NONEXISTENCE RESULTS FOR SEMILINEAR EQUATIONS IN CARNOT GROUPS

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Abstract. In this paper, following [3], we provide some nonexistence results for semilinear equations in the the class of Carnot groups of type $\star$.This class, see [19], contains, in particular, all groups of step 2, like the Heisenberg group, and also Carnot groups of arbitrarly large step. Moreover we prove some nonexistence results to semilinear equations in the Engel group which is the simplest Carnot group that is not of type $\star$.

1. Introduction

Aim of this paper is to apply the results contained in [3] to the Carnot groups setting (see Section 2 for the definitions). More precisely, we will deal with solutions $u : \mathbb{G} \to \mathbb{R}$ of

$$\Delta_{\mathbb{G}} u = f(u), \tag{1}$$

where $\Delta_{\mathbb{G}}$ is the sub-Laplacian associated with the Carnot group $\mathbb{G}$ and $f \in C^\infty(\mathbb{R})$. We prove, see Theorem 3.2 for the rigorous statement, that for every Carnot group $\mathbb{G}$ and for every solution u of (1) the following inequality holds

$$\int_{\mathbb{G}_0} \mathcal{D}\eta^2 \mathrm{d}x \leq \int_{\mathbb{G}} |\nabla_{\mathbb{G}}\eta|^2 |\tilde{\nabla}_{\mathbb{G}} u|^2 \mathrm{d}x \qquad \forall \eta \in C_c^\infty(\mathbb{G}), \tag{2}$$

where $\mathcal{D}$ denotes the so called *defect* of the level surfaces $\{u = c\}$, where c is constant, (see Section 3), while $\nabla_{\mathbb{G}}$, $\tilde{\nabla}_{\mathbb{G}}$ are respectively the horizontal gradient, the right-horizontal gradient (see Sections 2 and 3 for the definitions) and

$$\mathbb{G}_0 := \{x \in \mathbb{G} \mid \tilde{\nabla}_{\mathbb{G}} u(x) \neq 0\}.$$

We explicitely point out that (2) is a Poincaré-type formula, in the sense that the weighted L^2-norm of any test function is bounded by a weighted L^2-norm of its gradient. These type of inequalities were studied in [23, 24] and then refined and applied to several PDE's questions in [10, 11]. Applications in the subRiemannian setting of (2) were performed in [13, 3] for the Heisenberg group, in [14] for the Grushin plane and in [22] for the Engel group. In several cases, these type of weighted inequalities lead to rigidity results (such as classification, symmetry, or nonexistence, of solutions), see [12, 11] and also [4, 5, 16] for some other nonexistence results.

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One of the main consequences of (2) is that under a suitable growth condition of the energy

$$\eta(R) := \int_{B(0,R)} |\tilde{\nabla}_{\mathbb{G}} u|^2 \mathrm{d}x \tag{3}$$

the right-horizontal normal

$$\tilde{\nu} := \frac{\tilde{\nabla}_{\mathbb{G}} u}{|\tilde{\nabla}_{\mathbb{G}} u|} \tag{4}$$

of the level surfaces of $\{u = c\}$ is constant in $\mathbb{G}_0$. Using this observation we will provide a nonexistence result for (1) in Carnot groups of arbitrarly large step, precisely in those groups satisfying the so called flatness condition. To this aim, let us introduce the notion of *flatness condition.*

Definition 1.1. *We say that a Carnot group $\mathbb{G}$ of step k satisfies the flatness condition if for every $u \in C^\infty(\mathbb{G})$ with $\{\tilde{\nabla}_{\mathbb{G}} u = 0\} = \emptyset$ and $\tilde{\nu} = \frac{\tilde{\nabla}_{\mathbb{G}} u}{|\tilde{\nabla}_{\mathbb{G}} u|}$ constant in $\mathbb{G}$, then*

$$u(x) = u((x^{(1)}, \dots, x^{(k)})) = u_0\Big(\Big\langle \bar{a}, x^{(1)} \Big\rangle \Big)$$

for some unit vector $\bar{a} \in \mathbb{S}^{h_1 - 1}$ and $u_0 \in C^\infty(\mathbb{R})$.

In particular the class of Carnot group of type $\star$ satisfies the so called *flat condition*, see [19].

We recall that a Carnot group is of type $\star$ if there is a basis $(X_1, \dots, X_{h_1})$ of V_1 (the first layer of the stratification) such that

$$[X_j, [X_j, X_i]] = 0 \quad \text{for} \quad i, j = 1, \dots, h_1. \tag{5}$$

We prove the following result.

Theorem 1.2. *Let $\mathbb{G} = (\mathbb{R}^m, \cdot)$ be a Carnot group of step $k \geq 2$ satisfying the flatness condition with Jacobian basis $X_1, \dots, X_m$. If $f \in C^\infty(\mathbb{R})$ then there is no $u : \mathbb{G} \longrightarrow \mathbb{R}$, solution of $\Delta_{\mathbb{G}} u = f(u)$, such that*

(i) *$\tilde{X}_j u > 0$ in $\mathbb{G}$ for some $j \in \{1, \dots, m\}$;*

(ii) $\liminf_{R \to +\infty} \frac{2 \int_{\sqrt{R}}^R \frac{\eta(\tau)}{\tau^3} \mathrm{d}\tau + \frac{1}{R^2} \eta(R)}{\log(R)^2} = 0,$

where η is as in (3).

The results of [3] seem to indicate that functions whose level surfaces have constant right-horizontal normal seemed to be the right candidates to select families of surfaces useful to determine simple symmetries, exactly as it happens in the Euclidean case for the planes.

Nevertheless, we point out that in the Engel group there exist functions with constant right-horizontal normal $\tilde{\nu}$ which are not flat in the sense of Definition 1.1 (see for example the family of functions defined in (26)).This more complicate situation seems to be linked to the intrinsic structure of the Engel group which is not a group of type $\star$. However, using the geometric characterization of constant normal sets in Carnot groups of step three contained in [1, 2], we obtain, also in this setting, some nonexistence results.

We resume below our main results in the Engel group.

Proposition 1.3. *If $u \in C^\infty(\mathbb{E})$ is a solution of $\Delta_{\mathbb{E}} u = f(u)$ satisfying:*

(i) $\tilde{\nu} = (1, 0)$,
(ii) $\{x \in \mathbb{E} \mid \tilde{X}_1 u = 0\} = \emptyset$,

then there exists $u_0 \in C^\infty(\mathbb{R})$ such that

$$u(x_1, x_2, x_3, x_4) = u_0(x_1) \quad \forall x \in \mathbb{E}$$

and u_0 solves the following one-dimensional problem:

$$u_0'' = f(u_0), \quad u_0' > 0 \quad \text{in } \mathbb{R}. \tag{6}$$

Moreover, if $f \in C^\infty(\mathbb{R})$, then there is no solution u of $\Delta_{\mathbb{E}} u = f(u)$ such that (i), (ii) hold and u also satisfies $\liminf_{R \to +\infty} \frac{2\int_{\sqrt{R}}^{R} \frac{\eta(\tau)}{\tau^3} \mathrm{d}\tau + \frac{1}{R^2}\eta(\tau)}{\log(R)^2} = 0$.

