Ground states of simple bodies that may undergo brittle fractures

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Abstract

Equilibrium states of elastic-brittle solids that may suffer nucleation of cracks in finite deformation setting are analyzed. Crack patterns are described in terms of appropriate Radon measures, namely curvature varifolds with boundary. A new form of the energy is presented: it includes terms associated with the curvature of margins and tips of possible cracks. Existence of minima of the energy are established in classes of pairs of deformation and families of varifolds. Configurational balances in weak form are determined with reference to generic curvature varifolds with boundary. They include non-standard terms associated with the curvatures involved in the energy. Pointwise balances of configurational actions are also evaluated in a special case: new pointwise balances at the tips and along the margins of the crack pattern emerge.

Key words: Fracture mechanics, varifolds, currents, calculus of variations.

Introduction

By following the suggestions in the pioneer work by Griffith [1], a variational view on the mechanics of brittle fracture has been proposed in [2]. There the overall energy $\mathfrak{E}$ of an elastic-brittle simple solid is defined by

$$\mathfrak{E}(\mathfrak{b}, u) := \int_{\mathcal{B}} e(x, Du(x))\ dx + \int_{\mathfrak{b}} \phi\ d\mathcal{H}^2, \qquad (1)$$

where $\mathcal{B}$ is the region occupied in the three-dimensional ambient space by the body under analysis, $Du(x)$ the spatial derivative evaluated at $x \in \mathcal{B}$ of the deformation $x \longmapsto u(x) \in \mathbb{R}^3$, $\mathfrak{b}$ the representation in $\mathcal{B}$ of a surface-like crack occurring in the actual (deformed) place $u(\mathcal{B})$, e the elastic energy in the bulk, ϕ a constant surface energy along the margins of the crack, $d\mathcal{H}^2$ the two-dimensional Hausdorff measure. A cracking process is then a map $t \longmapsto (\mathfrak{b}, u)(t)$, $t \in [0, \bar{t}]$, with $\mathfrak{b}$ considered by definition an *admissible crack* when it is a rectifiable set, consequently with zero volume measure. Minimality of the energy at every time among all virtual crack-displacement pairs at that time is required. Energy conservation throughout the time evolution is also imposed (see [2] for all details).

In evaluating existence of minimizers of $\mathfrak{E}$ in terms of pairs $(\mathfrak{b}, u)$, the convergence of sequences of surfaces in three-dimensions cannot be controlled through the constraints imposed by the energy as it has been defined above. The convenient simplification of identifying cracks with the jump sets of the deformation has been then adopted variously in the subsequent literature. In the minimum problem, then, the sole unknown remains the deformation u. An accurate review of the essential aspects of the consequent results is presented in [3]. However, natural function spaces for minimizers u of the energy host deformations with discontinuity set having closure with positive Lebesgue measure. In this sense, the initial requirement that a crack must be a rectifiable subset of $\mathcal{B}$ to be admissible – as defined in [2] – is not assured this way. Recall that rectifiability [1] is chosen to allow the representation of even really complicated – although realistic – crack patterns, assuring the a.e. existence of the approximate tangent space [2] which describes locally the orientation of the crack. Also, the identification of a crack only with the jump set of u forbids the description of partially closed cracks. At a certain instant of a fracture process, in fact, it may happen that a crack occurred at a previous instant and, due to the updated boundary conditions, at the current instant it is only partially open: u jumps then across the opened part and is continuous across the closed part. Along the latter, however, material bonds placed orthogonally

[1] $\mathfrak{b}$ can be represented as the image of a finite or countable number of Lipshitz functions.

[2] In the sense of geometric measure theory.

to the crack facies do not exist. So, the identification of the crack with the jump set of u does not describe such a physical circumstance.

Moreover, by definition, $\mathfrak{E}$ does not include possible peculiar tip energy due to the (possibly critical) tip arrangement of the material bonds. Also, $\mathfrak{E}$ does not take into account geometric peculiarities of the crack lateral sides, besides their total area.

Here we take back the point of view proposed in [2] and maintain distinction between the deformation and the crack. However, we propose a *different description* of the crack patterns, obtained by means of Radon measures over appropriate fiber bundles, and adopt a *different expression of the energy*. We take, in fact, into account (i) possible tip energy and (ii) a configurational energy associated with the curvature of the crack margins.

Before going into technical details, it can be useful to discuss the general ideas justifying the approach that we propose here.

Consider the typical starting point in traditional continuum mechanics: A body occupies a region $\mathcal{B}$ – a region endowed with a certain regularity specified in various ways – and, starting from it, subsequent macroscopic placements of the same body are reached by means of standard deformations $x \longmapsto u(x)$, $x \in \mathcal{B}$. The map u is assumed to be bijective, (at least approximately) differentiable, and orientation-preserving (the spatial derivative of u has positive determinant). The picture is standard and does not require further details. It has to be stressed, however, that the body remains the *same* along this process. In other words, $\mathcal{B}$ does not change, or better, the body in $\mathcal{B}$ does not undergo mutations in its macroscopic material structure.

When a crack is induced by some prescribed deformation or, generically, loading program, the common statement is that bijectivity of u is lost over some subset of $\mathcal{B}$. The statement implies and is supported by the recognition that a crack changes the structure of a body. In other words, *a cracking body is a mutant body*. Mutation is the occurrence of a crack. Here we consider a cracking (in this sense mutant) body as a family of non-mutant bodies which occupy the same region $\mathcal{B}$ and differ just by the crack pattern. The mutation associated with some prescribed cracking process is then represented by some specific family of bodies.

As already mentioned, a crack pattern can be represented in $\mathcal{B}$ (fictiously in a sense) by a rectifiable subset $\mathfrak{b}$ which is the pre-image, consistent with u, of the set where there is rupture of atomic bonds in the deformed configuration $u(\mathcal{B})$. In other words, $u(\mathfrak{b})$ is the real physical crack. Along the process, the subset $\mathfrak{b}$ varies in a monotonous way: for $t > t'$, $\mathfrak{b}(t) \supseteq \mathfrak{b}(t')$. The assumption that $\mathfrak{b}$ is a rectifiable subset of $\mathcal{B}$ implies the existence of the approximate tangent space at almost every x in $\mathfrak{b}$.

Take a point x in $\mathcal{B}$ and consider the star of planes crossing x. Each plane can, in principle, describe locally a possible crack which could cross x under appropriate conditions. Considering the star of planes just mentioned is tantamount to attach at x the Grassmanian of planes associated with $\mathcal{B}$. Moreover, the point x could be locus of a *one-dimensional crack* – a thin tube of matter where the material bonds are broken. Such a type of crack is characterized at x by the approximate tangent line (rather than plane). The star of straight lines at x has then to be considered. It includes all possible directions that a linear crack could take at x. As before, it can be substituted by the Grassmanian of lines associated with $\mathcal{B}$. If the procedure is repeated at every x, in three-dimensional ambient space two fiber bundles can be constructed. Each one is based on $\mathcal{B}$, the characteristic fiber being the Grassmanian of planes in one case and the Grassmanian of lines in the other one.

Radon measures [3] can be defined over such fiber bundles. These special Radon measures are called *varifolds* (see [7], [8], [9]). Among all possible choices, we are here interested in measures over sub-fiber bundles based on rectifiable subsets of $\mathcal{B}$ with zero volume measure – our aim is, in fact, to represent crack patterns. In particular, the measures used are connected to Hausdorff measures $\mathcal{H}^2$ and $\mathcal{H}^1$, in $3D-$ambient space, admit integer valued density, and (generalized) notions of *boundary* and *curvature* (see [10], [11]), and are called *integer rectifiable varifolds with boundary.* Let V be one of these varifolds, A its (generalized) curvature tensor (a third-rank tensor), ∂V the boundary of V. If V is associated with the two-dimensional Hausdorff measure $\mathcal{H}^2$, then ∂V is connected with $\mathcal{H}^1$.

For example, consider $\mathfrak{b}$ as a smooth bounded surface in $\mathcal{B}$, and take an integer rectifiable two-dimensional varifold $V_{(2)}$, associated with the fiber bundle constructed over $\mathfrak{b}$, by attaching at every x in $\mathfrak{b}$ the Grassmanian of planes crossing $\mathcal{B}$. $V_{(2)}$ is function of places $x \in \mathfrak{b}$ and tangent planes $\Pi(x)$. The projection of $V_{(2)}$ over $\mathfrak{b}$ is a measure there; when it is evaluated over the whole $\mathfrak{b}$, its value is the area of $\mathfrak{b}$ itself, and coincides with the so-called *mass* of $V_{(2)}$, a number indicated by $\mathbf{M}\left(V_{(2)}\right)$. The projection of $\partial V_{(2)}$ over $\mathcal{B}$ measures the intersection between $\mathfrak{b}$ and $\partial\mathcal{B}$, and the margin of $\mathfrak{b}$ in $\mathcal{B}$, namely the tip of the crack represented by $\mathfrak{b}$. At every x in $\mathfrak{b}$, the value $A(x)$ gives information on the curvature of $\mathfrak{b}$ at x.

[3] A *Radon measure* is a measure on the σ-algebra of Borel sets of a Hausdorff topological space X that is *locally finite* and *inner regular*. The presence at every point of a neighborhood of finite measure is the property for the measure to be locally finite. It is also said to be inner regular when the measure of every Borel set in X is the supremum of the set of measures of all compact subsets of it. The two properties allow one to define in a topological space measures with a defined support compatible with the topology.

The same considerations can be made in $n-$dimensional ambient space, for a $k-$dimensional crack. The role of $V_{(2)}$ is played there by a $k-$dimensional varifold $V_{(k)}$.

By restricting for a while the treatment to the three-dimensional case, in the expression of the energy, the part associated with the crack can be then described in terms of $V_{(2)}$. The option with respect to the standard way is not only formal: the use of $V_{(2)}$ allows the natural introduction of an energy contribution depending on the (generalized) curvature tensor A. Such an energy is purely configurational because A 'lives' completely in $\mathcal{B}$: with $\mathcal{B}$ in $\mathbb{R}^3$, in fact, we find $A(x) \in \mathbb{R}^3 \otimes \mathbb{R}^{3*} \otimes \mathbb{R}^3$, $x \in \mathfrak{b} \subset \mathcal{B}$, with $\mathfrak{b}$ a rectifiable set which is the basis of the fiber bundle where $V_{(2)}$ is defined[4]. In breaking material bonds, in fact, bending effects may occur. They appear in the current configuration $u(\mathcal{B})$ when a crack occurs there. However, the rupture of material bonds induces a mutation in the body – a cracks nucleates or grows – a configurational effect which can be described then by a configurational part of the energy. This energy has to be then associated with the curvature of the lateral margins of the crack. The introduction of the (generalized) curvature-dependent configurational part of the energy can be then considered as a manner to account indirectly for the effects due to the presence of latent microstructures at low scales along the crack margins. The interpretation is enforced by indirect connections – even analogies – with other works dealing with cracks in materials where adsorption of atoms is permitted between crack margins [4], in generic complex bodies [5] or in the special case of second-grade materials [6].

The appearance of the curvature dependent parts of the energy has also nontrivial analytical consequences. There is, in fact, an interplay in the expression of the energy between the presence of varifolds on $\mathfrak{b}$ and their curvature tensors. Take, in fact, a sequence $\left\{V_{(2),k}\right\}$ of two-dimensional integer rectifiable varifolds with boundary. A sequence $\{\mathfrak{b}_k\}$ of $\mathcal{H}^2-$rectifiable sets is associated with it. When all the elements of $\left\{V_{(2),k}\right\}$ are taken with bounded curvature, regularity to the topological properties of $\{\mathfrak{b}_k\}-$ members is imposed. This is regularity on the possible shapes of cracks. In this sense, the notion of admissibility for the generic $\mathfrak{b}$ (to be a rectifiable set) is restricted. Sets describing admissible cracks are not only $\mathcal{H}^2-$rectifiable subsets of $\mathcal{B}$, but, *in addition*, they must admit two-dimensional curvature varifold with *bounded* curvature. The restriction in the admissibility criterion produces advantages: the convergence of sequences of curvature varifolds with bounded curvature can be assured, the relevant space is closed. Compactness problems can be then naturally avoided.

However, there is something more.

[4] The situation is more general than considering a smooth surface $\mathcal{C}$ as in the previous example.

By using varifolds, in fact, it is possible first to assign peculiar energy to the tip and to other linear defects which can be in principle placed around the crack and/or connected with it [5]. If a one-dimensional varifold $V_{(1)}$ is associated with the tip, it is necessary to take into account that the tip is part of the boundary of the set $\mathfrak{b}$ describing in $\mathcal{B}$ the crack. That boundary is connected with the boundary measure ∂V_2. So a geometrical link between $V_{(2)}$ and $V_{(1)}$ is necessary and is given here by

$$\pi_{\#}|\partial V_{(2)}| \leq \pi_{\#}\left|V_{(1)}\right|, \tag{2}$$

where $\pi_{\#}$ is the projection of measures over $\mathfrak{b}$, and $|\cdot|$ indicates the total variation, when referred to a measure. The geometrical implications of the condition (2) are not all self-evident. However, it is the sole necessary condition to assure that the varifold $V_{(1)}$ describes (be supported by) the tip of the crack, that is the part of the boundary of $\mathfrak{b}$ enclosed in $\mathcal{B}$. In fact, the varifold $V_{(1)}$ can have boundary $\partial V_{(1)}$. It is based on the points where the tip has corners or cusps. The set includes also the intersection points between the tip and the external boundary of $\mathcal{B}$.

We call the pair $\left(V_{(2)}, V_{(1)}\right)$ a *stratified family of varifolds* over $\mathcal{B}$ – or, better, for the sake of conciseness, a *stratified varifold*. The terminology can be extended also to an analogous family of varifolds in dimension n – in that case we have a family $\left\{V_{(k)}\right\}$, $k = 1, ..., n-1$, with the elements of it linked by the extended version of (2), namely

$$\pi_{\#}|\partial V_{(k)}| \leq \pi_{\#}\left|V_{(k-1)}\right|. \tag{3}$$

The word 'stratified' reminds that the set considered includes integer rectifiable varifolds with bounded curvature, all based on submanifolds of $\mathcal{B}$, characterized by decreasing Hausdorff dimensions, and linked by the relation (3).

> *This way, to the purpose of assigning peculiar different energies to various parts of the crack, it is expedient to describe a crack by means of a stratified family of varifolds, more than a single varifold.*

The stratification of varifolds allows one to distribute the energy over submanifolds stratified at dimensions lower than the one of the ambient space. The technique can be then used in principle for describing clusters of defects of various dimensions.

By looking only at a three-dimensional case, the energy $\mathcal{E}\left(u, \left\{V_{(k)}\right\}, \mathcal{B}\right)$ of an

[5] That dislocations may be produced in front of a crack tip in crystalline materials is a well known phenomenon. The description of such dislocations or other possible linear defects in terms of one-dimensional varifolds seem to be natural.

elastic-brittle solid, with $\left\{V_{(k)}\right\}$ a stratified varifold, $k=1,2$, is then expressed by

$$\mathcal{E}\left(u,\left\{V_{(k)}\right\},\mathcal{B}\right):=\int_{\mathcal{B}} e\left(x,u\left(x\right),Du\left(x\right)\right)\ dx+\sum_{k=1}^{2}\alpha_k\int_{\mathcal{G}_k(\mathcal{B})}\left|A_{(k)}\right|^{p_k}\ dV_{(k)}+$$
$$+\sum_{k=1}^{2}\beta_k\mathbf{M}\left(V_{(k)}\right)+\gamma\mathbf{M}\left(\partial V_{(1)}\right), \tag{4}$$

where α_k, β_k, and γ are constitutive constants adjusting dimensions; p_1 and p_2 have also constitutive character. The density of energy $e\left(\cdot\right)$ is the difference between the elastic energy density $\hat{e}\left(x,Du\right)$ – it cannot depend on the field u for reasons of objectivity – and the potential $\hat{w}\left(u\right)$ of external body forces, namely $e\left(x,u,Du\right)=\hat{e}\left(x,Du\right)-\hat{w}\left(u\right)$. The addenda containing varifolds have only configurational nature because they are defined on $\mathcal{B}$ and the deformation is not involved – the gradient of deformation appears, in fact, only in the expression of the bulk elastic energy.

The energy $\mathfrak{E}\left(\mathfrak{b},u\right)$ in (1), analyzed in [2] and subsequent literature (see [3] for a critical review) is a special case of $\mathcal{E}\left(u,\left\{V_{(k)}\right\},\mathcal{B}\right)$. The surface energy appearing in the explicit expression of $\mathfrak{E}$ (see (1)) is here the term

$$\beta_2\mathbf{M}\left(V_{(2)}\right).$$

Besides the bulk energy which is common to the one in (1), the other terms in (4) mark the difference with respect to (1), and account from energy contributions from

1) the (generalized) curvature of the crack,

$$\alpha_2\int_{\mathcal{G}_2(\mathcal{B})}\left|A_{(2)}\right|^{p_2}\ dV_{(2)}$$

2) the curvature of the tip,

$$\alpha_1\int_{\mathcal{G}_1(\mathcal{B})}\left|A_{(1)}\right|^{p_1}\ dV_{(1)},$$

3) the length of the tip,

$$\beta_1\mathbf{M}\left(V_{(1)}\right),$$

4) possible corners and cusps of the tip,

$$\gamma\mathbf{M}\left(\partial V_{(1)}\right).$$

This way we presume that, to change or create a crack, it is necessary to pay energy associated with the above listed geometric characteristics of the geometry of the crack itself. The reason of our choice relies on the idea that, in producing a crack, part of the energy is dissipated, while another part re-distributes along the material margins of the crack in accord with the local geometry of the crack pattern and the nature of the material around. Such an energy assures the stability of the matter. In this sense, the energy $\mathcal{E}\left(u, \left\{V_{(k)}\right\}, \mathcal{B}\right)$ is a *refinement* – so an evolution – of Griffith's energy $\mathfrak{E}\left(\mathfrak{b}, u\right)$.

