STATIONARY CONFIGURATIONS FOR THE AVERAGE DISTANCE FUNCTIONAL AND RELATED PROBLEMS

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ABSTRACT. For a functional defined on the class of closed one-dimensional connected subsets of $\mathbb{R}^n$ we consider the corresponding minimization problem and we give suitable first order necessary conditions of optimality. The cases studied here are the average distance functional arising in the mass transportation theory, and the energy related to an elliptic PDE.

1. INTRODUCTION

In this paper we consider functionals $\mathcal{F}(\Sigma)$ defined on the class of all closed connected subsets of $\mathbb{R}^n$ and the corresponding minimization problems

$$\min\left\{\mathcal{F}(\Sigma)\ :\ \Sigma \text{ closed connected subset of } \mathbb{R}^n\right\}. \tag{1}$$

Due to the fact that the class of closed connected sets has good compactness properties with respect to the Hausdorff convergence, mild coercivity assumptions on $\mathcal{F}$ give the existence of minimizers for problem (1). We are interested in finding "first order" necessary optimality conditions satisfied by the minimizers Σ of (1).

The case we consider is the *average distance functional*

$$\mathcal{F}(\Sigma) := \int_{\mathbb{R}^n} \operatorname{dist}(x,\Sigma)\, d\mu(x) + \lambda\mathcal{H}^1(\Sigma), \tag{2}$$

where μ is a given finite nonnegative Borel measure over $\mathbb{R}^n$ with compact support, and the penalization term $\lambda\mathcal{H}^1(\Sigma)$ with $\lambda > 0$ is added to give a suitable coercivity to $\mathcal{F}$ and to prevent minimizing sequences to spread over all the space. A simple and standard argument involving Blaschke and Gołąb theorems gives the existence of minimizers of $\mathcal{F}$. Of particular interest for us will be situations when μ is a uniform measure over some open set $\Omega \subset \mathbb{R}^n$, i.e. $\mu = \mathcal{L}^n \llcorner \Omega$.

The average distance term in (2) comes from mass transport theory and describes for instance the total transportation cost to move a mass μ of residents to a public transport network Σ. This last is the unknown of the problem and has to be designed in order to minimize $\mathcal{F}$, also taking into account the construction costs which here are taken as proportional to $\mathcal{H}^1(\Sigma)$. The minimization problem (1), as well as some qualitative properties of its minimizers, have been studied in several recent papers (see e.g. [4, 5, 6, 7, 10, 11, 12]) to which we refer the interested reader. Our goal is to find "first order" conditions of differential character satisfied by the minimizers of (2). Such conditions will open the way to define a natural notion of stationary (or critical) points of (2). The main difficulty, which is quite common in shape optimization problems, is that the domain of definition of this

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functional (i.e. the class of closed connected subsets of $\mathbb{R}^n$) does not possess any natural differentiable structure, and the usual "first variation" argument has to be intended in a suitable way.

In the last section we consider a similar case arising from the theory of elliptic equations:

$$\mathcal{F}(\Sigma) := \int_\Omega u_\Sigma(x) f(x)\, dx + \lambda \mathcal{H}^1(\Sigma), \tag{3}$$

where $\Omega \subset \mathbb{R}^2$ is a given bounded open subset, f is a given $L^2(\Omega)$ function, and u_Σ is the unique solution of the PDE

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \setminus \Sigma, \\ \quad u &= 0 \quad \text{on } \partial\Omega \cup \Sigma. \end{cases}$$

One has to remark that while a lot of properties are known for minimizers of the average distance functional (see [4, 6, 10, 12]), like partial regularity, absence of loops, topological properties (finite number of branching points, each of which is a regular tripod), no such property has been studied for minimizers of (3).

2. The Euler equation for the average distance functional

For a compact set $\Sigma \subset \mathbb{R}^n$ we denote by π^Σ the projection map to Σ (i.e. such that $\pi^\Sigma(x) \in \Sigma$ is one of the nearest point in Σ to $x \in \mathbb{R}^n$). This map is uniquely defined everywhere except the *ridge* set $\mathfrak{R}_\Sigma$, which is defined as the set of all $x \in \mathbb{R}^n$ for which the minimum distance to Σ is attained at more than one point. It is well known that $\mathfrak{R}_\Sigma$ is the set of non differentiability points of the distance function to Σ (that is, of the map $x \in \mathbb{R}^n \mapsto \operatorname{dist}(x, \Sigma)$), and since the latter map is semiconcave, this set is an $(\mathcal{H}^{n-1}, n-1)$-rectifiable Borel set [9, proposition 3.7].

We will denote by $B_r(x) \subset \mathbb{R}^n$ the open ball with radius $r > 0$ and center $x \in \mathbb{R}^n$. The line segment with endpoints A and B will be denoted by $\overline{AB}$, the arc of a curve with the same endpoints will be denoted by $\widetilde{AB}$ (usually in this paper we will deal with arcs of circle).

To be begin with, we estimate the ascending local slope of (2) defined by

$$|\mathcal{F}'|(\Sigma) := \limsup_{d_H(\Sigma',\Sigma)\to 0} \frac{(\mathcal{F}(\Sigma') - \mathcal{F}(\Sigma))^+}{d_H(\Sigma', \Sigma)},$$

where d_H stands for Hausdorff distance between sets. The following simple assertion is valid.

Proposition 2.1. *If $\mu(\mathfrak{R}_\Sigma) = 0$, one has $|\mathcal{F}'|(\Sigma) \geq \lambda$.*

Proof. Let $x \in \Sigma$ be such that $\mu((\pi^\Sigma)^{-1}(\{x\})) = 0$ (all but a countable number of points of Σ have this property). Let then $\Sigma_\varepsilon := \Sigma \cup I_\varepsilon$, where I_ε stands for the line segment of length $\varepsilon > 0$, with one of the endpoints x and such that $\pi^\Sigma(I_\varepsilon) = x$. Then $d_H(\Sigma_\varepsilon, \Sigma) = \varepsilon$ and $\mathcal{H}^1(\Sigma_\varepsilon) = \mathcal{H}^1(\Sigma) + \varepsilon$. On the other hand, denoting

$$G_\varepsilon := \{z \in \mathbb{R}^n \,:\, \operatorname{dist}(z, \Sigma) \geq \operatorname{dist}(z, I_\varepsilon)\},$$

we have that

$$\int_{\mathbb{R}^n} \operatorname{dist}(x, \Sigma)\, d\mu(x) \geq \int_{\mathbb{R}^n} \operatorname{dist}(x, \Sigma_\varepsilon)\, d\mu(x) \geq \int_{\mathbb{R}^n} \operatorname{dist}(x, \Sigma)\, d\mu(x) - \varepsilon\mu(G_\varepsilon).$$

Thus

$$|\mathcal{F}'|(\Sigma) \geq \limsup_{d_H(\Sigma_\varepsilon,\Sigma)\to 0} \frac{(\mathcal{F}(\Sigma_\varepsilon)-\mathcal{F}(\Sigma))^+}{d_H(\Sigma_\varepsilon,\Sigma)} \geq \lim_{\varepsilon\to 0^+} \frac{(\lambda\varepsilon - \varepsilon\mu(G_\varepsilon))^+}{\varepsilon},$$

and to conclude the proof it suffices to mind that $\mu(G_\varepsilon) = o(1)$, because $G_\varepsilon \searrow \{x\}$ as $\varepsilon \to 0^+$. ☐

The above proposition in fact means that for the functional (2) no set Σ (not even a minimizer) is stationary in the strong sense, i.e. is such that

$$\mathcal{F}(\Sigma') = \mathcal{F}(\Sigma) + o(d_H(\Sigma, \Sigma'))$$

as $\Sigma' \to \Sigma$ in Hausdorff distance. Therefore, in search for the natural notion of stationary points of $\mathcal{F}$ we have to restrict the set of admissible variations of Σ. For this purpose let $\phi_\varepsilon \colon \mathbb{R}^n \to \mathbb{R}^n$ be a one parameter group of diffeomorphisms satisfying

$$\phi_\varepsilon(x) = x + \varepsilon X(x) + o(\varepsilon), \tag{4}$$

as $\varepsilon \to 0$, where $X \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^n)$. We will write Euler equation for the functional (2) by considering admissible variations of the type $\Sigma_\varepsilon := \phi_\varepsilon(\Sigma)$.

