (better) determined from some physically-motivated first principle. Here, we do not investigate time-varying processes. We introduce a function space containing fellows which can describe the transplacement of the body under scrutiny, compatibly with the constraint imposed on the stress and avoiding fractures. Then, as selection criterion, we require minimality of a certain functional over this space and prove existence of minimizers.

In the setting that we discuss here, the stress appears naturally as a vector-valued measure on a three-dimensional surface in a six dimensional space, which is the product of reference and actual spaces. Thinking of the surface generated by the pairs of reference and actual places is a natural
way of considering the transplacement when it is no more a one-to-one map everywhere.

The stress measure corresponds to the standard stress in the reference configuration in the region where the surface just mentioned is the graph of a map. The rest is concentrated on a set of zero volume measure in the reference place. It may describe concentrated actions along dislocations which can be nucleated once the stress reaches the admissibility threshold. Their presence – if it occurs – is the prodrome for a possible phase transition toward the elastic-plastic behavior which has to be described, however, by resorting to an enriched setting.

2 A commentary on large strain kinematics

Deformation is a relative concept. A body in a certain shape is deformed with respect to another shape that we decide be the reference one, typically described by a region $B$ of the Euclidean three-dimensional point space $\mathbb{E}^3$. $B$ is selected to be a regularly open, connected set, with boundary of non-zero two-dimensional measure, a boundary where the normal is defined everywhere to within a finite number of corners and edges. $B$ is only a geometric setting that we use to make paragons in pairs between volumes, areas and/or lengths, just to define appropriately strain measures. So, it is not important that $B$ be occupied by the body under examination at a given instant. It is important just that it could be – even only in principle – occupied by the body. Such a point of view allows us to select $B$ not in the physical space but in an isomorphic copy of it. The symbol $\tilde{\mathbb{E}}^3$ will indicate the isomorphic copy of $\mathbb{E}^3$ chosen as physical space. Reasons for adopting such a choice are related with the action of changes in observers, above all when the body undergoes macroscopic irreversible mutations in its material structure – a topic not discussed here, however (see [17] for details) – and for questions connected with the evaluation of variations of the energy, a point that we shall meet later.

Maps $u : B \times [0,t] \rightarrow \tilde{\mathbb{E}}^3$ select all current places $u(B)$. They are called transplacements (or deformations). Reasons of physical plausibility suggest that each $u$ be

(i) one-to-one and differentiable,
(ii) orientation preserving, and
(iii) able to allow self contact of the body boundary and to exclude self-penetration.$^{1}$

$^{1}$ Although orientation preserving diffeomorphisms satisfy all conditions (i)-(iii), already the analysis of the existence of ground states for elastic bodies undergoing large strain requires an enlarged functional setting to develop consistent analyses. The enlargement implies then a constitutive choice, as already mentioned in the introduction of this paper (see also [10], [11], [18])
D indicates here the spatial derivative with respect to coordinates in $\mathcal{B}$. So that if $F := Du (x)$ maps then linearly the tangent space at $x$ to $\mathcal{B}$ onto the one to $u (\mathcal{B})$ at $y := u (x)$, so, at $x$, it is an element of $\text{Hom} (T_x \mathcal{B}, T_y u (\mathcal{B}))$ which is isomorphic to $\mathbb{R}^3 \otimes \mathbb{R}^3$, where $\mathbb{R}^3$ is the coordinate space over $\mathcal{E}^3$, $\mathbb{R}^3$ the one over $\mathcal{E}^3$. Both $\mathcal{E}^3$ and $\mathcal{E}^3$ are assumed to be isometrically related, the isometry being fixed once we choose (orthogonal) isometric frames in both spaces, a choice rendering both spaces isomorphic with the standard $\mathbb{R}^3$.

Assumption (ii) requires that $\mathcal{E}^3$ and $\mathcal{E}^3$ be oriented, for instance by the choice of the frames in the two ambient spaces, frames defining observers at the end. So, the assumption that $\mathcal{E}^3$ and $\mathcal{E}^3$ be isometrically related allows us to say that the orientation preserving restriction reads $\det F > 0$, the determinant being computed after mapping the basis of the frame in $\mathcal{E}^3$ onto the one in $\mathcal{E}^3$ – this operation is not associated with the transplacement: it is just implicit in the calculation of the determinant.

Select a basis $e_1, e_2, e_3$ in the translation space of $\mathcal{E}^3$. This way we identify the reference space with $\mathbb{R}^3$. The base vector determine a prism – denote it by $\text{prism} (e_1, e_2, e_3)$ – that we assume with unitary volume.

Take then three linearly independent vectors $a_1, a_2, a_3$, attached at a point $x$ in $\mathcal{B}$. They determine another prism, $\text{prism} (a_1, a_2, a_3)$, with edges represented by the three vectors themselves. Let us consider the relevant oriented volume. Meaning that if we change the orientation of one vector, the volume changes sign. The volume of the second parallelepiped is given by the determinant

$$vol (a_1, a_2, a_3) := \begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{vmatrix},$$

where $a_j^i$ is the $i$–th component of the $j$–th vector $a$. By definition the map

$$(a_1, a_2, a_3) \mapsto vol (a_1, a_2, a_3)$$

is skew-symmetric and three-linear, meaning linear in each factor $a_j$, that is in each column. For this reason $vol (a_1, a_2, a_3)$ can be considered as the value reached by a third-rank skew-symmetric tensor generated by $a_1, a_2,$ and $a_3$, that is a tensor of the type $(3,0)$, called 3–vector, over the three-plet $a_1, a_2$, and $a_3$. Of course we can define skew-symmetric tensors of any rank $k$ of the type $(k,0)$ on some linear space, or else of the type $(0,k)$, which are called $k$–covectors.

A special role is played in the sequel by 1–covectors modeled over $\mathbb{R}^3$. They are in fact natural representatives of forces. This point, however, will be discussed later. In this section, in fact, the attention is focused on kinematics only, which is the primary primitive issue.

Always with reference to the linearly independent vectors $a_1, a_2,$ and $a_3$, map one of them in the current place $\mathcal{B}_a := u (\mathcal{B})$: say $a_1$ becomes $Fu_1$. It is then possible to construct another (ideal) prism, namely $\text{prism} (Fu_1, a_2, a_3)$,
living on the product $B_u \times B$, because the vector $F a_1$ is attached at $y := u \,(x) \in B_u$, while $a_2$ and $a_3$ are still on $B$. Difference between the two prisms is then determined by the change $a_1 \rightarrow F a_1$ and is then related to the strain measures defined by means of $F$ itself. To recall briefly the issue, consider $a_1$ as a tangent vector to $B$ at $x$. Write $\dot{a}_1$ for $F a_1$. A way to evaluate how lengths change from $B$ to $B_u$ along $u$ – it is well known – is to measure the difference between the lengths of $a_1$ and $\dot{a}_1$. Such a difference is significant when we put $a_1$ and $\dot{a}_1$ in the same space that we use as a paragon setting. Use the notation $a_1 = \ell \hat{n}$, with $n$ an element of the unit sphere $S^2$, indicating direction and orientation of $a_1$, and $\ell := |a_1|$. With the same meaning of symbols, the hat apart, put also $\dot{a}_1 = \ell \hat{n}$. If we choose $B$ as a paragon setting, it is common routine to compute that

$$\frac{\hat{l}^2 - l^2}{l^2} = (F^* F - g) \cdot (n \otimes n) = (C - g) \cdot (n \otimes n), \quad (1)$$

where $F^*$ is the formal adjoint of the derivative of the transplacement, namely $F^* \in \text{Hom} \left( T_y^* u (B), T_x^* B \right)$, $g$ the peculiar metric of $B$, and $C := F^* F$, the left Cauchy-Green tensor, is the pull-back in the reference place of the metric $\tilde{g}$ pertaining to the space where actual places are evaluated. In fact, in components referred to a local frame, we get

$$(C - g) \cdot (n \otimes n) = (C_{AB} - g_{AB}) n^A n^B,$$

with $C_{AB} = F_A^i \tilde{g}_{ij} F_B^j$, leaving the summation over repeated indices understood. Commonly, the second-rank symmetric tensor $\frac{1}{2} \left( \dot{C} - \dot{1} \right)$, with $\dot{1}$ the second-rank identity with components $\delta_B^A$, and $\dot{C} := g^{-1} C$ is accepted as a strain measure. It is obtained from $\frac{1}{2} (C - g)$ by pre-multiplication by the inverse metric $g^{-1}$, with components $g^{AB}$, and reads in components $\frac{1}{2} \left( \dot{C}^A_B - \delta_B^A \right)$, with $\dot{C}^A_B = g^{AD} C_{DB} = g^{AD} F_D^i \tilde{g}_{ij} F_B^j$ and $\delta_B^A := g^{AD} g_{DB}$. In other words $C_{AB} - g_{AB}$ is just a difference between metrics while $\dot{C}^A_B - \delta_B^A$ is a relative difference.

Analogous reasoning can be developed if we consider how the area of one face of prism $(a_1, a_2, a_3)$ and the entire volume of it change along $u$. The latter is almost immediate: prism $(a_1, a_2, a_3)$ changes in prism $(F a_1, F a_2, F a_3)$ and, if we compare volumes, namely $V := \text{vol} \,(a_1, a_2, a_3) = a_1 \cdot (a_2 \times a_3)$ and $\dot{V} := \text{vol} \,(F a_1, F a_2, F a_3) = F a_1 \cdot (F a_2 \times F a_3)$, we get first that $\det F > 0$ to maintain unchanged the orientation of prism $(a_1, a_2, a_3)$, which is given by the orientation of the vectors determining it, and then

$$\frac{\dot{V} - V}{V} = \det F - 1.$$ 

For changes in the area we can consider for example the transformation

$$\text{prism} \,(a_1, a_2, a_3) \rightarrow \text{prism} \,(F a_1, F a_2, a_3).$$
Once again prism \((Fa_1,Fa_2,a_3)\) is a prism on the product \(B \times B_0\), with \(B_0 := u(B)\). The cofactor of \(F\), commonly indicated by \(\text{cof} F\), is then the tool measuring how much the area \(A := |a_1 \times a_2|\) changes when it is transformed into \(\hat{A} := |Fa_1 \times Fa_2|\). Let \(\hat{n}\) be a unit normal to the plane spanned by \(Fa_1\) and \(Fa_2\) and \(n\) the corresponding normal to the plane determined by \(a_1\) and \(a_2\), so that \(\hat{n} \hat{A} = Fa_1 \times Fa_2\) and \(nA = a_1 \times a_2\). Nanson formula tells us that

\[
\hat{n} \hat{A} = (\text{cof} F) nA,
\]

where, thanks to the positiveness of \(\det F\), \(\text{cof} F := (\det F)^{-1}\). With \(F := \nabla u(x)\) instead of \(Du(x)\), we have \(\text{cof} F := (\det F)^{-T}\). Consequently, we get

\[
\frac{\hat{A} - A}{A} = |(\text{cof} F) n| - 1.
\]

The issue can be re-discussed by invoking exterior algebra. Preliminarily, remind that, with \(\mathcal{L}\) a linear spaces, with dimension \(m\), the symbol \(\wedge\) indicates a map \(\wedge: \mathcal{L} \times \mathcal{L} \rightarrow \text{Skw}(\mathcal{L}^*, \mathcal{L})\), with \(\text{Skw}(\mathcal{L}^*, \mathcal{L})\) the space of skew-symmetric tensors from the dual of \(\mathcal{L}\) to \(\mathcal{L}\). The space of skew-symmetric tensors of type \((k,0)\), that are fully contravariant tensors of order \(k\), also called \(k\)-vectors is then indicated by \(A_k(\mathcal{L})\), its dual by \(A^k(\mathcal{L})\). In particular, for \(k = 2\), \(\Xi \in A_2(\mathcal{L})\) means \(\Xi = \xi^j e_i \otimes e_j\), where \(e_k\) is a vector of the basis in \(\mathcal{L}\).