In accord with [2], we require minimality of the energy at any time step of some given loading program.

> *To us, assigned boundary conditions, the occurrence of a possible crack pattern is such that the pair* $\left(u, \left\{V_{(k)}\right\}\right)$ *minimizes the energy* $\mathcal{E}\left(u, \left\{V_{(k)}\right\}, \mathcal{B}\right)$ *over an assigned region* $\mathcal{B}$*.*

In fact, we do not prescribe any initial pre-existing crack. For technical reasons, we prefer to assign a initial *comparison stratified varifold* $\left\{\tilde{V}_{(k)}\right\}$ such that all competing stratified varifolds $\left\{V_{(k)}\right\}$ satisfy the bound $\pi_{\#}\tilde{V}_{(k)} \leq \pi_{\#}V_{(k)}$. The bound prescribes that, in presence of a pre-existing crack, the minimization procedure of the energy can produce only a crack which is bigger or equal to the existing one. In three-dimensions, for example, the condition becomes $\pi_{\#}\tilde{V}_{(2)} \leq \pi_{\#}V_{(2)}$, and $\pi_{\#}\tilde{V}_{(1)} \leq \pi_{\#}V_{(1)}$.

> *However, the initial comparison varifold can be zero. In this case, the minimality requirement describes the* **nucleation** *of possible cracks in the elastic-brittle solid considered.*

The quest for minimizers of $\mathcal{E}\left(u, \left\{V_{(k)}\right\}, \mathcal{B}\right)$ has to be developed in some functional class hosting the pairs $\left(u, \left\{V_{(k)}\right\}\right)$. The choice of such a class has strictly constitutive nature. We have already mentioned that we choose sequences of minimizing stratified varifolds in a space of integer rectifiable varifolds with boundary and *bounded* curvature. This is a choice restricting the possible geometries of the crack patterns. We have not yet mentioned functional assumptions about the deformation field u. Outside the crack, u should describe a standard non-linear deformation: it must be one-to-one, (at least approximately) differentiable, and orientation preserving. u may admit jumps which are concentrated on the crack only. Moreover, since the crack is represented in $\mathcal{B}$ by the set $\mathfrak{b}$ where the stratified varifold $\left\{V_{(k)}\right\}$ is defined. The jump set of u *does not necessarily coincide* with $\mathfrak{b}$. So, in contrast with other treatments (see e.g. [21]), for us the jump set of u does not necessarily coincide with the crack but, when it is not void, *it is included in the cracked region* $\mathfrak{b}$ and at most may coincide with it. This way, we are able to describe situation where a crack is produced and is only partially open along the deformation.

The constraint that the jump set of u be contained or at most coincide with $\mathfrak{b}$ can be expressed by a link between u and $\left\{V_{(k)}\right\}$. The reason is that $\mathfrak{b}$ is not known a priori, rather it is determined by sequences of stratified varifolds minimizing the energy. Different functional links between the jumps of u and $\left\{V_{(k)}\right\}$ are possible. Their choice has constitutive nature. Here we adopt a specific choice; its description, however, requires some preliminary remarks.

To have a control on the discontinuities of the map $u : \mathcal{B} \to \mathbb{R}^3$, it is expedient to consider the graph of the map itself. Since we have assumed that u be (at least approximately) differentiable, a $3-$vector $M\left(Du\right)$ can be associated with the spatial derivative Du. The entries of $M\left(Du\right)$ are all the minors of Du, namely Du itself, $\mathrm{adj}Du$, $\det Du$. When calculated at a certain point, they characterize the tangent plane to the graph at the same point. By means of $M\left(Du\right)$ it is possible to define a linear functional over the graph of u. Such a functional is indicated by G_u and defined by

$$G_u\left(\omega\right) := \int_{\mathcal{B}} < \omega(x, u(x))\, ,\, M(Du(x)) > \, dx$$

for all forms ω which are compactly supported on $\mathcal{B} \times \mathbb{R}^3$. It is the so-called *current* associated with u (precisely the *integer rectifiable 3-current with multiplicity 1*). $G_u\left(\omega\right)$ can be considered as an extended internal work. In fact, the elements of $M(Du(x))$ characterize completely the deformation: Du, $\mathrm{adj}Du$, and $\det Du$ describe respectively the deformation of lines, volume and oriented areas. Dual to them are respectively the stress (the first Piola-Kirchhoff representation of it), and related averages over volume and oriented areas. However, besides the possible physical interpretations, a geometrical property is essential for the developments presented here. In fact, a notion of *boundary* can be associated with G_u. The boundary of G_u is indicated by ∂G_u and defined by

$$\partial G_u(\omega) := G_u(d\omega)$$

for all $2-$forms which are compactly supported on $\mathcal{B} \times \mathbb{R}^3$. The key point is that u has no 'discontinuities' when $\partial G_u(\omega) = 0$ for any ω. In our case, u may admit discontinuities but only over $\mathfrak{b}$. As anticipated previouly, this constraint can be then expressed by connecting the varifold over $\mathfrak{b}$ with ∂G_u:

$$\pi_\# \left|\partial G_u\right| \leq \sum_{j=1}^{2} \pi_\# V_{(j)} + \pi_\# \partial V_{(1)}$$

Outside of the crack pattern, the additional constraint $\partial G_u(\omega) = 0$ is imposed.

A new space then arises. It is called the space of *extended weak diffeomorphisms*. We are able to prove its closure with respect to converging sequencies.

Under the assumption that the bulk energy be polyconvex in Du, and the positivity of the constitutive coefficients α_k, β_k, γ, we then prove an existence theorem of a pair $\left(u, \left\{V_{(k)}\right\}\right)$ realizing the minimum of $\mathcal{E}\left(u, \left\{V_{(k)}\right\}, \mathcal{B}\right)$. In fact, the treatment has its natural extension in $n-$dimensional ambient space and we furnish the details leading to the existence theorem to this case in the sequel, leaving the three-dimensional one to this introduction. Preliminarily, we discuss also the two-dimensional case which allows us to put in evidence some peculiar technical aspects.

> *The minimum pair* $\left(u, \left\{V_{(k)}\right\}\right)$ *can be such that* $\left\{V_{(k)}\right\}$ *is the null stratified varifold when the initial comparison stratified varifold is also null. In this case the body under scrutiny remains in a purely elastic phase, else it is elastic-brittle. In the case of null initial comparison stratified varifold, the transition from elastic to elastic-brittle behaviour is governed just by the energy and the boundary condition, and does not need additional external criteria prescribing a threshold.*

Besides the possibility to treat cases of partially opened cracks, and to describe naturally the nucleation of cracks, our treatment has another essential advantage: *it is possible to evaluate the first variation of the energy when the crack pattern is described by a stratified family of curvature varifolds*. This possibility is in constrast with approaches where only a field u is involved as a variable, is considered as a special function with bounded variation, and the crack is considered to be coincident with the jump set of u itself. In that approaches, in fact, special additional regularity assumptions have to be made on the topology of the crack pattern (see [3] for a detailed review of the relevant literature). We do not need such assumptions to evaluate *weak forms of the balance equations*. After evaluating them in general, we specialize such equations in a case of a regular crack pattern. In this case we explicit pointwise balances of interactions: they include standard balances and an additional one which can be used in evaluating curving and kinking processes. Moreover, when the crack geometry is regular – let say the crack is a smooth surface – our balance equations contains the configurational contributions associated with the curvature of the lateral margins of the crack and the tip itself. Ancillary results arise: the main one is the proof that the vector form of the J-integral is directed along the normal to the crack tip at a given point of the tip itself, under some conditions of regularity up to the tip of the Hamilton-Eshelby stress.

Details are presented in the ensuing sections.

1 Curvature varifolds with boundary

Definition of and notions on varifolds are collected in the present section. Two examples show explicit forms of varifolds related with specific simple cracks. Besides the definition, the compactness Theorem 4 is an essential tool for the existence results presented later. Theorems 2 and 3 inform just about essential geometrical properties of varifolds. They enforce the decision of taking varifolds as descriptors of crack patterns. At a first reading they can be skipped altogether. Meaning skipped without serious loss.

$\mathbb{R}^n$, $n \geq 2$, is the space selected as geometrical ambient, although we are conscious that $n = 3$ and $n = 2$ are the dimensions with physical stringent significance.

Let $\mathcal{B}$ be an open, bounded subset of $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary. For a positive integer k, $1 \leq k \leq n$, the Grassmanian manifold of k-planes through the origin in $\mathbb{R}^n$ is indicated by $\mathcal{G}_{k,n}$ and is also identified with the set of projectors $\Pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ onto k-planes, characterized by

$$\Pi^2 = \Pi, \qquad \Pi^* = \Pi, \qquad \text{Rank},\Pi = k,$$

a set which is a compact subset of $\mathbb{R}^n \otimes \mathbb{R}^n$. Consider also the trivial bundle $\mathcal{G}_k(\mathcal{B}) := \mathcal{B} \times \mathcal{G}_{k,n}$ with natural projection $\pi : \mathcal{G}_k(\mathcal{B}) \longrightarrow \mathcal{B}$. A *$k$-varifold* on $\mathcal{B}$ is a nonnegative Radon measure V over $\mathcal{G}_k(\mathcal{B})$, namely $V \in \mathcal{M}(\mathcal{G}_k(\mathcal{B}))$. The *weight measure* of V is the Radon measure $\pi_\# V$ where $\pi_\#$ is the natural projection of measures associated with the projection π, and the *mass* of V is $\mathbf{M}(V) := V(\mathcal{G}_k(\mathcal{B})) = \pi_\# V(\mathcal{B})$.

Denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure in $\mathbb{R}^n$. If $\mathfrak{b}$ is an $\mathcal{H}^k$-measurable, countably k-rectifiable subset of $\mathcal{B}$ and θ is a density function in $L^1(\mathfrak{b}, \mathcal{H}^k)$, for $\theta\,\mathcal{H}^k \llcorner \mathfrak{b}$ a.e. $x \in \mathcal{B}$ there exists the approximate tangent k-space $T_x\mathfrak{b}$ to $\mathfrak{b}$ at x. Define

$$V_{\mathfrak{b},\theta}(\varphi) := \int_{\mathcal{G}_k(\mathcal{B})} \varphi(x, \Pi)\, dV_{\mathfrak{b},\theta}(x, \Pi) := \int_{\mathfrak{b}} \theta(x)\varphi(x, \Pi(x))\, d\mathcal{H}^k(x) \qquad (5)$$

for any $\varphi \in C_c^0(\mathcal{G}_k(\mathcal{B}))$, where $\Pi(x)$ is the orthogonal projection of $\mathbb{R}^n$ onto $T_x\mathfrak{b}$. The formula in (5) makes sense because the map

$$x \longmapsto \varphi(x, \Pi(x))$$

is $\theta\mathcal{H}^k \llcorner \mathfrak{b}$-measurable. $V_{\mathfrak{b},\theta}$ is called the *rectifiable k-varifold* associated with $\mathfrak{b}$, with density θ.

A special case is the choice of a nonnegative integer valued function θ. In this case, $V_{\mathfrak{b},\theta}$ is called the *integer rectifiable varifold* associated with $\mathfrak{b}$ with density θ.

Rectifiable varifolds have been introduced in [7] as generalized surfaces. A restricted class of varifolds for which a notion of (generalized) mean curvature vector is available is analyzed in [8,9], where related regularity results are proven. A more restricted class of varifolds without boundary having a (generalized) *second fundamental form* has been introduced in [10], while the class of the *integer rectifiable curvature varifolds* that allow for a notion of second fundamental form and for a notion of *boundary* has been analyzed in [11] (see also [12]).

Definition 1 *V is called a* curvature k-varifold with boundary *if*

(1) $V = V_{\mathfrak{b},\theta}$ is the integer rectifiable k-varifold associated with $(\mathfrak{b}, \theta, \mathcal{H}^k)$,
(2) there exist a function $A \in L^1(\mathcal{G}_k(\mathcal{B}), \mathbb{R}^{n*} \otimes \mathbb{R}^n \otimes \mathbb{R}^{n*})$, $A = (A_j^{\ell i})$, and a vector Radon measure $\partial V \in \mathcal{M}(\mathcal{G}_k(\mathcal{B}), \mathbb{R}^n)$ such that

$$\int_{\mathcal{G}_k(\mathcal{B})} (\Pi D_x \varphi + A^t D_\Pi \varphi + \varphi\, AI)\, dV(x, \Pi) = -\int_{\mathcal{G}_k(\mathcal{B})} \varphi\, d\partial V(x, \Pi)$$

for every $\varphi \in C_c^\infty(\mathcal{G}_k(\mathcal{B}))$.

Moreover, for $p \geq 1$ the subclass of curvature k-varifolds with boundary such that $A \in L^p(\mathcal{G}_k(\mathcal{B}))$ is indicated by $CV_k^p(\mathcal{B})$.

The function $x \mapsto A(x, \Pi(x)) \in \mathbb{R}^{n*} \otimes \mathbb{R}^n \otimes \mathbb{R}^{n*}$ is called the *curvature* of the varifold V. The vector measure ∂V is called the *varifold boundary measure.*

In components, by considering $\Pi\,(x)$ as the orthogonal projection onto $T_x\mathfrak{b}$, the integral density in the previous definition can be written as

$$\begin{aligned} &\int_{\mathfrak{b}} \left(\Pi_j^i D_{x_j} \varphi(x, \Pi) + A_j^{\ell i}(x, \Pi)\, D_{\Pi_j^\ell} \varphi(x, \Pi) + A_j^{ij}(x, \Pi)\, \varphi(x, \Pi) \right) \theta d\mathcal{H}^k \\ &\qquad = -\int_{\mathcal{G}_k(\mathcal{B})} \varphi(x, \Pi)\, d\partial^i V \end{aligned} \tag{6}$$

for every $\varphi \in C_c^\infty(\mathcal{G}_k(\mathcal{B}))$, $i = 1, \ldots, n$. Summation over repeated indices is understood.

The relation (6) is reminiscent of the formulas

$$\begin{aligned} &\int_{\mathfrak{b}} \left(\Pi_j^i(x) D_{x_j} \varphi(x, \Pi(x)) + A_j^{\ell i}(x, \Pi(x))\, D_{\Pi_j^\ell} \varphi(x, \Pi(x)) \right) d\mathcal{H}^k \\ &\qquad\qquad + \int_{\mathfrak{b}} \mathbf{H}^i(x) \varphi(x, \Pi(x))\, d\mathcal{H}^k \\ &\qquad = -\int_{\partial\mathfrak{b}} \varphi(x, \Pi(x)) m^i(x)\, d\mathcal{H}^{k-1} \qquad \forall \varphi \in C_c^\infty(\mathcal{G}_k(\mathcal{B})) \end{aligned} \tag{7}$$

for $i = 1, \ldots, n$, that hold true for smooth k-manifolds $\mathfrak{b}$ with smooth boundary $\partial\mathfrak{b}$. Here $\mathbf{H} = (\mathbf{H}^1, \ldots, \mathbf{H}^n)$ is the mean curvature vector of $\mathfrak{b}$ at x, i.e. $\mathbf{H}^i := (\nabla^{\mathfrak{b}} \Pi_j^i)^j$, $\nabla^{\mathfrak{b}}$ is the tangential gradient operator to $\mathfrak{b}$ and $m = (m^1, \ldots, m^n)$ is the inward unit normal to $\partial\mathfrak{b}$ in $T_x\mathfrak{b}$. Finally, the third-rank tensor A has components defined by $A_j^{\ell i} := (\nabla^{\mathfrak{b}} \Pi_j^\ell)^i$.

The previous integral relation is justified by applying to vector fields

$$X_i(x) := \varphi(x, \Pi(x)) e_i, \qquad i = 1, 2, \ldots, n,$$

with $e_1, e_2, \ldots, e_n$ the canonical basis in the ambient space $\mathbb{R}^n$, the integration by parts formula on the manifold $\mathfrak{b}$ (see e.g. [13]), namely

$$\int_{\mathfrak{b}} \mathrm{div}^{\mathfrak{b}} X \, d\mathcal{H}^k = -\int_{\mathfrak{b}} X \cdot \mathbf{H} \, d\mathcal{H}^k - \int_{\partial\mathfrak{b}} X \cdot m \, d\mathcal{H}^{k-1} \,, \tag{8}$$

where $\mathrm{div}^{\mathfrak{b}}$ indicates, from now on, the divergence operator *along* $\mathfrak{b}$.