We recall the notion of a generalized mean curvature from [3]. The generalized mean curvature H_Σ of a $(\mathcal{H}^k, k)$-rectifiable set $\Sigma \subset \mathbb{R}^n$ (or, in terms of [3], of the measure $\mathcal{H}^k \llcorner \Sigma$) is the vector-valued distribution defined by the relationship

$$\langle X, H_\Sigma\rangle := -\int_\Sigma \operatorname{div}^\Sigma X \, d\mathcal{H}^k$$

for all $X \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^n)$, where $\operatorname{div}^\Sigma$ stands for the tangential divergence operator (i.e. projection of the divergence to the approximate tangent space of Σ at $\mathcal{H}^k$-a.e. point of Σ). We have then the following result.

Theorem 2.2. *Let μ be a Borel measure such that $\mu(E) = 0$ whenever $\mathcal{H}^{n-1}(E) < +\infty$. Then for all $X \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^n)$ one has*

$$\begin{aligned} \frac{\partial}{\partial\varepsilon}\mathcal{F}(\Sigma_\varepsilon)\Big|_{\varepsilon=0} &= \int_{\mathbb{R}^n} \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x)-x}{|\pi^\Sigma(x)-x|}\right\rangle d\mu - \lambda\langle H_\Sigma, X\rangle \\ &= \int_{\mathbb{R}^n} \big\langle X(\pi^\Sigma(x)), \nabla \operatorname{dist}(x,\Sigma)\big\rangle \, d\mu - \lambda\langle H_\Sigma, X\rangle. \end{aligned} \tag{5}$$

In particular, if Σ is a minimizer of $\mathcal{F}$, then

$$\int_{\mathbb{R}^n} \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x)-x}{|\pi^\Sigma(x)-x|}\right\rangle d\mu = \lambda\langle H_\Sigma, X\rangle \tag{6}$$

for all $X \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^n)$.

Proof. First of all, we perform the variation for the first term. We adopt the method of calculation of the derivative of the distance function with respect to the variation of the set, used in [2, lemma 4.5]. Clearly, for $z := \phi_\varepsilon(\pi^\Sigma(x))$ one has

$$\begin{aligned} \operatorname{dist}(x,\Sigma) &= |\pi^\Sigma(x) - x|, \\ \operatorname{dist}(x,\Sigma_\varepsilon) &\leq |z - x|. \end{aligned}$$

From (4) we get, for $\varepsilon \to 0$,

$$
\begin{aligned}
|z-x|^2 &= \left\langle \pi^\Sigma(x) - x + \varepsilon X(\pi^\Sigma(x)), \pi^\Sigma(x) - x + \varepsilon X(\pi^\Sigma(x)) \right\rangle + o(\varepsilon) \\
&= |\pi^\Sigma(x) - x|^2 + 2 \left\langle \pi^\Sigma(x) - x, \varepsilon X(\pi^\Sigma(x)) \right\rangle + o(\varepsilon) \\
&= |\pi^\Sigma(x) - x|^2 \left(1 + 2 \left\langle \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|^2}, \varepsilon X(\pi^\Sigma(x)) \right\rangle + o(\varepsilon) \right).
\end{aligned}
$$

Then

$$
\begin{aligned}
\operatorname{dist}(x, \Sigma_\varepsilon) - \operatorname{dist}(x, \Sigma) &\leq |z - x| - |\pi^\Sigma(x) - x| \\
&= \varepsilon \left\langle \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|}, X(\pi^\Sigma(x)) \right\rangle + o(\varepsilon),
\end{aligned}
$$

and we deduce

$$
\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} (\operatorname{dist}(x, \Sigma_\varepsilon) - \operatorname{dist}(x, \Sigma)) \leq \left\langle \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|}, X(\pi^\Sigma(x)) \right\rangle. \tag{7}
$$

On the other hand, consider a sequence $\varepsilon_\nu \to 0^+$ for $\nu \to \infty$. The set of points $x \in \mathbb{R}^n$ for which both $\pi^\Sigma(x)$ and $\pi^{\Sigma_{\varepsilon_\nu}}(x)$ are singletons for any $\nu \in \mathbb{N}$ is of full measure μ in $\mathbb{R}^n$ (the complement is a countable union of ridge sets $\mathfrak{R}_{\Sigma_\nu}$ and $\mathfrak{R}_\Sigma$ which are all $(\mathcal{H}^{n-1}, n-1)$-rectifiable, hence μ-negligible). For all such x, since ϕ_ε is invertible for all sufficiently small ε, let $\zeta := \phi_{\varepsilon_\nu}^{-1}(\pi^{\Sigma_{\varepsilon_\nu}}(x))$, so that

$$
\begin{aligned}
\operatorname{dist}(x, \Sigma_{\varepsilon_\nu}) &= |\phi_{\varepsilon_\nu}(\zeta) - x|, \\
\operatorname{dist}(x, \Sigma) &\leq |\zeta - x|.
\end{aligned}
$$

Again we have

$$
\begin{aligned}
|\phi_{\varepsilon_\nu}(\zeta) - x| - |\zeta - x| &= |\zeta - x| \left(\sqrt{1 + 2 \left\langle \frac{\zeta - x}{|\zeta - x|^2}, \varepsilon_\nu X(\zeta) \right\rangle + o(\varepsilon_\nu)} - 1 \right) \\
&= \varepsilon_\nu \left\langle \frac{\zeta - x}{|\zeta - x|}, X(\zeta) \right\rangle + o(\varepsilon_\nu).
\end{aligned}
$$

Therefore,

$$
\operatorname{dist}(x, \Sigma_{\varepsilon_\nu}) - \operatorname{dist}(x, \Sigma) \geq \varepsilon_\nu \left\langle \frac{\zeta - x}{|\zeta - x|}, X(\zeta) \right\rangle + o(\varepsilon_\nu).
$$

Passing to the limit as $\nu \to \infty$, we get

$$
\left\langle \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|}, X(\pi^\Sigma(x)) \right\rangle \leq \liminf_{\nu \to \infty} \frac{1}{\varepsilon_\nu} \left(\operatorname{dist}(x, \Sigma_{\varepsilon_\nu}) - \operatorname{dist}(x, \Sigma) \right). \tag{8}
$$

Combining (7) with (8), we get for μ-a.e. $x \in \mathbb{R}^n$,

$$
\lim_{\nu \to \infty} \frac{1}{\varepsilon_\nu} (\operatorname{dist}(x, \Sigma_{\varepsilon_\nu}) - \operatorname{dist}(x, \Sigma)) = \left\langle \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|}, X(\pi^\Sigma(x)) \right\rangle,
$$

so that, by Lebesgue dominated convergence theorem,

$$
\lim_{\nu \to \infty} \frac{1}{\varepsilon_\nu} \int_\Omega (\operatorname{dist}(x, \Sigma_{\varepsilon_\nu}) - \operatorname{dist}(x, \Sigma))\, d\mu = \int_\Omega \left\langle \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|}, X(\pi^\Sigma(x)) \right\rangle d\mu.
$$

Since the sequence ε_ν is arbitrary, one has

$$
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_\Omega (\operatorname{dist}(x, \Sigma_\varepsilon) - \operatorname{dist}(x, \Sigma))\, d\mu = \int_\Omega \left\langle \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|}, X(\pi^\Sigma(x)) \right\rangle d\mu.
$$

Finally, we observe that according to the theorem 7.31 of [1] one has

$$\frac{d}{d\varepsilon}\mathcal{H}^k(\Sigma_\varepsilon)\bigg|_{\varepsilon=0} = \int_\Sigma \operatorname{div}^\Sigma X \, d\mathcal{H}^k = -\langle H_\Sigma, X\rangle,$$

which concludes the proof. $\square$

Remark 2.3. The assumptions of the above theorem are satisfied, in particular, when $\mu \ll \mathcal{L}^n$.

We are in a position to give the following definition.

Definition 2.4. A closed connected set $\Sigma \subset \mathbb{R}^n$ will be called stationary for the functional $\mathcal{F}$, if (6) holds.

Clearly, every stationary point depends on the problem data, which in this case is the measure μ. To emphasize this dependence, we will further sometimes say about stationary points for the functional $\mathcal{F}$ that they are stationary with respect to μ. In the most important particular case we will be interested in, μ is a uniform measure over some open $\Omega \subset \mathbb{R}^n$ (i.e. $\mu = \mathcal{L}^n \llcorner \Omega$) with $\Sigma \subset \Omega$. In such a situation we will be speaking of stationary points with respect to the set Ω.

3. Examples of regular stationary points

We will first show that, in sharp contrast with minimizers, stationary points may contain closed loops (i.e. homeomorphic images of S^1).