The strain measures summarized above can be related, in fact, to maps of the type

\[
\begin{align*}
a_1 \wedge a_2 \wedge a_3 &\mapsto Fa_1 \wedge a_2 \wedge a_3, \\
a_1 \wedge a_2 \wedge a_3 &\mapsto Fa_1 \wedge Fa_2 \wedge a_3, \\
a_1 \wedge a_2 \wedge a_3 &\mapsto Fa_1 \wedge Fa_2 \wedge Fa_3.
\end{align*}
\]

The 3-vectors obtained by this scheme are linearly independent on \(A_3(\mathbb{R}^3 \times \mathbb{R}^3)\). Thus, by adding them to \(a_1 \wedge a_2 \wedge a_3\), we define a 3-vector \(M(F)\), namely

\[
M(F) := a_1 \wedge a_2 \wedge a_3 + 
+ Fa_1 \wedge a_2 \wedge a_3 + a_1 \wedge Fa_2 \wedge a_3 + a_1 \wedge a_2 \wedge Fa_3 + 
+ Fa_1 \wedge Fa_2 \wedge a_3 + Fa_1 \wedge a_2 \wedge Fa_3 + a_1 \wedge Fa_2 \wedge Fa_3 + 
+ Fa_1 \wedge Fa_2 \wedge Fa_3 = 
= (a_1,Fa_1) \wedge (a_2,Fa_2) \wedge (a_3,Fa_3) \in A_3(\mathbb{R}^3 \times \mathbb{R}^3).
\]

\(M(F)\) has 20 components, which are 1 and the entries of \(F\), \(\text{cof} F\), and \(\det F\), namely the entities determining, at the point where \(F\) is evaluated, the measures of deformation.

Consider a three-plet \((a_1,a_2,a_3)\) in \(\mathbb{R}^3\) and another three-plet \((\hat{a}_1,\hat{a}_2,\hat{a}_3)\) in \(\mathbb{R}^3\), with both triads of linearly independent vectors respecting the rule of the right hand, or both the one of the left-hand, if you want. There is always a linear map, say \(F\), projecting arbitrarily one edge of prism \((a_1,a_2,a_3)\) onto another edge of prism \((\hat{a}_1,\hat{a}_2,\hat{a}_3)\). There is also another linear map, say \(A\),
between one face of the first prism to another face of the second one, and they can be selected to be both orientation preserving. The two volumes are also proportional. The coefficient of proportionality and the two linear maps associated with edges and faces are in principle independent and can be collected in generic elements $M$ of $A_3(\mathbb{R}^3 \times \mathbb{R}^3)$ to within an (isometric) isomorphism. A special case is when, given the orientation preserving linear map $F$, which acts on edges, we choose for the faces of the prism the linear map $\text{cof} F$ and the relevant $\det F$ for volumes, so that $M = M(F)$, with the addition of a component $1$. In particular, we write $M(Du(x))$ when the linear operator $F$ is the spatial derivative of a map $u$, as indicated in defining transplacements. In this case, the 3–vector $M(Du(x))$ describes the tangent hyperplane to the graph of $u$ at the point $(x, u(x))$. This is a well known result (see, e.g., [11]).

The components of a generic element $M$ of $A_3(\mathbb{R}^3 \times \mathbb{R}^3)$ are determined, in general, by two constants, say $\zeta$ and $a$, and two independent linear operators, e.g. $F$ and $A$. In fact, let $(e_1, e_2, e_3)$ be a basis in $\mathbb{R}^3$ and $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ another basis in $\mathbb{R}^3$. Every 3–vector $M \in A_3(\mathbb{R}^3 \times \mathbb{R}^3)$ has the form

$$
M = \zeta e_1 \wedge e_2 \wedge e_3 + \sum_{i,j} (-1)^{j-1} F^{i\ell} e_j \wedge \tilde{e}_i + 
$$

$$
+ \sum_{i,j} (-1)^{i-1} A^{i\ell} e_j \wedge \tilde{e}_i + a \tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3,
$$

where $\ell$ is the complementary multi-index to $J$ with respect to $(1, 2, 3)$ and $i$ has an analogous relation with $\ell$ (for example, if $J = 1$, then $\ell = (2, 3)$ and $e_j = e_2 \wedge e_3$, and the same holds for the index $i$ and its pertinent $\ell$).

For the sake of conciseness we write $M = (\zeta, F, A, a)$ or $M = (\zeta, M)$, with $M$ indicating the three-plet $(F, A, a)$, if we want to emphasize the rôle of the first component of $M$. The latter notation suggests also the orthogonal decomposition $A_3(\mathbb{R}^3 \times \mathbb{R}^3) = \mathbb{R}^3 \times \mathcal{V}$. The generic element of $\mathcal{V}$ is then of the type

$$
M = \sum_{i,j} (-1)^{j-1} F^{i\ell} e_j \wedge \tilde{e}_i + \sum_{i,j} (-1)^{i-1} A^{i\ell} e_j \wedge \tilde{e}_i + a \tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3.
$$

Of course, when $M = M(F) = (1, M(F))$, we have $M(F) = (F, \text{cof} F, \det F)$.

2 The construction leading to $M(F)$ can be naturally extended. To this aim, consider a vector basis $e_1, \ldots, e_n$ in $\mathbb{R}^n$ and another basis $\tilde{e}_1, \ldots, \tilde{e}_N$ in $\mathbb{R}^N$. For any linear map $G : \mathbb{R}^n \to \mathbb{R}^N$, $M(G)$ is defined by

$$
M(G) := A_n(id \times G) (e_1 \wedge \ldots \wedge e_n) = (e_1, G(e_1)) \wedge \ldots \wedge (e_n, G(e_n))
$$

which is analogous to

$$
M(G) = \sum_{k=0}^{\min(n,N)} M(k) (G),
$$
Our remarks do not indicate an alternative way to define strain measures, because the map \( F \mapsto M(F) \) is one-to-one. They put just in evidence further geometric aspects of the manner we measure strain, paving the way to the possibility of using methods of geometric measure theory and functional analysis to deduce at least qualitative information on the solutions of appropriate boundary value problems, the ones emerging in the description of the mechanics of deformable bodies.

A number of sets appear in the subsequent developments. They are all subsets of \( \Lambda_3(\mathbb{R}^3 \times \mathbb{R}^3) \). We summarize here the relevant notations:

\[
V := \{ M = (F, A, a) \},
\]
\[
V_+ := \{ M = (F, A, a) \mid a > 0 \},
\]
\[
V_{+,F} := \{ M(F) = (F, \text{cof}F, \det F) \mid a > 0 \},
\]
\[
\Sigma_1 := \{1\} \times V, \quad \Sigma_{1,+} := \{1\} \times V_+,
\]
\[
\Sigma_{1,+} = \{1\} \times V_{+,F}.
\]

3 Contact actions and 1-forms

Our previous remarks put in evidence that \( M \in V_+ \) contains in principle all the ingredients necessary to define standard measures of strain, with orientation preserving nature. The subset \( V_{+,F} \) of \( V_+ \) includes elements that we use to determine the strain along lines, surfaces and volumes: information carried on by the standard strain measures. The distinction between \( V_+ \) and \( V_{+,F} \) is motivated by the consciousness that, at least in principle, we could locally deform independently lines and surfaces (a possibility not excluded in \( V_+ \), in fact), by exploiting a sort of (extended) notion of plastic strain incompatibility.

In standard continuum mechanics contact actions between neighboring parts of a body are defined by the power they develop. Since stresses are dual entities with respect to strain, we can figure fellows of the dual space of \( V_+ \), as entities containing information on stresses over lines, surfaces, and volume (pressure in this last case).

where, by indicating by \( \alpha \) a multi-index in \((1, 2, ..., n)\) with length \(|\alpha|\) and by \( \bar{\alpha} \) its complement always in \((1, 2, ..., n)\), \( \beta \) another multi-index, with sign \( \text{sign} (\alpha, \bar{\alpha}) \) the sign of the permutation from \((1, 2, ..., n)\) into \((\alpha_1, ..., \alpha_k, \bar{\alpha}_1, ..., \bar{\alpha}_{n-k})\), the component \( M_{(k)} (G) \) is given by

\[
M_{(k)} (G) = \sum_{|\alpha| + |\beta| = n \atop |\beta| = k} \text{sign} (\alpha, \bar{\alpha}) M_\alpha (G) e_\alpha \wedge \bar{e}_\beta.
\]

\( G \) indicates the matrix associated with \( G \), so that \( M_\alpha (G) \) is the determinant of the submatrix of \( G \) made of the rows and the columns indexed by \( \beta \) and \( \bar{\alpha} \) respectively. By definition \( M_0 (G) := 1 \), and \( M (G) \) is a simple \( n \)-vector in \( \Lambda_n (\mathbb{R}^n \times \mathbb{R}^N) \), a linear space with dual indicated by \( \Lambda^* (\mathbb{R}^n \times \mathbb{R}^N) \).
The same remarks apply to $\Sigma_{1,+}$, $\Sigma_{1,+F}$ and their dual spaces. In fact, the component $1$ marking the difference between $\mathcal{V}_+$ and $\Sigma_{1,+}$ has no role in the definition of strain measures.

Reasons justifying such a point of view are collected below.

Consider an oriented plane cutting ideally the current shape $\mathcal{B}_a$ in two pieces $\mathcal{B}_a^+$ and $\mathcal{B}_a^-$, where the algebraic signs distinguish between the positive and negative orientation of the normal to the plane. According to Cauchy’s view, at a generic point $y$ of the section, the contact interaction between $\mathcal{B}_a^+$ and $\mathcal{B}_a^-$ is locally described only by a force $t$ – the tension, indeed – which depends on $y$ and $n_a$, the latter being the normal to the plane with positive orientation. No applied couples or more exotic forms of contact actions, as the ones occurring when changes in the finer scale architecture of the matter are taken into account, occur. Time would enter the list if the analysis would be developed including motions, that are one-parameter families of transplacements.

The action-reaction principle suggests the equality $t(y, n_a) = -t(y, -n_a)$. Continuity of $t$, equilibrium conditions, and the standard tetrahedron argument assures the validity of Cauchy theorem about the existence of a tensor $\sigma$, depending only on $y$, the stress tensor, such that (Cauchy theorem)

$$t(y, n_a) = \sigma(y) n_a.$$  

Assumptions of Cauchy theorem have been variously weakened in the available literature (see, e.g., results and comments in [21], [22], [23], [25]). However, more than analyzing the way the conditions assuring Cauchy theorem can be weakened, here our attention is focused on some aspects of the geometry which is implicit in the tetrahedron argument used in the standard proof.

The tension $t(y, n_a)$ is a covector. Its nature is due to the decision of describing every material element only as a point in space. In fact, $t$ can be defined in abstract way by considering actions as operators over the algebra of bodies (see [28]). However, since in laboratory we measure directly velocities (variations of placement in time) and powers, rather than forces, a natural way of defining $t$ is to consider it as the entity that furnishes the power needed to obtain a change of place of a material element at a point, once it is multiplied by the relevant velocity. Once a plane oriented by the normal $n_a$ is selected within $\mathcal{B}_a$, given a velocity field $y \mapsto v(y)$, which can be considered even virtual, the tension $t$ is defined at $y$, belonging to the plane just selected, by the power $t(y, n_a) \cdot v(y)$ developed along $v$. So, since $v(y)$ is a vector, to have $t(y, n_a) \cdot v(y)$ defined without resorting to the additional structure of scalar product, $t(y, n_a)$ must be considered a covector, an element of $A^1(\mathbb{R}^3)$. Then, $t(y, n_a) \cdot v(y)$ is the value taken over $v(y)$ by the covector $t(y, n_a)$, a value that we also indicate commonly by $(t, v)$.

Take two linearly independent vectors over the plane considered in previous lines, say $\tilde{a}_1$ and $\tilde{a}_2$. The normal $n_a$ at $y \in \mathcal{B}_a$ is determined then by $\tilde{a}_1 \wedge \tilde{a}_2$ by means of the Hodge operator $\ast : A_2(\mathbb{R}^3) \to A_1(\mathbb{R}^3)$, namely
exists and (above all) is independent of
or the standard formula
mount to write
with

\[ P := \sigma \circ \text{det } F \circ j^{-1}. \]

The selection of a tetrahedron at \( y \) for proving that the stress tensor \( \sigma \)
exists and (above all) is independent of \( n_a \) – it is the standard argument –
requires at $y$ the choice of three linearly independent vectors determining the tetrahedron itself. Better, if we consider the normal $n_a$ as a covector, like we have done so far, we can choose to select three linearly independent covectors, say $c^1, c^2, n_a$, instead vectors. They have a counterpart in the reference place, say the three-plet $c^1, c^2, n$. We can construct a third-rank skew-symmetric tensor $c^1 \wedge c^2 \wedge Pn$ which ‘lives’ on the product $B \times B_a$. Such a tensor generates a linear map

$$c^1 \wedge c^2 \wedge Pn : \mathbb{R}^3 \rightarrow A^2(\mathbb{R}^3).$$

Connected with it there is the third-rank skew-symmetric tensor $a_1 \wedge a_2 \wedge F\alpha_3$, which induces a linear map

$$a_1 \wedge a_2 \wedge F\alpha_3 : \mathbb{R}^{3*} \rightarrow A_2(\mathbb{R}^3).$$

In this point of view the two tensors act on spaces which are dual one another. In this sense we can interpret the standard result that $P$ is dual$^3$ to $F$.