Moreover, if $\mathfrak{b}$ is a smooth manifold, the curvature functions $A_\ell^{ji}(x)$ in (7) are also linked with the second fundamental form II_x of $\mathfrak{b}$ at x. More precisely, by writing

$$B_{ij}^\ell(x) = \mathrm{II}_x(\Pi e_i, \Pi e_j)^\ell,$$

we find (see [11])

$$B_{ij}^\ell = \Pi_i^h A_h^{\ell j}, \qquad A_\ell^{ji} = B_{ij}^\ell + B_{i\ell}^j \,. \tag{9}$$

Therefore, *bounds to A are actually bounds to the curvature of* $\mathfrak{b}$. This aspect addresses the physical interpretation of the terms containing A in the explicit expression (4) of the energy.

Example 1 Let $\mathfrak{b}$ the segment joining $(-1,0)$ to $(1,0)$ in $\mathbb{R}^2$. Then the curvature 1-varifold V associated with $\mathfrak{b}$ and density 1 exists. It is endowed with zero curvature and boundary

$$\partial V = \begin{pmatrix} -\delta_{((-1,0),\Pi_1)} + \delta_{((1,0),\Pi_1)} \\ 0 \end{pmatrix}$$

where $\Pi_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. In fact equation (6) reduces to

$$\begin{cases} \int_{-1}^1 \varphi_x(x, 0, \Pi_1)\, dx = \varphi((1,0),\Pi_1) - \varphi((-1,0),\Pi_1) \,, \\ \int_{-1}^1 0 \, dx = 0 \,. \end{cases}$$

Example 2 Let $\mathfrak{b}$ be the union of the two segments joining respectively $(-1,0)$ with $(1,0)$ and $(1,0)$ with $(1,1)$. The curvature varifold associated with $\mathfrak{b}$ has zero curvature and boundary given by the sum of the boundaries of the two segments, namely

$$\partial V = \begin{pmatrix} -\delta_{((-1,0),\Pi_1)} + \delta_{((1,0),\Pi_1)} \\ -\delta_{((1,0),\Pi_2)} + \delta_{((1,1),\Pi_2)} \,. \end{pmatrix}$$

where $\Pi_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\Pi_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Notice that the projection of the mass of the boundary, namely $\pi_{\#}|\partial V|$, is concentrated at the endpoints $(-1,0)$ and $(1,1)$ and at the corner point $(1,0)$, where the multiplicity is 2.

Similarly, consider several segments meeting at a point from different directions. The boundary of the corresponding curvature 1-varifold concentrates also at the junction. In three dimensions, take cracks meeting at a curve. The junction (that is the curve now) can be described by affirming that the 2-varifolds associated with the cracks have in common parts of the boundary. An additional one-dimensional varifold can be also associated with the junction.

The following results proved in [10] and [11] collect basic geometrical properties of curvature k-varifolds with boundary.

Theorem 2 *Let $V = V_{\mathfrak{b},\theta}$ be a k-varifold with boundary ∂V and curvature A, with $A_j^{\ell i} \in L^1(\mathcal{G}_k(\mathcal{B}))$.*

(1) The following symmetry properties hold:

$$A_j^{\ell i} = A_\ell^{ji}, \quad A_j^{ji} = 0, \quad A_j^{\ell i} = \Pi_h^\ell A_j^{hi} + \Pi_j^h A_h^{\ell i}, \quad V - a.e.$$

(2) $\Pi_h^i A_j^{\ell h} = A_j^{\ell i}$ V-a.e. in such a way that, by setting $\mathbf{H}^i(x) := A_j^{ij}(x, \Pi(x))$, one gets $\Pi_i^h \mathbf{H}^h = 0$ V-a.e.; in particular,

$$\mathbf{H}(x) \perp T_x\mathfrak{b} \qquad \mu_V - a.e.$$

(3) The projection map $x \mapsto \Pi(x)$ is μ_V-a.e. approximately differentiable and

$$(\nabla^{\mathfrak{b}} \Pi_j^\ell(x))^i = A_j^{\ell i}(x, \Pi(x))$$

for μ_V-a.e. x.

(4) The support of $|\partial V|$ is contained in the support of V and $|\partial V| \perp V$.

(5) ∂V is tangential to $\mathfrak{b}$ in the sense that

$$(\Pi_j^i)_{\#} \partial^j V = \partial^i V$$

as measures on $\mathcal{G}_k(\mathcal{B})$.

(6) V is a varifold with locally bounded first variation and generalized mean curvature $\mathbf{H}(x)$ in the sense of Allard and generalized boundary $\pi_{\#}\partial V$.

Theorem 3 (Rectifiability of the boundary) *Let V be a curvature varifold of dimension k with boundary ∂V. There exists an $\mathcal{H}^{k-1}$-countably rectifiable set $\mathcal{C}$ and a function $\sigma \in L^1(\mathcal{C}, \mathcal{H}^{k-1})$ such that $\pi_{\#}|\partial V| = \sigma \mathcal{H}^{k-1} \llcorner \mathcal{C}$. Moreover, one has*

$$\int \varphi(x, \Pi(x))\, d\partial V(x, \Pi) = \int_{\mathcal{C}} \left(\int_{\mathcal{G}_{k,n}} \varphi(x, \Pi)\, d\tau_x(\Pi) \right) d\mathcal{H}^{k-1}(x)$$

for every $\varphi \in C_c^\infty(\mathcal{G}_k(\mathcal{B}))$*, where for* $\mathcal{H}^{k-1}$*-a.e.* $x \in \mathcal{C}$ *the vector valued measure* τ_x *on* $\mathcal{G}_{k,n}$ *has the structure*

$$\tau_x = \sum_{i=1}^{i_x} m_i^x \, \alpha_i^x \, \delta_{p_i^x} \,, \tag{10}$$

where $i_x \in \mathbb{N}$*,* $\delta_{p_i^x}$ *is the Dirac delta supported by a* k*-plane* p_i^x *of the Grassmanian* $\mathcal{G}_{k,n}$*; moreover, the* α_i^x*'s are positive integers and the* m_i^x*'s are unit vectors in* $\mathbb{R}^n$*. In addition* p_i^x *contains the tangent* $(k-1)$*-space* $T_x\mathcal{C}$ *to* $\mathcal{C}$ *at* x *and*

$$p_i^x = \mathrm{Span}, \Big\{ T_x\mathcal{C}, m_i^x \Big\}.$$

In the special case of one-dimensional curvature varifolds V *with boundary, the formula* (10) *reduces to*

$$\tau_x := \sum_{j=1}^{j_x} \alpha_j t_j \delta_{P_j}$$

where δ_{P_j} *is the Dirac delta function supported by a straight line* P_j *in* $\mathcal{G}_{1,n}$*,* t_j *is a unit vector that orients* P_j *and* α_j *a positive integer. As a consequence, for the boundary of a curvature* 1*-varifold one gets*

$$\partial V(x, P) = \sum_{i=1}^{\infty} \delta_{x_i}(x) \times \tau_{x_i}(P) \,.$$

Since varifolds in $CV_k^p(\mathcal{B})$ have mean curvature $\mathbf{H}$ in L^p, as a consequence of Allard's compactness result, the theorem below can be stated.

Theorem 4 (Compactness [11]) *For* $1 < p < \infty$*, let* $\{V^{(r)}\} \subset CV_k^p(\mathcal{B})$ *be a sequence of curvature* k*-varifolds* $V^{(r)} = V_{\mathfrak{b}_r, \theta_r}$ *with boundary. The corresponding curvatures and boundaries are indicated by* $A^{(r)} = \{A^{(r)}{}_j^{\ell i}\}$ *and* $\partial V^{(r)}$*, respectively. Assume that for every open set* $\Omega \subset\subset \mathcal{B}$ *there exists a constant* $c = c(\Omega) > 0$ *such that for every* r

$$\mu_{V^{(r)}}(\Omega) + |\partial V^{(r)}|(\mathcal{G}_k(\Omega)) + \int_{\mathcal{G}_k(\Omega)} |A^{(r)}|^p \, dV^{(r)} \le c(\Omega).$$

Then, there exists a subsequence $\{V^{(r_s)}\}$ *of* $\{V^{(r)}\}$ *and a* k*-varifold* $V = V_{\mathfrak{b},\theta} \in CV_k^p(\mathcal{B})$*, with curvature* A *and boundary* ∂V*, such that*

$$V^{(r_s)} \rightharpoonup V, \quad A^{(r_s)} \, dV^{(r_s)} \rightharpoonup A \, dV, \qquad \partial V^{(r_s)} \rightharpoonup \partial V,$$

in the sense of measures. Moreover, for any convex and l.s.c. function $f : \mathbb{R}^{n*} \otimes \mathbb{R}^n \otimes \mathbb{R}^{n*} \to [0, +\infty]$*, one gets*

$$\int_{\mathcal{G}_k(\mathcal{B})} f(A) \, dV \le \liminf_{s \to \infty} \int_{\mathcal{G}_k(\mathcal{B})} f(A^{(r_s)}) \, dV^{(r_s)}.$$

Allard's regularity results [8] also apply. In fact, if $V_{\mathfrak{b},\theta} \in CV^p$ for some $p > n$, then $\mathfrak{b}$ is locally the graph of a $C^{1,\alpha}$ multivalued function, with $\alpha = 1 - n/p$, far from boundaries [6]. The remark indicates the broad variety of possible crack shapes that can be described by varifolds. For sufficiently large p, shapes involved in traditional analyses can be recovered.

As it shall be explained in the ensuing sections, curvature varifolds are useful tools to describe cracks in continuous deformable media.

2 Deformations and currents

Matter appears discrete at very low dimensional scales. Its description in terms of continuum models is a valid approximation at macroscopic and/or mesoscopic scales. If the representation is at that scales, a body can be considered as an abstract set whose members – call them the *material elements* – are commonly seen as the smallest atomic aggregates bringing peculiar features of the matter constituting the body under analysis. This definition is extremely vague. Dimensions of the material elements, numbers of atoms inside them, nature of the mentioned 'peculiar features' remain not specified. In standard continuum mechanics, however, such a vagueness is accepted. Whatever the material elements be, in fact, only their placement in space is chosen as descriptor of their morphology. This way, each of them is in a certain sense 'collapsed' in a point. So, a minimalistic description of the morphology of a body is adopted. Only the region $\mathcal{B}$ occupied in the ambient space is considered, leaving out the inner geometry of the molecular shapes and entanglements.

$\mathcal{B}$ is considered here an open set in $\mathbb{R}^n$ with Lipschitz boundary (concrete physical dimensions are $n = 2$ and $n = 3$). Other configurations are reached in $\hat{\mathbb{R}}^n$ (an isomorphic copy of $\mathbb{R}^n$) [7] by means of *deformations*

$$u : \mathcal{B} \longrightarrow \hat{\mathbb{R}}^n,$$

which maps $\mathcal{B}$ in the current configuration $u(\mathcal{B})$, a set that is presumed to be always open and endowed with Lipschitz boundary.

By leaving out the possible formation of fracture, standard assumptions are as follows:

(i) u is a one-to-one, (at least approximately) differentiable map.

[6] Approximation results for curvature varifolds without boundaries can be found in [14].

[7] The distinction between the space containing the reference place and the one where all other configuration are reached has handy nature.

(ii) u is an orientation-preserving map, i.e. $\det Du(x) > 0$ for x in $\mathcal{B}$.
(iii) For any compactly supported function f on $\mathbb{R}^n \times \hat{\mathbb{R}}^n$, the following inequality holds:

$$\int_{\mathcal{B}} f(x, u(x)) \det Du(x) \ dx \leq \int_{\hat{\mathbb{R}}^n} \sup_{x \in \mathcal{B}} f(x, z) \ dz.$$

The last condition permits along deformations self-contact between distant portions of the boundary of $\mathcal{B}$, but still prevents self-penetration of the matter. It is a 'global' one-to-one condition proposed in [15]. It amounts to

$$\int_{\mathcal{B}'} \det Du(x)\, dx \leq \mathcal{L}^2(\tilde{u}(\widetilde{\mathcal{B}}')),$$

for any subbody $\mathcal{B}'$ of $\mathcal{B}$ (a condition proposed in [18]). $\tilde{u}$ denotes Lusin representative of u, and $\widetilde{\mathcal{B}}'$ is the intersection of $\mathcal{B}$ with the domain of $\tilde{u}$.

Possibility of fracture weakens the bijectivity prescribed in (i). In that case, the map u has to be considered just *piecewise* one-to-one. Its jump set, in addition, must be included in the sub-region of $\mathcal{B}$ where the material bonds are broken along a cracking process. Control of the discontintinuities of approximately differentiable maps can be exerted by means of currents defined over the graphs of such maps. Mention has been made in the introduction. Additional details are then necessary.

2.1 Graphs as currents

For reader's convenience, some notions about currents are summarized below. The material is presented in a bit more general way than the strictly necessary one: in fact, $\mathbb{R}^n$ and $\mathbb{R}^N$ are involved, rather than only $\mathbb{R}^n$ and its isomorphic copy. Reference is the treatise [16].

Let $I(k,n)$ be the space of multi-indices in $(1,\ldots,n)$ of length k. Denote also by 0 the empty multi-index of length 0. For any α, the *complementary* multi-index to α in $(1,...,n)$ is indicated by $\bar{\alpha}$, $\bar{\alpha} \in I(n-k,n)$, and $\sigma(\alpha,\bar{\alpha})$ is the sign of the permutation from $(1,\ldots,n)$ into $(\alpha_1,\ldots,\alpha_k,\bar{\alpha}_1,...,\bar{\alpha}_{n-k})$.

For $(e_1, e_2, \ldots, e_n)$ and $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ bases in $\mathbb{R}^n$ and $\mathbb{R}^N$, respectively, $\Lambda_r(\mathbb{R}^n \times \mathbb{R}^N)$ denotes the vector space of skew-symmetric tensors over $\mathbb{R}^n \times \mathbb{R}^N$ of the form

$$\xi = \sum_{|\alpha|+|\beta|=r} \xi^{\alpha\beta} e_\alpha \wedge \epsilon_\beta = \sum_{\max(0,r-n)}^{\min(r,N)} \xi_{(k)},$$

where

$$\xi_{(k)} = \sum_{\substack{|\alpha|+|\beta|=r \\ |\beta|=k}} \xi^{\alpha\beta} e_\alpha \wedge \epsilon_\beta.$$

The decomposition $\xi = \sum_k \xi_{(k)}$ does not depend on the choice of the bases.

For any linear map $G : \mathbb{R}^n \to \mathbb{R}^N$, the notation $M(G)$ is used for the simple n-vector in $\Lambda_n(\mathbb{R}^n \times \mathbb{R}^N)$ which is tangent to the graph of G and is defined by

$$M(G) := \Lambda_n(\mathrm{Id} \times G)(e_1 \wedge \cdots \wedge e_n) = (e_1, G(e_1)) \wedge \cdots \wedge (e_n, G(e_n)).$$

Expression in coordinates is given by

$$M(G) = \sum_{k=0}^{\min(n,N)} M_{(k)}(G),$$

where

$$M_{(k)}(G) := M(G)_{(k)} = \sum_{\substack{|\alpha|+|\beta|=n \\ |\beta|=k}} \sigma(\alpha, \bar{\alpha}) M_{\bar{\alpha}}^{\beta}(\mathbf{G}) e_\alpha \wedge \epsilon_\beta.$$

$\mathbf{G}$ indicates the $N \times n$ matrix in $\mathbb{M}_{N\times n}$ associated with G and $M_{\bar{\alpha}}^{\beta}(\mathbf{G})$ is the determinant of the submatrix of $\mathbf{G}$ made of the rows and the columns indexed by β and $\bar{\alpha}$ respectively. It is also convenient to put $M_0^0(\mathbf{G}) := 1$. In the special case in which $n = N = 3$, the components of $M(G)$ are 1 and the entries of $\mathbf{G}$, $\operatorname{adj} \mathbf{G}$ and $\det \mathbf{G}$.

Remind that $\mathcal{B}$ is in $\mathbb{R}^n$. Consider $u : \mathcal{B} \to \mathbb{R}^N$ an a.e. approximately differentiable map, and its approximate gradient Du. It is well-known that u has Lusin representative on the subset $\widetilde{\mathcal{B}}$ of Lebesgue points of both u and Du, and that $|\mathcal{B} \setminus \widetilde{\mathcal{B}}| = 0$. Let $\widetilde{u}(x)$ and $D\widetilde{u}(x)$ be the Lebesgue values of u and Du at $x \in \widetilde{\mathcal{B}}$, respectively.

Assume that $|M(Du)| \in L^1(\mathcal{B})$, where

$$|M(G)|^2 := \sum_{|\alpha|+|\beta|=n} |M_{\bar{\alpha}}^{\beta}(\mathbf{G})|^2$$

is the Euclidean norm of $M(G)$. The *graph* of u, defined by

$$\mathcal{G}_u := \left\{ (x, y) \in \mathcal{B} \times \mathbb{R}^N \,\middle|\, x \in \widetilde{\mathcal{B}},\ y = \widetilde{u}(x) \right\},$$

is a n-rectifiable subset of $\mathcal{B} \times \mathbb{R}^N$ with approximate tangent vector n-space at $(x, \widetilde{u}(x))$ generated by the vectors $(e_1, D\widetilde{u}(x)e_1), \ldots, (e_n, D\widetilde{u}(x)e_n)$ in $\mathbb{R}^n \times \mathbb{R}^N$ (see [16]). The *n-current integration over the graph* of u is defined as the linear functional G_u acting on smooth n-forms $\omega = \omega(x, y)$ with compact support in $\mathcal{B} \times \mathbb{R}^N$ as

$$
\begin{aligned}
G_u(\omega) : &= \int_{\mathcal{B}} (\mathrm{Id} \times \widetilde{u})^{\#}(\omega) = \int_{\mathcal{B}} < (\mathrm{Id} \times \widetilde{u})^{\#}(\omega) \,,\, e_1 \wedge \cdots \wedge e_n \, > dx \\
&= \int_{\mathcal{B}} < \omega(x, \widetilde{u}(x)) \,,\, M(D\widetilde{u}(x)) \, > dx \\
&= \int_{\mathcal{B}} < \omega(x, u(x)) \,,\, M(Du(x)) \, > dx,
\end{aligned}
$$

for every $\omega \in \mathcal{D}^n(\mathcal{B} \times \mathbb{R}^N)$. Here $\#$ indicates pull-back of forms ω and $(\mathrm{Id} \times \widetilde{u})(x) := (x, \widetilde{u}(x))$.