Proposition 3.1. *Let $\mu := \mathcal{L}^2 \llcorner B_1(0)$. There exist $r < 1$ such that the circumference $\partial B_r(0)$ is a stationary point for functional (2) if and only if $\lambda < \frac{1}{2}$. Nevertheless, no circumference is a minimizer of (2), since minimizers cannot contain closed loops.*

Proof. We set $\Sigma := \partial B_r(0)$ and impose (6). We choose X to be normal to Σ without loss of generality, since the normal part only plays a role in (6). If we write the integral term in polar coordinates, the integrand depends only on the angle. Setting $A = B_r(0)$ and $B = B_1(0)\backslash B_r(0)$, and letting $\nu(x)$ be the outward unit normal to $\partial B_r(0)$, we get

$$\begin{aligned}\int_\Omega \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle dx &= \int_A \langle X(\pi^\Sigma(x)), \nu(\pi^\Sigma(x))\rangle \, dx \\ &\quad - \int_B \langle X(\pi^\Sigma(x)), \nu(\pi^\Sigma(x))\rangle \, dx,\end{aligned}$$

and we can compute

$$\begin{aligned}\int_A \langle X(\pi^\Sigma(x)), \nu(\pi^\Sigma(x))\rangle \, dx &= \int_0^r \int_0^{2\pi} |X(\theta)| \, \rho \, d\rho d\theta \\ &= \frac{1}{2} r^2 \int_0^{2\pi} |X(\theta)| d\theta,\end{aligned}$$

and similarly for the integral over B. Moreover,

$$\langle X, H_\Sigma\rangle = -\int_{\partial B_r(0)} |H_\Sigma(x)| \langle X(x), \nu(x)\rangle d\mathcal{H}^1(x) = -\frac{1}{r}\int_0^{2\pi} |X(\theta)| r d\theta.$$

So the Euler equation reads

$$\left(r^2 - \frac{1}{2} + \lambda\right) \int_0^{2\pi} |X(\theta)| d\theta = 0. \tag{9}$$

This equation is identically satisfied, if and only if $\lambda < 1/2$, for $r = \sqrt{1/2 - \lambda}$ (of course, $\lambda = 1/2$ would also suite for (9), but it corresponds to a degenerate case when the circumference reduces to a point).

To show that minimizers of (2) cannot contain closed loops, and hence the above stationary points are not minimizers, we may act as in the proof of absence of loops in minimizers of average distance functionals with length constraint (see e.g. [10] or [4, 6]). In fact, suppose that Σ is a minimizer containing a closed loop. Then there is a set of positive length $C \subset \Sigma$ such that for every $x \in C$ and for every $\varepsilon > 0$ there is a closed connected subset $D_\varepsilon \subset \Sigma$ such that $x \in D_\varepsilon$, $\operatorname{diam} D_\varepsilon = \varepsilon$ (hence $\mathcal{H}^1(D_\varepsilon) \geq \varepsilon$) and $\Sigma_\varepsilon := \Sigma \setminus D_\varepsilon$ is connected. We may suppose without loss of generality that $\mu((\pi^\Sigma)^{-1}(\{x\})) = 0$ for all $x \in C$ (since the set of atoms of the latter measure is clearly at most countable). One has then by triangle inequality

$$\int_{\mathbb{R}^n} \operatorname{dist}(x, \Sigma_\varepsilon)\, d\mu(x) \leq \int_{\mathbb{R}^n} \operatorname{dist}(x, \Sigma)\, d\mu(x) + \varepsilon\mu((\pi^\Sigma)^{-1}(D_\varepsilon)),$$

and hence

$$\mathcal{F}(\Sigma_\varepsilon) \leq \mathcal{F}(\Sigma) + \varepsilon\mu((\pi^\Sigma)^{-1}(D_\varepsilon)) - \lambda\varepsilon.$$

Minding that $D_\varepsilon \searrow \{x\}$ as $\varepsilon \to 0^+$, we get

$$\mu((\pi^\Sigma)^{-1}(D_\varepsilon)) \to \mu((\pi^\Sigma)^{-1}(\{x\})) = 0,$$

and thus

$$\mathcal{F}(\Sigma_\varepsilon) \leq \mathcal{F}(\Sigma) + o(\varepsilon) - \lambda\varepsilon$$

as $\varepsilon \to 0^+$, which means that $\mathcal{F}(\Sigma_\varepsilon) < \mathcal{F}(\Sigma)$ for small $\varepsilon > 0$ concluding the proof. □

Let us now consider another example of a stationary point for (2) given by Figure 1, where the radii of the semicircles are equal to $\sqrt{\lambda}$. Here, as well as in all the other figures, the arrows starting at the endpoints of Σ indicate the directions of $-H_\Sigma$ in these points.

Proposition 3.2. *There exists a line segment which is stationary for the region Ω shown on Figure 1.*

Proof. In the example of Figure 1, points belonging to regions A and B are projected to the line segment Σ along the perpendicular, and it is clear that the symmetry of the domain yields

$$\int_A \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle dx + \int_B \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle dx = 0$$

for any vector field $X \in C_0^\infty(\mathbb{R}^n; \mathbb{R}^n)$.

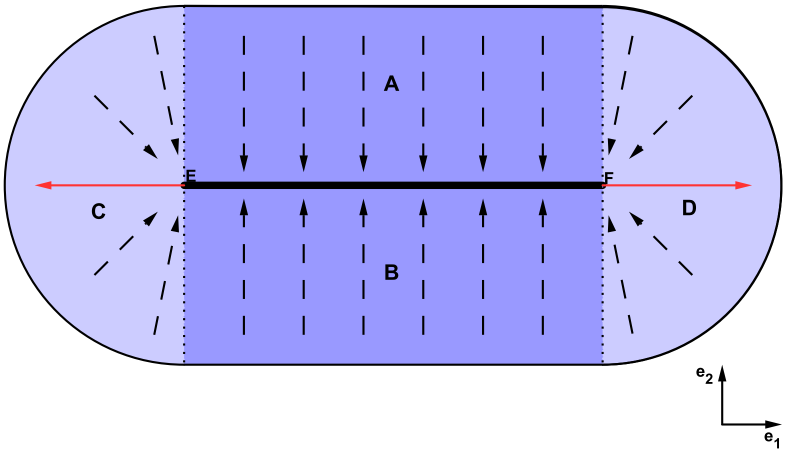

FIGURE 1. Construction of the proof of Proposition 3.2

Set $X_1 := \langle X, \mathbf{e}_1 \rangle$ and $X_2 := \langle X, \mathbf{e}_2 \rangle$, where $\mathbf{e}_1$, $\mathbf{e}_2$ stand for the base vectors in $\mathbb{R}^2$. Let us compute the contribution of the right unit semicircle:

$$\begin{aligned}\int_D \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle dx &= -\int_0^{\sqrt{\lambda}} \int_{-\pi/2}^{\pi/2} X_1(F) \cos\theta \rho d\rho d\theta \\ &= -2X_1(F) \int_0^{\sqrt{\lambda}} \rho d\rho \\ &= -\lambda X_1(F).\end{aligned}$$

In the same way, the contribution of semicircle C is given by

$$\int_C \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle dx = \lambda X_1(E).$$

Therefore,

$$\int_\Omega \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle dx = -\lambda X_1(F) + \lambda X_1(E).$$

On the other hand, at the endpoints E and F of the segment, the distributional curvature is given by $\delta_E \mathbf{e}_1$, $-\delta_F \mathbf{e}_1$ where δ_x stands for the Dirac mass concentrated at the point x (see [3]), while at all the other points of the segment the curvature is zero. Thus the curvature term of the Euler equation reduces to $\lambda(X_1(F) - X_1(E))$, and hence (6) is satisfied. □

We now show an example of a set which is never stationary (i.e. it is not stationary for any ambient set Ω).