We can also do more and construct another third-rank skew-symmetric tensor ‘living’ on the product $B \times B_a$, namely $c^1 \wedge Pc^2 \wedge Pn$, which is the dual counterpart of $a_1 \wedge F\alpha_2 \wedge F\alpha_3$, exactly in the sense rendered precise in the previous case. Namely, $c^1 \wedge Pc^2 \wedge Pn$ generates a linear map

$$c^1 \wedge Pc^2 \wedge Pn : A^2(\mathbb{R}^3) \rightarrow \mathbb{R}^{3*},$$

while $a_1 \wedge F\alpha_2 \wedge F\alpha_3$ the map (always linear)

$$a_1 \wedge F\alpha_2 \wedge F\alpha_3 : A_2(\mathbb{R}^{3*}) \rightarrow \mathbb{R}^3.$$

Tensor $c^1 \wedge Pc^2 \wedge Pn$ gives us information about the ‘averaged’ stress on the planar region $F\alpha_2 \wedge F\alpha_3$ in $B_a$, an information carried by the second-rank minors of $P$. The role of them is dual of the one of cof$F$.

Finally, we can write $Pc^1 \wedge Pc^2 \wedge Pn$, which ‘lives’ in $B_a$. The ratio between the volume of the prism $Pc^1 \wedge Pc^2 \wedge Pn$ and the one of the prism $c^1 \wedge c^2 \wedge n$ is dual of the analogous ratio between the relevant volumes of $F\alpha_1 \wedge F\alpha_2 \wedge F\alpha_3$ and $a_1 \wedge a_2 \wedge a_3$, so it is dual to det $F$. In this sense it can be interpreted as a pressure associated with the change in volume.

Since $c^1 \wedge c^2 \wedge Pn$, $c^1 \wedge Pc^2 \wedge Pn$, and $Pc^1 \wedge Pc^2 \wedge Pn$, are linearly independent, given a covector basis $dx^1, dx^2, dx^3$ in $\mathbb{R}^3$, we can then construct a 3-covector – denote it by $\omega(P)$ – defined by

$$\omega(P) := dx^1 \wedge dx^2 \wedge dx^3 +$$

$$+Pdx^1 \wedge dx^2 \wedge dx^3 + dx^1 \wedge Pdx^2 \wedge dx^3 + dx^1 \wedge dx^2 \wedge Pdx^3 +$$

$$+Pdx^1 \wedge Pdx^2 \wedge dx^3 + Pdx^1 \wedge dx^2 \wedge Pdx^3 + dx^1 \wedge Pdx^2 \wedge Pdx^3 +$$

$^3$ The result is confirmed when energetic issues enter the stage by the link given by the energy between $P$ (at least the energetic part of $P$) and $F$, as prescribed by Clausius-Duhem inequality, a version of the second law of thermodynamics.
\[ +Pdx^1 \wedge Pdx^2 \wedge Pdx^3 = \]
\[ = (dx^1, Pdx^1) \wedge (dx^2, Pdx^2) \wedge (dx^3, Pdx^3) \in \Lambda^3(\mathbb{R}^3 \times \mathbb{R}^3). \]

\( \omega(P) \) contains all entities giving us information on stresses on lines, their ‘averaged’ values over surfaces, and a pressure associated with changes in volume. Such entities are \( P \), its second-rank minors, and the pressure \( p \) appearing as a component \(-pdy^1 \wedge dy^2 \wedge dy^3\), where \( dy^1, dy^2, dy^3 \) are the elements defining the covector basis in \( \mathbb{R}^3 \). Specifically, \( \omega(P) \) belongs to the space \( \Lambda^3(\mathbb{R}^3 \times \mathbb{R}^3) \) which is dual of \( A_3(\mathbb{R}^3 \times \mathbb{R}^3) \).

More in general, as above, let \( dx^1, dx^2, dx^3 \) and \( dy^1, dy^2, dy^3 \) be the dual bases of \( \mathbb{R}^3 \) and \( \mathbb{R}^3 \), respectively. Any \( \omega \in \Lambda^3(\mathbb{R}^3 \times \mathbb{R}^3) \) writes as

\[ \omega : = \beta dx^1 \wedge dx^2 \wedge dx^3 + \sum_{i,j=1}^3 (-1)^{j-1} r_{i,j} dx^j \wedge dy^i + \]
\[ + \sum_{i,j=1}^3 (-1)^{i-1} s_{i,j} dx^i \wedge dy^j + \zeta dy^1 \wedge dy^2 \wedge dy^3, \]

where, as in the formula defining \( M, J \) is the complementary multi-index to \( J \) with respect to \((1, 2, 3)\) and there is the same relation between \( i \) and \( \bar{i} \). For example, if \( J = 1 \) then \( \bar{J} = (2, 3) \) and we have \( dx^J = dx^2 \wedge dx^3 \).

Moreover, if \( i = 3 \) then \( \bar{i} = (1, 2) \) and we write \( dy^i = dy^1 \wedge dy^2 \), and so on. \( \beta \) and \( \delta \) are scalars, \( r \) and \( s \) are linear operators.

Even in the case of 3−covectors, for the sake of conciseness we write \( \omega = (\beta, \gamma, \zeta) \) or \( \omega = (\beta, \varpi) \), with \( \varpi \) indicating the three-plet \((r, s, \delta)\). The latter notation suggests also the orthogonal decomposition \( \Lambda^3(\mathbb{R}^3 \times \mathbb{R}^3) = \mathbb{R}^3 \times \mathcal{V}^* \). The generic element of \( \mathcal{V}^* \) is then of the type

\[ \varpi = \sum_{i,j=1}^3 (-1)^{j-1} r_{i,j} dx^j \wedge dy^i + \sum_{i,j=1}^3 (-1)^{i-1} s_{i,j} dx^i \wedge dy^j + \zeta dy^1 \wedge dy^2 \wedge dy^3. \]

For \( \omega \in \Lambda^3(\mathbb{R}^3 \times \mathbb{R}^3) \) and \( M \in A_3(\mathbb{R}^3 \times \mathbb{R}^3) \), the product \( \omega \cdot M \) is defined by

\[ \omega \cdot M = \langle \omega, M \rangle := \beta \zeta + \sum_{i,j=1}^3 r_{i,j} F^{iJ} + \sum_{i,j=1}^3 s_{i,j} A^{iJ} + \zeta a. \]

4 Energy

We call Cauchy’s bodies (or materials) those with mechanical behavior represented by a scheme in which the morphology is just described by the place that they occupy in space, and the inner contact actions are only

\[ ^4 \text{By } \langle \cdot, \cdot \rangle \text{ we denote the natural duality pairing between } \Lambda^3(\mathbb{R}^3 \times \mathbb{R}^3) \text{ and } A_3(\mathbb{R}^3 \times \mathbb{R}^3). \text{ For the sake of conciseness, sometimes we shall adopt simply the dot } ^*\cdot \text{ for } \langle \cdot, \cdot \rangle. \]
standard stresses. The denomination is selected to remind (see discussions in [17]) that such a scheme is not the sole possible option: the morphology requires a refined description in appropriate cases, with consequent changes in the representation of the inner actions.

In common treatises, bodies are called simple when they are Cauchy’s and the constitutive structures depend just on the deformation gradient $F$. The point of view is refined when we realize that reasons of objectivity under $SO(3)$-based changes in observers prescribe that the constitutive structures depend on strain measures such as the right Cauchy-Green tensor $C := F^* F$, rather than $F$ alone. Objectivity means only that any given entity which is sensitive to changes in observers, intended as alterations of the atlas in the physical space, transforms according with the geometric rules which pertain to its tensor nature.

Here we want to give from the beginning a view on simple bodies anticipating at the level of definition of the energy what is commonly obtained a posteriori for analytical reasons. Our construction is not merely a simple shift of the point where we introduce some objects. Rather, it seems to us a natural consequence of the remarks we have collected when we have expressed both kinematics and representation of actions in terms of 3-vectors and 3-covectors.

Both spaces $V_+ \text{ and } \Sigma_{1,+}$ contain, as we have pointed out previously, elements which are essential for constructing standard strain measures and to describe situations where strain incompatibility occurs. Both $V_+$ and $\Sigma_{1,+}$ are path connected, i.e. any two points selected in each of them can be connected by a piecewise-$C^1$ curve.

For any piecewise-$C^1$ curve $\gamma : [-1, 1] \rightarrow \Sigma_{1,+}$, consider a continuous form $t \mapsto \omega (t) \in \Sigma_{1,+}$, with $t \in [-1, 1]$. Along $\gamma$ we define a functional $w(\omega, \gamma)$ that we call extended virtual internal (or inner) work, namely

$$w(\omega, \gamma) := \int_0^1 \langle \omega (\gamma (t)), \dot{\gamma} (t) \rangle \, dt.$$ 

Since $$\omega = \sum_{k=0}^3 \omega_{(k)}, \quad \gamma (t) = \sum_{k=0}^3 \gamma_{(k)},$$

the product $\langle \omega (\gamma (t)), \dot{\gamma} (t) \rangle = \sum_{k=0}^3 \omega_{(k)} (\gamma (t)) \cdot \dot{\gamma}_{(k)} (t)$ involves

1. the power density $\omega_{(1)} \cdot \dot{\gamma}_{(1)}$ related to line deformations,
2. a power determined by volume changes, and given by $\langle \omega_{(3)} \cdot \dot{\gamma}_{(3)} (t) \rangle = \langle -p \dot{\gamma}_{(3)} (t) \rangle$, with $p$ the scalar appearing in $\omega_{(3)} = -p \, dy^1 \wedge dy^2 \wedge dy^3$ with the meaning of a pressure$^5$, and

$^5$ In our previous notations, $-p = c$. 
(3) terms given by the multiplication of the components of $\omega_{(2)}$ with the ones of $\dot{\gamma}_{(2)}(t)$, and representing the power over coordinate planes in a local frame, and for every local frame.

The requirements that (i) $w$ be non-negative along closed paths and (ii) any path in $\Sigma_{1,+}$ be physically realizable are tantamount to impose that $w$ vanishes along all closed paths, a requirement of conservativeness, indeed. These conditions are, in fact, equivalent to the existence of a function $e : \Sigma_{1,+} \rightarrow \mathbb{R}$ such that

$$\omega(M) = de(M), \quad \forall M \in \Sigma_{1,+}. \quad (2)$$

In this setting we consider then $e$ as stored energy density. The assumption that such an energy be defined on the whole $\Sigma_{1,+}$ may induce immediate criticisms. Fellows of $\Sigma_{1,+}$ may be, in fact, of the type $(1;F;A;a)$, with the linear operator $A$ not coinciding with $\text{cof} F$, and $a$ not with $\det F$. These elements of $\Sigma_{1,+}$ may describe in principle strain incompatibility, as already mentioned. So, doubts of the admissibility of perfect physical accessibility of one point in $\Sigma_{1,+}$ from any other deformation state in the same space may be correctly formulated. However, the availability of equilibrium configurations (ground states) for deforming bodies in nature would suggest us to imagine that the energy be a convex function of what we use to define strain. However, objectivity prevents convexity with respect to the deformation gradient, as it is well known. Also, convexity cannot be defined on $\Sigma_{1,+;F}$, which is the image of the set of $3 \times 3$ matrices with real entries and positive determinant, obtained through the map $F \mapsto M(F) = (1;F;\text{cof} F;\det F)$, for such a space is not convex. So, when we choose as elastic energy a polyconvex function of $F$, by following [4], we are selecting in essence a convex function defined on $\Sigma_{1,+}$, which is the convex hull of $\Sigma_{1,+;F}$. However, when we evaluate the existence of minimizers of an energy with the aim of finding ground states (equilibrated configurations), the minimization process furnishes strain measures in $\Sigma_{1,+;F}$, saving the physical significance of the result. So, this is the reason pushing us to consider $e$ as a function over $\Sigma_{1,+;F}$, having in mind the determination of ground states.

Consider now a piecewise-$C^1$ path $\gamma$ in $\Sigma_{1,+;F}$. The extended virtual internal power density at a given $t$ reads

$$\langle \omega(\gamma(t)), \dot{\gamma}(t) \rangle = \langle \omega(M(F)), \dot{M}(F) \rangle =$$

$$= \omega \cdot \frac{dM}{dF}(F) \cdot \dot{F} = \omega \frac{dM}{dF}(F) \cdot \dot{F}.$$

In the last product in the previous expression, namely $\omega \frac{dM}{dF}(F)$, the fifth-rank tensor $\frac{dM}{dF}(F)$, which is skew-symmetric in the first three indices, acts – it is applied from the right – as an element of the space

$$\text{Hom} \left( \text{Hom} \left( T_x^* \mathcal{B}, T_y^* \mathcal{B}_a \right), \Lambda^3(\mathcal{B} \times \mathbb{R}^3) \right).$$
At $x \in B$, it maps linearly 3-covectors in $\Lambda^3(B \times \mathbb{R}^3)$ onto elements of $\text{Hom}(T^*_x B, T^*_x B)$, which is the space of second-rank tensors of the type of the first Piola-Kirchhoff stress $P$. So, we write

$$P = \omega \frac{dM(F)}{dF},$$

and we can identify the product $\omega \frac{dM(F)}{dF} \cdot \hat{F}$ with the standard density of internal power of simple bodies, namely $P \cdot \hat{F}$. The result is then summarized below.