The number

$$
\mathbf{M}(G_u) := \sup_{||\omega||_\infty \leq 1} G_u(\omega)
$$

is called the *mass* of G_u. The area formula implies

$$
\mathbf{M}(G_u) = \int_{\mathcal{B}} |M(Du(x))| \, dx = \mathcal{H}^n(\mathcal{G}_u),
$$

and

$$
G_u(\omega) = \int < \omega \,,\, \xi \, > d\mathcal{H}^n \llcorner \mathcal{G}_u \,,
$$

where $\xi(x) := \frac{M(D\widetilde{u}(x))}{|M(D\widetilde{u}(x))|}$, for $x \in \widetilde{\mathcal{B}}$, is the unit n-vector orienting the approximate tangent $n-$space to $\mathcal{G}_u$ at $(x, \widetilde{u}(x))$. Moreover, G_u has finite mass:

$$
\mathbf{M}(G_u) < \infty \,.
$$

G_u can be seen as a vector valued measure over $\mathcal{B} \times \mathbb{R}^N$. Since the graph $\mathcal{G}_u$ is rectifiable, G_u is an *integer rectifiable $n-$current with multiplicity 1* on $\mathcal{B} \times \mathbb{R}^N$.

A functional ∂G_u, defined by

$$
\partial G_u(\omega) := G_u(d\omega)
$$

on compactly supported smooth $(n-1)-$forms ω in $\mathcal{B} \times \mathbb{R}^N$, namely $\omega \in \mathcal{D}^{n-1}(\mathcal{B} \times \mathbb{R}^N)$, with $d\omega$ the differential of ω, is called the *boundary* of G_u.

By Stokes theorem, if u is of class C^2, then

$$
\partial G_u = 0
$$

on $\mathcal{D}^{n-1}(\mathcal{B} \times \mathbb{R}^N)$. Such a relation holds also for Sobolev maps $u \in W^{1,n}(\mathcal{B}, \mathbb{R}^N)$ by approximation. In general ∂G_u does not vanish in $\mathcal{B} \times \mathbb{R}^N$. A typical example is the map $u(x) := \frac{x}{|x|}$, that belongs to $W^{1,p}(B^3(0,1), \mathbb{R}^3)$ for every $p < 3$. It is possible to compute that $\partial G_u = -\delta_0 \times [\![\, S^2 \,]\!]$ on $\mathcal{D}^2(B^3(0,1) \times \mathbb{R}^3)$, with δ_0 the Dirac delta at 0.

In the sequel, π will indicate both the orthogonal projection $\pi : \mathbb{R}^n \times \hat{\mathbb{R}}^n \to \mathbb{R}^n$ and the natural projections $\pi : \mathcal{G}_k(\mathcal{B}) \to \mathcal{B}$, confusion being avoided by the context.

3 Bulk energy

The energy (4), proposed for elastic-brittle bodies, is the sum of bulk energy with density $e\left(x, u\left(x\right), Du\left(x\right)\right)$ and configurational terms. The latter may vary not only as a consequence of constitutive choices, but also depending on the dimensions of the ambient space. For example, in two dimensions, line energy at the tip is not available. Apart from these aspects, the bulk part of the energy has the same formal expression at all dimensions considered. So, since we treat different cases having common bulk energy (indicate it by $\mathcal{E}^b\left(u, \mathcal{B}\right)$), we find advantageous to fix, once and for all, basic assumptions. Conservative external bulk actions are considered only, so that

$$\mathcal{E}^b\left(u, \mathcal{B}\right) := \int_{\mathcal{B}} e\left(x, u\left(x\right), Du\left(x\right)\right)\, dx,$$

where $e(\cdot)$ is the sum of the elastic energy and the potential of external actions. By assumption, $e = e(x, u, F)$ satisfies the common properties listed below:

(H1) $e : \mathcal{B} \times \hat{\mathbb{R}}^n \times \mathbb{M}^+_{n\times n} \to [0, +\infty]$ is continuous, where $\mathbb{M}^+_{n\times n}$ is the class of real $(n \times n)$-matrices F such that $\det F > 0$.

(H2) The map $F \mapsto e(x, u, F)$ is *polyconvex*, i.e. there exists a function

$$Pe(x, u, \xi) : \mathcal{B} \times \hat{\mathbb{R}}^n \times \Lambda_n(\mathbb{R}^n \times \hat{\mathbb{R}}^n) \to [0, +\infty]$$

which is continuous in (x, u) for every ξ, convex and lower semicontinuous in ξ for every (x, u), such that

$$e(x, u, F) = Pe(x, u, M(F)) \qquad \forall F \in \mathbb{M}^+_{n\times n}, \quad \forall (x, u) \in \mathcal{B} \times \hat{\mathbb{R}}^n\,.$$

(H3) $e = e(x, u, F)$ satisfies the growth conditions

$$e(x, u, F) \geq c_4\, |M(F)|^q \qquad \forall F \in \mathbb{M}^+_{n\times n}, \quad \forall (x, u) \in \mathcal{B} \times \hat{\mathbb{R}}^n\,,$$

for some $c_4 > 0$ and $q > 1$.

(H4) For every $x \in \mathcal{B}$ and $F \in \mathbb{M}^+_{n\times n}$ if for some $u \in \hat{\mathbb{R}}^n$ the inequality $e(x, u, F) < +\infty$ is satisfied, then $\det F > 0$.

The assumptions (H1) and (H4) are essentially suggested by physical plausibility. The hypothesis (H2) may be regarded as a prescription of material stability while the growth condition (H3) is dictated by technical needs.

4 Analysis in two-dimensional ambient space

Attention is focused first on two-dimensional ambient space to focus further ideas in a simple setting, because only 1−dimensional curvature varifolds with curvature in L^p, $p > 1$, need to be involved, and stratified families of varifolds are here not necessary to describe crack patterns. Varifolds in $CV_1^p(\mathcal{B})$ can be described as (the integration over) a locally finite union of $C^{1,1-1/p}$ curves counted with integer multiplicities, and their boundaries are just Dirac measures concentrated at endpoints and junctions, with their tangential directions (see [12]).

The situation is summarized in the following definition.

Definition 5 *A macroscopic configuration of a two-dimensional body with possible cracks is a pair composed by a bounded connected open set $\mathcal{B} \subset \mathbb{R}^2$ with Lipschitz boundary and a curvature* 1*-varifold with boundary V, such that $V = V_{\mathfrak{b},\theta} \in CV_1^p(\mathcal{B})$ for some $p > 1$.*

In the previous definition, the gross place occupied by the body and the crack are treated as distinct objects. The crack is not part of the initial boundary: it is selected by a measure over $\mathcal{B}$, namely a curvature varifold. The distinction is made because we do not want to analyze a body with a specific crack, rather we are considering elastic-brittle bodies which may nucleate fractures, depending on boundary conditions (loading programs, if you want) and constitutive structures. Along the deformation, when a crack occurs, the crack faces may loose contact completely or partially. So, as already recalled, the deformation graph may have non-zero boundary. Then an appropriate class of admissible deformations has to be defined.

Weak diffeomorphisms have been found to be natural descriptors of deformations of standard elastic bodies, see [15]. They are orientation-preserving maps. They allow frictionless contact of parts of the boundary while still prevent self-penetration of the matter. However, they satisfy a condition of zero boundary in the sense of currents, which avoids the formation of 'holes' of various nature. An extended class of weak diffeomorphisms is then necessary here. To define it, only the condition of zero boundary current is relaxed. Orientation preserving maps with graphs admitting currents with non-zero boundaries are considered. A condition is, however, imposed: the total variation of the considered boundary currents must be bounded from above by the measure of the varifold describing the possible crack pattern. In other words, we try to define deformation that may have discontinuities that are contained in the crack and may coincide at most with it. The subsequent definition formalizes the idea.

Definition 6 *Take $\mathcal{B} \subset \mathbb{R}^2$ and $V \in CV_1^p(\mathcal{B})$. An extended weak diffeomorphism on $\mathcal{B}$ according to V is an a.e. approximately differentiable map*

$u : \mathcal{B} \to \hat{\mathbb{R}}^2$ *such that*

(1) $|M(Du)| \in L^1(\mathcal{B})$, *i.e.* $|Du|, \det Du \in L^1(\mathcal{B})$*;*
(2) $\pi_{\#}|\partial G_u| \leq \pi_{\#}V$*;*
(3) $\det Du(x) > 0$ *for a.e.* $x \in \mathcal{B}$*;*
(4) for every compactly supported smooth function $f : \mathcal{B} \times \hat{\mathbb{R}}^2 \to [0, +\infty)$

$$\int_{\mathcal{B}} f(x, u(x)) \, \det Du(x) \, dx \leq \int_{\hat{\mathbb{R}}^2} \sup_{x \in \mathcal{B}} f(x, y) \, dy \, .$$

The space of maps satisfying the previous definition is indicated by $\mathrm{Dif}^{1,1}(\mathcal{B}, V, \hat{\mathbb{R}}^2)$. Moreover, for $q > 1$, the class $\mathrm{Dif}^{q,1}(\mathcal{B}, V, \hat{\mathbb{R}}^2)$ is defined by

$$\mathrm{Dif}^{q,1}(\mathcal{B}, V, \hat{\mathbb{R}}^2) := \left\{ u \in \mathrm{Dif}^{1,1}(\mathcal{B}, V, \hat{\mathbb{R}}^2) \,\middle|\, |M(Du)| \in L^q(\mathcal{B}) \right\} .$$

The maps just defined are reasonable descriptors of non-classical deformations of bodies allowing the formation of fractures. Condition (2) implies that the Green formulas hold true in $\mathcal{B}$ outside the crack pattern, and prescribes that the boundary current has finite mass, namely $\mathbf{M}(\partial G_u) < \infty$. Therefore, since G_u is integer rectifiable, conditions (1) and (2) imply that ∂G_u is integer rectifiable too. Since by (2) the projection $\pi_{\#} \, |\partial G_u|$ is controlled by $\pi_{\#}V$, it follows that $\pi_{\#}|\partial G_u|$ is absolutely continuous with respect to $\mathcal{H}^{n-1} \llcorner \mathfrak{b}$. Condition (2) implies also that u is a special function with bounded variation, an element of $SBV(\mathcal{B}, \hat{\mathbb{R}}^2)$ with jump set J_u contained in $\mathfrak{b}$. Actually, u belongs to the class $SBV_0(\mathcal{B}, \hat{\mathbb{R}}^2)$, hence its *traces* u^+, u^- are $\mathcal{H}^1$-approximately differentiable in the jump set J_u as discussed in [17]. The standard orientation preserving requirement for the map u is stated in the item (3). Requirement (4) is the global one-to-one condition discussed in Section 2.

The essential properties of extended weak diffeomorphisms in Definition 6 are collected in the ensuing closure theorem.

Theorem 7 *Let* $\{V^{(r)}\} \subset CV_1^p(\mathcal{B})$*, with* $p > 1$*, be a sequence of curvature varifolds on* $\mathcal{B}$ *with equibounded total variations, i.e.* $\sup_r \mu_{V^{(r)}}(\mathcal{B}) < \infty$*, and equibounded total variations of* $\pi_{\#}|\partial V^{(r)}|$*. Moreover, assume* $u_r \in \mathrm{Dif}^{1,1}(\mathcal{B}, V^{(r)}, \hat{\mathbb{R}}^2)$*. Suppose also that there exist functions* $u \in L^1(\mathcal{B}, \hat{\mathbb{R}}^2)$*,* $v \in L^1(\mathcal{B}, \Lambda_2(\mathbb{R}^2 \times \hat{\mathbb{R}}^2))$*, and a curvature varifold* $V \in CV_1^p(\mathcal{B})$ *such that* $u_r \rightharpoonup u$*,* $M(Du_r) \rightharpoonup v$ *weakly in* L^1*, and* $V^{(r)} \rightharpoonup V$ *as measures. Then* $v = M(Du)$ *and, moreover, if* $\det Du > 0$ *a.e., then* $u \in \mathrm{Dif}^{1,1}(\mathcal{B}, V, \hat{\mathbb{R}}^2)$*.*

Proof. The hypotheses imply that $\sup_r (\mathbf{M}(G_{u_r}) + \mathbf{M}(\partial G_{u_r})) < \infty$. In particular, the sequence u_r is bounded in $BV(\mathcal{B}, \hat{\mathbb{R}}^2)$, so that, by passing eventually to subsequences, $u_r \to u$ in L^1 and a.e.. Thus $v = M(Du)$ (see [16, Vol. I, Section 3.3.1]). Moreover, (1), (2), and (4) in Definition 6 hold true by lower semicontinuity. Thus, if u satisfies (3), it follows that $u \in \mathrm{Dif}^{1,1}(\mathcal{B}, V, \hat{\mathbb{R}}^2)$. ■

In two-dimensional setting, the energy, indicated here by $\mathcal{E}(u, V)$ for the sake of conciseness, reads

$$\begin{aligned}\mathcal{E}(u, V) : &= \mathcal{E}(u, V, \mathcal{B}) \\ &= \int_{\mathcal{B}} e(x, u, Du)\, dx + c_1 \int_{\mathcal{G}_1(\mathcal{B})} |A|^p\, dV + c_2 \mathbf{M}(V) + c_3 \mathbf{M}(\partial V)\end{aligned} \tag{11}$$

where the c_i's are positive constants and the hypotheses (H1) (H2), (H3) and (H4) of Section 3 on the bulk energy density $e = e(x, u, F)$ are satisfied.

As regards the crack energy term, the $p-$power $|A|^p$ of the curvature can be replaced by $\phi(|A|)$ where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a convex function satisfying $\phi(t) \geq c_5\, t^p$. After the replacement, the results obtained in the sequel still hold with unessential modifications.

The term $c_2 \mathbf{M}(V)$ is the Griffith-like part of the surface energy of the crack. Non-standard terms are

$$c_1 \int_{\mathcal{G}_1(\mathcal{B})} |A|^p \; dV \quad \text{and} \quad c_3 \mathbf{M}\left(\partial V\right),$$

where $\mathcal{G}_1(\mathcal{B})$ is the Grassmanian of straigth lines over $\mathcal{B}$. The nature of the term involving the (generalized) curvature A has been already discussed. Also, crucial technical consequences of its presence have been mentioned. The term $c_3 \mathbf{M}(\partial V)$ takes into account the energy associated with the material bonds in the microscopic process region around the tip (see [19]).

Two simple examples show that $\mathcal{E}(u, V)$, with non-null varifold V, can be smaller that the energy $\mathcal{E}(u, \emptyset)$ associated with a null varifold $\emptyset$, interpreted as the energy of a simple elastic body. With the same boundary conditions, the gap between $\mathcal{E}(u, V)$ and $\mathcal{E}(u, \emptyset)$ is the energy losed in the nucleation of the crack represented by V.

Example 3 Consider $\mathcal{B}$ to be the unit disc $B(0, 1)$ in $\mathbb{C}$. Let $u : B(0, 1) \subset \mathbb{C} \to B(0, R) \subset \mathbb{C}$ be the linear dilatation $z \mapsto Rz$, $R > 1$, and $v : B(0, 1) \to B(0, R)$ be the dilatation with crack at $\mathfrak{b} := \partial B(0, \alpha)$, $0 \leq \alpha \leq 1$, precisely

$$v(z) := \begin{cases} z & \text{if } |z| \leq \alpha \\ Rz & \text{if } \alpha < |z| < 1\,. \end{cases}$$

Consider the energy of the two maps as given by

$$\mathcal{E}(u, V) := \frac{1}{2} \int_{B(0,1)} |Du|^2\, dx + \frac{1}{2} \int_{B(0,1)} \frac{|Du|^2}{\det Du}\, dx + c_1 \mathbf{M}(V)\,.$$

Set $V = V_{\mathfrak{b},2}$. Since $z \mapsto Rz$ is biholomorphic, it follows that

$$\begin{aligned}\mathcal{E}(u,\emptyset) &= \pi(R^2+1)\,,\\ \mathcal{E}(v,V) &= \pi 2\alpha^2 + \pi(R^2+1)(1-\alpha^2) + 4c_1\pi\alpha\\ &= \pi(R^2+1) + \pi(\alpha^2(1-R^2) + 4c_1\alpha)\,,\end{aligned}$$

so that $\mathcal{E}(v,V) < \mathcal{E}(u,\emptyset)$ if and only if $R > \sqrt{1+4c_1\alpha^{-1}}$. Therefore in this case the body admits fractures and the optimal placement for them is at the boundary, i.e. the solution loses Dirichlet boundary condition. The addition of the curvature term $\int_{\mathfrak{b}} |A|^p dV$ augments the dilatation needed for a crack to be convenient, but the crack is still at the boundary of $B(0,1)$ since the curvature term is a decreasing function of α ($p > 1$).