Proposition 3.3. *The line Σ made of two segments (not reduced to a single segment), is not stationary for any open set $\Omega \subset \mathbb{R}^2$.*

Proof. Let P be the common vertex of the two segments (with the aperture $2\varphi < \pi$), R be a point on one of the two edges, with $z := |P - R|$. Let moreover S be a point on the normal to the same segment passing through R, with $y := |S - R|$ located in the region B in Figure 2. Since the whole polygonal line Σ, and hence P, is in the interior of Ω, it is clear that the rectangle $B := PRST$ (with sidelengths z and y), is all contained in Ω for all sufficiently small y and z. Let finally Q be a point of the intersection of the line passing through S and R, with the bisector of the angle formed by the two segments of Σ (see Figure 2). Choose now a regular vector field X compactly supported in the open segment $\overline{PR}$, and normal to it, pointing towards the region B in Figure 2. It is clear that there is no contribution from the curvature term in the Euler equation, since the curvature of the line segment is zero outside its endpoints. So it remains to check the integral term. Since $|Q - R| = z \tan\varphi$, an easy computation in the suitable coordinate system yields

$$\int_B \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle dx = -\int_B |X(\pi^\Sigma(x))|\, dx = -y \int_0^z |X(\zeta)| d\zeta$$

and

$$\int_A \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle dx = \int_A |X(\pi^\Sigma(x))|\, dx = z \tan\varphi \int_0^z |X(\zeta)| d\zeta.$$

Notice that z can be chosen small enough such that the sum of the above terms is strictly negative, while

$$\begin{aligned} \int_\Omega \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle dx \leq & \int_A \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle dx \\ & + \int_B \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle dx, \end{aligned}$$

that is, for sufficiently small z the equation (6) is not satisfied. $\square$

It is worth emphasizing that it is still quite easy to find a measure μ such that the given polygonal line is stationary with respect to μ.

4. Examples of irregular stationary points

In this section we will show that there exist Ω and Σ stationary in Ω such that Σ has angular points.

From now on, we will consider sets Σ made of two arcs of circumference with a common end point O. We will refer to such sets simply as curved corners. We will say that a curved corner is convex, if it is a convex curve (i.e. it intersects every line in at most two points).

Proposition 4.1. *There exists a convex curved corner Σ stationary with respect to some open $\Omega \subset \mathbb{R}^2$.*

Proof. Let $\lambda > 0$ be fixed. Our construction is that shown on Figure 3. Namely, the set Σ is made by two arcs $\widetilde{QO}$ and $\widetilde{PO}$ of circumferences with the same radius R and with centers C_1 and C_2 respectively. The points P and Q are chosen in such a way that both belong to the line v containing the centers of the circumferences. We denote by $2\varphi \in [0, \pi]$ the angle between the normals in O to the respective arcs,

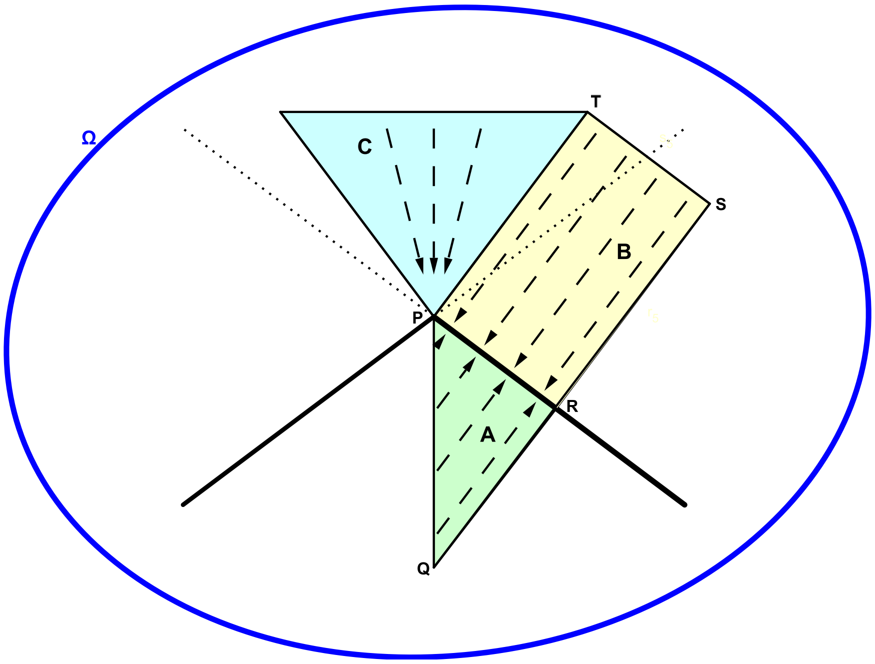

FIGURE 2. Construction of the proof of Proposition 3.3

pointing away from v. Then $\alpha = \pi/2 - \varphi$ is the angle between v and the ray C_1O (and also, by symmetry, between v and the ray C_2O). We also assume the unit coordinate vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ to be directed as in Figure 3.

Now let

$$\begin{aligned} b &:= \sqrt{R^2 + 2\lambda} - R, \\ f(\theta) &:= \sqrt{2R^2 + 2\lambda - \left(\frac{R\cos\alpha}{\cos\theta}\right)^2}, \quad \theta \in [0, \alpha], \\ r &:= \sqrt{2\lambda}. \end{aligned}$$

Notice that $r > b$. Moreover, fix a $k \in (0, R(1 - \cos\alpha))$ and an $h > 0$ such that

$$-\int_{-k}^{k} \left(\int_{-h}^{0} y(z^2 + y^2)^{-1/2} dy \right) dz = \lambda. \tag{10}$$

Consider now the region bounded by Σ and the segment $\overline{PQ}$. It is divided symmetrically in two regions A and B by the line u passing through O perpendicular to v. Let C indicate the region identified by the arc $\widetilde{QO}$, the ray C_1O, the ray C_1Q and the curve defined by the equation $\rho = f(\theta)$ in polar coordinates with center C_1 and the angle θ counted counterclockwise increasing from 0 to α. Define D to be the region symmetric to C with respect to u. Let E and G be equal rectangles with an edge on v of length k, centered in P and Q respectively, with another edge of length h, and belonging to the half space bounded by v and not containing O. Finally, let F stand for the circular sector with center O and with the radius r bounded by the normals to $\widetilde{QO}$ and $\widetilde{PO}$ as in Figure 3.

Define now $\Omega := A \cup B \cup C \cup D \cup E \cup F \cup G$. We will show that Σ is optimal with respect to such Ω. Let ν be the outward normal to $\widetilde{QO}$. Points in B and C are projected on Σ to the arc $\widetilde{QO}$, and since $f(\alpha) = R + b$ and $f(\theta) > R + b$ for $\theta \in [0, \alpha)$, we have

$$\begin{aligned}
\int_C \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle dx &= -\int_0^\alpha \int_R^{f(\theta)} \langle X(\theta), \nu(\theta)\rangle \rho d\rho d\theta \\
&= -\int_0^\alpha \int_{R+b}^{f(\theta)} \langle X(\theta), \nu(\theta)\rangle \rho d\rho d\theta + \\
&\quad - \int_0^\alpha \int_R^{R+b} \langle X(\theta), \nu(\theta)\rangle \rho d\rho d\theta,
\end{aligned}$$

but, by the definition of b and f,

$$\begin{aligned}
\int_0^\alpha \int_R^{R+b} \langle X(\theta), \nu(\theta)\rangle \rho d\rho d\theta &= \left(\frac{1}{2}(R + b^2) - \frac{1}{2}R^2\right) \int_0^\alpha \langle X(\theta), \nu(\theta)\rangle d\theta \\
&= \lambda \int_0^\alpha \langle X(\theta), \nu(\theta)\rangle d\theta,
\end{aligned}$$

$$\int_{R+b}^{f(\theta)} \rho d\rho = \frac{1}{2}(f(\theta))^2 - \frac{1}{2}(R+b)^2 = \frac{1}{2}R^2 - \frac{1}{2}\left(\frac{R\cos\alpha}{\cos\theta}\right)^2.$$

For the computation of the integral in the region B, it is easily seen that

$$B = \left\{(\rho, \theta) \colon 0 \le \theta \le \alpha\ ,\ \frac{R\cos\alpha}{\cos\theta} \le \rho \le R\right\}, \tag{11}$$

so it follows that

$$\begin{aligned}
\int_B \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle dx &= \int_0^\alpha \langle X(\theta), \nu(\theta)\rangle \int_{\frac{R\cos\alpha}{\cos\theta}}^R \rho d\rho d\theta \\
&= \frac{1}{2}\int_0^\alpha \langle X(\theta), \nu(\theta)\rangle \left(R^2 - \left(\frac{R\cos\alpha}{\cos\theta}\right)^2\right) d\theta.
\end{aligned}$$

Hence one obtains

$$\int_{B\cup C} \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle dx = -\lambda \int_0^\alpha \langle X(\theta), \nu(\theta)\rangle d\theta. \tag{12}$$