**Proposition 1** The first Piola-Kirchhoff stress admits a multiplicative decomposition into a third-rank skew-symmetric tensor (the value at $x$ of a form over the reference place $B$) and a fifth-rank tensor which is skew-symmetric in the first three indices. In the elastic setting, the first tensor, namely $\omega$, is the only part related with the energy by the relation (2), while the second factor, namely $\frac{dM(F)}{dF}$ is just a geometric projector depending only on the specific deformation along which the first Piola-Kirchhoff stress is evaluated.

In particular, we have

$$P(F) = \omega(M(F)) \frac{dM}{dF}(F) =$$

$$= \omega(1)(M(F))I + \omega(2)(M(F)) \frac{\text{cof}F}{dF}(F) +$$

$$+ \omega(3)(M(F)) \frac{\text{det}F}{dF}(F),$$

where $I$ is now the identity in $\text{Hom}(\mathbb{R}^3, \mathbb{R}^3)$. The derivative $\frac{\text{det}F}{\partial F}$ equals $\text{cof}F$. Moreover, by identifying covariant with contravariant components (that is $\mathbb{R}^3$ with its dual), for the component $\frac{\partial(\text{cof}F)}{\partial F_{AB}}$ we get

$$\frac{\partial (\text{cof}F)}{\partial F_{iA}} = e_{ijm}F_{mC}e_{CAB},$$

with $e$ Ricci’s permutation index.

For the composition map $\tilde{e}(F) := e(M(F))$, with $F \in M_{3 \times 3}$, we then have

$$P = \partial_F \tilde{e}(F)$$

which is

$$P(F) = \text{det}(M(F)) \frac{dM(F)}{dF}.$$

As a consequence of the construct presented so far, we then have

(1) a variational characterization of the stress form $\omega$,

(2) a natural framework for evaluating ground states (equilibrated configurations) with stress form $\omega$, constrained to take values in some convex admissibility set, a circumstance that we find in discussing the incoming occurrence of the phase transition from elastic to plastic or brittle behavior.
5 A few elements of convex analysis

Before discussing the existence of ground states with stress constraints, we find appropriate to recall certain elementary notions in convex analysis that are useful for what we present later. For further details the reader can refer to standard treatises like, e.g., [15] and [20].

We refer to a finite-dimensional vector space, and we take \( V \) for it. We consider also its dual \( V^* \) and the natural duality pairing \( \langle \cdot, \cdot \rangle \) between the two spaces.

Given a convex and lower semicontinuous (l.s.c.) function \( f : V \to (-\infty, +\infty] \), we call conjugate the map \( f^* : V^* \to (-\infty, +\infty] \), defined by

\[
f^*(w) = \sup_{M \in V} \langle (w, M) - f(M) \rangle, \quad \forall w \in V^*.
\]

Again, \( f^* \) is convex and l.s.c.; and also the conjugation operation is involutive: \( f^{**} \) coincides with \( f \). An element \( w \) of the dual space \( V^* \) is called a subgradient of \( f \) at \( M \in V \) if and only if \( f(Z) \geq f(M) + \langle w, Z-M \rangle \) for any \( Z \in V \). The element \( w \) of \( V^* \) satisfying the previous inequality is not necessarily unique, when it exists. The set of all subgradients of \( f \) at \( M \) is a subset of \( V^* \) called subdifferential of \( f \) at \( M \) and denoted by \( \partial f(M) \). Since the conjugation operation is involutive, we get \( w \in \partial f(M) \) if and only if \( M \in \partial f^*(w) \). If \( f \) is differentiable at \( M \), then the subgradient of \( f \) at \( M \) contains only the differential \( df(M) \), so that \( \partial f(M) = \{ df(M) \} \).

**Fenchel inequality**, namely

\[
f(M) + f^*(w) \geq \langle w, M \rangle, \quad \forall (M, w) \in V \times V^*,
\]

with the equality sign holding if and only if \( w \in \partial f(M) \) (or, equivalently, \( M \in \partial f^*(w) \)), is the classical way to express what is usually called convex duality. Another way can be followed by introducing the Lagrangian of \( f \), precisely a function \( \mathcal{L} : V \times V^* \to \mathbb{R} \) defined by

\[
\mathcal{L}(M, w) := f(M) + f^*(w) - \langle w, M \rangle, \quad (M, w) \in V \times V^*.
\]

Fenchel inequality can be then expressed by affirming that \( \mathcal{L}(M, w) \geq 0 \) and the pair \((M, w)\) is a minimizer for the Lagrangian, if and only if \( w \in \partial f(M) \) (or, equivalently, \( M \in \partial f^*(w) \)).

For \( K \subset V^* \), denote by \( I_K \) the indicator function \( I_K : V^* \to [0, +\infty] \) defined by

\[
I_K(w) = \begin{cases} 
0 & \text{if } w \in K, \\
+\infty & \text{if } w \notin K.
\end{cases}
\]

\( I_K(w) \) is a convex function if and only if \( K \) is a convex set and is also l.s.c. if and only if \( K \) is closed.

Let \( f : V \to \mathbb{R} \cup \{ +\infty \} \) be convex and l.s.c., and \( K \) be a bounded convex set in \( V^* \), containing the origin in its interior. Define

\[
w^*(w) := f^*(w) + I_K(w), \quad w \in V^*.
\]

Trivially, \( w^*(w) = f^*(w) \) if \( w \in K \) and moreover, \( w^*(w) = f^*(w) \) implies \( w \in K \), provided \( f^*(w) < \infty \).
**Proposition 2** $\partial w^*(\varpi) \neq \emptyset$ if $\varpi \notin K$ and $\partial w^*(\varpi) = \partial f^*(\varpi)$, if $\varpi \in \text{int}K$. For $\varpi \in \partial K$ we get just $\partial w^*(\varpi) \supset \partial f^*(\varpi)$.

**Proof** The first two claims are trivial because $w^*$ and $f^*$ agree on a neighborhood of $\xi$. Moreover, if $\varpi \in K$ and $M \in \partial f^*(\varpi)$, then $f^*(\eta) \geq f^*(\varpi) + \langle \eta - \varpi, M \rangle$. Consequently, $w^*(\eta) \geq f^*(\eta) \geq w^*(\varpi) + \langle \eta - \varpi, M \rangle$, i.e. $M \in \partial w^*(\varpi)$. □

Consider now the conjugate function of $w^*$, namely $w : \mathcal{V} \rightarrow \mathbb{R}$, given obviously by

$$w(M) := w^{**}(M) = \sup_{\varpi \in \mathcal{V}^*}(\langle \varpi, M \rangle - w^*(\varpi)), \quad M \in \mathcal{V}.$$  

**Proposition 3** The following statements hold true.

1. $w(M) = f(M)$ if and only if $K \cap \partial f(M) \neq \emptyset$.
2. If $K$ is bounded, there exist positive constants $c_1$ and $c_2$ such that

$$w(M) \leq c_1 |M| + c_2, \quad \forall M \in \mathcal{V}.$$

3. If $K$ contains 0 in its interior and $\sup_{\varpi \in \mathcal{V}^*}f^*(\varpi) < \infty$, then there exist positive constants $c_3$ and $c_4$ such that

$$w(M) \geq c_3 |M| + c_4, \quad \forall M \in \mathcal{V}.$$

4. $\partial w(M) \subset K$ and $\text{int}K \cap \partial f(M) = \text{int}K \cap \partial w(M)$.

5. Assume that $f$ be superlinear at infinity, that is $f(M) / |M| \rightarrow \infty$ as $|M| \rightarrow \infty$. Then $\partial w(\lambda M) \in \partial K$ for $\lambda$ a number large enough.

**Proof** 1. Assume that $\varpi \in K \cap \partial f(M)$, i.e. $\varpi \in K$ and $f(M) + f^*(\varpi) = \langle \varpi, M \rangle$. Since $w^*(\varpi) \geq f^*(\varpi)$ for any $\varpi$, we get $w(M) = w^{**}(M) \leq f^{**}(M) = f(M)$ for any $M$. On the other hand,

$$w(M) = \sup_{\varpi \in K}(\langle \varpi, M \rangle - f^*(\varpi)) \geq \langle \varpi, M \rangle - f^*(\varpi) = f(M)$$

by Fenchel inequality. Conversely, assume that $w(M) = f(M)$. Since $K$ is compact, there exists $\varpi \in K$ such that $w(M) = \langle \varpi, M \rangle - f^*(\varpi)$. If $w(M) = f(M)$, then $f(M) + f^*(\varpi) = \langle \varpi, M \rangle$, hence $\varpi \in \partial f(M)$ by Fenchel theorem.

2. Since $K$ is compact,

$$w(M) = \sup_{\varpi \in K}(\langle \varpi, M \rangle - f^*(\varpi)) \leq c_1 |M| + c_2,$$

where $c_1 := \sup_{\varpi \in K} |\varpi|$ and $c_2 := \sup_{\varpi \in K} (-f^*(\varpi))$.

3. By assumption, the ball $B(0, c_3)$, centered at 0 and with radius $c_3$, is contained in $K$, namely $B(0, c_3) \subset K$, for some $c_3 > 0$. Therefore, $c_3 \frac{\varpi}{|\varpi|} \in K$, so that

$$w(M) = \sup_{\varpi \in K}(\langle \varpi, M \rangle - f^*(\varpi)) \geq c_3 |M| + c_4,$$
with $c_4 := \sup_{\varphi \in K} f^* (\varphi)$.

4. The statements follows from Proposition 2.

5. By contradiction, if $\varphi \in \text{int} K \cap \partial f (\lambda M)$ for some $\lambda$, it follows that $\varphi \in \partial f (\lambda M)$ as a consequence of the previous item. If $\lambda$ is large enough, the last conclusion is a contradiction because $f$ is superlinear at infinity. \hfill \Box

Let $w : \mathcal{V} \to \mathbb{R}$ be convex and l.s.c. For $t > 0$ and $M \in \mathcal{V}$, define

$$W_t (M) := tw \left( \frac{M}{t} \right).$$

Evidently, $W_1 (M) = w (M)$. The function $W (t, M) := W_t (M)$ is a convex, l.s.c. function on $\mathbb{R}^+ \times \mathcal{V}$, which is also positively homogeneous of degree 1, i.e.

$$W (\lambda t, \lambda M) = \lambda W (t, M), \quad \forall \lambda \in \mathbb{R}^+, \ (t, M) \in \mathbb{R}^+ \times \mathcal{V}. \quad (3)$$

In particular, fixing $t$, the function $W_t$ is convex and l.s.c. on $\mathcal{V}$ and, for $t > 0$, $\partial W_t (M) = \partial w (M/t)$. The so-called \textit{recession function} of $W_t$, namely

$$W_0 (M) := \lim_{t \to 0^+} W_t (M),$$

is then well defined, convex, and such that $\text{dom} W_0 (M) = \mathcal{V}$, if $w$ has linear growth at infinity. Moreover, if both $W_1$ and $W_0$ are of class $C^1$, from $W_t (M) \to W_0 (M)$ we get $dW_t (M) \to dW_0 (M)$ for any $M \in \mathcal{V}$.

We state below a proposition holding in some assumptions listed variously above that we recall here. The ingredients are then (1) a convex and l.s.c. function $f : \mathcal{V} \to \mathbb{R} \cup \{+\infty\}$ with superlinear growth at infinity, such that $\text{dom} f^* = \mathcal{V}^*$, (2) a bounded, closed, convex subset $K$ of $\mathcal{V}^*$, containing the origin in its interior, (3) the function $w$ and $w^*$ defined previously, so that $w$ has linear growth at infinity.