Proposition 1.4. *Let $f \in C^\infty(\mathbb{R})$. Then, there is no solutions of $\Delta_{\mathbb{E}} u = f(u)$ satisfying the following conditions:*

(i) $u(x_1, x_2, x_3, x_4) = g(x_2, x_4)$ *with* $g \in C^\infty(\mathbb{R}^2)$,
(ii) $\tilde{\nu} = (0, 1)$,
(iii) $\{x \in \mathbb{E} \mid \tilde{X}_2 u = 0\} = \emptyset$,
(iv) $\liminf_{R \to +\infty} \frac{2\int_{\sqrt{R}}^{R} \frac{\eta(\tau)}{\tau^3} \mathrm{d}\tau + \frac{1}{R^2}\eta(\tau)}{\log(R)^2} = 0$.

Proposition 1.5. *Let p be a polynomial of degree $1, 2$ or 3 then there are no solutions of $\Delta_{\mathbb{E}} u = p(u)$ satisfying the following conditions:*

(i) $u(x_1, x_2, x_3, x_4) = g_1(x_2) + g_2(x_3, x_4)$,
(ii) $\tilde{\nu} = (0, 1)$,
(iii) $\{x \in \mathbb{E} \mid \tilde{X}_2 u = 0\} = \emptyset$,

where, $g_1 \in C^\infty(\mathbb{R})$, $g_2 \in C^\infty(\mathbb{R}^2)$ with $\partial_4 g_2 \neq 0$ in $\mathbb{R}^2$. Moreover, if $p(s) = a \in \mathbb{R} \setminus \{0\}$ for all $s \in \mathbb{R}$ then the same conclusion holds assuming only (ii) and (iii). Finally, if $p(s) = 0$ for all $s \in \mathbb{R}$ then there are no solutions of $\Delta_{\mathbb{E}} u = 0$ satisfying (ii) and

(iii) $\liminf_{R \to +\infty} \frac{2\int_{\sqrt{R}}^{R} \frac{\eta(\tau)}{\tau^3} \mathrm{d}\tau + \frac{1}{R^2}\eta(\tau)}{\log(R)^2} = 0$.

Obviously, our results in the Engel group setting are not optimal, in the sense that we have to impose some a priori restrictions on the structure of the solutions. Nevertheless, we believe that they can be considered as a first attempt toward the classification of solutions of semilinear problems with constant normal in Carnot groups not satisfying the flatness condition.

The paper is organized as follows. In Section 2 we introduce Carnot groups and we briefly recall their main properties. In Section 3 we introduce the tools to deal with our semilinear PDEs in Carnot groups, while Section 4 is devoted to the proof of Theorem 1.2. In the last Section 5 we discuss the Engel group case.

2. Carnot groups

We briefly recall some standard facts on Carnot groups, see [6, 9, 7, 17, 20] for further details.

Definition 2.1. *A finite dimensional Lie algebra $\mathfrak{g}$ is said to be stratified of step $k \in \mathbb{N}$ if there exist $V_1, \dots, V_k$ subspaces of $\mathfrak{g}$ with linear dimension $v_k := \dim V_k$ such that:*

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_k;$$
$$[V_1, V_i] = V_{i+1} \quad i = 1, \dots, k-1; \quad [V_1, V_k] = \{0\}.$$

A connected and simply connected Lie group $\mathbb{G}$ is said to be a Carnot group if its Lie algebra $\mathfrak{g}$ is finite dimensional and stratified. We also denote by $h_0 := 0$, $h_i := \sum_{j=1}^{i} v_j$ and $m := h_k$

Using the classical exponential map (see [6] for the definition) every Carnot group $\mathbb{G}$ of step k is isomorphic as a Lie group to $(\mathbb{R}^m, \cdot)$ where $\cdot$ is the group operation obtained projecting on $\mathbb{G}$ the Baker-Campbell-Hausdorff formula. For each $\lambda > 0$ and each $x \in \mathbb{G}$ we denote by $\delta_\lambda : \mathbb{G} \longrightarrow \mathbb{G}$ and $\tau_x : \mathbb{G} \longrightarrow \mathbb{G}$ the mappings defined respectively by:

$$\delta_\lambda(x) = \delta_\lambda(x_1, \dots, x_m) := (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_k} x_m) \tag{7}$$

$$\tau_y(x) := x \cdot y, \tag{8}$$

where $\sigma_i \in \mathbb{N}$ is called the homogeneity of the variable x_i in $\mathbb{G}$ and it is defined by

$$\sigma_j := i \quad \text{whenever} \quad h_{i-1} < j \leq h_i.$$

We endow $\mathbb{G}$ with a homogeneous norm and a pseudo-distance defining

$$|x|_{\mathbb{G}} := |(x^{(1)}, \dots, x^{(k)})|_{\mathbb{G}} := \Big(\sum_{j=1}^{k} \|x^{(j)}\|^{\frac{2k!}{j}} \Big)^{\frac{1}{2k!}}, \tag{9}$$

$$\mathrm{d}(x, y) := |y^{-1} \cdot x|_{\mathbb{G}}, \tag{10}$$

here $x^{(j)} := (x_{h_{j-1}+1}, \dots, x_{h_j})$ and $\|x^{(j)}\|$ denotes the standard Euclidean norm in $\mathbb{R}^{h_j - h_{j-1}}$. We define the gauge ball centered at $x \in \mathbb{G}$ of radius $R > 0$ by

$$B(x, R) := \{y \in \mathbb{G} \mid |y^{-1} \cdot x|_{\mathbb{G}} < R\}.$$

We also recall that for every Carnot group $\mathbb{G} = (\mathbb{R}^m, \cdot)$ the $m-$dimensional Lebesgue measure (denoted by $\mathcal{L}^m$) is the Haar measure associated to $\mathbb{G}$. Finally a basis $X = (X_1, \dots, X_m)$ of $\mathfrak{g}$ is called Jacobian basis if $X_j = J(e_j)$ where $(e_1, \dots, e_m)$ is the canonical basis of $\mathbb{R}^m$ and $J : \mathbb{R}^m \longrightarrow \mathfrak{g}$ is defined by

$$J(\eta)(x) := \mathcal{J}_{\tau_x}(0) \cdot \eta$$

here $\mathcal{J}_{\tau_x}$ denotes the Jacobian matrix of τ_x.

The following Proposition is standard, see [6, 15] for a proof.

Proposition 2.2. *Let $\mathbb{G} = (\mathbb{R}^{h_k}, \cdot)$ be a Carnot group of step $k \in \mathbb{N}$. Then the Jacobian basis $X_1, \dots, X_m$ have polynomial coefficients and if $h_{l-1} < j \leq h_l$, $1 \leq l \leq k$,*

$$X_j(x) = \partial_j + \sum_{i>h_l}^{h_k} a_i^{(j)}(x)\partial_i$$

where $a_i^{(j)}(\delta_\lambda(x)) = \lambda^{\sigma_i - \sigma_j} a_i^{(j)}(x)$ and if $h_{l-1} < i \leq h_l$ then $a_i^{(j)}(x) = a_i^{(j)}(x_1, \dots, x_{h_{l-1}})$. We point out that if $k = 2$ then

$$X_j(x) = \partial_j \quad \forall\ h_1 < j \leq h_2. \tag{11}$$

Let $X = (X_1, \dots, X_m)$ be a Jacobian basis of $\mathbb{G} = (\mathbb{R}^m, \cdot)$ we define for any function $u : \mathbb{G} \longrightarrow \mathbb{R}$ for which the partial derivative $X_j u$ exist ($j = 1, \dots, h_1$), the horizontal gradient by

$$\nabla_{\mathbb{G}} u := \sum_{i=1}^{h_1} (X_i u) X_i. \tag{12}$$

Moreover, we define the horizontal laplacian of $u : \mathbb{G} \longrightarrow \mathbb{R}$ and we denote it by $\Delta_{\mathbb{G}} u$, the following function

$$\Delta_{\mathbb{G}} u := \sum_{i=1}^{h_1} X_i X_i u.$$

We end this preliminary part recalling a well known result which we will use in Section 3.