Now consider the curvature term of the Euler equation. Let $H_\Sigma(\widetilde{QO})$ indicate the nonatomic part of the curvature of the arc $\widetilde{QO}$, i.e. the part not involving the contribution of endpoints. The term $\langle H_\Sigma(\widetilde{QO}), X\rangle$ is clearly equal to

$$-\int_0^\alpha \langle X(\theta), \nu(\theta)\rangle d\theta.$$

We end up with

$$\int_{B\cup C} \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle dx - \lambda\langle H_\Sigma(\widetilde{QO}), X\rangle = 0 \tag{13}$$

By symmetry, the integral over region $A\cup D$ can be computed in polar coordinates with respect to C_2 and v, with angle θ' counted clockwise increasing from 0 to α, and has exactly the same form. Reasoning in the same way, one sees the analogy between the terms $\langle H_\Sigma(\widetilde{PO}), X\rangle$ and $\langle H_\Sigma(\widetilde{QO}), X\rangle$. It follows that

$$\int_{A\cup D} \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x)-x}{|\pi^\Sigma(x)-x|}\right\rangle dx - \lambda\langle H_\Sigma(\widetilde{PO}), X\rangle = 0. \tag{14}$$

Let us now compute the integrals over E and G. These two regions are disjoint thanks to the choice of k. By (10) we get

$$\begin{aligned}
&\int_E \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x)-x}{|\pi^\Sigma(x)-x|}\right\rangle dx \\
&\qquad = -X_2(P)\int_{-k}^{k}\left(\int_{-h}^{0} y(z^2+y^2)^{-1/2}dy\right)dz \\
&\qquad = \lambda X_2(P).
\end{aligned} \tag{15}$$

Analogously the integral over G is given by

$$\int_G \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x)-x}{|\pi^\Sigma(x)-x|}\right\rangle dx = \lambda X_2(Q). \tag{16}$$

For the integral over F, we consider polar coordinates referred to the center O with the angle θ measured counterclockwise starting from the direction parallel to the ray C_1Q, so that

$$\frac{\pi^\Sigma(x)-x}{|\pi^\Sigma(x)-x|} = -(\cos\theta, \sin\theta), \quad x\in F.$$

Then, since in F the minimum distance from Σ is always attained in the point O, we get

$$\begin{aligned}
&\int_F \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x)-x}{|\pi^\Sigma(x)-x|}\right\rangle dx \\
&\qquad = -\int_{\frac{\pi}{2}-\varphi}^{\frac{\pi}{2}+\varphi}\int_0^r \langle X(O), (\cos\theta,\sin\theta)\rangle \rho d\rho d\theta \\
&\qquad = -X_2(O)\int_{\frac{\pi}{2}-\varphi}^{\frac{\pi}{2}+\varphi}\int_0^r \sin\theta \rho d\rho d\theta \\
&\qquad = -X_2(O)r^2\sin\varphi = -2\lambda X_2(O)\sin\varphi
\end{aligned} \tag{17}$$

Finally consider the curvature terms at the endpoints P and Q. We have respectively

$$\langle H_\Sigma(P), X\rangle = X_2(P), \quad \langle H_\Sigma(Q), X\rangle = X_2(Q). \tag{18}$$

For the point O, we have

$$H_\Sigma(O) = -2\cos\alpha\,\delta_O\mathbf{e}_2,$$

yielding

$$\langle H_\Sigma(O), X\rangle = -2\sin\varphi X_2(O). \tag{19}$$

Since $\Omega = A\cup B\cup C\cup D\cup E\cup F\cup G$ and

$$\langle H_\Sigma, X\rangle = \langle H_\Sigma(\widetilde{QO}), X\rangle + \langle H_\Sigma(\widetilde{PO}), X\rangle + \langle H_\Sigma(P), X\rangle + \langle H_\Sigma(O), X\rangle + \langle H_\Sigma(Q), X\rangle,$$

combining (13), (14), (15), (16), (17), (18) and (19) we see that the Euler equation (6) is identically satisfied. $\square$

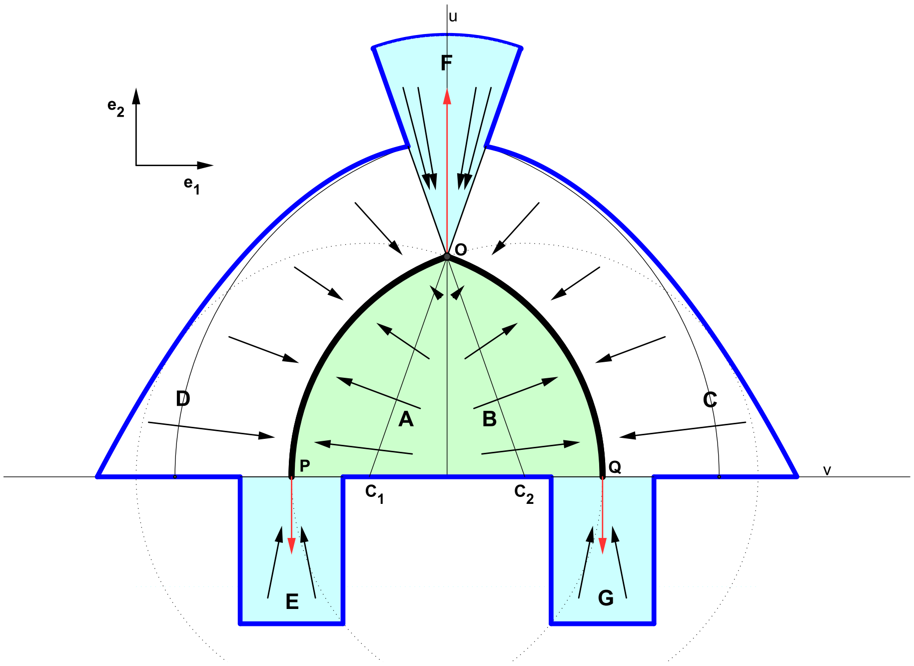

Figure 3. Construction of the proof of Proposition 4.1

Next we will show that, for a convex domain Ω, if the amplitude of the corner is not too large, then a set composed of two arcs of circle is not stationary.

We first introduce the notation similar to that used in the proof of Proposition 4.1, but for a generic curved corner Σ made by two arcs $\widetilde{QO}$ and $\widetilde{PO}$ of circumferences with different radii R_1 and R_2 and with centers C_1 and C_2 respectively. Again $2\varphi \in [0, \pi]$ is the angle between the normals in O which bound the set of points (we will call the bisector ray of the latter angle u) in $\mathbb{R}^2$ having O as the unique point of minimum distance to Σ. Let v' be a ray starting at C_1 forming the angle $\alpha \leq \pi/2 - \varphi$ with the ray C_1O. We assume that α is sufficiently small so that v' meets $\widetilde{QO}$ in some point M. In this way the rays v', C_1O and the arc $\widetilde{QO}$ form a sector of area $\alpha R_1^2/2$. We also assume the unit coordinate vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ to be directed as in Figures 4 and 5.

Fix α small enough such that v' meets the continuation of u. Consider the ridge set $\mathfrak{R}_\Sigma$ of Σ (i.e. the set of points of equal distance from the two arcs). Note that there exists a segment $\overline{OZ}$ with $Z \in v'$ which intersects $\mathfrak{R}_\Sigma$ only in O (this assertion is implied by the fact that $\mathfrak{R}_\Sigma$ is a regular curve tangent in O to u). Denote by W the curvilinear triangle bounded by $\widetilde{MO}$, $\overline{ZO}$ and $\overline{ZM}$. Clearly it contains the set of points T having the projection to Σ on $\widetilde{MO}$. Moreover, they are all projected to $\widetilde{MO}$ from the same side (i.e. either from outside of the circle $B_{R_1}(C_1)$ as in Figure 4, or from the inner part of the circle $B_{R_1}(C_1)$ as in Figure 5). It is important to observe that there are no points with such a property outside of W. We denote by C the set of points having the projection to Σ on $\widetilde{MO}$ but from the different side with respect to T.

In this section we will consider a vector field X supported in a small neighborhood of a subset of $\widetilde{MO}$ (in polar coordinates with respect to C_1 and v', the points of the support are contained in the set with angular coordinate $\theta \in [\theta_0, \alpha]$). We assume that X be vanishing in O and have restriction to $\widetilde{MO}$ directed towards the outward normal ν to the circle $B_{R_1}(C_1)$. Thus in the first member of (6) the only nonzero terms are the integrals in the regions T and C and the curvature term restricted to $\widetilde{MO}$.