**Proposition 4** If $w$ and $W_0$ are differentiable, then

$$\partial W_0 (M) = \{dW_0 (M)\} \subset \partial K, \quad \forall M \in \mathcal{V}.$$ 

**Proof** Trivially, $dW_t (M) = dw (M/t) \in K$, so that we get the inclusion $\partial W_t (M) \subset K$ for $t > 0$. Since $W_t (M) \to W_0 (M)$ pointwise, and $W_t$ and $W_0$ are convex and of class $C^1$, we have $dW_t (M) \to dW_0 (M)$. Therefore, $dW_0 (M) \in K$ since $dW_t (M) \in K$ and $K$ is closed. If $dW_t (M) \in \text{int} (K)$, then, since in the interior of $K$ we have $dW_t (M) = dw (M/t) = df (M/t)$, we would have $df (M/t) \in \text{int} (K)$ which is impossible for $t$ small enough, thanks to item 5 of Proposition 3. By evaluating then the limit as $t \to 0^+$, we then get $dW_0 (M) \notin \text{int} (K)$, so it belongs to the boundary of $K$. \hfill \Box
6 Transplacements with constrained stress

Constraints on the possible values of the stress appear in standard schemes pursuing a description of elastic-perfectly-plastic or elastic-brittle behavior of continuous bodies. A constraint of this type corresponds to what is called admissibility criterion for the stress to be sustained point by point by the material under scrutiny. The number of available criteria in the current literature is large. Each criterion refers to certain aspects of the material behavior, so the choice of a specific one restricts the class of materials or situations that we can analyze profitably.

In small strain regime (namely when $|Du| \ll 1$), requirements like maximum dissipation principle (which is a natural tool also in presence of hardening, an aspect not treated here) or Drucker’s postulate, all with thermodynamic origin, imply convexity for the admissibility region in the stress space when we describe elastic-plastic behavior. If we select the admissibility region in the strain space, its convexity is implied by the acceptance of Il’yushin’s postulate. So, in small strain regime, convexity of the energy and the analogous property for the stress admissibility region allow a natural and synthetic description of the related mechanics in terms of convex duality (see, e.g., [2], [3], [7], [12], [13], [16], [19], [26], [27]).

The setting does not translate entirely in large strain regime where one admits Kröner-Lee decomposition of the deformation gradient, $F$, into elastic, $F^e$, and plastic, $F^p$, components, namely $F = F^e F^p$. Such a decomposition is based on a picture of elastic-plastic processes in which we imagine first to change irreversibly the inner structure of the material from place to place, without any specific link among different places, except we assure continuity of the matter, then we simply crowd and/or shear the mutated material elements. The decomposition itself is attributed to the deformation gradient only, not to the transplacement map, so that we have $\text{curl } F = 0$ (in case $x \mapsto F$ be differentiable) but $\text{curl } F^p$ and $\text{curl } F^e$ do not vanish unless at least one of them is the identity. So, $F^p$ individualize just locally the so-called intermediate configuration, which is not known globally, because we do not recognize and extract plastic and elastic components of the transplacement. In this context, Drucker’s postulate has no counterpart. In contrast Il’yushin’s one admits a finite strain version. It implies convexity of the region containing admissible deformation gradients $F$. As a consequence, it is possible to prove (see the proof in [24]) just the convexity of an admissibility set in the space containing the deviatoric part of the (point-by-point) pull-back in the local intermediate configuration of the fully covariant version of the Kirchhoff stress, which is the standard Cauchy stress multiplied by $\det F$. Always in this setting, the maximum dissipation principle applies but it is useful to find just (associated) flow rules, as in small strain regime, not properly the convexity of the admissibility region of the stress (in contrast with the small strain regime). The convexity of such a region is then often just postulated.
However, if we do not have a look to evolutionary aspects, we recognize that a stress admissibility criterion individualizes just elastic and critical states, nothing more. We know, in fact, just that we are in an elastic state when we are in the interior of the admissibility region (so the region itself is connected to assure path accessibility of states, one another). We reach a critical state when we are at the boundary of it. No matter about we are in an elastic-plastic or elastic-brittle process when we depart from the critical state. For the post-critical behavior, additional tools have to be introduced.

Here, in contrast with common treatments, we assign a generic admissibility convex criterion in terms of the stress form only. We generate, then, indirectly an admissibility region for the first Piola-Kirchhoff stress, thanks to the left-action of the linear operator $\frac{dM}{dF}(F)$, with a counterpart in terms of Cauchy stress, obtained by the use of Nanson formula.

Our approach extends schemes for elastic-plastic materials in small strain regime with linear elastic phase, based for example on Huber-von Mises-Hencky criteria or else (see [1], [2], [3], [7], [12], [14], [26], [27]). Essentially, we analyze the problem of selecting the deformation satisfying the stress constraint with less stored energy. To this aim, since the energy density can be considered a convex function of $M$, we can exploit in large strain setting properties of convex duality. So, we take the elastic energy density as a function defined over $V$ by extension from $V_+$, rather than on $\Sigma_{1,4}$. We do not consider then, the component 1. The basic ingredients of our analysis are then the energy density $e : V \rightarrow \mathbb{R}$ and the stress constraint, which is given assigning a convex set $K \subset V^*$ with $0 \in \text{int}\, K$. We then define $w^* (\omega) := e^* (\omega) + 1_K (\omega)$, with $\omega \in V^*$ and compute $w : V \rightarrow \mathbb{R}$, given by $w = w^{**}$.

By Proposition 4 it follows that $w$ (we can call it stored energy density) grows linearly at infinity so it is possible to have strain concentration (localization). Two steps are in general necessary for determining existence of minimizers of energies by direct methods in calculus of variations: (i) the extension of the class of competitors to some topological space in such a way that energy bounded level sets are compact, and (ii) a companion re-definition of the functional under scrutiny over such an extended class as a lower semicontinuous function – this is the so-called relaxed energy. Once the first step has been completed, the need of enlarging the functional setting to non-smooth functions appears often. It is hard to compute explicitly the relaxed energy which is in general not know even in traditional settings such as traditional non-linear elasticity of simple bodies. So, heuristic choices for the two steps above are in a sense necessary.

For the first one, and with reference to the problem we have at hands here, subsets of the class of Cartesian currents, introduced and discussed in [10] and [11], are appropriate spaces of competitors. Within their setting we analyze below the existence of minimizers for the energy with constrained stresses in case of Dirichlet boundary conditions. To go into details, some functional notions need to be called upon.
First we remind that a $k$-current $T$ on $\mathbb{R}^n$, $k \leq n$, is a linear continuous operator on the space of $k$-forms $\omega$ on $\mathbb{R}^n$ with $C^\infty$ coefficients. As pointed out previously, $T$ has the meaning of inner power in the setting that we discuss in these pages.

We call **boundary** of $T$, indicating it by $\partial T$, a $(k - 1)$-current on $\mathbb{R}^n$ defined by duality by the exterior differential operator $d$ on $(k - 1)$-forms, namely $\partial T (\eta) := T (d\eta)$ for any $(k - 1)$-form $\eta$ on $\mathbb{R}^n$. The **total variation** of a current $T$ is then defined as the nonnegative measure

$$
\|T\| (f) := \sup \{ T (\eta) \mid |\eta (z)| \leq f (z) \}, \ \eta \in C^0_c (\mathbb{R}^n),
$$

and we call **mass** of $T$ the value of the total variation over 1, and we indicate it by $M(T)$, namely

$$
M (T) := \|T\| (1).
$$

We affirm that a sequence of $k$-currents $\{T_i\}$ weakly converges to a current $T$, and we write $T_i \rightharpoonup T$, if $T_i (\omega) \to T (\omega)$ for every $k$-form $\omega$. Observe that

$$
M (T) \leq \liminf_{i \to \infty} M (T_i),
$$

if $T_i \to T$.

With $H^k$ the $k$-dimensional Hausdorff measure over $\mathbb{R}^n$, consider a $k$-dimensional $H^k$-rectifiable subset $T$ of $\mathbb{R}^n$, and denote by $\overline{T} (x)$ the unit tangent $3$-vector orienting it. Over $T$, define then a nonnegative, integer-valued, $H^k$-summable map $\theta$. All these ingredients allow us to define, with respect to the triple $(T, \theta, \overline{T})$, what is called $k$-**dimensional integer rectifiable current** – write for it $T = \tau (T, \theta, \overline{T})$ just to remind the reference to the triple above – by

$$
T (\omega) := \int_T \langle \omega (z), \overline{T} (z) \rangle \theta (z) H^k (dz).
$$

Its total variation $\|T\|$ is then given by $\|T\| := \theta H^k$, i.e.,

$$
\|T\| (f) = \int_T f (z) \theta (z) H^k (dz) \ \forall f \in C^\infty_c (\mathbb{R}^n),
$$

so that the mass $M$ of $T$ is

$$
M (T) = \int_T \theta (z) dH^k (z).
$$

Closure and compactness of this class of currents are assured by the fundamental Federer-Fleming theorem.

**Theorem 1 (Federer-Fleming)** Let $\{T_i\}$ be a sequence of $k$-dimensional integer rectifiable currents such that $\sup_i \{ M (T_i) + M (\partial T_i) \} < \infty$. Then there exists a $k$-dimensional integer rectifiable current $T$ and a subsequence $\{T_{i_k}\}$ of $\{T_i\}$ such that $T_{i_k} \to T$. Moreover, $\partial T$ is also a $(k - 1)$-dimensional integer rectifiable current.
For a 3–dimensional integer rectifiable current $T = \tau(T, \theta, \overrightarrow{T})$ in $\mathbb{R}^3 \times \overline{\mathbb{R}}^3$, the unit 3–vector $\overrightarrow{T} \in A_3(\mathbb{R}^3 \times \overline{\mathbb{R}}^3)$ writes

$$\overrightarrow{T}(x, y) = (\zeta, F, A, a)(x, y)$$

with the constraint

$$\zeta^2 + |F|^2 + |A|^2 + a^2 = 1.$$

Consider two bounded, connected, open sets, $B \subset \mathbb{R}^3$ and $\overline{B} \subset \overline{\mathbb{R}}^3$, with $\mathcal{L}^3(\partial B) = \mathcal{L}^3(\partial \overline{B}) = 0$ ($\mathcal{L}^3$ is Lebesgue volume measure). In the Cartesian product $\mathbb{R}^3 \times \overline{\mathbb{R}}^3$ indicate also by $\pi : \mathbb{R}^3 \times \overline{\mathbb{R}}^3 \to \mathbb{R}^3$ the orthogonal projection onto $\mathbb{R}^3$.

**Definition 1** For $M$ a 3–dimensional $H^3$–rectifiable subset of $\mathbb{R}^3 \times \overline{\mathbb{R}}^3$, a 3–dimensional integer rectifiable current $T := (\overrightarrow{T}, \theta, \overrightarrow{T})$ is said to be a weak diffeomorphism in the class $\text{dif}(B, \overline{B})$ if $T = (\zeta, F, A, a)$ is characterized by $\zeta \geq 0$ and $a \geq 0$, for $H^3$ a.e. $z \in T$

$$\int_{T \cap \pi^{-1}(x)} \zeta(z) \theta(z) \, dH^k(z) = 1 \quad \text{for } \mathcal{L}^3 \text{ a.e. } x \in B,$$

$$\int_{T \cap \overline{\pi}^{-1}(y)} a(z) \theta(z) \, dH^k(z) = 1 \quad \text{for } \mathcal{L}^3 \text{ a.e. } y \in \overline{B},$$

and $\text{spt}(T) \subset B \times \overline{B}$ and $\text{spt}(\partial T) \subset \partial B \times \partial \overline{B}$.

The last requirement, namely $\text{spt}(\partial T) \subset \partial B \times \partial \overline{B}$, is tantamount to impose that $\partial T = 0$ in $B \times \overline{B}$. Such a condition excludes maps describing transplacements with formation of voids or discontinuities like the ones associated with the formation of fractures (see also related discussions in [9]).

The connection of the notions above with the mechanics discussed in previous sections is evident. In fact, once $B$ is the reference place of a body, we can take $\overline{B}$ to be coincident with the actual placement $B_a$, obtained through a transplacement. As a consequence, $B$ and $B_a$ can be viewed as the projections of the graph $G_u$ of $u$ connecting one another. $G_u$ is a subset of the Cartesian product $\mathbb{R}^3 \times \overline{\mathbb{R}}^3$.

If $u$ is a $C^1$ orientation preserving diffeomorphism mapping $\partial B$ into $\partial \overline{B}$, integration over the graph $G_u$ is a $k$–dimensional integer rectifiable current $G_u = \tau(G_u, \theta, \overrightarrow{G_u})$ in $\text{dif}(B, \overline{B})$. In particular, we have

$$\overrightarrow{G_u} := \frac{M(Du(x))}{|M(Du(x))|}$$

and

$$\pi_{\#}(H^3) = |M(Du(x))| \, dx,$$
where \( \pi_\# \) is the counterpart in terms of measures of \( \pi \). Then, we write

\[
G_u(\omega) = \int_B \langle \omega(x,u(x)), M(Du(x)) \rangle \, dx
\]

for every smooth 3–form with compact support in \( \mathbb{R}^3 \times \tilde{\mathbb{R}}^3 \).