Lemma 2.3. *Let $\mathbb{G} = (\mathbb{R}^m, \cdot)$ be a Carnot group. There exist $C > 0$ such that for each $f \in C^1(\mathbb{R})$ the following relation holds*

$$|\nabla_{\mathbb{G}} f(|x|_{\mathbb{G}})| \leq C |f'(|x|_{\mathbb{G}})| \quad \mathcal{L}^m - a.e\ x \in \mathbb{G}.$$

Proof. By definition the map $\mathbb{G} \ni x \mapsto |x|_{\mathbb{G}}$ is $1-$ Lipschitz with respect to d (defined in (10)). Denoting by d_{cc} the Carnot Carathéodory distance in $\mathbb{G}$ (we refer to [6] and references therein for the definition), by Proposition 5.1.4 in [6] there exist $C > 0$ such that

$$||x|_{\mathbb{G}} - |y|_{\mathbb{G}}| \leq C \mathrm{d}_{cc}(x, y) \quad \forall x, y \in \overline{B}(0, 1) \tag{13}$$

For each $x, y \in \mathbb{G}$, let $R > 0$ such that $\delta_R(x), \delta_R(y) \in \overline{B}(0, 1)$. Using (13) and the homogeneity of $|\cdot|_{\mathbb{G}}$ we obtain

$$||x|_{\mathbb{G}} - |y|_{\mathbb{G}}| = \frac{1}{R}\Big||\delta_R x|_{\mathbb{G}} - |\delta_R y|_{\mathbb{G}}\Big| \leq \frac{C}{R} \mathrm{d}_{cc}(\delta_R x, \delta_R y) = C \mathrm{d}_{cc}(x, y). \tag{14}$$

By Theorem 2.2.1 in [21] we conclude that

$$|\nabla_{\mathbb{G}} |x|_{\mathbb{G}}| \leq C \quad \mathcal{L}^m - \text{a.e}\ x \in \mathbb{G}.$$

The thesis follows observing that

$$|\nabla_{\mathbb{G}} f(|x|_{\mathbb{G}})| = |f'(|x|_{\mathbb{G}})| |\nabla_{\mathbb{G}} |x|_{\mathbb{G}}| \quad \mathcal{L}^m - a.e\ x \in \mathbb{G}. \tag{15}$$

□

2.1. **Right-invariant vector fields.**

Proposition 2.4. *Let $\mathbb{G} \equiv (\mathbb{R}^m, \cdot)$ be a Carnot group with Jacobian basis $X_1, \dots, X_m$. Then there exists a family of vector fields $\tilde{X} = (\tilde{X}_1, \dots, \tilde{X}_m)$ such that*

$$[\tilde{X}_i, X_j] = [X_j, \tilde{X}_j] = 0 \qquad \forall\ i, j = 1, \dots, m. \tag{16}$$

Proof. For each $x \in \mathbb{G}$ let us define $\tilde{\tau}_x : \mathbb{G} \longrightarrow \mathbb{G}$ by $\tilde{\tau}_x(y) := y \cdot x$. We claim that the family of vector fields $\tilde{X} = (\tilde{X}_1, \dots, \tilde{X}_m)$ defined by

$$\tilde{X}_i(x) := \mathcal{J}_{\tilde{\tau}_x}(0) \cdot e_i$$

is such that $[\tilde{X}_i, X_j] = 0\ \forall i, j = 1, \dots, m$.
Let us observe that for every smooth function φ on $\mathbb{R}^m$ and for every $i = 1, \dots, m$

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \varphi((t\tilde{\eta}_i) \cdot x) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \varphi(\tilde{\tau}_x(t\tilde{\eta}_i)) = \nabla\varphi(x) \cdot \mathcal{J}_{\tilde{\tau}_x}(0) \cdot \tilde{\eta}_i = (\tilde{X}_i\varphi)(x) \tag{17}$$

where $\tilde{\eta}_i \in \mathbb{R}^m$ is the vector whose components are the component functions of $\tilde{X}_i$ calculated in $x = 0$. Moreover, by a computation similar to the one presented in (17) we also obtain that for each $j = 1, \dots m$

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \varphi(x \cdot (t\eta_j)) = (X_j\varphi)(x) \tag{18}$$

where $\eta_j \in \mathbb{R}^m$ is the vector whose components are the component functions of X_j computed in $x = 0$. By (17) and (18) we infer that for every smooth function φ on $\mathbb{R}^m$

$$\begin{aligned} [\tilde{X}_j, X_i]\varphi(x) &= (\tilde{X}_j(X_i\varphi))(x) - (X_i(\tilde{X}_j\varphi))(x) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \Big[(X_i\varphi)(t\tilde{\eta}_j \cdot x) - (\tilde{X}_j\varphi)(x \cdot (t\eta_i)) \Big] \\ &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}_{|t=0} \Big[\varphi((t\tilde{\eta}_j) \cdot x \cdot (t\eta_i)) - \varphi((t\tilde{\eta}_j) \cdot x \cdot (t\eta_i)) \Big] = 0 \end{aligned} \tag{19}$$

which is the thesis. □

Remark 2.5. *By (17) it immediately follows that $\tilde{X}_j$ is right invariant for each $j = 1, \dots, m$ (see* [6] *for the definition).*

Example 2.6. *We recall that the $n-$dimensional Heisenberg group, $\mathbb{H}^n$, is a Carnot group of step 2 with $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \cdot)$ and for each $(x, y, t), (\bar{x}, \bar{y}, \bar{t}) \in \mathbb{H}^n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$*

$$(\bar{x}, \bar{y}, \bar{t}) \cdot (x, y, t) := (\bar{x} + x, \bar{y} + y, \bar{t} + t + 2(\langle\ \bar{y}, x\rangle - \langle\ \bar{x}, y\rangle)),$$

where $\langle\ \cdot\ ,\ \cdot \rangle$ is the standard Euclidean scalar product in $\mathbb{R}^n$. It is well known that the Jacobian basis is $X_j = \partial_j + 2y_j\partial_t$ and $X_{j+n} = \partial_{j+n} - 2x_j\partial_t$, for $j = 1, \dots, n$ and $X_{2n+1} = \partial_t$ (see for example [6]*). In order to find the family $\tilde{X}_j$ we observe that*

$$\mathcal{J}_{\tilde{\tau}_{(x,y,t)}}(0) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ -2y_1 & -2y_2 & \dots & 2x_n & 1 \end{pmatrix}$$

and, using Proposition 2.4, we conclude that $\tilde{X}_j = \partial_j - 2y_j\partial_t$, $\tilde{X}_{j+n} = \partial_{j+n} + 2x_j\partial_t$ *for* $j = 1, \dots, n$ *and* $\tilde{X}_{2n+1} = \partial_t$.