Proposition 4.2. *A non convex curved corner is not stationary, for any $\Omega \subset \mathbb{R}^2$.*

Proof. Let Σ be a non convex curved corner. In this case one of the centers belongs to one of the rays bounding the cone of points for which the projection to Σ coincides with O (let it be C_1). So the region C is inside the sector bounded by the arc $\widetilde{QO}$ (see Figure 4). If β is the angle formed by $\overline{OZ}$ and $\overline{OC_1}$, it is easily seen that one can choose the point Z so that $\beta \in (\pi/2, \pi)$. In polar coordinates with respect to C_1 and v', for small α, one has then

$$W = \left\{ (\rho, \theta) \colon 0 < \theta < \alpha, R_1 < \rho < \frac{R_1 \sin \beta}{\sin(\beta + \alpha - \theta)} \right\} \tag{20}$$

(observe that $\sin \beta / \sin(\beta + \alpha - \theta) > 1$ since $\beta \in (\pi/2, \pi)$, and $\alpha - \theta > 0$ is small enough). We obtain also

$$-\lambda \langle H_\Sigma, X \rangle = \lambda \int_{\theta_0}^{\alpha} \langle X(\theta), \nu(\theta) \rangle d\theta = \lambda \int_{\theta_0}^{\alpha} |X(\theta)| d\theta.$$

Moreover, thanks to (20), we have

$$\begin{aligned} \left| \int_T \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle dx \right| &\le \int_W \left| \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle \right| dx \\ &= \frac{1}{2} \int_{\theta_0}^{\alpha} |X(\theta)| \left(\frac{R_1^2 \sin^2 \beta}{\sin^2(\beta + \alpha - \theta)} - R_1^2 \right) d\theta. \end{aligned}$$

But for $\theta \to \alpha$, with β fixed, we get

$$\frac{R_1^2 \sin^2 \beta}{\sin^2(\beta + \alpha - \theta)} - R_1^2 = o(1),$$

implying that

$$\left| \int_T \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle dx \right| \le o \left(\int_{\theta_0}^{\alpha} |X(\theta)| \, d\theta \right).$$

Therefore it is clear that, for θ_0 close enough to α, the Euler equation (6) is never satisfied for Σ. In fact, since

$$\int_C \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle \ge 0,$$

we have that

$$\begin{aligned} \int_\Omega \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle - \lambda \langle H_\Sigma, X \rangle \ge \int_T &\left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle \\ &- \lambda \langle H_\Sigma, X \rangle. \end{aligned}$$

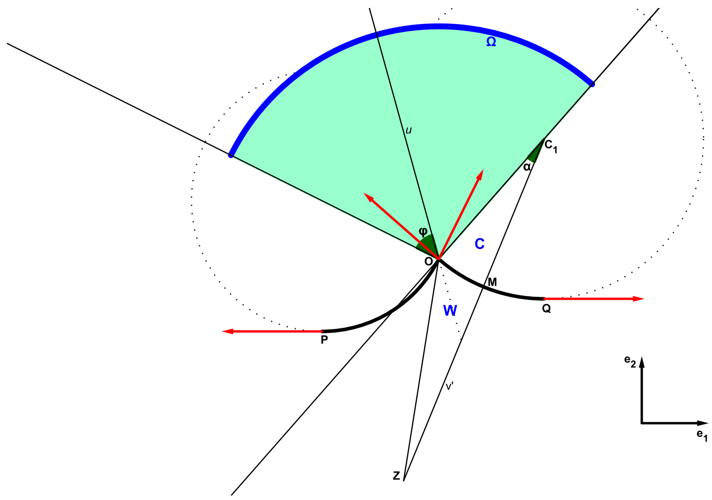

FIGURE 4. Construction of the proof of Proposition 4.2

Hence, for $\theta_0 \to \alpha$ one has

$$\int_\Omega \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|} \right\rangle - \lambda \langle H_\Sigma, X\rangle \geq \lambda \int_{\theta_0}^{\alpha} |X(\theta)|\, d\theta - o\left(\int_{\theta_0}^{\alpha} |X(\theta)|\, d\theta\right),$$

that is, the right hand side of the above inequality is always strictly positive once θ_0 is sufficiently close to α. $\square$

Finally, we show that the condition for a curved corner to be stationary with respect to a *convex* Ω is even more restrictive.

Proposition 4.3. *Let Ω be convex. A curved corner is not stationary with respect to Ω if*

$$\frac{4\lambda}{b_1^2 + b_2^2} \geq h(\varphi), \tag{21}$$

where $b_i := \sqrt{R_i^2 + 2\lambda} - R_i$, $i = 1, 2$,

$$h(\varphi) := \frac{1}{\sin\varphi} \int_0^\varphi \frac{\cos(\varphi - \theta)}{\cos^2\theta} d\theta.$$

In particular, there are no curved corners of amplitude less than or equal to 2γ, *where* $\gamma \in (0, \pi/2)$ *is the angle that satisfies*

$$\int_0^\gamma \frac{\cos(\gamma - \theta)}{\cos^2\theta} d\theta = \sin\gamma,$$

so $\gamma \simeq 54°$.

Proof. If the curved corner is not convex, we refer to the previous Proposition 4.2. Otherwise, let β be the angle between $\overline{OZ}$ and $\overline{OC_1}$. This time $\beta < \varphi$, so that once

α is sufficiently small, one has $\beta + \alpha < \pi/2$ (see Figure 5). In polar coordinates with respect to C_1 and v', we have

$$W = \left\{(\rho, \theta)\colon 0 < \theta < \alpha, \frac{R_1 \sin\beta}{\sin(\beta + \alpha - \theta)} < \rho < R_1\right\}. \tag{22}$$

Notice that $\alpha - \theta > 0$ is small and the bound on β gives $\sin\beta / \sin(\beta + \alpha - \theta) < 1$. The curvature term in the Euler equation (6) is given by

$$-\lambda\langle H_\Sigma, X\rangle = \lambda \int_{\theta_0}^{\alpha} \langle X(\theta), \nu(\theta)\rangle d\theta = \lambda \int_{\theta_0}^{\alpha} |X(\theta)|\, d\theta. \tag{23}$$

Using (22), we get

$$\begin{aligned}\int_T \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|}\right\rangle dx &\le \int_W \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|}\right\rangle dx \\ &= \int_{\theta_0}^{\alpha} \frac{|X(\theta)|}{2}\left(R_1^2 - \frac{R_1^2 \sin^2\beta}{\sin^2(\beta + \alpha - \theta)}\right) d\theta,\end{aligned}$$

and again for $\theta \to \alpha$, with β fixed, we have

$$R_1^2 - \frac{R_1^2 \sin^2\beta}{\sin^2(\beta + \alpha - \theta)} = o(1),$$

and hence

$$\int_T \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|}\right\rangle dx \le o\left(\int_{\theta_0}^{\alpha} |X(\theta)| d\theta\right). \tag{24}$$

Notice that C contains a region formed by v', the ray C_1O, the arc $\widetilde{MO}$ and some arc concentric to $\widetilde{MO}$ but of bigger radius. We express the subset of the boundary of Ω bounding C in polar coordinates (ρ, θ) with respect to C_1 and v' by the equation $\rho = b_1 + R_1 + g_1(\theta)$, where $g_1(\theta) \to 0$ as $\theta \to \alpha$, and b_1 is the distance between O and the intersection between $\partial\Omega$ and the ray C_1O, which we denote by S. Then

$$\begin{aligned}\int_C \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|}\right\rangle dx &= -\int_{\theta_0}^{\alpha} \int_{R_1}^{R_1 + b_1 + g_1(\theta)} |X(\theta)| \rho d\rho d\theta \\ &= -\frac{1}{2}\int_{\theta_0}^{\alpha} |X(\theta)| \left(2R_1 b_1 + b_1^2 + g_1(\theta)^2 + 2(R_1 + b_1) g_1(\theta)\right) d\theta \\ &= -\frac{1}{2}(2R_1 b_1 + b_1^2) \int_{\theta_0}^{\alpha} |X(\theta)| d\theta \\ &\quad - \frac{1}{2}\int_{\theta_0}^{\alpha} |X(\theta)| \left(g_1(\theta)^2 + 2(R_1 + b_1) g_1(\theta)\right) d\theta.\end{aligned} \tag{25}$$