Example 4 Take $0 < \epsilon < 1$ and denote by Ω_ϵ the ellipsoid

$$\Omega_\epsilon := \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, \frac{x^2}{(1+\epsilon^2)^2} + \frac{y^2}{(1-\epsilon^2)^2} < 1 \right\}.$$

Let u be a biholomorphic map from Ω_ϵ onto $B(0,R)$, $R > 0$. Consider also the map $v : \Omega_\epsilon \setminus ([-2\epsilon, 2\epsilon] \times \{0\}) \to B(0,R) \setminus B(0,\epsilon R)$ defined as the inverse of the Youkowski map $w(z) := \frac{z}{R} + \frac{\epsilon^2 R}{z}$. Since both u and v are biholomorphic, the computation of the energy

$$\mathcal{E}(u,V) := \frac{1}{2}\int_{\Omega_\epsilon} |Du|^2\,dx + \frac{1}{2}\int_{u(\Omega_\epsilon)} |Du^{-1}|^2\,dy + c_1\mathbf{M}(V)$$

at $(u,\emptyset)$ and (v,V) with $V = V_{\mathfrak{b},2}$ and $\mathfrak{b} = [-2\epsilon, 2\epsilon] \times \{0\}$ furnishes

$$\begin{aligned}\mathcal{E}(u,\emptyset) &= \pi(1-\epsilon^4) + \pi R^2 + 0\,,\\ \mathcal{E}(v,V) &= \pi(1-\epsilon^4) + \pi R^2(1-\epsilon^2) + 8c_1\epsilon\,,\end{aligned}$$

so that $\mathcal{E}(v,V) < \mathcal{E}(u,\emptyset)$ iff $R^2 > \frac{8c_1}{\pi\epsilon}$. Again, by adding $\mathbf{M}(\partial V) = 4$ to the energy, it follows that the creation of a crack is energetically convenient when the dilatation of ratio R is greater than the one foreseen in absence of $\mathbf{M}(\partial V)$. The essential reason is that the boundary ∂V of the varifold describes the tips of the crack so that, if a peculiar energetic contribution to the tip is considered, a sort of energetic threshold to generate a fracture is introduced.

4.2 Ground states: existence theorems

In a purely elastic setting, large deformations are admitted at will (leave a part extreme deformation here). When the possible formation of cracks is accounted for, lack of bounds for the L^q-norm of u may occur. For example, consider a crack dividing $\mathcal{B}$ into two connected components. It may happen that a sequence of deformations may involve the rigid displacement of one of the two pieces to infinity, without affecting the energy. Moreover, even if the body is not disconnected by the crack, it is still possible that the body extends itself arbitrarily far away. Boundary conditions are essentially irrelevant in these phenomena, either Dirichlet boundary conditions or mixed ones, with a zero traction force on a part of $\partial\mathcal{B}$. Technical needs and, in a sense, physical plausibility suggest constraints to deformations in order to have bounds to the L^∞ norm. In a different setting, the previous requirement has been relaxed in [20], [21].

As mentioned in the introduction, it may also be convenient to prescribe a *comparison varifold* $\widetilde{V} \in CV_1^p(\mathcal{B})$ such that all competing varifolds V satisfy the bound $\pi_\# \widetilde{V} \leq \pi_\# V$. Such a requirement is analogous to the monotonicity requirement for cracks sequences in time imposed in [2]. The comparison varifold $\widetilde{V}$ can be also null when an initial crack is absent. In the opposite case, $\widetilde{V}$ describes a crack from which the competing cracks may at most extend.

The space

$$\mathcal{A}_{q,p,K,\tilde{V}}(\mathcal{B}) := \Big\{ (u,V) \,|\, V \in CV_1^p(\mathcal{B}), u \in \mathrm{Dif}^{1,1}(\mathcal{B}, V, \hat{\mathbb{R}}^2), \\ \|u\|_{L^\infty(\mathcal{B})} \leq K, \pi_\# \tilde{V} \leq \pi_\# V \Big\},$$

with $K > 0$, is then the natural functional environment for investigating the existence of minimizers (u, V) for the energy $\mathcal{E}$.

Previous closure and compactness properties, the ones summarized in Theorems 4, 7, and classical lower semicontinuity theorems (see e.g [16]) imply an existence result.

Theorem 8 *Consider $\mathcal{B} \subset \mathbb{R}^2$, $q, p > 1$, $K > 0$, $\widetilde{V} \in CV_1^p(\mathcal{B})$. Assume that there exists an element $(u, V) \in \mathcal{A}_{q,p,K,\widetilde{V}}(\mathcal{B})$ that satisfies prescribed Dirichlet boundary conditions for u. In these conditions, the energy functional* (11) *attains its minimum in the subclass of $\mathcal{A}_{q,p,K,\widetilde{V}}(\mathcal{B})$ of couples (u, V) with u satisfying the prescribed boundary conditions.*

The constant K is selected at will for purposes of physical plausibility: it is only necessary for establishing the boundedness of the L^∞ norm of u. In contrast, the constants p and q and the comparison varifold $\widetilde{V}$ have constitutive nature.

The simple description of the boundary measure of one-dimensional curvature varifolds allows us to state another existence theorem with a different growth condition for the bulk energy. It includes the functionals in Examples 3 and 4.

Consider the energy functional (11) with bulk energy density $e(x, u, F)$ satisfying (H1), (H2), (H4) of Section 3, and impose a different growth condition, indicated here by

(H3-1)

$$e(x,u,F) \geq c_4|F|^2 \qquad \forall F \in \mathbb{M}^+_{2\times 2}, \quad \forall (x,u) \in \mathcal{B} \times \hat{\mathbb{R}}^2\,,$$

for some $c_4 > 0$.

For $K > 0$ and $\widetilde{V} \in CV_1^p(\mathcal{B})$ the class

$$\mathcal{A}_{q,p,K,\tilde{V}}(\mathcal{B}) := \Big\{ (u,V)\,|V \in CV_1^p(\mathcal{B})\,, p > 1, u \in \mathrm{Dif}^{1,1}(\mathcal{B}, V, \hat{\mathbb{R}}^2),$$

$$Du \in L^2(\mathcal{B})\,, \|u\|_{L^\infty(\mathcal{B})} \leq K, \pi_\# \tilde{V} \leq \pi_\# V \Big\}\,, \tag{12}$$

is then the natural functional setting for another existence result.

Theorem 9 *Assume that the bulk energy density in* (11) *satisfies (H1), (H2) (H4) of Section 3 and (H3-1). Suppose that there is at least one element* (u_0, V_0) *in the class* (12) *with* u_0 *satisfying given Dirichlet data. Under these assumption, the functional* (11) *has a minimizer in the subclass of* (12) *of couples* (u, V) *with* u *satisfying the prescribed Dirichlet boundary conditions.*

Proof. A path presented in [12] for obtaining an analogous result in a different setting is followed here. Let (u_n, V_n) be a minimizing sequence in $\mathcal{A}_{p,\widetilde{V},K}$. By taking subsequences one can and does suppose that $V_n \rightharpoonup V$ as measures, and also that $u_n \rightharpoonup u$ and $Du_n \rightharpoonup v$ in $L^2(\mathcal{B})$. Moreover $||u||_\infty \leq K$. The energy estimate furnishes also $||\det Du_n||_{L^1} \leq \frac{1}{2}||Du_n||^2_{L^2} \leq C$ independently on n.

By using the regularity result for one-dimensional curvature varifolds with $p > 1$, it is possible to assume that $\pi_\# V_n = \mathcal{H}^1 \llcorner \Gamma_n$ for every n and $\pi_\# V = \mathcal{H}^1 \llcorner \Gamma$, where Γ_n and Γ are closed sets in $\mathcal{B}$ with finite $\mathcal{H}^1$ measure. By using Lemma 5.5 in [12], one may affirm that there exists a finite set $\mathcal{D}$ such that if Ω is an open set such that $\Omega \subset\subset \mathcal{B} \setminus (\Gamma \cup \mathcal{D})$, then $\Omega \subset\subset \mathcal{B} \setminus \Gamma_n$ for infinitely many n. Therefore, if Ω is as above, by taking eventually subsequences, one finds $u_n \in W^{1,2}(\Omega)$ for every n. It then follows that $Du_n \rightharpoonup Du$ in $L^2(\Omega)$, i.e. $v = Du$ a.e. on Ω.

Moreover, from the integrability result presented in [22] (see also [23]), the sequence $\{\det Du_n\}$ results equibounded in $L \log L_{loc}(\Omega)$, hence $\det Du_n \rightharpoonup w$ in $L^1_{loc}(\Omega)$. Finally, the closure theorem yields $w = \det Du$.

By energy semicontinuity, it then follows that, for every compact set $\mathcal{A} \subset\subset \Omega$,

$$\mathcal{E}(u, V, \mathcal{A}) \leq \liminf_{n\to\infty} \mathcal{E}(u_n, V_n, \mathcal{A}) \leq \liminf_{n\to\infty} \mathcal{E}(u_n, V_n, \mathcal{B})\,,$$

hence, since $\Omega \subset\subset \mathcal{B} \setminus (\Gamma \cup \mathcal{D})$ is arbitrary,

$$\mathcal{E}(u, V, \mathcal{B}) = \mathcal{E}(u, V, \mathcal{B} \setminus (\Gamma \cup \mathcal{D})) \leq \liminf_{n\to\infty} \mathcal{E}(u_n, V_n, \mathcal{B}) < +\infty\,.$$

(H4) then yields $\det Du > 0$ a.e., hence (u, V) belongs to $\mathcal{A}_{p,\widetilde{V},K}$ and minimizes the energy in its class. ■

In the previous scheme, a sequence of varifolds accumulating at the boundary of $\mathcal{B}$ vanishes at the limit.

It is possible to consider a different situation where the propagation of cracks at the boundary of the body $\mathcal{B}$ is taken into account, and a term involving the crack at the boundary may contribute to the limit energy of minimizing sequences. Meaning is rather clear in presence of Dirichlet boundary conditions, where the limit crack may be seen as a rupture of the kinematic constraint given by the boundary condition.

A variant of Theorem 8 can be then stated. Let $\Gamma \subset \partial\mathcal{B}$ be a piece of the boundary, where Dirichlet-type data are assigned. Let Ω be an open set such that $\mathcal{B} \subset \Omega$ and $\Omega \cap \partial\mathcal{B} = \Gamma$. Consider cracks represented by $V \in CV_1^p(\Omega)$ such that $\pi_\# V = 0$ on $\Omega \setminus \overline{\mathcal{B}}$, and a deformation represented by an extended weak diffeomorphism $u \in \mathrm{Dif}^{q,1}(\Omega, V, \hat{\mathbb{R}}^2)$ such that u agrees with a given function u_0 on $\Omega \setminus \overline{\mathcal{B}}$. The setting under examination is then the class

$$\overline{\mathcal{A}}_{q,p,K,\widetilde{V},u_0} := \Big\{ (u, V) \,|\, V \in CV_1^p(\mathcal{B}), \pi_\# \tilde{V} \leq \pi_\# V, \pi_\# V = 0 \text{ on } \Omega \setminus \overline{\mathcal{B}}$$

$$u \in \mathrm{Dif}^{q,1}(\mathcal{B}, V, \hat{\mathbb{R}}^2), u = u_0 \text{ on } \Omega \setminus \overline{\mathcal{B}}, \|u\|_{L^\infty(\mathcal{B})} \leq K \Big\}.$$

If $q, p > 1$, $K > 0$, and Γ and u_0 are sufficiently smooth, the class $\overline{\mathcal{A}}_{q,p,K,\widetilde{V},u_0}$ satisfies closure and compactness properties. As a consequence, a minimizer of the energy functional (11) exists on $\overline{\mathcal{A}}_{q,p,K,\widetilde{V},u_0}$, provided that this class is non-empty.

If $1 < p < 2$, the boundary condition $u = u_0$ on $\Omega \setminus \overline{\mathcal{B}}$ can be then enforced by a *strong anchorage* prescribing that $G_u = G_{u_0}$ on $(\Omega \setminus \overline{\mathcal{B}}) \times \hat{\mathbb{R}}^2$. A related existence theorem again follows.

5 Analysis in n-dimensional ambient space

Previous results have natural generalizations in $n-$dimensional ambient space. Stratified families of varifolds are implicated in the description of crack pattern geometry. In principle, they can be used also to represent arrangements of different defects, above all when certainty of nucleation and placement of these defects are not know a priori.

Definition 10 *A stratified family of curvature varifolds with boundary in $\mathcal{B}$ is a family $\{V_{(k)}\}_{k=1}^{n-1}$ of curvature k-varifolds with boundary $V_{(k)} = V_{\mathfrak{b}_k,\theta_k}$*

in $CV_k^{p_k}(\mathcal{B})$*, with corresponding boundaries* $\partial V_{(k)}$ *and curvatures* ${A_{(k)}}_j^{\ell i}$ *in* $L^{p_k}(\mathcal{G}_{n-1}(\mathcal{B}))$*, for some* $p_k > 1$*, such that*

$$\pi_{\#}|\partial V_{(k)}| \leq \pi_{\#} V_{(k-1)} \qquad \forall\, k = 2, \ldots, n-1\,. \tag{13}$$

Condition (13) appears also in [11] and [24].

Definition 11 *An extended weak diffeomorphism on* $\mathcal{B}$ *according with* $\left\{V_{(k)}\right\}$ *is an a. e. approximately differentiable map* $u : \mathcal{B} \to \hat{\mathbb{R}}^n$ *such that (1), (3) and (4) in Definition 6 hold true and, in addition,*

(2) $\pi_{\#}|\partial G_u| \leq \sum_{j=1}^{n-1} \pi_{\#} V_{(j)} + \pi_{\#}|\partial V_{(1)}|$.

The space of maps satisfying Definition 11 is indicated by $\mathrm{Dif}^{1,1}(\mathcal{B}, \{V_{(k)}\}, \hat{\mathbb{R}}^n)$. Moreover, for $q > 1$, the class $\mathrm{Dif}^{q,1}(\mathcal{B}, \{V_{(k)}\}, \hat{\mathbb{R}}^n)$ is defined by

$$\mathrm{Dif}^{q,1}(\mathcal{B}, \{V_{(k)}\}, \hat{\mathbb{R}}^n) := \left\{ u \in \mathrm{Dif}^{1,1}(\mathcal{B}, \{V_{(k)}\}, \hat{\mathbb{R}}^n) \,\middle|\, |M(Du)| \in L^q(\mathcal{B}) \right\}.$$

Condition (2) implies that Green formulas hold true outside the cracks $V_{(k)}$. It indicates again that the boundary current ∂G_u has finite mass. In particular, u belongs to the class $SBV_0(\mathcal{B}, \hat{\mathbb{R}}^n)$ (see [17]).

The energetic scenario described here involves various aspects of fracturing phenomena. Consider for example a three-dimensional crystalline body where cracks may develop. In front of the tip, dislocations and vacancies can appear (see e.g. [25]). They are line and point defects which accompany the cracking process and are endowed with their own energy. The stratification of the varifolds is then useful to represent the energetic scenario associated with the surface of the crack, the tip, and the appearance of low-dimensional defects. More specifically, when a process zone 'around' the crack tip is significant, the stratification of energies associated with the varifolds can constitute a reasonable description of the 'fragmented' critical events near the tip, i.e. the various small ruptures determining steps forward in the evolution of the macroscopic (two-dimensional) crack.

The energy functional $\mathcal{E}\left(u, \left\{V_{(k)}\right\}, \mathcal{B}\right)$ is here indicated simply by $\mathcal{E}\left(u, \left\{V_{(k)}\right\}\right)$ and is given by

$$\begin{aligned} \mathcal{E}(u, \{V_{(k)}\}) :&= \int_{\mathcal{B}} e(x, u, Du)\, dx + \sum_{k=1}^{n-1} \alpha_k \int_{\mathcal{G}_k(\mathcal{B})} |A_{(k)}|^{p_k}\, dV_{(k)} \\ &\quad + \sum_{k=1}^{n-1} \beta_k\, \mathbf{M}(V_{(k)}) + \sum_{k=1}^{n-1} \gamma_k\, \mathbf{M}(\partial V_{(k)})\,, \end{aligned} \tag{14}$$

where $\alpha_k, \beta_k, \gamma_k$ are positive constants. By taking into account (13), it is possible to select $\gamma_k = 0$ for every $k = 2, \ldots, n-1$ in (14).

For one-dimensional 'defects' ($k = 1$), the contribution of their possible bending energy is accounted for through their curvature. This way, line defects are considered like beams [26]. In the case of linear dislocations, for example, this is a way to model the defect core (see e.g. [27]). In a coarse grained way, the line energy along the tip takes into account in a coarse grained way the intricate mixture of broken and integer materials bonds constituting the crack tip itself.

The varifold $V_{(1)}$ describes the tip and all linear defects 'around' the crack. The boundary $\partial V_{(1)}$ of $V_{(1)}$ represents 'corners' and 'edges' along the tip where the entanglement of atomic bonds may cause concentration of energy – see relevant remarks in [28] about the possible presence of sparse atomic (or molecular) bonds near the tip of a crack, in the terminal part of the crack surfaces. $\partial V_{(1)}$ is also associated with a set of discrete points where possible linear defects meet the tip.