The remark allows us to interpret the current associated with a bijective orientation preserving differentiable map as an internal work in the extended sense suggested by the natural expression of the mechanics of simple bodies in terms of forms that we have developed here.

In going back to the class of weak diffeomorphisms, however, we notice that it includes degenerate diffeomorphisms. Set \( \mathcal{B} = \tilde{\mathcal{B}} \) and consider a point \( x_0 \in \partial \mathcal{B} \). The integration over the 3–dimensional surface \( (\mathcal{B} \times \{x_0\}) \cup \left(\{x_0\} \times \tilde{\mathcal{B}}\right) \) belongs, in fact, to \( \text{dif}(\mathcal{B}, \tilde{\mathcal{B}}) \).

Since the unit tangent vector \( \vec{T} = (\zeta, F, A, a) \) that orients the integration domain of a generic weak diffeomorphism (the concept, in fact, is not strictly associated with a transplacement) may possibly have \( \zeta = 0 \) and/or \( a = 0 \), we need to extend the stored energy density to the whole \( A_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3) \).

To this aim we use the extension \( W: \mathbb{R}^+ \times \mathcal{V} \longrightarrow \mathbb{R} \) of the stored energy function \( w: \mathcal{V} \longrightarrow \mathbb{R} \), as defined in previous sections. The space \( \mathbb{R}^+ \times \mathcal{V} \), we remind, is defined by

\[
\mathbb{R}^+ \times \mathcal{V} := \left\{ M \in A_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3) \mid M = (\zeta, M), \zeta \geq 0 \right\}.
\]

And we put, as above,

\[
W(\zeta, M) := \begin{cases} 
W_\zeta(M) := \zeta w \left( \frac{M}{\zeta} \right), & \text{if } \zeta > 0, \\
W_0(M) & \text{as } \zeta \longrightarrow 0^+, 
\end{cases}
\]

with \( W_0(M) \) the recession function of \( w(M) \). We call \( W(\zeta, M) \) extended stored energy density. Then, for \( T = \tau(T, \theta, \vec{T}) \in \text{dif}(\mathcal{B}, \tilde{\mathcal{B}}) \), the extended global energy is given by

\[
\mathcal{E}(T) := \int_T W(\vec{T}(z)) d\|T\| (z) = \int_T W(\vec{T}(z)) \theta(z) d\mathcal{H}^3(z).
\]

Let us identify as before \( \tilde{\mathcal{B}} \) with the current shape \( \mathcal{B}_a \) of the body under scrutiny.

**Theorem 2** Let \( \Gamma \) be a two-dimensional current of finite mass and zero boundary on \( \partial \mathcal{B} \times \partial \mathcal{B}_a \). Assume that the class

\[
\mathcal{A} := \{ T \in \text{dif}(\mathcal{B}, \mathcal{B}_a) \mid \partial T = \Gamma \}
\]

is not empty. In this class the functional \( \mathcal{E}(T) \) attains a minimum.
Proof Define \( \lambda := \inf_{T \in \mathcal{A}} \mathcal{E}(T) \). Let \( \{T_k\} \subset \mathcal{A} \), with \( T_k = \tau(T_k, \theta_k, \overline{T}_k) \), be a minimizing sequence. Then \( \mathbf{M}(\partial T_k) = \mathbf{M}(T) < \infty \) for every \( k \), and, by (3),

\[
\mathbf{M}(T_k) = \int_{T_k} \theta_k(z) d\mathcal{H}^k(z) \leq \int_T W(\overline{T}_k(z)) \theta_k(z) d\mathcal{H}^k(z) = \mathcal{E}(T)
\]

Hence the masses of all \( T_k \) are equibounded. Federer-Fleming closure-compactness theorem then applies. So, there exists a 3-dimensional integer rectifiable current \( T = \tau(T, \theta, \overline{T}) \), with \( \overline{T} = (\zeta, F, A, a) \), and a subsequence \( \{T_{k_n}\} \) of \( \{T_k\} \) such that \( T_{k_n} \rightarrow T \). Since \( W \) is lower semicontinuous

\[
\mathcal{E}(T) \leq \liminf_{k \to \infty} \mathcal{E}(T_k) = \lambda.
\]

Moreover, since weak convergence implies \( \zeta \geq 0, a \geq 0 \), and the validity of the equalities in Definition 1 for \( T \), we can conclude that \( T \in \mathcal{A} \). \( \square \)

The theorem assures the existence of minimizers, that are weak diffeomorphisms, for a boundary value problem where the datum \( \Gamma \) is expressed by the boundary of a weak diffeomorphism, say \( \gamma \), namely \( \Gamma = \partial\gamma \) (strong anchoring condition in the terminology of [11]), i.e. it is assigned in terms of extended internal work. The result is then reconciled with the standard view on continuum mechanics of simple bodies (about it see [11]). It also assures us that minimizers are weak diffeomorphisms with zero boundary in \( \mathcal{B} \times \mathcal{B}_a \) as long as the stress constraint is satisfied, so even critical states are described by weak diffeomorphisms, so they do not display fractures or holes such as the ones due to cavitation in liquids.

7 Stress form associated with a deformation

Once we have determined the existence of minimizers, we find natural to check the form of the stress associated with them, above all the interest is on the reactive part of the stress itself, the one determined by the constraint. To this aim, let us remind first some steps of the procedure followed so far.

Once we have the energy \( \tilde{\mathcal{E}}(Du) \), a polyconvex function of \( Du \), we evaluate its conjugate counterpart \( e^*(\tilde{\mathcal{E}}) \), and we add the indicator function of the admissibility set in the space of \( \mathcal{W}_a \), constructing this way \( \mathcal{W}_a \). From it we derive \( w \) and construct \( W : \mathbb{R}^+ \times \mathcal{V} \rightarrow \mathbb{R} \), the extended stored energy defined in (4).

Assume that the components of \( W \), namely \( W_\zeta \) and \( W_0 \), defined previously, be differentiable functions. Hence, for the corresponding subdifferential we get

\[
\partial W_\zeta(M) = \zeta \partial w \left( \frac{M}{\zeta} \right) \subset K, \quad \partial W_0(M) \subset \partial K.
\]
Let $T = \tau(T, \theta, \overline{T}) \in \text{dif}(\overline{B}, \overline{B})$ be a weak diffeomorphism such that

$$\mathcal{E}(T) := \int_T W(\overline{T}(z)) \|T\| (dz) < \infty,$$

where $\|T\| := \theta H^3$ denotes the mass measure of $T$, with density $\theta$. To every unit 3-vector $\overline{T} \in \mathbb{R}^+ \times \mathcal{V}$, with $\overline{T} = (\zeta, \overline{T}_s)$, $\overline{T}_s \in \mathcal{V}$, we associate the corresponding stress form taking values on $\mathcal{V}^*$, namely

$$\varpi(\overline{T}) := dW(\zeta, \overline{T}_s).$$

Therefore, since $T = \tau(T, \theta, \overline{T}) \in \text{dif}(\overline{B}, \overline{B})$ contains (in the generalized sense described so far) the elements necessary to construct standard strain measures, the product $\overline{T}(z) \, d\|T\|(z)$ can be considered as a generalized strain measure, with a corresponding stress measure given by

$$S := \varpi(\overline{T}) \|T\|(z).$$

$S$ is a vector-valued measure with values in $\mathcal{V}^*$, with bounded coefficients with respect to $\|T\|$.

Once again, the traditional view is recovered when we remind that for every diffeomorphism $T = (T, \overline{T})$, one can uniquely associate with it a $BV(\mathbb{R}, \mathbb{R}^3)$ map $u$ with some peculiar properties (see [10] and [11] for further details). Specifically, with $Du = Du^{(a)} + Du^{(s)}$ the decomposition of the gradient measure $Du$ into absolutely continuous and singular components,

$$\left| M \left(Du^{(a)}\right) \right| := \sqrt{1 + \left|M \left(Du^{(a)}\right)\right|^2},$$

with

$$M \left(Du^{(a)}\right) := \left(Du^{(a)}, \text{cof} Du^{(a)}, \text{det} Du^{(a)}\right),$$

is summable. Moreover, by denoting by $G_u$ the integration over the graph of $u$ over $\mathcal{B}_+$, which is the set of Lebesgue points of $u$ and $Du^{(a)}$, we have

$$\mathcal{T}_+ = G_u := (T_+, 1, G_u),$$

where

$$T_+ = \{z = (x, u(x)) \mid x \in \mathcal{B}_+\} \quad H^3 - a.e.,$$

$$G_u = \overline{T} (x, u(x)) = \frac{1}{|M \left(Du^{(a)}(x)\right)|} \left(1, M \left(Du^{(a)}(x)\right)\right).$$

Moreover, with $\pi : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathcal{B}$ the projection onto the reference configuration and $\pi_\#$ its counterpart in terms of measures, we have also

$$\pi_\# \|T\| \, T_+ = \left| M \left(Du^{(a)}(x)\right) \right| dx.$$
and
\[
\begin{align*}
\pi_{\#} \left( \varpi(\overline{T}) \|T\| \right) &= \varpi(x, u(x)) \left| M \left( Du^{(a)}(x) \right) \right| dx = \\
&= dW_1(T_0) dx = dw \left( M \left( Du^{(a)}(x) \right) \right) dx.
\end{align*}
\]

In other words, the restriction of \( u \) to \( B_+ \), namely \( u : B_+ \rightarrow \mathbb{R}^3 \), represents the `part' of the transplacement endowed with the common properties, suggested by physical evidence, that we impose in starting the construction of standard continuum mechanics. The complementary set \( B\setminus B_+ \) has zero measure.

The global stored energy on \( B_+ \), \( \mathcal{E}(T) \), is given by
\[
\int_{B_+} W(\overline{T}(z)) \|T\|(dz) = \int_{B_+} w \left( M \left( Du^{(a)}(x) \right) \right) dx. \tag{5}
\]

The projection of the corresponding stress form over the reference configuration reveals that the first Piola-Kirchhoff stress is well defined just on \( B_+ \) and is given by the map
\[
F \mapsto \frac{\partial w}{\partial M} \left( M \left( Du^{(a)}(x) \right) \right) \frac{dM}{dF} \left( Du^{(a)}(x) \right).
\]

8 Weak balances involving the stress form

Once again, let us identify the set \( \tilde{B} \) in \( \mathbb{R} \), involved in the definition of weak diffeomorphism, with the current shape \( B_+ \) of the body under scrutiny. Also, when a weak diffeomorphism \( T := \tau(T, \theta, \overline{T}) \) appears in the following lines, it is presumed to be a fellow of \( \mathcal{A} \), the class appearing in the existence theorem.

For every \( h \in C^\infty_c(\mathcal{B}, \mathbb{R}^3) \), \( k \in C^\infty_c(B_+, \mathbb{R}^3) \), and \( \varepsilon \in \mathbb{R} \), set
\[
\rho(z) := (h(x), k(y)),
\]
with \( z = (x, y), x \in \mathcal{B}, y \in B_+ \), and define the map
\[
\phi_\varepsilon : B \times B_+ \rightarrow \mathbb{R}^3 \times \mathbb{R}^3
\]
by
\[
\phi_\varepsilon(z) := z + \varepsilon \rho(z).
\]
For \( |\varepsilon| \) small enough, the two components of \( \phi_\varepsilon \), namely \( x \mapsto x + \varepsilon h(x) \) and \( y \mapsto y + \varepsilon k(y) \), are orientation preserving diffeomorphisms of both the reference and current places \( \mathcal{B} \) and \( B_+ \), respectively. So, \( \phi_\varepsilon \) is univalent on \( T \). Moreover, the current \( \phi_\varepsilon\# T \), the push forward of \( T \) along \( \phi_\varepsilon \# \), belongs to \( \mathcal{A} \).