Example 2.7. *Let us now consider the Engel group, which is the Carnot group usually denoted by* $\mathbb{E}$*, whose Lie algebra* $\mathfrak{e}$ *is such that* $\mathfrak{e} = V_1 \oplus V_2 \oplus V_3$ *with* $V_1 = \mathrm{span}\{X_1, X_2\}$, $V_2 = \mathrm{span}\{X_3\}$ *and* $V_4 = \mathrm{span}\{X_4\}$ *and the only nonvanishing commutators are*

$$[X_1, X_2] = X_3 \quad , \quad [X_1, X_3] = X_4.$$

Using exponential coordinates of second kind for each $x = (x_1, x_2, x_3, x_4)$, $y = (y_1, y_2, y_3, y_4) \in \mathbb{E}$ *we get*

$$x \cdot y = \Big(x_1 + y_1, x_2 + y_2, x_3 + y_3 - y_1x_2, x_4 + y_4 + \frac{1}{2}y_1^2x_2 - y_1x_3\Big) \tag{20}$$

consequently the Jacobian basis is

$$\begin{array}{cc} X_1 = \partial_1 - x_2\partial_3 - x_3\partial_4 & X_2 = \partial_2 \\ X_3 = \partial_3 & X_4 = \partial_4. \end{array}$$

Moreover, by (20) and Proposition 2.4, it follows that

$$\begin{array}{cc} \tilde{X}_1 = \partial_1 & \tilde{X}_2 = \partial_2 - x_1\partial_3 + \frac{x_1^2}{2}\partial_4 \\ \tilde{X}_3 = \partial_3 - x_1\partial_4 & \tilde{X}_4 = \partial_4 \end{array}$$

and the only nonvanishing commutators are

$$[\tilde{X}_1, \tilde{X}_2] = -\tilde{X}_3 \quad [\tilde{X}_1, \tilde{X}_3] = -\tilde{X}_4 \tag{21}$$

3. Semilinear PDEs in Carnot groups

In the following section we recall some results contained in [3]. In particular, using the results contained in Section 2, it follows that the abstract setting described in [3] can be applied to every Carnot group. Using this observation, we prove a classification result for stable solutions of $\Delta_{\mathbb{G}}u = f(u)$ in every Carnot group. Throughout this section, we denote by $\mathbb{G} = (\mathbb{R}^m, \cdot)$ a Carnot group with Jacobian basis $X = (X_1, \dots, X_m)$ and by $\tilde{X}_1, \dots, \tilde{X}_m$ the family of right invariant vector fields associated to X as in Proposition 2.4. Moreover, we call right horizontal gradient of a function $u : \mathbb{G} \longrightarrow \mathbb{R}$ and we denote it by $\tilde{\nabla}_{\mathbb{G}}u$ the operator $\tilde{\nabla}_{\mathbb{G}}u := \sum_{i=1}^{h_1}(\tilde{X}_iu)\tilde{X}_i$.

We start recalling that for any fixed $f \in C^\infty(\mathbb{R})$ a solution u of $\Delta_{\mathbb{G}}u = f(u)$ in an open $\Omega \subset \mathbb{G}$ is said to be stable if

$$\int_\Omega |\nabla_{\mathbb{G}}\varphi(x)|^2 + f'(u(x))\varphi^2(x)\mathrm{d}x \geq 0 \quad \forall \varphi \in C_c^\infty(\Omega).$$

The previous stability condition has been widely studied in the calculus of variation setting. Indeed, it states that the second variation of the energy functional associated with (1) is nonnegative at the critical point u hence, for instance, minimal solutions are always stable, but, in principle, stability is a weaker condition than minimality (we refer the interested reader to [12] for further motivations about the stability condition).

Lemma 3.1. *Let $f \in C^\infty(\mathbb{R})$ and u be a solution of $\Delta_{\mathbb{G}} u = f(u)$ in an open $\Omega \subseteq \mathbb{G}$ then $u \in C^\infty(\Omega)$. Moreover if for some $j \in \{1, \dots, h_1\}$*

$$\tilde{X}_j u > 0 \quad \text{in } \Omega$$

then u is stable. Here h_1 is the same of Definition 2.1.

Proof. The first part directly follows from the celebrated Hörmander's Theorem (see [18]). For the second part, we start observing that, by (16), $\tilde{X}_j u$ is a positive solution of $\Delta_{\mathbb{G}} \xi = f'(u)\xi$ in Ω. Hence, for each $\varphi \in C_c^\infty(\Omega)$ if $\xi \neq 0$ then $\varphi^2/\xi \in C_c^\infty(\Omega)$ and

$$\begin{aligned}
0 &= \int_\Omega \left\langle \nabla_{\mathbb{G}} \xi, \nabla_{\mathbb{G}}(\varphi^2/\xi) \right\rangle + f'(u)(\varphi^2/\xi) \mathrm{d}x \\
&= \int_\Omega \frac{2\varphi}{\xi} \left\langle \nabla_{\mathbb{G}} \xi, \nabla_{\mathbb{G}} \varphi \right\rangle - \frac{\varphi^2}{\xi^2} |\nabla_{\mathbb{G}} \xi|^2 + f'(u)\varphi^2 \mathrm{d}x \\
&= \int_\Omega |\nabla_{\mathbb{G}} \varphi|^2 - |\nabla_{\mathbb{G}} \varphi - \frac{\varphi}{\xi} \nabla_{\mathbb{G}} \xi|^2 + f'(u)\varphi^2 \mathrm{d}x \\
&\leq \int_\Omega |\nabla_{\mathbb{G}} \varphi|^2 + f'(u)\varphi^2 \mathrm{d}x.
\end{aligned}$$

□

Using Proposition 2.4, the following result, which was originally proved in [3] for a general manifold M endowed with smooth vector fields $X_1, \dots X_k$ and a family of vector fields $\tilde{X}_1, \dots \tilde{X}_m$ such that

$$[X_i, \tilde{X}_j] = 0 \quad \forall i, j = 1, \dots, m,$$

also holds in every Carnot group.

Theorem 3.2. *Let $\Omega \subseteq \mathbb{G}$ be an open set and $u \in C^\infty(\Omega)$. Let*

$$\Omega_0 := \Omega \cap \{\tilde{\nabla}_{\mathbb{G}} u \neq 0\}.$$

Denote $\tilde{\nu}(x) := \frac{\tilde{\nabla}_{\mathbb{G}} u(x)}{|\tilde{\nabla}_{\mathbb{G}} u(x)|}$ and

$$\mathcal{D}(x) := \sum_{j=1}^{h_1} \left[|\tilde{\nabla}_{\mathbb{G}} X_j u|^2 - \left\langle \tilde{\nu}, \tilde{\nabla}_{\mathbb{G}} X_j u \right\rangle^2 \right](x)$$

for $x \in \Omega_0$. Then, $\mathcal{D} \geq 0$ and the following conditions are equivalent:

(1) *$\mathcal{D}(x) = 0$ at some point of $x \in \Omega_0$,*
(2) *$\tilde{\nabla}_{\mathbb{G}} X_j u$ is parallel to $\tilde{\nu}$ at such point for any $j = 1, \dots, h_1$,*
(3) *$X_j \tilde{\nu}$ is parallel to $\tilde{\nu}$ at such point for any $j = 1, \dots, h_1$.*

Moreover, if u is a stable solution of $\Delta_{\mathbb{G}} u = f(u)$ in Ω then

$$\int_{\Omega_0} \mathcal{D}\varphi^2 \mathrm{d}x \leq \int_\Omega |\nabla_{\mathbb{G}} \varphi|^2 |\tilde{\nabla}_{\mathbb{G}} u|^2 \mathrm{d}x \qquad \forall \varphi \in C_c^\infty(\Omega). \tag{22}$$

Using Theorem 3.2 and some ideas contained in [13] and [22] we can prove the following proposition.