The tip energy may be interpreted as the indicator of the presence of a local threshold that is necessary to overcome to allow the crack to grow.

A *comparison stratified family of curvature varifolds* $\left\{\widetilde{V}_{(k)}\right\}$ in $CV_k^{p_k}(\mathcal{B})$ and a constant $K > 0$ are introduced. The analysis of the existence of minimizers for $\mathcal{E}\left(u, \left\{V_{(k)}\right\}\right)$ is then limited to the class

$$\mathcal{A}_{q,p,K,\{\widetilde{V}_{(k)}\}}(\mathcal{B}) := \left\{(u, \{V_{(k)}\}) \,\middle|\, V_{(k)} \in CV_k^{p_k}(\mathcal{B}),\ u \in \mathrm{Dif}^{q,1}(\mathcal{B}, \{V_{(k)}\}, \hat{\mathbb{R}}^n) \right.$$
$$\left. (13) \text{ holds},\ ||u||_{L^\infty(\mathcal{B})} \leq K,\ \mu_{\widetilde{V}_{(k)}} \leq \mu_{V_{(k)}}\ \forall\, k = 1, \ldots, n-1 \right\}$$

where $q, p_k > 1$, $K > 0$ and $k = 1, \ldots, n-1$.

The previous closure and compactness properties discussed earlier, Theorem 4, 7, condition (13), and the classical lower semicontinuity theorem imply an existence result.

Theorem 12 *Take $q, p_k > 1$, $K > 0$, $\widetilde{V}_{(k)} \in CV_k^{p_k}(\mathcal{B})$ for $k = 1, \ldots, n-1$. Assume that there exists an element $(u, \{V_{(k)}\}) \in \mathcal{A}_{q,\{p_k\},K,\{\widetilde{V}_{(k)}\}}(\mathcal{B})$ satisfying prescribed boundary conditions. In these conditions, the energy functional* (14) *attains a minimum in the class $\mathcal{A}_{q,\{p_k\},K,\{\widetilde{V}_{(k)}\}}(\mathcal{B})$ with the prescribed boundary conditions.*

Remind that the explicit expression of $\mathcal{E}\left(u, \left\{V_{(k)}\right\}\right)$ reduces to (4) when $n = 3$.

When $\{\widetilde{V}_{(k)}\}$ is not zero, it represents an initial crack pattern from which the competing cracks may possibly extend only.

Finally, we notice that the condition $\|u\|_{L^\infty(\mathcal{B})} \leq K$ can be replaced by $\|u\|_{L^r(\mathcal{B})} \leq K$, $r \geq 1$, or simply eliminated by adding, however, a term $\varepsilon \int_{\mathcal{B}} |u|^r \, dx$ to the energy.

6 The first variation: the weak form of configurational balances

The first variation of $\mathcal{E}\left(u, \left\{V_{(k)}\right\}\right)$ can be calculated. For the sake of simplicity, we evaluate it in three-dimensional ambient space, so the explicit expression of the energy considered is (4).

We point out that the evaluation of the first variation is made *without* considering additional regularity hypotheses on crack pattern geometry. So, the *weak form* of the balance equations, summarized later in Theorem 13, holds for crack patterns described by generic stratified families of curvature varifolds. This is a peculiar aspect of our approach.

Geometric regularity is imposed to crack patterns only in the next section, where we calculate in a special case *pointwise* balances of configurational actions on the crack.

When $\mathcal{B}$ is three-dimensional, a stratified family of varifolds over it is a couple $\left(V_{(2)}, V_{(1)}\right)$ of curvature k-varifolds $V_{(k)} = V_{\mathfrak{b}_k,\theta_k}$ in $CV_k^{p_k}(\mathcal{B})$, $k = 1, 2$, with boundary $\partial V_{(k)}$ and curvature $A_{(k)}{}_j^{\ell i}$ in $L^{p_k}(\mathcal{G}_k(\mathcal{B}))$, for some $p_k > 1$, such that $\pi_\# |\partial V_{(2)}| \leq \pi_\# V_{(1)}$. Less formally, the two varifolds $V_{(2)}$ and $V_{(1)}$ describe respectively the lateral margins of a two-dimensional crack and its tip, and possible dislocations in $\mathcal{B}$. Moreover, as mentioned earlier, $V_{(1)}$ may describe also linear defects far from the crack. For example, in the case of crystalline bodies, such defects can be not only dislocations, as just recalled above, but even a linear coalescence of vacancies generating what we can call 'linear cracks'.

Denote by $\Pi_{(k)}(x)$ the orthogonal projection over $T_x\mathfrak{b}_k$, for $k = 1, 2$. For every smooth vector field $\lambda \in C_c^\infty(\mathcal{B}, \mathbb{R}^n)$ and for $|\epsilon|$ sufficiently small, consider also the family of diffeomorphisms

$$\xi_\epsilon : \overline{\mathcal{B}} \to \overline{\mathcal{B}}, \qquad \xi_\epsilon(z) := z + \epsilon\, \lambda(z)$$

where λ vanishes on all sets $\widetilde{\mathfrak{b}}_k$ supporting the family $\left\{\widetilde{V}_{(k)}\right\}$ of comparison curvature varifolds introduced previously. Meaning is as follows: if there is a pre-existing crack $\widetilde{\mathfrak{b}}_k$, the variation of $\mathcal{B}$ leaves unaltered $\widetilde{\mathfrak{b}}_k$. Set $u_\epsilon(z) :=$

$u(\xi_\epsilon(z))$. Observe that $G_{u_\epsilon} = \psi_{\epsilon\#} G_u$, where $\psi_\epsilon(x, y) := (\eta_\epsilon(x), y)$ and the map η_ϵ is the inverse of ξ_ϵ. For $k = 1, 2$, set $\mathfrak{b}_{k,\epsilon} := \eta_{\epsilon\#}\mathfrak{b}_k$ and $\theta_{k,\epsilon}(z) := \theta_k(\xi_\epsilon(z))$. Remind that $\{\mathfrak{b}_k\}$, $k = 1, 2$, is the support of the stratified varifold $\left\{V_{(k)}\right\}$ minimizing $\mathcal{E}$. The deformed varifold of $V_{\mathfrak{b}_k,\theta_k}$ is then given by $V_{(k)\epsilon} = V_{\mathfrak{b}_{k,\epsilon},\theta_{k,\epsilon}} \in CV_k^{p_k}(\mathcal{B})$. The first variation of the energy functional at a minimizer $(u, \{V_{\mathfrak{b}_k,\theta_k}\})$ is defined formally by

$$\frac{d}{d\epsilon}\mathcal{E}(u_\epsilon, \{V_{(k)\epsilon}\})\,|_{\epsilon=0} = 0$$

where only vector fields $\lambda \in C_c^\infty(\mathcal{B}, \mathbb{R}^n)$ that vanish on all $\widetilde{\mathfrak{b}}_k$ have to be considered, due to the condition $\pi_\#\widetilde{V}_{(k)} \leq \pi_\# V_{(k)}$ on the possibly nonzero comparison varifolds.

It is expedient to discuss the differentiability of each term of the energy functional (4) separately.

6.1 Variation of the bulk energy

As pointed out in [29] and [30], the differentiability of the bulk energy map $\epsilon \mapsto \int_{\mathcal{B}} e(x, u_\epsilon, Du_\epsilon)\, dx$ holds under the weak energy estimate

$$|\partial_x e(x, u, F)| + |F^T \partial_F e(x, u, F)| \leq \widetilde{c}_1\, e(x, u, F) + \widetilde{c}_2 \tag{15}$$

for every (x, u, F) such that $F \in \mathbb{M}^+_{n\times n}$. It follows that

$$\begin{aligned} &\frac{d}{d\epsilon}\left(\int_{\mathcal{B}} e(x, u_\epsilon, Du_\epsilon)\, dx\right)_{|\epsilon=0} \\ &\qquad = -\int_{\mathcal{B}} \left(\mathbb{P}^\beta_\alpha(x, u, Du)\, \lambda^\alpha_{x^\beta} - e_{x^\alpha}(x, u, Du)\, \lambda^\alpha\right) dx\,, \end{aligned} \tag{16}$$

where $\mathbb{P}^\beta_\alpha$ is the *Hamilton–Eshelby tensor*

$$\mathbb{P}^\beta_\alpha := e\delta^\beta_\alpha - F^i_\alpha\, e_{F^i_\beta} \qquad e = e(x, u, F)\,,$$

and $\lambda^\alpha_{x^\beta}$ indicates the partial derivatives of λ^α with respect to the coordinate x^β, i.e. the α-controvariant β-covariant component of the Jacobian matrix of λ. Also, $e_{F^i_\beta}$ and e_{x^α} are partial derivatives of e with respect to F^i_β and x^α, respectively, and δ^β_α is Kronecker symbol.

The main task here is to compute the curvature tensor $A_{(2)\epsilon}$ of the two dimensional varifold $V_{(2)\epsilon} := V_{\mathfrak{b}_{2,\epsilon}}, \theta_{2,\epsilon} \in CV_2^{p_2}(\mathcal{B})$. In order to simplify the notation, set $p := p_2$, $A_{(\epsilon)} := A_{(2)\epsilon}$, $V_\epsilon := V_{(2)\epsilon}$, $\mathfrak{b}_\epsilon := \mathfrak{b}_{2,\epsilon}$, $\theta_\epsilon := \theta_{2,\epsilon}$ and $\Pi(x) := \Pi_{(2)}(x)$.

By (5), the equality

$$\int_{\mathcal{G}_2(\mathcal{B})} |A|^p \, dV = \int_{\mathfrak{b}} \theta(x)|A(x, \Pi(x))|^p \, d\mathcal{H}^2(x) \tag{17}$$

follows. The corresponding energy term on the deformed varifold is given by

$$\int_{\mathcal{G}_2(\mathcal{B})} |A_{(\epsilon)}|^p \, dV_\epsilon = \int_{\mathfrak{b}_\epsilon} \theta_\epsilon(z)|A_{(\epsilon)}(z, \Pi_{(\epsilon)}(z))|^p \, d\mathcal{H}^2(z)\,,$$

where $\Pi_{(\epsilon)}(z)$ is the orthogonal projection onto $T_z\mathfrak{b}_\epsilon$. Moreover, by (3) in Theorem 2

$$A^{\ell i}_{(\epsilon)j}(z, \Pi_{(\epsilon)}(z)) = \left(\nabla^{\mathfrak{b}_\epsilon}\Pi^\ell_{(\epsilon)j}(z)\right)^i \qquad \text{for } \pi_\# V_{(\epsilon)}\text{- a.e. } z,$$

where $\nabla^{\mathfrak{b}_\epsilon}$ is the gradient along $\mathfrak{b}_\epsilon$. For $\mathcal{H}^2$-a.e. $z \in \mathfrak{b}_\epsilon$ set

$$A_{(\epsilon)}(x) := \nabla^{\mathfrak{b}_\epsilon}\Pi_{(\epsilon)}(z), \qquad z = \eta_\epsilon(x) \in \mathfrak{b}_\epsilon\,. \tag{18}$$

Proposition 1 *For small ϵ and for $\mathcal{H}^2$-a.e. $x \in \mathfrak{b}$ one gets*

$$A_{(\epsilon)}(x) = A(x, \Pi(x)) + \epsilon\, A'(x) + O(\epsilon^2)\,,$$

where $|O(\epsilon^2)| \le c\,\epsilon^2$ and the constant $c = c(||\lambda||_{C^2})$ does not depend on $x \in \mathfrak{b}$. Moreover, it follows that

$$A'^{\ell i}_{j}(x) := \left(B(x)\nabla_x\Pi^\ell_j(x) + \Pi(x)L^T\nabla_x\Pi^\ell_j(x) + \Pi(x)\nabla_x B^\ell_j(x)\right)^i, \tag{19}$$

where $L := \nabla\lambda(x)$ and $B = B(x)$ is the symmetric 3×3 matrix

$$B(x) := 2(L\nu \cdot \nu)\,\nu \otimes \nu - L^T\nu \otimes \nu - \nu \otimes L^T\nu \tag{20}$$

where $\nu = \nu(x)$ is a unit normal to $\mathfrak{b}$ at x.

Proof. Since $\Pi_{(\epsilon)}(z) = Id - \nu_\epsilon(z) \otimes \nu_\epsilon(z)$, the dyad $\nu_\epsilon(z) \otimes \nu_\epsilon(z)$ is evaluated first. As $\nu = \nu(x)$ is a unit normal to $\mathfrak{b}$ at $x = \xi_\epsilon(z)$, and $F^T(z)\nu$ is orthogonal to $T_z\mathfrak{b}_\epsilon$, with $F(z) := D\xi_\epsilon(z)$, unit normal ν_ϵ to $\mathfrak{b}_\epsilon$ at z is given by

$$\nu_\epsilon = \frac{F^T(z)\nu}{|F^T(z)\nu|}\,.$$

Since $F^T(z)\nu(x) = (Id + \epsilon D\lambda(z))^T\nu = \nu + \epsilon\, L^T\nu + O(\epsilon^2)$, it follows that

$$(F^T\nu) \otimes (F^T\nu) = \nu \otimes \nu + \epsilon\,(L^T\nu \otimes \nu + \nu \otimes L^T\nu) + O(\epsilon^2)$$

and hence

$$\nu_\epsilon \otimes \nu_\epsilon = \frac{F^T\nu \otimes F^T\nu}{|F^T\nu \otimes F^T\nu|} = \frac{a + \epsilon\, b + o(\epsilon)}{|a + \epsilon\, b + o(\epsilon)|}$$

where $a := \nu \otimes \nu$ and $b := (L^T\nu \otimes \nu + \nu \otimes L^T\nu)$.

By setting $\Phi^\ell_j(\epsilon) := (\nu_\epsilon \otimes \nu_\epsilon)^\ell_j$, one gets $\Phi^\ell_j(\epsilon) = \Phi^\ell_j(0) + \Phi^{\ell\prime}_j(0)\,\epsilon + O(\epsilon^2)$. Since $|a| = 1$, it follows that

$$\Phi^\ell_j(0) = a^\ell_j, \qquad \Phi^{\ell\prime}_j(0) = b^\ell_j - a^\ell_j\, b^h_k\, a^h_k\,,$$

where $a^\ell_j := \nu^\ell\nu^j$ and $b^h_k := \Big(L^T\nu \otimes \nu + \nu \otimes L^T\nu\Big)^h_k$. A straightforward computation leads to

$$\begin{aligned} b^h_k\, a^h_k &= ((L^T)^h_\alpha\, \nu^\alpha\nu^k + \nu^h(L^T)^k_\beta\, \nu^\beta)\nu^h\nu^k \\ &= (\nu^k)^2((L^T)^h_\alpha\, \nu^\alpha\nu^h) + (\nu^h)^2(\nu^k(L^T)^k_\beta\, \nu^\beta) \\ &= (L^T\nu)^h\, \nu^h + \nu^k(L^T\nu)^k \\ &= (L^T\nu \cdot \nu) + (\nu \cdot L^T\nu) = 2\, L\nu \cdot \nu\,, \end{aligned}$$

which yields $\Phi^{\ell\prime}_j(0) = -\Big(2\,(L\nu \cdot \nu)\, a^\ell_j - b^\ell_j\Big)$, whence

$$\nu_\epsilon(z) \otimes \nu_\epsilon(z) = \nu(x) \otimes \nu(x) - \epsilon\, B(x) + O(\epsilon^2)\,.$$

Thus it follows that

$$\Pi_{(\epsilon)}(z) = \Pi(x) + \epsilon\, B(x) + O(\epsilon^2), \qquad x = \xi_\epsilon(z)\,, \tag{21}$$

since $\Pi(x) = Id - \nu(x) \otimes \nu(x)$.

Let now $\widetilde{f} : \mathbb{R}^3 \to \mathbb{R}$ be a smooth map. Define $f(z) := \widetilde{f}(x)$, $x = \xi_\epsilon(z)$, so that $D_z f(z) = D_x\widetilde{f}(x)\, F(z)$ and $\nabla_z f(z) = F(z)^T\, \nabla_x\widetilde{f}(x)$. Then, for $\mathcal{H}^2$-a.e. $z \in \mathfrak{b}_\epsilon$ one gets

$$\nabla^{\mathfrak{b}_\epsilon} f(z) = \Pi_{(\epsilon)}(z)\, \nabla_z f(z) = \Pi_{(\epsilon)}(z)\, F(z)^T\, \nabla_x\widetilde{f}(x)\,,$$

where $x = \xi_\epsilon(z)$.