Since the integrand \( W \) in (5) is positively homogeneous of degree 1 – it is a parametric integrand, indeed – the area formula yields
\[
\mathcal{E}(\phi_\varepsilon\# T) = \int \overline{W(\phi_\varepsilon\# T)} \ d\|\phi_\varepsilon\# T\| = \int_T W(\xi_\varepsilon(z)) \ d\|T\|(z),
\]
where \( \xi_z (z) := A_3 (D \phi_z (z)) \overrightarrow{T} (z) \), i.e.

\[
\xi_z (z) := D \phi_z (z) v_1 \wedge D \phi_z (z) v_2 \wedge D \phi_z (z) v_3.
\]

if we express \( \overrightarrow{T} (z) \) as \( v_1 \wedge v_2 \wedge v_3 \). Moreover, since \( D \phi_z (z) := I + \varepsilon D \rho (z) \)
I the identity in \( M_{6 \times 6} \), we compute

\[
\xi_z (z) = (I + \varepsilon D \rho (z)) v_1 \wedge (I + \varepsilon D \rho (z)) v_2 \wedge (I + \varepsilon D \rho (z)) v_3
\]

then, for every \( R \in M_{6 \times 6} \), \( L(R) \) denotes the linear operator

\[
L (R) : A_3 (\mathbb{R}^3, \mathbb{R}^3) \rightarrow A_3 (\mathbb{R}^3, \mathbb{R}^3)
\]

with values

\[
L (R) t := Rv_1 \wedge v_2 \wedge v_3 + v_1 \wedge Rv_2 \wedge v_3 + v_1 \wedge v_2 \wedge Rv_3
\]

if \( t = v_1 \wedge v_2 \wedge v_3 \in A_3 (\mathbb{R}^3, \mathbb{R}^3) \). So, we conclude that the map \( \varepsilon \mapsto \xi_z \) is differentiable at \( \varepsilon = 0 \) and

\[
\frac{\partial \xi_z (z)}{\partial \varepsilon} |_{\varepsilon = 0} = L (D \rho (z)) \overrightarrow{T} (z).
\]

As a special case, consider now variations only in the actual shape of the simple body under scrutiny, which is tantamount to set \( h (x) = 0 \) so that \( \phi_z (z) := (x, y + \varepsilon k (y)) \). So, the components of the six-dimensional vector \( (I + \varepsilon D (0, k (y))) v \), with \( v \) any vector of the three-plet \( v_1, v_2, v_3 \), in the reference place \( \mathcal{B} \), are independent of \( \varepsilon \). As a consequence, the component of \( \xi_z (z) \) proportional to the element \( e_1 \wedge e_2 \wedge e_3 \), which is constructed with basis vectors \( e_1, e_2, e_3 \) in \( \mathbb{R}^3 \), so the ones referred to the reference place, is also independent of \( \varepsilon \). Hence, the \( e_1 \wedge e_2 \wedge e_3 \) component of \( L (D \rho (z)) \overrightarrow{T} (z) \) is zero, and the following proposition holds.

**Proposition 5** Assume that the extended stored energy density \( W(t, M) : \mathbb{R}^+ \times \mathcal{V} \rightarrow \mathbb{R} \) is differentiable in \( M \) for all \( t \geq 0 \), which is tantamount to require that the stored energy \( w(M) \) and the relevant recession function are differentiable. Then

\[
\delta E (T, (0, k)) = \int_T (0, \omega (\overrightarrow{T} (z))) L (D (0, k (y))) \overrightarrow{T} (x, y) d \| T \| (x, y).
\]

To be more explicit, take once again the three-plet \( e_1, e_2, e_3 \) as basis in \( \mathbb{R}^3 \), and write \( \hat{e}_1, \hat{e}_2, \hat{e}_3 \) for the one in \( \mathbb{R}^3 \). Consider also two \( 2 \)-multi-indices \( \alpha \) and \( \beta \) such that the sum of their lengths is \( 3 \), namely \( |\alpha| + |\beta| = 3 \). As a matter of notation, \( 0 \) will denote the empty multi-index, while \( 0 \) will be used for the full multi-index, namely \( 0 = \{1, 2, 3\} \), so that \( e_0 \wedge \hat{e}_0 = e_1 \wedge \hat{e}_2 \wedge \hat{e}_3 \) and \( e_0 \wedge \hat{e}_0 = \hat{e}_1 \wedge \hat{e}_2 \wedge \hat{e}_3 \). This way \( \{e_\alpha \wedge \hat{e}_\beta\}_{|\alpha| + |\beta| = 3} \) is a basis for \( A_3 (\mathbb{R}^3, \mathbb{R}^3) \) which can be perhaps more explicit written as \( A_3 (\mathbb{R}^3 \times \mathbb{R}^3) \).
Denote by $W_{e_\alpha \wedge e_\beta}(t)$ the derivative of $W$ at $t$ in the ‘direction’ $e_\alpha \wedge e_\beta$. The stress form at $t$ has components listed in the $1 \times 19$ matrix

$$[W_{e_\alpha \wedge e_\beta}(t)], \quad |\alpha| + |\beta| = 3, \quad |\beta| > 0.$$ 

Moreover, $L(D(0,k))\overline{T}$ has components listed in the $20 \times 1$ matrix

$$[L(D\rho)^{\alpha\beta}], \quad |\alpha| + |\beta| = 3.$$ 

It is also immediate to compute $L(D(0,k))^{00} = 0$, so that

$$\delta \mathcal{E}(T,(0,k)) = \int_T \sum_{|\alpha|+|\beta|=3 \atop |\beta|>0} W_{e_\alpha \wedge e_\beta}(\overline{T}) \langle (L(D(0,k))_{\overline{T}})^{\alpha\beta} \rangle d\|T\|(x,y).$$

Similarly, we may consider variations only in the reference place by setting $\rho(z) = (h(x),0)$. In this case, the components of the six-dimensional vector $(1 + \varepsilon D(h(x),y)) \nu$ referred to the actual place are independent of $\varepsilon$. Consequently, $L(D(h,0))\overline{T}$ is zero with respect to $e_1 \wedge e_2 \wedge e_3$, constructed with the basis vectors in the actual place. By assuming that the extended stored energy $W$ be differentiable on $\mathbb{R}^+ \times \mathcal{V}$, we conclude that

$$\delta \mathcal{E}(T,(0,k)) = \int_T \langle dW(\overline{T}), L(D(h,0)) \overline{T} \rangle d\|T\|(x,y),$$

or, more explicitly, by taking into account that $(L(D(h,0)) \overline{T})^{00} = 0$,

$$\delta \mathcal{E}(T,(h,0)) = \int_T \sum_{|\alpha|+|\beta|=3 \atop |\beta|>0} W_{e_\alpha \wedge e_\beta}(\overline{T}) \langle (L(D(h,0))_{\overline{T}})^{\alpha\beta} \rangle d\|T\|(x,y).$$

Both variations $\delta \mathcal{E}(T,(0,k))$ and $\delta \mathcal{E}(T,(h,0))$ are elliptic integrals for the partial derivatives of $W$ are homogeneous functions of degree zero and the map $\overline{T} \mapsto L(D\rho(z))\overline{T}$ is linear. Such an aspect is useful in explicit computations.

Assume that the set $T$ in $\mathbb{R}^3 \times \tilde{\mathbb{R}}^3$, called upon every time we have constructed a weak diffeomorphism $T = (T,1,\overline{T})$, can be parametrized by using another set, say $\mathcal{S}$, in $\mathbb{R}^3$ and a Lipschitz map $\psi: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \times \tilde{\mathbb{R}}^3$ such that $\psi(\mathcal{S}) = T$, so $\psi(s) = (x,y) \in T$, for any $s \in \mathcal{S}$, and $\psi|\mathcal{S}$ is injective. Moreover, assume that we have over $\mathcal{S}$ a $k$-dimensional integer rectifiable current $S = \tau(\mathcal{S},1,\overline{S})$ such that $T$ be the push-forward of $S$ along $\psi$, namely $T := \psi_#S$. In this case, by setting $\xi(s) := A_3(D\psi(s))\overline{S}$, the variations $\delta_{(0,k)}\mathcal{E}(T)$ and $\delta_{(h,0)}\mathcal{E}(T)$ rewrite as

$$\delta \mathcal{E}(T,(0,k)) = \int_{\mathcal{S}} \sum_{|\alpha|+|\beta|=3 \atop |\beta|>0} W_{e_\alpha \wedge e_\beta}(\xi) \langle (L(D(0,k))_{\xi})^{\alpha\beta} \rangle d\|S\|(s),$$

---

6 The symbol $\delta_{(h,0)}$ indicates that the variation is performed just by altering the actual place.
\[ \delta \mathcal{E} (T, (h, 0)) = \int_{S} \sum_{|\alpha|+|\beta|=3 \atop |\beta|>0} W_{\varepsilon_{\alpha} \varepsilon_{\beta}} (\xi) (L (D (h, 0))) \xi^\alpha \beta (s) \, d \|S\| (s) . \]

9 Examples

Example 1 (Smooth diffeomorphisms.) Let \( u : \mathcal{B} \longrightarrow \mathbb{R}^{3} \) be a smooth diffeomorphism. Take \( T = G_{u} \), with \( G_{u} \) the integration current over \( u \). Then, by taking into account (5),

\[ \mathcal{E} (T) = \int_{\mathcal{B}} w (M (Du (x))) \, dx = \int_{\mathcal{B}} \tilde{w} (Du (x)) \, dx \]

once we write \( \tilde{w} (F) := w (F, \text{cof} F, \text{det} F) \). As a consequence,

\[ \delta \mathcal{E} (T, (0, k)) = \int_{\mathcal{B}} \tilde{w}_{,F_{A}} (Du (x)) D_{A} u^{i} (x) D_{j} k^{i} (u (x)) \, dx , \]

with \( D \) the derivative with respect to \( y \). Capital indices refer to coordinates in \( \mathcal{B} \) while the lower-case ones are related with coordinates in \( u (\mathcal{B}) \). The equality \( \delta \mathcal{E} (T, (0, k)) = 0 \), for any \( k \in C_{1}^{1} (u (\mathcal{B}), \mathbb{R}^{3}) \) is a weak form of the standard balance of forces (bulk actions are not considered from the beginning of our analysis) written in terms of Kirchhoff stress, which is Cauchy stress when multiplied by \( \text{det} Du \). Moreover, we get

\[ \delta \mathcal{E} (T, (h, 0)) = \int_{\mathcal{B}} (\tilde{w} (Du (x)) \delta_{AB} - D_{A} u^{i} (x)) \tilde{w}_{,F_{B}} (Du (x)) D_{A} h^{B} (x) \, dx . \]

The equality \( \delta \mathcal{E} (T, (h, 0)) = 0 \) for any \( h \in C_{1}^{1} (\mathcal{B}, \mathbb{R}^{3}) \) is the balance of configurational actions in absence of body forces in conservative setting (see [11], pp. 264-268, for further analytical details).

The subsequent examples show that balance equations may involve stresses in the ‘plasticized’ region, although we have no inserted any additional thermodynamic ingredient related with hardening.

Example 2 (Blow up at a point in a ball.) We imagine here that the reference place \( \mathcal{B} \) coincides with the ball \( B_{2} \) of radius 2. We consider a smooth diffeomorphism \( u : \mathcal{B} \longrightarrow \mathbb{R}^{3} \), and we indicate by \( B_{u} \) the current shape \( u (\mathcal{B}) \), as usual. As parametrization set for the graph we take also the three-dimensional ball \( B_{2} \) and introduce a map \( \psi : B_{2} \longrightarrow \mathcal{B} \times \mathbb{R}^{3} \) defined by

\[
\psi (s) := \begin{cases} 
(0, u (2s)) & \text{if } |s| < \frac{1}{2}, \\
(2|s|^{-1} s, u \left( \frac{\pi}{2} \right)) & \text{if } \frac{1}{4} \leq |s| < 1, \\
(s, u (s)) & \text{if } 1 \leq |s| < 2.
\end{cases}
\]

By construction, \( \psi \) is injective and Lipschitz. By indicating by \( ||B_{2}|| \) the integration over \( B_{2} \), we find that \( T := \psi_{\#} ||B_{2}|| \) is a 3-dimensional integer rectifiable current on \( \mathbb{R}^{3} \times \mathbb{R}^{3} \), with multiplicity 1, which is, in fact, a weak
diffeomorphism which coincides with the graph of \( u \) at \( \partial B \times \partial B_u \). For this reason, we write
\[
\psi_\# \left[ [B_2] \right] = \tau \left( T, 1, \bar{T} \right).
\]
Moreover, \( \psi_\# \left[ [B_2 \setminus B_{1/2}] \right] \), with \( B_{1/2} \) the ball of radius \( \frac{1}{2} \) is the integration over the graph of the map \( u : B \setminus \{0\} \to \mathbb{R}^3 \) defined by
\[
u(x) := \begin{cases} \frac{u(x)}{|x|} & \text{if } |x| < 1, \\ u(x) & \text{if } 1 \leq |x| < 2, \end{cases}
\]
while \( \psi_\# \left[ [B_{1/2}] \right] \) is the integration over the 3–dimensional set \( \{0\} \times u(B_1) \), a subset of \( \mathbb{R}^3 \times \mathbb{R}^3 \) with tangent vector \( \xi \) at \( z = (0, y) \) given by
\[
\xi(z) = \tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3,
\]
for all \( y \in B_a \). Moreover, by setting \( R = D (0, k(y)) \) we get
\[
R \tilde{e}_j = \frac{3}{|j|} D_j k^1(y) \tilde{e}_1,
\]
while, if \( R = D (h(x), 0) \), we find \( R \tilde{e}_i = 0 \), so that
\[
L(D(0, k(y))) \xi = R \tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3 + \tilde{e}_1 \wedge R \tilde{e}_2 \wedge \tilde{e}_3 + \tilde{e}_1 \wedge \tilde{e}_2 \wedge R \tilde{e}_3
\]
\[
= \operatorname{div} k(y) \tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3,
\]
and
\[
L(D(h(x), 0)) \xi = 0.
\]
As a consequence, we find
\[
E(T) = \int_{B_a} w(Du(x)) \, dx + \frac{4}{3} \pi W(\tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3)
\]
where \( \frac{4}{3} \pi \) is the volume of the ball with unit radius, and
\[
\delta E(T, (0, k)) = \delta E(G_u, (0, k)) + W(\tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3 (\tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3) \int_{B_a} \operatorname{div} k(y) \, dy,
\]
and
\[
\delta E(T, (h, 0)) = \delta E(G_u, (h, 0)),
\]
for every \( k \in C^1_c(B_a, \mathbb{R}^3) \) and every \( k \in C^1_c(B, \mathbb{R}^3) \).