Proposition 3.3. *Let $\mathbb{G} = (\mathbb{R}^m, \cdot)$ be a Carnot group of step $k \in \mathbb{N}$. Let u be a stable solution of $\Delta_{\mathbb{G}} u = f(u)$ in the whole $\mathbb{G}$. If*

$$\liminf_{R\to+\infty} \frac{2\int_{\sqrt{R}}^{R} \frac{\eta(\tau)}{\tau^3}\mathrm{d}\tau + \frac{1}{R^2}\eta(R)}{\log(R)^2} = 0 \tag{23}$$

then, for all $j = 1, \dots, m$, $\tilde{\nabla}_{\mathbb{G}} X_j u$, $X_j \tilde{\nu}$ and $\tilde{\nu}$ are all parallel at any point of $\mathbb{G}_0 := \{x \in \mathbb{G} \mid \tilde{\nabla}_{\mathbb{G}} u(x) \neq 0\}$ and $\tilde{\nu}$ is constant in $\mathbb{G}_0$.

Before giving this proof, we recall the following useful result proved in [13].

Lemma 3.4. *Let $g \in L^\infty_{loc}(\mathbb{R}^n, [0, +\infty))$ and let $q > 0$. Let also, for any $\tau > 0$,*

$$\kappa(\tau) := \int_{B(0,\tau)} g(x)\mathrm{d}x \tag{24}$$

Then, for every $0 < r < R$,

$$\int_{B(0,R)\setminus B(0,r)} \frac{g(x)}{|x|^q}\,\mathrm{d}x \le q\int_r^R \frac{\kappa(\tau)}{\tau^{q+1}}\,\mathrm{d}\tau + \frac{1}{R^q}\kappa(R).$$

Proof of Proposition 3.3. For any $R > 1$, we define

$$\varphi_R(x) := \begin{cases} 1 & \text{if } x \in B(0, \sqrt{R}) \\ 2(\log R)^{-1}\log(R/|x|_{\mathbb{G}}) & \text{if } x \in B(0,R)\setminus B(0,\sqrt{R}) \\ 0 & \text{if } x \in \mathbb{G}\setminus B(0,R) \end{cases}$$

Using Lemma 2.3 with $f(s) := s^{2k!}$ we obtain[1]

$$|\nabla_{\mathbb{G}}|x|_{\mathbb{G}}^{2k!}| \le C|x|_{\mathbb{G}}^{2k!-1}. \tag{25}$$

Using the explicit expression of φ_R and (25) we obtain that for every $x \in \mathbb{G}$

$$|\nabla_{\mathbb{G}}\varphi_R(x)| = \frac{|\nabla_{\mathbb{G}}|x|_{\mathbb{G}}^{2k!}|}{4k!\log(R)|x|_{\mathbb{G}}^{2k!}} \le \frac{C}{\log(R)|x|_{\mathbb{G}}}.$$

Therefore, by Theorem 3.2,

$$\int_{G_0} \mathcal{D}\varphi_R^2\mathrm{d}x \le \int_{\mathbb{G}} |\nabla_{\mathbb{G}}\varphi_R|^2|\tilde{\nabla}_{\mathbb{G}}u|^2\mathrm{d}x \le C(\log(R))^{-2}\int_{B(0,R)\setminus B(0,\sqrt{R})} \frac{|\tilde{\nabla}_{\mathbb{G}}u|^2}{|x|_{\mathbb{G}}^2}\mathrm{d}x.$$

By Lemma 3.4 we obtain

$$\int_{B(0,R)\setminus B(0,\sqrt{R})} \frac{|\tilde{\nabla}_{\mathbb{G}}u|^2}{|x|_{\mathbb{G}}^2}\mathrm{d}x \le 2\int_{\sqrt{R}}^{R} \frac{\eta(\tau)}{\tau^3}\mathrm{d}\tau + \frac{1}{R^2}\eta(R)$$

where η is as in (3). The thesis follows by sending $R \to +\infty$ and using Theorem 3.2. □

[1] here C is a positive constant which could be different from line to line.

4. Proof of the main result in Carnot group of type $\star$

In the rest of this section we will focus on the class of Carnot group satisfying the flatness condition, see Definition 1.1.

In [15] and [3] it is proved that every Carnot group of step 2 satisfies the flatness condition. Recently, in [19], the class of Carnot group satisfying Definition 1.1 has been extended to the so called Carnot group of type $\star$, see the Introduction for the precise Definition. Obviously, every Carnot group of step 2 is of type $\star$. It is also interesting to note that the class of group of type $\star$ contains Carnot groups of arbitrarily large step. Indeed, for every $m \in \mathbb{N}$, the Lie group of unit upper triangular $(m+1)\times(m+1)$ matrices is a Carnot group of type $\star$ (see [19, Example 2.3]). Moreover, as pointed out in [19], if a Carnot group of step greater than 2 is of type $\star$ then the dimension of its first layer V_1 is at least 3, therefore the Engel group is not of type $\star$. We also observe that the Engel group does not satisfy the flatness condition at all. Indeed, let us consider the functions $u_\alpha : \mathbb{E} \longrightarrow \mathbb{R}$

$$u_\alpha(x_1, x_2, x_3, x_4) := e^{x_2} + x_4 + x_4^3 + \alpha, \quad \alpha \in \mathbb{R} \tag{26}$$

then $u_\alpha \in C^\infty(\mathbb{E})$ and it is a matter of calculations that

$$\tilde{X}_1 u_\alpha = 0, \quad \tilde{X}_2 u_\alpha = e^{x_2} + \frac{x_1^2}{2}(3x_4^2 + 1) > 0$$

in particular, $\tilde{\nu} = (0, 1)$.

Using Propositions 3.3 and Lemma 3.1 we can give the proof of Theorem 1.2.

Proof of Theorem 1.2. The proof is by contradiction. Let us suppose that there exists $u \in C^\infty(\Omega)$ solution of $\Delta_{\mathbb{G}} u = f(u)$ satisfying (i) and (ii). By Lemma 3.1, u is stable. Moreover, by Proposition 3.3, we have $\tilde{\nu}$ constant in $\mathbb{G}$ and hence

$$u(x^{(1)}, \dots, x^{(k)}) = u_0\Big(\Big\langle a, x^{(1)} \Big\rangle\Big),$$

for some $a \in \mathbb{S}^{h_1 - 1}$ and $u_0 \in C^\infty(\mathbb{R})$ non constant. Since u_0 is non constant there exists $\epsilon > 0$ and an open interval $I \subset \mathbb{R}$ such that $|u_0'(s)| \geq \epsilon$ for any $s \in I$. Moreover, for each $R > 0$ define

$$\mathcal{A}_R^k := \Big\{ x = (x^{(1)}, \dots, x^{(k)}) \in \mathbb{G} \mid \|x^{(j)}\| \leq \frac{R^j}{k^{\frac{j}{2k!}}}, j = 1, \dots, k \Big\}$$

then

$$\mathcal{A}_R^k \subset B(0, R).$$

Hence

(27)