By applying previous formula to each component $\Pi^\ell_{(\epsilon)j}(z)$ of $\Pi_{(\epsilon)}(z)$, from (21) it follows that

$$
\begin{aligned}
&\nabla^{\mathfrak{b}_\epsilon}\Pi_{(\epsilon)}{}_j^\ell(z) = \\
&\qquad = (\Pi(x) + \epsilon\, B(x) + O(\epsilon^2))(Id + \epsilon L^T + O(\epsilon^2)) \\
&\qquad\qquad\qquad\qquad \cdot (\nabla_x \Pi_j^\ell(x) + \epsilon\, \nabla_x B_j^\ell(x) + O(\epsilon^2)) \\
&\qquad = \Pi(x)\nabla_x\Pi_j^\ell(x) \\
&\qquad\quad + \epsilon\left(B(x)\nabla_x\Pi_j^\ell(x) + \Pi(x)L^T\nabla_x\Pi_j^\ell(x) + \Pi(x)\nabla_x B_j^\ell(x)\right) + O(\epsilon^2)\,,
\end{aligned}
$$

where $\Pi(x)\nabla_x\Pi_j^\ell(x) = \nabla^{\mathfrak{b}}\Pi_j^\ell(x)$ and $A_j^{\ell i}(x) = (\nabla^{\mathfrak{b}}\Pi_j^\ell(x))^i$. Finally, the estimate on the term $O(\epsilon^2)$ is obtained by computing the derivatives w.r.t. ϵ in the formula

$$
(F^T\nu \otimes F^T\nu) = \nu\otimes\nu + \epsilon\,(L_\epsilon^T\nu\otimes\nu + \nu\otimes L_\epsilon^T\nu) + \epsilon^2(L_\epsilon^T\nu\otimes L_\epsilon^T\nu)\,,
$$

where $L_\epsilon(z) := D\lambda(z)$ and $z = \eta_\epsilon(x)$. Further details are omitted. ∎

Compute now

$$
\begin{aligned}
&\frac{d}{d\epsilon}\left(\int_{\mathcal{G}_2(\mathcal{B})} |A_{(\epsilon)}|^p\, dV_\epsilon\right)_{|\epsilon=0} \\
&\qquad = \frac{d}{d\epsilon}\left(\int_{\mathfrak{b}_\epsilon} \theta_\epsilon(z)|A_{(\epsilon)}(z,\Pi_{(\epsilon)}(z))|^p\, d\mathcal{H}^2(z)\right)_{|\epsilon=0}\,.
\end{aligned}
$$

By taking into account the area formula and (18), it follows that

$$
\int_{\mathfrak{b}_\epsilon} \theta_\epsilon(z)\,|A_{(\epsilon)}(z,\Pi_{(\epsilon)}(z))|^p\, d\mathcal{H}^2(z) = \int_{\mathfrak{b}} \theta(x)\,|A_{(\epsilon)}(x)|^p\, J^{\mathfrak{b}}\eta_\epsilon(x)\, d\mathcal{H}^2(x)\,,
$$

where $z = \eta_\epsilon(x)$ and $J^{\mathfrak{b}}\eta_\epsilon(x)$ is the tangential Jacobian to $\mathfrak{b}$ of the map η_ϵ. Moreover, Proposition 1 yields

$$
\frac{d}{d\epsilon}\left(\int_{\mathfrak{b}} \theta(x)\,|A_{(\epsilon)}(x)|^p\, J^{\mathfrak{b}}\eta_\epsilon(x)\, d\mathcal{H}^{n-1}\right) = \int_{\mathfrak{b}} \frac{d}{d\epsilon}\left(\theta(x)\,|A_{(\epsilon)}(x)|^p\, J^{\mathfrak{b}}\eta_\epsilon(x)\right) d\mathcal{H}^2
$$

whereas

$$
\frac{d}{d\epsilon}A_{(\epsilon)}(x)_{|\epsilon=0} = A'(\lambda)(x)\,.
$$

Finally, since $\eta_\epsilon(x) = x - \epsilon\lambda(x) + o(\epsilon)$, we get $D\eta_\epsilon(x) = I - \epsilon D\lambda(x) + o(\epsilon)$ and $J^{\mathfrak{b}}\eta_\epsilon = 1 - \epsilon\,\mathrm{div}^{\mathfrak{b}}\lambda + o(\epsilon)$, where $\mathrm{div}^{\mathfrak{b}}$ is the divergence along $\mathfrak{b}_\epsilon$, so that

$$
\frac{d}{d\epsilon}(J^{\mathfrak{b}}\eta_\epsilon)_{|\epsilon=0} = -\mathrm{div}^{\mathfrak{b}}\lambda\,.
$$

The conclusion is that the curvature term (17) is differentiable at zero and

$$\frac{d}{d\epsilon}\left(\int_{\mathcal{G}_2(\mathcal{B})} |A_{(2)\epsilon}|^{p_2}\, dV_{(2)\epsilon}\right)_{|\epsilon=0}$$
$$= \int_{\mathfrak{b}_2\setminus\widetilde{\mathfrak{b}}_2} \theta_2(x)\, p_2 |A_{(2)}|^{p_2-2}\, A_{(2)}{}_j^{\ell i} A'_{(2)}(\lambda)_j^{\ell i}\, d\mathcal{H}^2 - \int_{\mathfrak{b}_2\setminus\widetilde{\mathfrak{b}}_2} \theta_2(x)\, |A_{(2)}|^{p_2}\, \mathrm{div}_x^{\mathfrak{b}_2}\lambda\, d\mathcal{H}^2 \tag{22}$$

where $A_{(2)} = A_{(2)}(x, \Pi_{(2)}(x))$, and $A'_{(2)}(\lambda)(x) = A'(\lambda)(x)$ is given by (19), with $\Pi(x) = \Pi_{(2)}(x)$.

Remark 1 Let us analyze the dependence of the first variation of the curvature term on the test field λ. First observe that, by (20), $B = CL$, where $L_k^h = D_k\lambda^h(x)$ and

$$C_{jh}^{\ell k} := 2\nu^\ell \nu^j \nu^h \nu^k - \delta_{\ell k}\nu^j\nu^h - \delta_{jk}\nu^\ell\nu^h\,,$$

so that $B_j^\ell = C_{jh}^{\ell k} D_k \lambda^h$.

On the other hand, for a smooth map f

$$\nabla_x f = \Pi\nabla_x f + (Id - \Pi)\nabla_x f = \Pi\nabla_x f + \nu\otimes\nu\nabla_x f = \Pi\nabla_x f + D_\nu f \nu$$

where $D_\nu f := \frac{\partial}{\partial\nu} f$. Thus, by setting $A_j^\ell(x) := (A_j^{\ell 1}(x,\Pi(x)), \dots, A_j^{\ell n}(x,\Pi(x)))^T$, one computes

$$\begin{aligned}
&B\nabla_x\Pi_j^\ell = (CL)(A_j^\ell + D_\nu\Pi_j^\ell\,\nu)\ ,\\
&\Pi L^T\nabla_x\Pi_j^\ell = \Pi L^T A_j^\ell + D_\nu\Pi_j^\ell\,\Pi L^T\nu\\
&\Pi\nabla_x B_j^\ell = \Pi\nabla_x(CL)_j^\ell = \Pi(\nabla_x CL)_j^\ell + \Pi(C\nabla_x L)_j^\ell\,.
\end{aligned}$$

Therefore, by (19) we obtain

$$\begin{aligned}
A'^\ell_j = {} & (CL)A_j^\ell + D_\nu\Pi_j^\ell\,(CL)\nu + \Pi L^T A_j^\ell\\
& + D_\nu\Pi_j^\ell\,\Pi L^T\nu + \Pi(\nabla_x CL)_j^\ell + \Pi(C\nabla_x L)_j^\ell\,.
\end{aligned}$$

Remark 2 The tensor B_j^ℓ defined in (20) depends on the gradient of λ and the unit normal ν. By assuming for a while that $\mathfrak{b}$ is smooth, in case of normal variation with a constant amplitude $v\in\mathbb{R}$, one gets $B(x) = 0$. In fact, if $\lambda(x) = v\,\nu(x)$, from $L(x) = D\lambda(x) = vD\nu(x)$ and $D|\nu|^2 = 0$ one obtains that $L\nu\cdot\nu = 0$ and $L^T\nu = 0$, so that $B = 0$. Consequently (19) reduces to

$$A'^\ell_j = \Pi L^T\nabla_x\Pi_j^\ell = \Pi L^T A_j^\ell + D_\nu\Pi_j^\ell\,\Pi L^T\nu = \Pi L^T A_j^\ell\,.$$

6.3 Variation of the term $\mathbf{M}(V_{(2)})$

Since the mass $\mathbf{M}(V_{(2)})$ is equal to $\pi_{\#}V_{(2)}(\mathcal{B})$, with $\pi_{\#}V_{(2)} = \theta_2\,\mathcal{H}^2\llcorner\mathfrak{b}_2$, the variational formula

$$\frac{d}{d\epsilon}\mathbf{M}(V_{(2)\epsilon})_{|\epsilon=0} = \frac{d}{d\epsilon}(\eta_{\epsilon\#}\pi_{\#}V_{(2)}(\mathcal{B}))_{|\epsilon=0} = -\int_{\mathfrak{b}_2\setminus\widetilde{\mathfrak{b}_2}} \theta_2(x)\,\mathrm{div}^{\mathfrak{b}_2}\lambda\, d\mathcal{H}^2 \qquad (23)$$

holds for every smooth vector field $\lambda \in C_c^\infty(\mathcal{B},\mathbb{R}^3)$ (see e.g. [13]).

6.4 Variation of the one-dimensional curvature term of $\mathcal{E}\left(u,\left\{V_{(k)}\right\}\right)$

The computation of the curvature $A_{(1)\epsilon}$ of $V_{(1)\epsilon}$ is summarized in the following proposition.

Proposition 2 *For $\mathcal{H}^1$-a.e. $z \in \mathfrak{b}_{1,\epsilon}$ and $x = \xi_\epsilon(z) := z + \epsilon\lambda(z)$, $\lambda \in C_c^\infty(\mathcal{B},\mathbb{R}^3)$, set $L = L(x) := \nabla\lambda(x)$ and consider the 3×3 matrix $\widetilde{B} = \widetilde{B}(x)$ defined by*

$$\widetilde{B} := 2(L\mathbf{t}\cdot\mathbf{t})\,\mathbf{t}\otimes\mathbf{t} - L\mathbf{t}\otimes\mathbf{t} - \mathbf{t}\otimes L\mathbf{t}\,,$$

that is, $\widetilde{B}^\ell_j = \widetilde{C}^{\ell h}_{jk}L^k_h$, where

$$\widetilde{C}^{\ell h}_{jk} := 2\mathbf{t}^\ell\mathbf{t}^j\mathbf{t}^h\mathbf{t}^k - \delta_{\ell k}\mathbf{t}^j\mathbf{t}^h - \delta_{jk}\mathbf{t}^\ell\mathbf{t}^h\,,$$

$\mathbf{t} = \mathbf{t}(x)$ denoting a unit tangent vector to $\mathfrak{b}_1$ at x. Then the following equality holds:

$$\begin{aligned} A_{(1)\epsilon}{}^{\ell i}_j(z) = {} & A_{(1)}{}^{\ell i}_j(x) \\ & + \epsilon\Big(\widetilde{B}(x)\nabla_x\Pi_{(1)}{}^\ell_j(x) + \Pi_{(1)}(x)L^T\nabla_x\Pi_{(1)}{}^\ell_j(x) + \Pi_{(1)}(x)\nabla_x\widetilde{B}^\ell_j(x)\Big)^i \\ & + O(\epsilon^2), \end{aligned}$$

with $|O(\epsilon^2)| \le K\epsilon^2$ uniformly and K depending on the C^2 norm of λ.

Proof. Let $\mathbf{t}_\epsilon$ be a unit tangent vector to $\mathfrak{b}_{1,\epsilon}$ at z. Since $\mathfrak{b}_{1,\epsilon}$ has dimension one, the projection onto $T_z\mathfrak{b}_{1,\epsilon}$ is $\Pi_{(\epsilon)}(z) = \mathbf{t}_\epsilon(z)\otimes\mathbf{t}_\epsilon(z)$. Moreover,

$$\mathbf{t}_\epsilon(z) = \frac{F(x)\mathbf{t}(x)}{|F(x)\mathbf{t}(x)|}, \qquad F(x) := D\eta_\epsilon(x)\,, \qquad z = \eta_\epsilon(x)\,.$$

Trivially, the relation

$$F(x)\mathbf{t}(x) = (Id - \epsilon L)\mathbf{t}(x) + o(\epsilon) \qquad \text{with} \qquad |o(\epsilon)| \le C\epsilon^2, \quad C = C(||\lambda||_{C^2})\,,$$

holds, so that

$$F\mathbf{t}\otimes F\mathbf{t} = \mathbf{t}\otimes\mathbf{t} - \epsilon\,(L\mathbf{t}\otimes\mathbf{t} + \mathbf{t}\otimes L\mathbf{t}) + O(\epsilon^2)$$

and

$$\Pi_\epsilon(z) = \mathbf{t}_\epsilon\otimes\mathbf{t}_\epsilon = \frac{F\mathbf{t}\otimes F\mathbf{t}}{|F\mathbf{t}\otimes F\mathbf{t}|} =: \frac{a+\epsilon b + O(\epsilon^2)}{|a+\epsilon b+O(\epsilon^2)|}\,,$$

where $a := \mathbf{t}\otimes\mathbf{t}$ and $b := -(L\mathbf{t}\otimes\mathbf{t}+\mathbf{t}\otimes L\mathbf{t})$. By setting then $\Phi^\ell_j(\epsilon) := (\mathbf{t}_\epsilon\otimes\mathbf{t}_\epsilon)^\ell_j$, one gets $\Phi^\ell_j(\epsilon) = \Phi^\ell_j(0) + \Phi^{\ell\prime}_j(0)\,\epsilon + o(\epsilon)$, and, since $|a| = 1$, it follows that

$$\begin{aligned} -b^h_k\,a^h_k &= (L^h_\alpha\,\mathbf{t}^\alpha\mathbf{t}^k + \mathbf{t}^h L^k_\beta\,\mathbf{t}^\beta)\mathbf{t}^h\mathbf{t}^k \\ &= (\mathbf{t}^k)^2(L^h_\alpha\,\mathbf{t}^\alpha\mathbf{t}^h) + (\mathbf{t}^h)^2(\mathbf{t}^k L^k_\beta\,\mathbf{t}^\beta) \\ &= (L\mathbf{t})^h\,\mathbf{t}^h + \mathbf{t}^k(L\mathbf{t})^k \\ &= (L\mathbf{t}\cdot\mathbf{t}) + (\mathbf{t}\cdot L\mathbf{t}) = 2(L\mathbf{t}\cdot\mathbf{t}) \end{aligned}$$

whence

$$\Phi^{\ell\prime}_j(0) = b^\ell_j - a^\ell_j\,b^h_k\,a^h_k = 2(L\mathbf{t}\cdot\mathbf{t})a^\ell_j - b^\ell_j\,,$$

so that

$$\Pi_{(\epsilon)}(z) = \mathbf{t}_\epsilon(z)\otimes\mathbf{t}_\epsilon(z) = \nu\otimes\nu + \epsilon\widetilde{B}(x) + O(\epsilon^2)\,,$$

with $|O(\epsilon^2)| \le C\epsilon^2$ uniformly, C being a constant depending on the C^2 norm of λ. The desired assertion then follows, by repeating the argument of Proposition 1. ■

As a consequence, the map $\epsilon \mapsto \int_{\mathcal{G}_1(\mathcal{B})} |A_{(1)\epsilon}|^{p_1}\,dV_{(1)\epsilon}$ is differentiable at 0 and

$$\begin{aligned} &\frac{d}{d\epsilon}\left(\int_{\mathcal{G}_1(\mathcal{B})} |A_{(1)\epsilon}|^{p_1}\,dV_{(1)\epsilon}\right)_{|\epsilon=0} \\ &\quad= \int_{\mathfrak{b}_1\setminus\widetilde{\mathfrak{b}}_1} \theta_1(x)\,p_1|A_{(1)}|^{p_1-2}\,A_{(1)}{}^{\ell i}_j A'_{(1)}(\lambda)^{\ell i}_j\,d\mathcal{H}^1 \\ &\qquad - \int_{\mathfrak{b}_1\setminus\widetilde{\mathfrak{b}}_1} \theta_1(x)\,|A_{(1)}|^{p_1}\mathrm{div}_{,}^{\mathfrak{b}_1}\lambda\,d\mathcal{H}^1 \end{aligned} \tag{24}$$

where $A_{(1)} = A_{(1)}(x, P(x))$ and

$$A'_{(1)}(\lambda)^{\ell i}_j = \left(\widetilde{B}(x)\nabla_x\Pi_{(1)}{}^\ell_j(x) + \Pi_{(1)}(x)L^T\nabla_x\Pi_{(1)}{}^\ell_j(x) + \Pi_{(1)}(x)\nabla_x\widetilde{B}^\ell_j(x)\right)^i.$$

Since $\Pi_{(1)} = \mathbf{t}\otimes\mathbf{t}$, it follows that

$$\begin{aligned} A'_{(1)}(\lambda)^{\ell i}_j &= ((\widetilde{C}L)\,\nabla_x(\mathbf{t}^\ell\mathbf{t}^j) + (\mathbf{t}\otimes\mathbf{t})\,L^T\nabla_x(\mathbf{t}^\ell\mathbf{t}^j) \\ &\quad + (\mathbf{t}\otimes\mathbf{t})(\nabla_x\widetilde{C} : L)^\ell_j + (\mathbf{t}\otimes\mathbf{t})(\widetilde{C}\nabla_x L)^\ell_j)^i\,. \end{aligned}$$

Remark 3 The tensor $\widetilde{B}^\ell_j$ depends on the gradient of λ and on the unit tangent $\mathbf{t}$ to $\mathfrak{b}_1$. By assuming for a while that $\mathfrak{b}_2$ and $\mathfrak{b}_1$ are smooth, in case of normal variation to $\mathfrak{b}_2$ of constant amplitude $v\in\mathbb{R}$, $\lambda(x) = v\,\nu(x)$, where

$\nu(x)$ is a unit normal to $\mathfrak{b}_2$ (see Remark 2), $L(x) = \nabla\lambda(x) = v\nabla\nu(x)$ and hence (see Proposition 2)

$$\widetilde{B}_j^\ell = (v\,\Omega(n,\mathbf{t}))_j^\ell$$

where

$$\Omega(n,\mathbf{t}) := 2(\mathbf{t}\cdot D_{\mathbf{t}}\nu)\mathbf{t}\otimes\mathbf{t} - D_{\mathbf{t}}\nu\otimes\mathbf{t} - \mathbf{t}\otimes D_{\mathbf{t}}\nu\,.$$

Then, it follows that

$$A'_{(1)}(v\nu)_j^{\ell i} = v\,(\Omega(n,\mathbf{t})\,\nabla_x(\mathbf{t}^\ell\mathbf{t}^j) + (\mathbf{t}\otimes\mathbf{t})\,(\nabla\nu)^T\nabla_x(\mathbf{t}^\ell\mathbf{t}^j) + (\mathbf{t}\otimes\mathbf{t})(\nabla_x\Omega(n,\mathbf{t})_j^\ell)^i\,.$$

6.5 Variation of the term $\mathbf{M}(V_{(1)})$

Since the mass $\mathbf{M}(V_{(1)})$ is equal to $\int_{\mathfrak{b}_1}\theta_1\,d\mathcal{H}^1$, it follows that

$$\frac{d}{d\epsilon}\mathbf{M}(V_{(1)\epsilon})_{|\epsilon=0} = \frac{d}{d\epsilon}(\eta_{\epsilon\#}\left(\mathcal{H}^1\llcorner\theta_1\right)(\mathcal{B}))\,|_{\epsilon=0} = -\int_{\mathfrak{b}_1\setminus\widetilde{\mathfrak{b}}_1}\theta_1(x)\,\mathrm{div}^{\mathfrak{b}_1}\lambda\,d\mathcal{H}^1\,. \quad (25)$$

The variation of the term $\mathbf{M}(\partial V_{(1)})$ is equal to zero, $\pi_\#\left|\partial V_{(1)}\right|$ being a finite set.