Suppose now that the Euler equation (6) holds, that is,

$$\begin{aligned}\int_T \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|}\right\rangle dx + \int_C \left\langle X(\pi^\Sigma(x)), \frac{\pi^\Sigma(x) - x}{|\pi^\Sigma(x) - x|}\right\rangle dx \\ = \lambda\langle H_\Sigma, X\rangle.\end{aligned} \tag{26}$$

Combining (23), (24) and (25) in the above relationship, by comparison of the first order terms with respect to $\int_{\theta_0}^{\alpha} |X(\theta)|\, d\theta$ as $\theta_0 \to \alpha$, we obtain that

$$b_1 = \sqrt{R_1^2 + 2\lambda} - R_1. \tag{27}$$

Moreover, by this choice of b_1 we have

$$-\frac{1}{2}(2R_1b_1+b_1^2)\int_{\theta_0}^{\alpha}|X(\theta)|d\theta=\lambda\langle H_\Sigma,X\rangle.$$

From (26) and (25), we conclude that

$$\int_T\left\langle X(\pi^\Sigma(x)),\frac{\pi^\Sigma(x)-x}{|\pi^\Sigma(x)-x|}\right\rangle dx \\ -\frac{1}{2}\int_{\theta_0}^{\alpha}|X(\theta)|\left(g_1(\theta)^2+2(R_1+b_1)g_1(\theta)\right)d\theta=0;$$

but since

$$\int_T\left\langle X(\pi^\Sigma(x)),\frac{\pi^\Sigma(x)-x}{|\pi^\Sigma(x)-x|}\right\rangle dx>0,$$

it follows

$$\int_{\theta_0}^{\alpha}|X(\theta)|\left(g_1(\theta)^2+2(R_1+b_1)g_1(\theta)\right)d\theta>0.$$

Minding that X has an arbitrary support in $[\theta_0,\alpha]$, this means that

$$g_1(\theta)^2+2(R_1+b_1)g_1(\theta)>0$$

and implies $g_1(\theta)>0$ for all $\theta\in[\theta_0,\alpha]$ whenever α is small enough (since otherwise $g_1(\theta)<-R_1-b_1$ which contradicts the fact that g should be vanishing as $\theta\to\alpha$). Hence, the part of $\partial\Omega$ corresponding to the angular coordinate $\theta\in[\theta_0,\alpha]$ is, for small α, more distant from C_1 than the arc σ of the circumference with center C_1 passing through S, thus satisfying the equation $\rho(\theta)=R_1+b_1$. Thanks to convexity of Ω we have then that any ray starting in S, directed inside the cone of points with projection to Σ in O, and belonging to a support line to $\partial\Omega$ in S, forms an angle not greater than $\pi/2$ with the segment $\overline{SO}$ (mind that the angle of $\pi/2$ corresponds to the case when the ray is tangent to σ). As a consequence, the part of Ω which lies in the angle (of value φ) bounded by u and the ray OS, is contained in the triangle V_1, formed by u, $\overline{OS}$ and the tangent in S to σ.

Now fix a new vector field $\hat{X}$, compactly supported in a small neighborhood of O and such that $\hat{X}(O)$ is directed along u. One has

$$H_\Sigma(O)=\delta_O(\tau_Q+\tau_P),$$

where τ_Q and τ_P are the unit vectors tangent in O to the arcs $\widetilde{PO}$ and $\widetilde{QO}$ respectively and directed towards P and Q respectively. Since

$$\langle\hat{X},\delta_O\tau_Q\rangle=\langle\hat{X},\delta_O\tau_P\rangle=-|\hat{X}(O)|\sin\varphi,$$

we get

$$-\lambda\langle\hat{X},H_\Sigma(O)\rangle=2\lambda|\hat{X}(O)|\sin\varphi. \tag{28}$$

Now compute the contribution given by triangle V_1 to the first term of the Euler equation (6). For this purpose we use polar coordinates with respect to O and the ray OS, with $\theta\in[0,\varphi]$. It is clear that

$$V=\left\{(\rho,\theta)\colon 0\le\theta\le\varphi, 0<\rho\le\frac{b}{\cos\theta}\right\}. \tag{29}$$

Therefore,

$$
\begin{aligned}
\int_{V_1} \left\langle \hat{X}(\pi^\Sigma(x)), \frac{\pi^\Sigma(x)-x}{|\pi^\Sigma(x)-x|} \right\rangle dx &= -|\hat{X}(O)| \int_0^\varphi \cos(\varphi-\theta) \int_0^{\frac{b_1}{\cos\theta}} \rho d\rho d\theta \\
&= -\frac{1}{2} b_1^2 |\hat{X}(O)| \int_0^\varphi \frac{\cos(\varphi-\theta)}{\cos^2\theta} d\theta \\
&= -\frac{1}{2} b_1^2 |\hat{X}(O)| h(\varphi) \sin\varphi.
\end{aligned} \tag{30}
$$

Reasoning in the same way with arc $\widetilde{PO}$ instead of the arc $\widetilde{PQ}$, we obtain the analogous triangle V_2 with

$$
\int_{V_2} \left\langle \hat{X}(\pi^\Sigma(x)), \frac{\pi^\Sigma(x)-x}{|\pi^\Sigma(x)-x|} \right\rangle dx = -\frac{1}{2} b_2^2 |\hat{X}(O)| h(\varphi) \sin\varphi, \tag{31}
$$

with $b_2 := \sqrt{R_2^2 + 2\lambda} - R_2$. But since Ω is convex, and one of the sides of V_1 (resp. V_2) is in the support line to Ω, we have

$$
(\pi^\Sigma)^{-1}(O) \cap \Omega \subset V_1 \cup V_2. \tag{32}
$$

Let us write the Euler equation (6) with respect to the vector field $\hat{X}$. Letting $\Gamma := (\Sigma \backslash \{O\}) \cap \operatorname{supp} \hat{X}$, thanks to (28) we get

$$
\int_{(\pi^\Sigma)^{-1}(O)} \left\langle \hat{X}(\pi^\Sigma(x)), \frac{\pi^\Sigma(x)-x}{|\pi^\Sigma(x)-x|} \right\rangle dx + 2\lambda |\hat{X}(O)| \sin\varphi + c_\Gamma = 0, \tag{33}
$$

where by c_Γ we denoted the sum of all the terms in the Euler equation which involve integrals over Γ. Minding the strict inclusion (32), and using (30) and (31), we obtain

$$
-\frac{1}{2}(b_1^2 + b_2^2) |\hat{X}(O)| h(\varphi) \sin\varphi + 2\lambda |\hat{X}(O)| \sin\varphi + c_\Gamma < 0. \tag{34}
$$

Since c_Γ contains only integral terms, we have that c_Γ can be made arbitrarily small by choosing a sufficiently small support of $\hat{X}$, and hence (34) may be satisfied, only if

$$
\frac{4\lambda}{b_1^2 + b_2^2} < h(\varphi), \tag{35}
$$

or, in other words, when $h(\varphi)$ is as in the statement being proven, then the Euler equation is not satisfied. Finally, to prove the second claim, it remains to observe that $4\lambda/(b_1^2+b_2^2) > 1$, and hence with $h(\varphi) \le 1$ the respective curved corner is not stationary. □

5. The compliance case

In this section we consider the case of a functional arising from the theory of elliptic equations:

$$
\mathcal{F}(\Sigma) := \int_\Omega u_\Sigma(x) f(x)\, dx + \lambda \mathcal{H}^1(\Sigma). \tag{36}
$$

Here $\Omega \subset \mathbb{R}^2$ is a given bounded open subset, f is a given function, and u_Σ is the unique solution of the PDE

$$
\begin{cases} -\Delta u = f & \text{in } \Omega \setminus \Sigma, \\ u = 0 & \text{on } \partial\Omega \cup \Sigma. \end{cases}
$$