$$\int_{B(0,R)} |\tilde{\nabla}_{\mathbb{G}} u(x)|^2 \mathrm{d}x \geq \int_{\mathcal{A}_R^k} |\tilde{\nabla}_{\mathbb{G}} u(x)|^2 \mathrm{d}x$$
$$= \int_{\{\|x^{(1)}\| \leq R/k^{\frac{1}{2k!}}\}} \int_{\{\|x^{(2)}\| \leq R^2/k^{\frac{2}{2k!}}\}} \cdots \int_{\{\|x^{(k)}\| \leq R^k/k^{\frac{k}{2k!}}\}} |\tilde{\nabla}_{\mathbb{G}} u(x)|^2 \mathrm{d}x$$
$$= \int_{\{\|x^{(1)}\| \leq R/k^{\frac{1}{2k!}}\}} \left| u_0'\Big(\left\langle a, x^{(1)} \right\rangle \Big) \right|^2 \mathrm{d}x^{(1)} \int_{\{\|x^{(2)}\| \leq R^2/k^{\frac{2}{2k!}}\}} \cdots \int_{\{\|x^{(k)}\| \leq R^k/k^{\frac{k}{2k!}}\}} \mathrm{d}x^{(k)} \dots \mathrm{d}x^{(2)}$$
$$= CR^{\sum_{i=2}^k i(h_i - h_{i-1})} \int_{\{\|x^{(1)}\| \leq R/k^{\frac{1}{2k!}}\}} \left| u_0'\Big(\left\langle a, x^{(1)} \right\rangle \Big) \right|^2 \mathrm{d}x^{(1)},$$

for some $C > 0$. Observing that $\sum_{i=2}^k i(h_i - h_{i-1}) = Q - h_1$ we conclude

$$\int_{B(0,R)} |\tilde{\nabla}_{\mathbb{G}} u(x)|^2 \mathrm{d}x \geq CR^{Q-h_1} \int_{\{\|x^{(1)}\| \leq R/k^{\frac{1}{2k!}}\}} \left| u_0'\Big(\left\langle a, x^{(1)} \right\rangle \Big) \right|^2 \mathrm{d}x^{(1)}. \tag{28}$$

Now we consider $a_1, \dots, a_{h_1}$ unit vectors such that $a_1 = a$ and $\{a_1, \dots, a_{h_1}\}$ is an orthonormal basis of $\mathbb{R}^{h_1}$ and we define the following change of variables

$$\tilde{x}_i := \left\langle a_i, x^{(1)} \right\rangle \quad i = 1, \dots, h_1$$

so that

$$\begin{aligned} &\int_{B(0,R)} |\tilde{\nabla}_{\mathbb{G}} u(x)|^2 \mathrm{d}x \\ &\geq CR^{\mathcal{Q}-h_1} \int_{\{\|\tilde{x}\| \leq h_1(R/k^{\frac{1}{2k!}})\}} |u_0'(\tilde{x}_1)|^2 \mathrm{d}\tilde{x}. \end{aligned} \tag{29}$$

Hence for each R sufficiently large the following inequality holds

$$\begin{aligned} &\int_{B(0,R)} |\tilde{\nabla}_{\mathbb{G}} u(x)|^2 \mathrm{d}x \\ &\geq CR^{\mathcal{Q}-h_1} \int_{\{\sum_{i=2}^{h_1} \tilde{x}_i^2 \leq h_1^2 (R^2/k^{2/2k!})\}} \int_{\{\tilde{x}_1 \in I\}} |u_0'(\tilde{x}_1)|^2 \mathrm{d}\tilde{x} \\ &\geq CR^{\mathcal{Q}-h_1} \epsilon^2 |I| \int_{\{\sum_{i=2}^{h_1} \tilde{x}_i^2 \leq h_1^2 (R^2/k^{2/2k!})\}} \mathrm{d}\tilde{x} \\ &= CR^{\mathcal{Q}-1} \epsilon^2 |I|, \end{aligned} \tag{30}$$

which is clearly incompatible with

$$\liminf_{R \to +\infty} \frac{2 \int_{\sqrt{R}}^R \frac{\eta(\tau)}{\tau^3} \mathrm{d}\tau + \frac{1}{R^2} \eta(R)}{\log(R)^2} = 0$$

since $\mathcal{Q} \geq 4$ for each Carnot group of step $k \geq 2$. □

Corollary 4.1. *Let $\mathbb{G}$ be a Carnot group satisfying the flatness condition. If $f \in C^\infty(\mathbb{R})$ then there are no solutions of $\Delta_{\mathbb{G}} u = f(u)$ satisfying $\tilde{X}_j u > 0$ for some $j \in \{1, \dots, h_1\}$ and $\tilde{\nu}$ constant in $\mathbb{G}$.*

5. Semilinear PDEs in the Engel group

In this section we provide some partial extensions to the Engel group of the results contained in Sections 3 and 4. It is well known that the Engel group's geometry is much more complicated than the one of Carnot groups of step 2 (see for example [1, 2, 15]). In particular, as pointed out in [15], there exist sets with constant normal $\tilde{\nu}$ which are not vertical halfspaces (see also the family of functions u_α defined in (26)). This fact implies that the Engel group does not satisfy the flatness condition. Nevertheless, as mentioned in the Introduction (see Proposition 1.3) $\mathbb{E}$ satisfies a partial flatness condition. We also recall that, proceeding exactly as in [2, Lemma 2.3], if $\tilde{\nu}$ is constant then it is not restrictive to suppose either $\tilde{\nu} = (1, 0)$ or $\tilde{\nu} = (0, 1)$.

Before giving the proof of Proposition 1.3 we recall a useful result proved in [1] and successively refined and improved in [2].

Proposition 5.1. *Let $\mathbb{G}$ be a Carnot group of step 3 with Lie algebra $\mathfrak{g}$. Let $X, Y \in \mathfrak{g}$ and $u \in C^\infty(\mathbb{G})$. If $Xu = 0$ and $Yu \geq 0$ in $\mathbb{G}$ then*

$$\Big(Y + [X, Y] + \frac{1}{2}[X, [X, Y]]\Big)u \geq 0 \quad \text{in } \mathbb{G}. \tag{31}$$

Proof of Proposition 1.3: This proof is inspired by analogous arguments of Lemma 2.1 in [2].

We start observing that defining $\tilde{\mathbb{E}} := (\mathbb{R}^4, \cdot)$ where

$$(x_1, x_2, x_3, x_4) \cdot (y_1, y_2, y_3, y_4) := \Big(x_1 + y_1, x_2 + y_2, x_3 + y_3 - y_1x_2, x_4 + y_4 + \frac{1}{2}y_1^2x_2 - y_1x_3\Big)$$

then, $\tilde{\mathbb{E}}$ is a Carnot group of step 3 with Jacobian basis $(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4)$. Here $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4$ are as in Remark 2.7. By (i) we know that

$$\tilde{X}_1 u > 0 \quad \text{in } \tilde{\mathbb{E}} \quad \text{and} \quad (t\tilde{X}_2)u = 0 \quad \forall t \in \mathbb{R}. \tag{32}$$

By Proposition 5.1 and (21) we have

$$\tilde{X}_1 u - (t\tilde{X}_3)u \geq 0 \quad \text{in } \tilde{\mathbb{E}}. \tag{33}$$

Hence letting $t \to +\infty$ and $t \to -\infty$ in (33) we get

$$\tilde{X}_3 u = 0 \quad \text{in } \tilde{\mathbb{E}}. \tag{34}$$

Analogously, applying Proposition 5.1 with $Y = \tilde{X}_1$ and $X = t\tilde{X}_3$, for each $t \in \mathbb{R}$ we get

$$\tilde{X}_1 u - (t\tilde{X}_4)u \geq 0 \quad \text{in } \tilde{\mathbb{E}}, \tag{35}$$

and again letting $t \to +\infty$ and $t \to -\infty$ in (35) we obtain

$$\tilde{X}_4 u = 0 \quad \text{in } \tilde{\mathbb{E}}. \tag{36}$$

By (32), (34) ,(36) and the explicit expression of $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4$ (see Example (2.7)) we conclude that $\partial_i u = 0$ for $i \in \{2, 3, 4\}$ which is the thesis. □

In the remaining part of this Section we provide some partial results about the non existence of solutions of semilinear equations with $\tilde{\nu} = (0, 1)$. To this end we start proving the following characterization Lemma.