The analysis of the blow-up of a point in a disc in a two-dimensional space and the one of a line in a cylinder wrapped around it in three-dimensional setting require the same analyses developed in the lines above.
Example 3 (Two-dimensional slip.) Preliminary notations are the following ones: \( B_1 := ]-1, 1[ \subset \mathbb{R}, B_2 := ]-2, 2[ \subset \mathbb{R} \). In two dimensions, take as a reference place \( B \) the rectangle \( B_1 \times B_2 \). A point in \( B_1 \times B_2 \) is then \( x := (x_1, x_2) \). To parametrize the graph, define the map 
\[
\psi : B_1 \times B_2 \longrightarrow (B_1 \times B_2) \times \mathbb{R}^2
\]
by
\[
\psi(s_1, s_2) := \begin{cases} 
(s_1, s_2 + 1; s_1, s_2 + 1) & \text{if } -2 \leq s_2 < -1, \\
(s_1, 0; s_1 + \beta(s_2 + 1), 0) & \text{if } -1 \leq s_2 < 1, \\
(s_1, s_2 - 1; s_1 + \beta, s_2 - 1) & \text{if } 1 \leq s_2 < 2,
\end{cases}
\]
with \( \beta > 0 \). The map \( \psi \) is Lipschitz and injective onto its image. The integration over the image, namely \( \psi_\# [[B_1 \times B_2]] \) is a 2-dimensional integer rectifiable current on \( \mathbb{R}^2 \times \mathbb{R}^2 \) with multiplicity 1, which is, in fact, a weak diffeomorphism. Then, we write
\[
\psi_\# [[B_1 \times B_2]] = \tau(T, 1, \overline{T}).
\]
Moreover, \( \psi_\# [[B_1 \times (B_2 \setminus B_1)]] \) is the integration over the graph of the map \( u : B_1 \times (B_2 \setminus \{0\}) \longrightarrow \mathbb{R}^2 \) defined by
\[
u(x_1, x_2) := \begin{cases} 
(x_1 + \beta, x_2) & \text{if } x_2 > 0, \\
(x_1, x_2) & \text{if } x_2 < 0.
\end{cases}
\]
The current shape \( B_u \) is here once again \( u(B) \). Specifically, \( \psi_\# [[B_1 \times B_2]] \) is the integration over the 2-dimensional parallelogram \( S \) in \( \mathbb{R}^4 \) with vertices \((-1, 0, -1, 0), (1, 0, 1, 0), (-1 + \beta, 0, -1 + \beta, 0), (1 + \beta, 0, 1 + \beta, 0)\). The relevant tangent plane is generated by the vectors \((1, 0, 1, 0)\) and \((0, 0, \beta/2, 0)\), so that
\[
\xi(s) = (e_1 + \bar{e}_1) \wedge \frac{\beta}{2} \bar{e}_1 = \frac{\beta}{2} e_1 \wedge \bar{e}_1,
\]
with \( e_1 \) in the basis over \( B_1 \times B_2 \) and \( \bar{e}_1, \bar{e}_2 \) the vectors of the basis in \( \mathbb{R}^2 \). Moreover,
\[
D(0, k(y)) e_1 = 0, \quad D(0, k(y)) \bar{e}_1 = \sum_{i=1}^2 D_ik^i(y) \bar{e}_i,
\]
\[
D(h(x), 0) e_1 = \sum_{i=1}^2 D_ikh^i(x) e_i, \quad D(h(x), 0) \bar{e}_1 = 0.
\]
Therefore, we find
\[
L(D(0, k(y))) (e_1 \wedge \bar{e}_1) = \sum_{i=1}^2 D_ik^i(y) e_1 \wedge \bar{e}_i,
\]
\[
L(D(h(x), 0)) (e_1 \wedge \bar{e}_1) = \sum_{i=1}^2 D_ikh^i(x) e_i \wedge \bar{e}_1,
\]
so that
\[
\mathcal{E}(T) = \int_{B_1 \times B_2} w(Du(x_1, x_2)) \, dx_1 \, dx_2 + W(e_1 \wedge \tilde{e}_1) \mathcal{H}^3(\mathcal{S}),
\]
with
\[
\delta \mathcal{E}(T, (0, k)) = \delta_{(0, k)} \mathcal{E}(G_u) + \frac{2}{5} \sum_{i=1}^3 W_{e_i \wedge \tilde{e}_i} \int e_1 \wedge \tilde{e}_1 \int_{-1}^1 dx_1 \int_{x_1}^{x_1 + \beta} D_k^i(y_1, 0) \, dy_1,
\]
and
\[
\delta \mathcal{E}(T, (h, 0)) = \delta_{(h, 0)} \mathcal{E}(G_u) + \beta \frac{2}{3} \sum_{i=1}^3 W_{e_i \wedge \tilde{e}_i} \int e_1 \wedge \tilde{e}_1 \int_{1}^{h + \beta} (h^i(1, 0) - h^i(-1, 0)),
\]
for every $k \in C^1_c(B, \mathbb{R}^2)$ and every $k \in C^1(B, \mathbb{R})$.

**Example 4 (Three-dimensional slip.)** Once again take $B_1 := [-1, 1] \subset \mathbb{R}$ and $B_2 := [-2, 2] \subset \mathbb{R}$. We identify the reference place $B$ with the product $B_1 \times B_1 \times B_2$. A point $x$ in $B$ will be then individualized by the triple $(x_1, x_2, x_3)$ while the generic one, namely $y$, in the current place will be specified by the triple $(y_1, y_2, y_3)$. Define the map
\[
\psi : B_1 \times B_1 \times B_2 \rightarrow (B_1 \times B_1 \times B_2) \times \mathbb{R}^3
\]
by
\[
\psi(s_1, s_2, s_3) := \begin{cases} 
(s_1, s_2, s_3 + 1; s_1, s_2, s_3 + 1) & \text{if } -2 \leq s_3 < -1, \\
(s_1, s_2, 0; s_1 + \frac{\beta(s_3 + 1)}{2}, s_2, 0) & \text{if } -1 \leq s_3 < 1, \\
(s_1, s_2, s_3 - 1; s_1 + \beta, s_2, s_3 - 1) & \text{if } 1 \leq s_3 < 2,
\end{cases}
\]
with $\beta > 0$. The map $\psi$ is injective and Lipschitz onto its image. The integration over the image, namely $T := \psi_\# [[B_1 \times B_1 \times B_2]]$ is a 3-dimensional integer rectifiable current on $\mathbb{R}^3 \times \mathbb{R}^3$ with multiplicity 1, which is, once again, a weak diffeomorphism. Then, we write
\[
\psi_\# [[B_1 \times B_1 \times B_2]] = \tau(T, 1, T),
\]
Moreover,
\[
\psi_\# [[B_1 \times B_1 \times (B_2 \setminus B_1)]]
\]
is the integration over the graph of the map $u : B_1 \times B_1 \times (B_2 \setminus \{0\}) \rightarrow \mathbb{R}^3$
defined by
\[
u(x_1, x_2, x_3) := \begin{cases} 
(x_1 + \beta, x_2, x_3) & \text{if } x_3 > 0, \\
(x_1, x_2, x_3) & \text{if } x_3 < 0.
\end{cases}
\]
The current shape $B_u$ is once again $u(B)$. Specifically, $\psi_\# [[B_1 \times B_1 \times B_2]]$ is the integration over the 3-dimensional prism $\mathcal{S}$ in $\mathbb{R}^3 \times \mathbb{R}^3$ with vertices
\[
(-1, -1, 0, -1, -1, 0), (-1, 1, 0, -1, 1, 0), (1, -1, 0, 1, -1, 0), (1, 1, 0, 1, 1, 0),
\]
\[
(-1, -1, 0, -1 + \beta, -1, 0), (-1, 1, 0, -1 + \beta, 1, 0), (1, -1 + \beta, 0, 1 + \beta, -1, 0),
\]
and
\[
(-1, -1, 1 + \beta, -1, 0), (-1, 1, 1 + \beta, -1, 0), (1, -1 + \beta, 1, 1, 0), (1, 1 + \beta, 0, 1 + \beta, -1, 0),
\]
and
\[
(-1, -1 + \beta, -1, 0), (-1, 1 + \beta, -1, 0), (1, -1 + \beta, 1, -1, 0), (1, 1 + \beta, 0, -1 + \beta, -1, 0).
\]
(1, 1, 0, 1 + β, 1, 0). The relevant tangent plane is generated by the vectors
(1, 0, 0, 1, 0, 0), (0, 1, 0, 1, 0, 0) and (0, 0, 0, β/2, 0, 0), so that

\[ \xi(s) = (e_1 + \tilde{e}_1) \wedge (e_2 + \tilde{e}_2) \wedge \frac{\beta}{2} e_1 = \]
\[ = \frac{\beta}{2} e_1 \wedge e_2 \wedge \tilde{e}_1 - \frac{\beta}{2} e_1 \wedge \tilde{e}_1 \wedge \tilde{e}_2, \]

with \( e_1, e_2 \) in the basis over \( B_1 \times B_1 \times B_2 \) and \( \tilde{e}_1, \tilde{e}_2 \) in the basis in \( \mathbb{R}^3 \).

Moreover, for every \( j = 1, 2, 3 \), we calculate

\[ D(0, k(y)) e_j = 0, \quad D(0, k(y)) \tilde{e}_j = \sum_{i=1}^{3} D_j k^i(y) \tilde{e}_i, \]
\[ D(h(x), 0) e_j = \sum_{i=1}^{3} D_j h^i(x) e_i, \quad D(h(x), 0) \tilde{e}_j = 0. \]

Therefore, we find

\[ L(D(0, k))(e_1 \wedge e_2 \wedge \tilde{e}_1) = \sum_{i=1}^{3} D_1 k^i(y) e_1 \wedge e_2 \wedge \tilde{e}_1, \]
\[ L(D(0, k))(e_1 \wedge \tilde{e}_1 \wedge \tilde{e}_2) = \sum_{i<j} (D_1 k^i D_2 k^j - D_1 k^j D_2 k^i) (e_1 \wedge \tilde{e}_1 \wedge \tilde{e}_j), \]
\[ L(D(h, 0))(e_1 \wedge e_2 \wedge \tilde{e}_1) = \sum_{i<j} (D_1 h^i D_2 h^j - D_1 h^j D_2 h^i) (e_1 \wedge e_j \wedge \tilde{e}_1), \]
\[ L(D(h, 0))(e_1 \wedge \tilde{e}_1 \wedge \tilde{e}_2) = \sum_{i=1}^{3} D_1 h^i(x) e_j \wedge \tilde{e}_1 \wedge \tilde{e}_2, \]

for every \( k \in C^1(B_0, \mathbb{R}^3) \) and every \( k \in C^1(B, \mathbb{R}^3) \), so that the energy \( E(T) \) and its variations can be explicitly evaluated. In Example 3 the 3-vector \( \xi \) orienting \( \psi_\# [[B_1 \times B_2]] \) includes just line deformations. In contrast, here, \( \xi \) which orienting \( \psi_\# [[B_1 \times B_1 \times B_2]] \) includes both line and surface deformations, the latter ones associated with \( (e_1 \wedge \tilde{e}_1 \wedge \tilde{e}_2) \).

In all previous examples a smooth diffeomorphism can be superimposed to the ones considered, breeding this way a process which would be analogous to what is described by Kröner-Lee decomposition of the deformation gradient in elasto-plasticity.

**Acknowledgements.** This work has been developed within the activities of the research group in ‘Theoretical Mechanics’ of the ‘Centro di Ricerca Matematica Ennio De Giorgi’ of the Scuola Normale Superiore in Pisa. The support of MIUR under the grant ‘Azioni integrate Italia-Spagna, 2009’ is acknowledged.
References