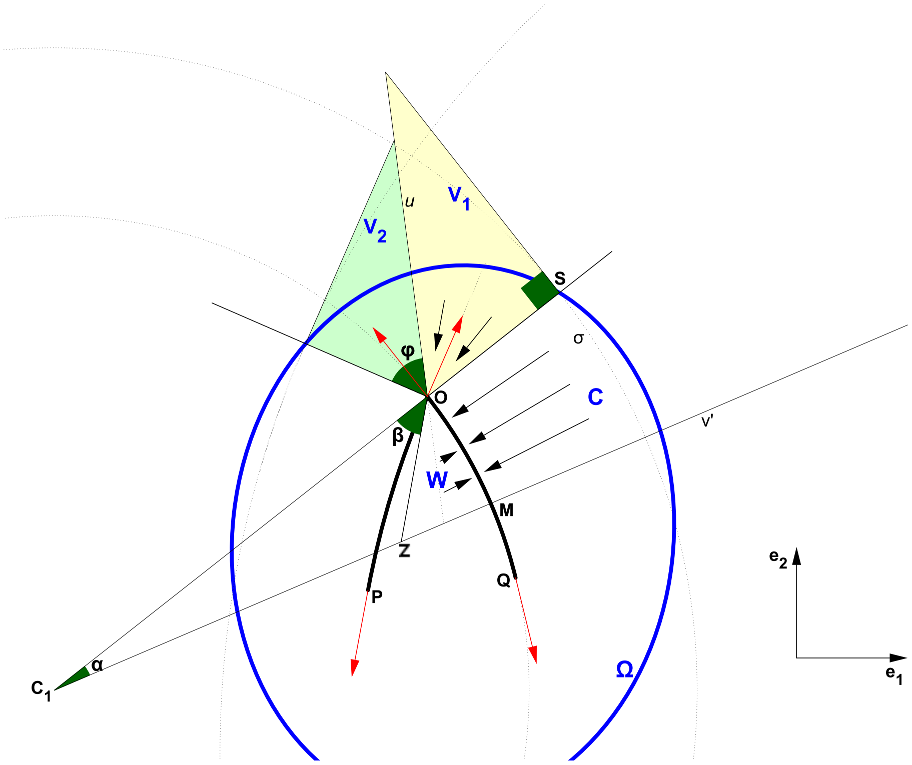

FIGURE 5. Construction of the proof of Proposition 4.3

An integration by parts in the PDE above gives that the *compliance* term $\int_\Omega u_\Sigma f\,dx$ appearing in the functional $\mathcal{F}$ can be expressed in an equivalent way:

$$\int_\Omega u_\Sigma(x) f(x)\,dx = \max\Big\{\int_\Omega \big(2fu - |\nabla u|^2\big)\,dx\ :\ u \in W_0^{1,2}(\Omega\setminus\Sigma)\Big\}.$$

For simplicity we assume that Ω has a Lipschitz boundary and that $f \in W^{1,2}(\mathbb{R}^2)$. In fact, we could also consider the case of a p-Laplace operator, and the similarity with the average distance functional consists in the fact (shown in [5]) that as $p \to +\infty$ the p-compliance problem converges to the one with the average distance functional. Here we limit ourselves to the case $p = 2$. Also for simplicity we have taken the Dirichlet condition $u = 0$ on $\partial\Omega$; all the arguments can be repeated for the Neumann case $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$.

The existence of a solution to the minimum problem

$$\min\big\{\mathcal{F}(\Sigma)\ :\ \Sigma \text{ closed connected subset of } \Omega\big\}$$

follows by an application of the Šverák compactness theorem (see [5]). Here we are interested, as before, in the first order necessary conditions of optimality.

Following [8, theorem 5.3.2], if ϕ_ε is a one parameter group of diffeomorphisms satisfying (4), setting $\Sigma_\varepsilon := \phi_\varepsilon(\Sigma)$, $u := u_\Sigma$ and $u_\varepsilon = u_{\Sigma_\varepsilon}$, we have as $\varepsilon \to 0$ that

$\frac{u_\varepsilon - u}{\varepsilon} \to u'$ in $L^2(\Omega)$, where u' satisfies the PDE

$$\begin{cases} -\Delta u' = 0 \text{ in } \Omega \setminus \Sigma, \\ u' = 0 \text{ on } \partial\Omega, \ u' = -\nabla u \cdot X \text{ on } \Sigma. \end{cases}$$

Note that the boundary conditions in the above equation are understood in the weak sense, i.e. $u' + \nabla u \cdot X \in W_0^{1,2}(\mathbb{R}^2)$. Therefore, the first variation argument applied to the functional $\mathcal{F}$ gives

$$\frac{\partial}{\partial \varepsilon}\mathcal{F}(\Sigma_\varepsilon)\Big|_{\varepsilon=0} = \int_\Omega u' f\, dx - \lambda \langle H_\Sigma, X\rangle.$$

Suppose now that $\Omega = \Omega^+ \cup \Omega^-$ with $\Sigma \subset \partial\Omega^+ \cap \partial\Omega^-$. Then, if Σ, $\partial\Omega$ and f provide sufficient regularity for u and u' so that the Green formula can be applied, we have

$$\begin{aligned} \int_{\Omega^+} u' f\, dx = -\int_{\Omega^+} u' \Delta u\, dx &= \int_{\Omega^+} \nabla u' \nabla u\, dx - \int_{\partial\Omega^+} u' \frac{\partial u}{\partial n}\, d\mathcal{H}^1 \\ &= \int_{\Omega^+} \nabla u' \nabla u\, dx + \int_\Sigma \nabla u \cdot X \frac{\partial u}{\partial n}\, d\mathcal{H}^1 \\ &\quad - \int_{\partial\Omega^+ \setminus (\partial\Omega \cup \Sigma)} u' \frac{\partial u}{\partial n}\, d\mathcal{H}^1, \end{aligned}$$

where n stands for the external normal to Ω^+. But

$$\begin{aligned} \int_{\Omega^+} \nabla u' \nabla u\, dx &= -\int_{\Omega^+} u \Delta u'\, dx + \int_{\partial\Omega^+} u' \frac{\partial u}{\partial n}\, d\mathcal{H}^1 \\ &= -\int_{\partial\Omega^+ \setminus (\partial\Omega \cup \Sigma)} u \frac{\partial u'}{\partial n}\, d\mathcal{H}^1. \end{aligned}$$

Thus,

$$\int_{\Omega^+} u' f\, dx = \int_\Sigma \nabla u^+ \cdot X \frac{\partial u^+}{\partial n}\, d\mathcal{H}^1 - \int_{\partial\Omega^+ \setminus (\partial\Omega \cup \Sigma)} \left(u \frac{\partial u'}{\partial n} + u' \frac{\partial u}{\partial n} \right) d\mathcal{H}^1, \tag{37}$$

where ∇u^+ stands for the trace on Σ of the gradient of u restricted to Ω^+, and $\frac{\partial u^+}{\partial n}$ stands for the trace of the respective normal derivative. Analogously, minding that the external normal to Ω^- over $\partial\Omega^+ \cap \partial\Omega^-$ is given by $-n$, we get

$$\int_{\Omega^-} u' f\, dx = -\int_\Sigma \nabla u^- \cdot X \frac{\partial u^-}{\partial n}\, d\mathcal{H}^1 + \int_{\partial\Omega^+ \setminus (\partial\Omega \cup \Sigma)} \left(u \frac{\partial u'}{\partial n} + u' \frac{\partial u}{\partial n} \right) d\mathcal{H}^1, \tag{38}$$

where ∇u^- stands for the trace on Σ of the gradient of u restricted to Ω^-, and $\frac{\partial u^-}{\partial n}$ stands for the trace of the respective normal derivative. From (37) and (38) we obtain

$$\int_\Omega u' f\, dx = \int_\Sigma \nabla u^+ \cdot X \frac{\partial u^+}{\partial n}\, d\mathcal{H}^1 - \int_\Sigma \nabla u^- \cdot X \frac{\partial u^-}{\partial n}\, d\mathcal{H}^1.$$

Recalling that

$$\nabla u^\pm = \frac{\partial u^\pm}{\partial n} n,$$

since the tangential derivatives of $u^\pm$ over Σ vanish (because $u^\pm = u = 0$ on Σ), we get

$$\int_\Omega u' f\, dx = \int_\Sigma \left(\left(\frac{\partial u^+}{\partial n} \right)^2 - \left(\frac{\partial u^-}{\partial n} \right)^2 \right) X \cdot n\, d\mathcal{H}^1.$$

Hence,

$$\frac{\partial}{\partial \varepsilon}\mathcal{F}(\Sigma_\varepsilon)\Big|_{\varepsilon=0} = \int_\Sigma \left(\left(\frac{\partial u^+}{\partial n}\right)^2 - \left(\frac{\partial u^-}{\partial n}\right)^2\right) X\cdot n\, d\mathcal{H}^1 - \lambda\langle H_\Sigma, X\rangle.$$

Since this holds for every vector field X, we deduce the Euler equation that must hold for every minimizer of $\mathcal{F}$:

$$\left(\frac{\partial u^+}{\partial n}\right)^2 - \left(\frac{\partial u^-}{\partial n}\right)^2 = \lambda\langle H_\Sigma, n\rangle.$$

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