Lemma 5.2. *Let $u \in C^1(\mathbb{E})$ with $\tilde{\nu} = (0,1)$ then $\partial_1 u = 0$, $\partial_2 u > 0$ in $\mathbb{E}$ and*

(i) *If $\partial_4 u(x) = 0$ then $\partial_3 u(x) = 0$,*

(ii) *If $\partial_4 u(x) \neq 0$ then $(\partial_3 u(x))^2 < 2(\partial_2 u(x))(\partial_4 u(x))$.*

Moreover, if $\partial_4 u(x) \neq 0$ then $\partial_4 u(x) > 0$. In addition, if $\partial_1 u = 0$, $\partial_2 u > 0$ in $\mathbb{E}$ and for every $x \in \mathbb{E}$ either (i) or (ii) holds then $\tilde{\nu} = (0,1)$.

Proof. By definition, $\tilde{\nu} = (0,1)$ if and only if

$$\tilde{X}_1 u = \partial_1 u = 0 \tag{37}$$

$$\tilde{X}_2 u = \partial_2 u - x_1 \partial_3 u + \frac{x_1^2}{2} \partial_4 u > 0 \quad \text{in } \mathbb{E}. \tag{38}$$

By (37) u is independent of x_1, hence choosing $x_1 = 0$ in (38) we get $\partial_2 u > 0$ in $\mathbb{E}$. If $\partial_4 u(x) = 0$ for some $x \in \mathbb{E}$ then by (38) the following inequality holds

$$\partial_2 u(x) - x_1 \partial_3 u(x) > 0 \quad \forall x_1 \in \mathbb{R} \tag{39}$$

which easily implies $\partial_3 u(x) = 0$. On the other hand, if $\partial_4 u(x) \neq 0$ then $\tilde{X}_2 u(x)$ is a second order polynomial in x_1. Therefore, inequality (38) is satisfied if and only if

$$(\partial_3 u(x))^2 < 2(\partial_2 u(x))(\partial_4 u(x)).$$

The reverse implication easily follows using the same argument. □

Now we recall the classical Liouville Theorem for the operator $\Delta_{\mathbb{E}}$, see [6, Theorem 5.8.1] for a more general version and the proof.

Theorem 5.3. *Let $u \in C^\infty(\mathbb{E})$ be a function satisfying $u \geq 0$ and $\Delta_{\mathbb{E}} u = 0$ in $\mathbb{E}$, then u is constant.*

We are now in position to give the proof of Proposition 1.4

Proof of Proposition 1.4: By definition, u solves $\Delta_{\mathbb{E}} u = f(u)$ if and only if

$$x_2 \partial_4 u + x_3^2 \partial_{4,4} u + \partial_{2,2} u = f(u). \tag{40}$$

By assumption, $f(u) - \partial_{2,2} u - x_2 \partial_4 u$ is independent of x_3, hence

$$2x_3 \partial_{4,4} u = \partial_3 (x_3^2 \partial_{4,4} u) = \partial_3 (f(u) - \partial_{2,2} u - x_2 \partial_4 u) = 0 \tag{41}$$

which implies $\partial_{4,4} u = 0$ in $\mathbb{E}$. From (41) we easily infer

$$u(x_2, x_4) = x_4 h_1(x_2) + h_2(x_2)$$

for some, $h_1, h_2 \in C^\infty(\mathbb{R})$. By Lemma 5.2,

$$\partial_2 u(x_2, x_4) = x_4 h_1'(x_2) + h_2'(x_2) > 0 \quad \forall (x_2, x_4) \in \mathbb{R}^2$$

and therefore $h_1' = 0, h_2' > 0$ in $\mathbb{R}$. All in all we proved that

$$u(x_2, x_4) = a x_4 + h_2(x_2) \quad a \in \mathbb{R}. \tag{42}$$

We claim that $a = 0$. Indeed, if $a \neq 0$, then by (40) and (42) we get

$$0 = \partial_4 (a x_2 + h_2''(x_2)) = \partial_4 (f(u)) = a f'(u),$$

and hence f is a constant. By Proposition 2.4 and (2)

$$\tilde{X}_2\Delta_{\mathbb{E}}u = \Delta_{\mathbb{E}}\tilde{X}_2u = 0,$$
$$\tilde{X}_2u > 0$$

therefore, by Theorem 5.3, $\tilde{X}_2u$ is a constant. Finally, recalling (21), we get

$$\tilde{X}_3u = [\tilde{X}_2, \tilde{X}_1]u = 0 \quad \text{and} \quad \tilde{X}_4u = [\tilde{X}_3, \tilde{X}_1]u = \partial_4 u = a = 0,$$

which is in contradiction with $a \neq 0$. This proves that

$$u(x_1, x_2, x_3, x_4) = h_2(x_2) \quad \text{in } \mathbb{E} \tag{43}$$

and the thesis follows arguing as in Theorem 1.2. □

Remark 5.4. *From Proposition 1.4 we get that if $u \in C^\infty(\mathbb{E})$ satisfies $\tilde{\nu} = (0,1)$ and it is a solution of $\Delta_{\mathbb{E}}u = f(u)$ then it has to depend also on x_3. We explicitely observe that, by Lemma 5.2, the same nonexistence result proved in Proposition 1.4 also holds if $u = u(x_2, x_3)$ or $u = u(x_3, x_4)$. Indeed, by Lemma 5.2, if u does not depend on x_4 then $\partial_3 u = 0$ in $\mathbb{E}$ and we conclude as in Theorem 1.2. Moreover, always by Lemma 5.2, $\partial_2 u > 0$ in $\mathbb{E}$ and hence u has to depend on x_2.*

Lemma 5.5. *Let $f \in C^\infty(\mathbb{R})$. Let $u \in C^\infty(\mathbb{E})$ be a solution of $\Delta_{\mathbb{E}}u = f(u)$ such that*

$$\partial_1 u = \partial_{3,3}\partial_2 u = \partial_{3,4}\partial_2 u = \partial_{4,4}\partial_2 u = 0 \quad \text{in } \mathbb{E},$$

then the following relation holds

$$f^{(3)}(u)(\partial_2 u)^3 + 3f^{(2)}(u)\partial_2 u\partial_2^{(2)}u + f^{(1)}(u)\partial_2^{(3)}u - \partial_2^{(5)}u - \partial_4\partial_2^{(2)}u = 0 \quad \text{in } \mathbb{E} \tag{44}$$

where $f^{(k)}$ denotes the $k-$th derivative of f and $\partial_2^{(k)}u := \underbrace{\partial_2 \dots \partial_2}_{k} u$.

Proof. Using the coordinate expressions of X_1 and X_2 and Lemma 5.2 we get that u solves $\Delta_{\mathbb{E}}u = f(u)$ if and only if

$$x_2^2\partial_{3,3}u + x_2\partial_4 u + 2x_2x_3\partial_{3,4}u + x_3^2\partial_{4,4}u + \partial_{2,2}u = f(u). \tag{45}$$

Since $x_3^2\partial_{4,4}u$ does not depend on x_2 we obtain

$$\partial_2(f(u) - x_2^2\partial_{3,3}u - 2x_2x_3\partial_{3,4}u - \partial_{2,2}u - x_2\partial_4 u) = 0$$

and hence,

$$2x_3\partial_{3,4}u + \partial_4 u = f^{(1)}(u)\partial_2 u - \partial_2^{(3)}u - 2x_2\partial_3^{(2)}u. \tag{46}$$

Using the same argument in the equality above we have

$$2\partial_3^{(2)}u + \partial_{2,4}u = f^{(2)}(u)(\partial_2 u)^2 + f^{(1)}(u)\partial_2^{(2)}u - \partial_2^{(4)}u \tag{47}$$

and deriving once more with respect to x_2 we obtain the thesis. □

Proof Proposition 1.5: We start observing that, by Lemma 5.5 and (i) the following relation holds

$$f^{(3)}(u)\Big(g_1^{(1)}\Big)^3 + 3f^{(2)}(u)g_1^{(1)}g_1^{(2)} + f^{(1)}(u)g_1^{(3)} - g_1^{(5)} = 0 \tag{48}$$

and hence,

$$0 = \partial_4\Big[f^{(3)}(u)(g_1^{(1)})^3 + 3f^{(2)}(u)g_1^{(1)}g_1^{(2)} + f^{(1)}(u)g_1^{(3)} - g_1^{(5)}\Big] = \partial_4 g_2\Big(f^{(4)}(u)(g_1)^3 + 3f^{(3)}(u)g_1^{(1)}g_1^{(2)} + f^{(2)}(u)g_1^{(3)}\Big).$$

which together with $\partial_4 u > 0$ implies

$$f^{(4)}(u)(g_1)^3 + 3f^{(3)}(u)g_1^{(1)}g_1^{(2)} + f^{(2)}(u)g_1^{(3)} = 0. \tag{49}$$

Let us start supposing $f(s) := as^3 + bs^2 + cs + d$, $a, b, c, d \in \mathbb{R}$, $a \neq 0$ then by (49) we get the following relation:

$$18ag_1^{(1)}g_1^{(2)} + 6ag_1g_1^{(3)} + 2bg_1^{(3)} = -6ag_2g_1^{(3)}. \tag{50}$$

Hence, the second member is independent to x_4, so that $(\partial_4 g_2)g_1^{(3)} = 0$. Moreover, since $\partial_4 g_2 > 0$, we conclude that

$$g_1^{(3)}(x_2) = 0 \quad \text{in } \mathbb{R}. \tag{51}$$

Therefore, by (50), it follows that

$$g_1^{(1)}(x_2)g_1^{(2)}(x_2) = 0 \quad \forall x_2 \in \mathbb{R}$$

and recalling that, by Lemma 5.2, $g^{(1)}(x_2) > 0$ we get

$$g_1^{(2)}(x_2) = 0 \quad \forall x_2 \in \mathbb{R}.$$

Hence, $g_1(x_2) = ex_2 + f$, $e, f \in \mathbb{R}$, $e > 0$ and using again (48) we conclude that $6ae^3 = 0$ which contradicts the fact that $a, e > 0$.

If $f(s) = as^2 + bs + c$, $a, b, c \in \mathbb{R}$, $a \neq 0$ then, by (49), we deduce $g_1^{(3)}(x_2) = 0$ in $\mathbb{R}$ and, by (48) and the fact that $\partial_4 g_2 > 0$, we have $g_1^{(1)}g_1^{(2)} = 0$ in $\mathbb{R}$. Finally, using (46), we conclude that $ae = 0$ which, as before, is contradiction with the fact that $a, e > 0$.

If $f(s) = as + b$, $a, b \in \mathbb{R}$, $a \neq 0$ then by (47) we get

$$2\partial_3^{(2)}g_2(x_3, x_4) = ag_1^{(2)}(x_2) - g_1^{(4)}(x_2) \quad \forall (x_2, x_3, x_4) \in \mathbb{R}^3$$

and hence,

$$\partial_3^{(2)}g_2(x_3, x_4) = c \in \mathbb{R}, \quad \forall (x_3, x_4) \in \mathbb{R}^2 \tag{52}$$

$$ag_1^{(2)}(x_2) - g_1^{(4)}(x_2) = c \in \mathbb{R} \quad \forall x_2 \in \mathbb{R}. \tag{53}$$

By (52) there are $h, \tilde{h} \in C^\infty(\mathbb{R})$ such that

$$g_2(x_3, x_4) = \frac{cx_3^2}{2} + x_3h(x_4) + \tilde{h}(x_4) \quad \forall (x_3, x_4) \in \mathbb{R}^2. \tag{54}$$

By (46) we obtain

$$3x_3h^{(1)}(x_4) + \tilde{h}^{(1)}(x_4) = ag_1(x_2) - g_1^{(3)}(x_2) - 2cx_2 \tag{55}$$

therefore h is constant in $\mathbb{R}$ and $\tilde{h}(x_4) = dx_4 + e$, $d, e \in \mathbb{R}$. We now claim that $c = 0$. Indeed, by Lemma 5.2, the following inequality has to be satisfied

$$(x_3 c + h(x_4))^2 < 2g_1^{(1)}(x_2)(x_3 h^{(1)}(x_4) + \tilde{h}^{(1)}(x_4)) \quad \forall (x_2, x_3, x_4) \in \mathbb{R}^3, \tag{56}$$

using the fact that h and $\tilde{h}$ are constant we easily deduce $c = 0$. Therefore, there are $k, e \in \mathbb{R}$ and $d \in \mathbb{R} \setminus \{0\}$ such that

$$g_2(x_3, x_4) = kx_3 + dx_4 + e \quad \forall (x_3, x_4) \in \mathbb{R}^2. \tag{57}$$

Recalling (45), by (57) we obtain

$$dx_2 + g_1^{(1)}(x_2) = f(u) \tag{58}$$

and hence

$$0 = \partial_4(dx_2 + g_1^{(1)}(x_2)) = \partial_4(f(u)) = ad \tag{59}$$

but this is in contradiction with $a, d > 0$.

If $f(s) = a$, $a \in \mathbb{R} \setminus \{0\}$ then

$$\tilde{X}_2 \Delta_{\mathbb{E}} u = \Delta_{\mathbb{E}} \tilde{X}_2 u = 0$$

and hence, by Theorem 5.8.1 in [6], we conclude that $\tilde{X}_2 u$ is constant. Consequentely, $\tilde{X}_3 u = [\tilde{X}_1, \tilde{X}_2]u = 0$ and $\partial_4 u = \tilde{X}_4 = [\tilde{X}_1, \tilde{X}_3]u = 0$ which implies $u(x_1, x_2, x_3, x_4) = \bar{a}x_2 + \bar{b}$, $\bar{a} > 0$, $\bar{b} \in \mathbb{R}$. Since, u solves $\Delta_{\mathbb{E}} u = a$ we have $a = 0$. Finally, if $a = 0$, then $u(x_1, x_2, x_3, x_4) = \bar{a}x_2 + \bar{b}$, $\bar{a} > 0$, $\bar{b} \in \mathbb{R}$ and the conclusion follows proceeding as in Theorem 1.2. □

Remark 5.6. *A complete caracterization of sets with constant normal has been given in* [2]. *Among the other things they proved that each open set in the Engel group with constant normal can be written as the sublevel of a function of the form* $u(x_1, x_2, x_3, x_4) = g_1(x_2) + g_2(x_3, x_4)$. *For this reason we belive that Proposition 1.5 represent a first step toward the classification of the solutions* u *of* $\Delta_{\mathbb{E}} u = f(u)$ *with constant normal.*

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