6.6 Summary

The following theorem summarizes all variations evaluated previously (see (16), (22), (23), (24), (25)).

Theorem 13 *Let $(u,\{V_{(k)}\})\in\mathcal{A}_{q,p_k,K,\{\widetilde{V}_{(k)}\}}(\mathcal{B})$, $V_{(k)} = V_{\mathfrak{b}_k,\theta_k}$, $k=1,2$, be a local minimizer of 4. Assume that the bulk density energy satisfies* (15)*. At equilibrium, the couple $(u,\{V_{(2)},V_{(1)}\})$ satisfies the* balance equation

$$\begin{aligned}\int_{\mathcal{B}}\Big(e_{x^\alpha}(x,u,Du)\lambda^\alpha - \mathbb{P}_\alpha^\beta(x,u,Du)\,\lambda_{x^\beta}^\alpha\Big)\,dx& \\ +\alpha_2\int_{\mathfrak{b}_2\setminus\widetilde{\mathfrak{b}}_2}\theta_2(x)\,p_2|A_{(2)}|^{p_2-2}\,A_{(2)}{}_j^{\ell i}A'_{(2)}(\lambda)_j^{\ell i}\,d\mathcal{H}^2& \\ -\int_{\mathfrak{b}_2\setminus\widetilde{\mathfrak{b}}_2}\theta_2(x)\,(\alpha_2\,|A_{(2)}|^{p_2}+\beta_2)\,\mathrm{div},^{\mathfrak{b}_2}\lambda\,d\mathcal{H}^2& \\ +\alpha_1\int_{\mathfrak{b}_1\setminus\widetilde{\mathfrak{b}}_1}\theta_1(x)\,p_1|A_{(1)}|^{p_1-2}\,A_{(1)}{}_j^{\ell i}A'_{(1)}(\lambda)_j^{\ell i}\,d\mathcal{H}^1& \\ -\int_{\mathfrak{b}_1\setminus\widetilde{\mathfrak{b}}_1}\theta_1(x)\,(\alpha_1\,|A_{(1)}|^{p_1}+\beta_1)\,\mathrm{div},^{\mathfrak{b}_1}\lambda\,d\mathcal{H}^1 = 0&\end{aligned} \quad (26)$$

for any $\lambda\in C_c^\infty(\mathcal{B},\mathbb{R}^n)$ with $\lambda=0$ on the support of $\pi_\#\widetilde{V}_{(2)}$ and $\pi_\#\widetilde{V}_{(1)}$.

Equation (26) is the *weak balance of configurational actions* at the tip for a stationary crack. It has variational origin so that the dissipative force driving the crack tip along its evolution is absent. In contrast with standard instances, the energetics of the crack is treated in a new way. The dependence of the energy on the curvature of the crack margins and the curvature of the tip implies the occurrence of new configurational terms.

7 Pointwise balances in a special case

As anticipated in previous section, we discuss pointwise balance equations in a *special case*. $\mathcal{B}$ is three-dimensional; the body is endowed with a single crack represented by a smooth surface $\mathfrak{b}_2$; the tip is indicated by $\mathfrak{b}_1$. It is assumed that $V_{(2)} = V_{\mathfrak{b}_2,\theta_2}$ is the curvature varifold, with *constant* density θ_2, of a *smooth* 2D-manifold $\mathfrak{b}_2$ with boundary, parametrized by $\chi : B^+ \subset \mathbb{R}^2 \to \mathfrak{b}_2$, where

$$B^+ := \{(v^1, v^2) \in \mathbb{R}^2 \,|\, |(v^1, v^2)| < 1\,,\ v^2 > 0\}\,,$$

such that $\chi(\Gamma) = \partial\mathfrak{b}_2 \cap \mathcal{B}$, where

$$\Gamma := \{(v^1, v^2) \in B^+ \,|\, v^2 = 0\}.$$

The surface $\mathfrak{b}_2$ is naturally oriented by the normal vector field $\nu := \frac{\chi_{,1}\wedge\chi_{,2}}{|\chi_{,1}\wedge\chi_{,2}|}$, and the transplacement field u is presumed to be smooth in $\mathcal{B} \setminus \overline{\mathfrak{b}}_2$, with smooth traces u^+ and u^- on both the positive and negative sides of $\mathfrak{b}_2$. Moreover, $V_{(1)} = V_{\mathfrak{b}_1,\theta_1}$ is the curvature varifold associated with the tip $\mathfrak{b}_1$, with *constant* density θ_1. Initial cracks are absent, $\widetilde{\mathfrak{b}}_2 = \widetilde{\mathfrak{b}}_1 = \emptyset$. Also, $\mathfrak{b}_1$ is naturally oriented, and at $x \in \mathfrak{b}_1$ the vector $\mathbf{t} = \mathbf{t}(x)$ denotes the oriented unit tangent to $\mathfrak{b}_1$. Moreover, $m(x)$ is the inward unit normal to $\mathfrak{b}_1$ in $T_x\mathfrak{b}_2$, so that $\mathbf{t} \wedge m = \nu$.

Since $\mathfrak{b}_2$ is a smooth manifold, $|A_{(2)}|^2$ is twice the norm of the second fundamental form. This follows using the second equation in (9) in an orthonormal frame where $\nu = (0, 0, 1)$. Thus, by denoting by k_1, k_2 the principal curvatures, and by H and K the *scalar mean curvature* and the *Gauss curvature* of $\mathfrak{b}_2$, respectively, one gets

$$|A_{(2)}|^2 = 2(k_1^2 + k_2^2) = 4(2H^2 - K)$$

hence the curvature term of the energy related to the 2-dimensional crack can be written as

$$\int |A_{(2)}|^{p_2}\, d\,V_{(2)} = 2^{p_2}\theta_2 \int_{\mathfrak{b}_2} f(H, K)\, d\mathcal{H}^2 \tag{27}$$

where $f(H, K) := (2H^2 - K)^{p_2/2}$.

For analogous reasons, it is possible to write

$$\int |A_{(1)}|^{p_1}\, d\,V_{(1)} = 2^{p_1}\theta_1 \int_{\mathfrak{b}_1} |k|^{p_1}\, d\mathcal{H}^1\,, \tag{28}$$

where k is the *curvature* of the curve $\mathfrak{b}_1$.

In order to integrate by parts the first variation of the bulk energy (16), it is also useful to introduce for every small ϵ a tubular neighborhood C_ϵ of radius ϵ of $\mathfrak{b}_1$ and the inward unit normal vector field $\nu_{\partial C_\epsilon}$ at boundary points in ∂C_ϵ. For $y \in \mathfrak{b}_1$, set

$$D_\epsilon(y) := C_\epsilon \cap N_y\mathfrak{b}_1\,, \qquad \partial D_\epsilon(y) = \partial C_\epsilon \cap N_y\mathfrak{b}_1\,,$$

where $N_y\mathfrak{b}_1$ is the *normal space* to $\mathfrak{b}_1$ at y. Assume also that, for $\mathcal{H}^1$-a.e. $y \in \mathfrak{b}_1$,

$$\mathbf{J}^\alpha(y) := \lim_{\epsilon\to 0^+} \int_{\partial D_\epsilon(y)} \mathbb{P}^\beta_\alpha(x,u,Du)\,\nu^\beta_{\partial C_\epsilon}\, d\mathcal{H}^1 \tag{29}$$

exists for $\alpha = 1,2,3$, with $\mathbf{J} := (\mathbf{J}^1, \mathbf{J}^2, \mathbf{J}^3) \in L^1(\mathfrak{b}_1, \mathbb{R}^3)$, and

$$\lim_{\epsilon\to 0} \int_{\partial C_\epsilon} \mathbb{P}^\beta_\alpha(x,u,Du)\,\nu^\beta_{\partial C_\epsilon}\,\lambda^\alpha\, d\mathcal{H}^2 = \int_{\mathfrak{b}_1} \mathbf{J}^\alpha\lambda^\alpha\, d\mathcal{H}^1 \tag{30}$$

for every $\lambda \in C^\infty_c(\mathcal{B}, \mathbb{R}^3)$. The component of the vector $\mathbf{J}$ along the normal m, namely $\mathbf{J}\cdot m$, is the well-known *J-integral.*

7.1 Variation of the bulk energy

From (16), with $n = 3$, it follows that

$$\begin{aligned}
&\frac{d}{d\epsilon}\left(\int_{\mathcal{B}} e(x,u_\epsilon,Du_\epsilon)\,dx\right)_{|\epsilon=0} \\
&\quad = \int_{\mathcal{B}} \Big(D_\beta\mathbb{P}^\beta_\alpha(x,u,Du) + e_{x^\alpha}(x,u,Du)\Big)\lambda^\alpha\,dx - \int_{\mathfrak{b}_2} [\![\mathbb{P}^\beta_\alpha]\!]\,\nu^\beta\,\lambda^\alpha\,d\mathcal{H}^2 \\
&\qquad - \lim_{\epsilon\to 0}\int_{\partial C_\epsilon} \mathbb{P}^\beta_\alpha(x,u,Du)\,\nu^\beta_{\partial C_\epsilon}\,\lambda^\alpha\,d\mathcal{H}^2
\end{aligned} \tag{31}$$

where $[[\mathbb{P}^\beta_\alpha]] := \mathbb{P}^\beta_\alpha(x,u^+,Du^+) - \mathbb{P}^\beta_\alpha(x,u^-,Du^-)$. Of course, the last term in (31) (see (30)), disappears if $u \in W^{2,2}(\mathcal{B})$.

7.2 Variation of the area term $\mathbf{M}\left(V_{(2)}\right)$

By using the integration by parts formula (8), with $\mathfrak{b} = \mathfrak{b}_2$ and $k = 2$, for every $\lambda \in C^\infty_c(\mathcal{B}, \mathbb{R}^3)$, it is possible to compute

$$\frac{d}{d\epsilon}\mathbf{M}\left(V_{(2)}\right) = \frac{d}{d\epsilon}\left(\eta_{\epsilon\#}\pi_{\#}V_{(2)}\left(\mathcal{B}\right)\right)|_{\epsilon=0} = -\theta_2\int_{\mathfrak{b}_2}\operatorname{div}^{\mathfrak{b}_2}\lambda d\mathcal{H}^2$$
$$= \theta_2\int_{\mathfrak{b}_2}\mathbf{H}^\alpha\lambda^\alpha d\mathcal{H}^2 + \theta_2\int_{\mathfrak{b}_1} m^\alpha\lambda^\alpha d\mathcal{H}^1 \tag{32}$$

where $\mathbf{H} = (\mathbf{H}^1, \mathbf{H}^2, \mathbf{H}^3)$ is the mean curvature vector of $\mathfrak{b}_2$ and $m = (m^1, m^2, m^3)$ is the inward unit normal to $\mathfrak{b}_1 = \partial\mathfrak{b}_2 \cap \mathcal{B}$.

7.3 Normal variations of the surface curvature term

Consider a parametric functional $\mathcal{F}(\mathfrak{b})$ with smooth integrand f, defined by an integral over a smooth surface $\mathfrak{b}$ with boundary parametrized by $\chi : B^+ \to \mathfrak{b}$, namely

$$\mathcal{F}(\mathfrak{b}) := \int_{\mathfrak{b}} f(H,K)\, d\mathcal{H}^2 = \int_{B^+} f(H,K)\sqrt{g}\, dv^1 dv^2. \tag{33}$$

By taking into account (27), the first variation of $\mathcal{F}(\mathfrak{b}_2)$ can be evaluated. With $\lambda : \mathcal{B} \to \mathbb{R}^3$, $\eta_\epsilon(x) = x + \epsilon\lambda(x)$, $\mathfrak{b}_\epsilon = \eta_\epsilon(\mathfrak{b})$, we write for the sake of convenience

$$\mathcal{F}(\mathfrak{b}_\epsilon, \lambda) \quad \text{instead of} \quad \mathcal{F}(\mathfrak{b}_\epsilon)\,.$$

Standard notations are listed below:

$$g_{\alpha\beta} := \chi_{,\alpha}\cdot\chi_{,\beta}\,, \qquad g := \det(g_{\alpha\beta})\,,\ \ (g^{\alpha\beta}) := (g_{\alpha\beta})^{-1}\,,$$
$$\nu := \frac{1}{\sqrt{g}}\left(\chi_{,1}\wedge\chi_{,2}\right),\ b_{\alpha\beta} := \nu\cdot\chi_{,\alpha\beta}\,,\ b := \det(b_{\alpha\beta})\,,$$
$$H = \frac{1}{2}g^{\alpha\beta}b_{\alpha\beta}\,, \qquad K = \frac{b}{g}\,, \quad K\,b^{\alpha\beta} := \frac{\operatorname{cof},(b_{\alpha\beta})}{g}\,,$$

where $\chi_{,\alpha} := \chi_{v^\alpha} = \dfrac{\partial\chi}{\partial v^\alpha}$ and $\chi_{,\alpha\beta} := \dfrac{\partial^2\chi}{\partial v^\alpha\partial v^\beta}$, and, as usual, the symbol f is always used for a map f defined on $\mathfrak{b}$ and the corresponding local representation $f\circ\chi$ defined on B^+ (see also the beginning of Section 7).

Consider first a family of *normal* variations of $\mathfrak{b}$, $\eta_\epsilon(x) := x + \epsilon\lambda(x)$, $\lambda(x) = \varphi(x)\nu(x)$ that in local coordinates reads

$$\chi_\epsilon(v) := \chi(v) + \epsilon\,\varphi(v)\,\nu(v)\,, \qquad v = (v^1, v^2)\,,$$

where $\varphi \in C^\infty(\overline{B}^+)$, $\varphi = 0$ near $\partial B \cap B^+$. By denoting by H_ϵ and K_ϵ the corresponding curvatures of $\mathfrak{b}_\epsilon := \chi_\epsilon(B^+)$, one computes (compare e.g. [31, p. 84])

$$\delta H := \frac{d}{d\epsilon}H_{\epsilon|\epsilon=0} = (2H^2 - K)\varphi + \frac{1}{2}\Delta\varphi\,,$$
$$\delta K := \frac{d}{d\epsilon}K_{\epsilon|\epsilon=0} = 2HK\varphi + \Box\varphi\,, \qquad \frac{\delta\sqrt{g}}{\sqrt{g}} = -2\,H\,\varphi\,,$$

where

$$\Delta\varphi := \frac{1}{\sqrt{g}} D_\alpha(\sqrt{g} g^{\alpha\beta} D_\beta \varphi)\,, \qquad \Box\varphi := \frac{1}{\sqrt{g}} D_\alpha(\sqrt{g} K b^{\alpha\beta} D_\beta \varphi)\,.$$

It follows that

$$\begin{aligned}
\frac{d}{d\epsilon}\mathcal{F}(\mathfrak{b}_\epsilon, \varphi)_{|\epsilon=0} &= \int_{B^+} \left\{ f_{,K}\delta K + f_{,H}\delta H + f\frac{\delta\sqrt{g}}{\sqrt{g}} \right\} \sqrt{g}\, dv^1 dv^2 \\
&= \int_{B^+} \left\{ 2f_{,K} HK f_{,H} + (2H^2 - K) - 2fH \right\} \varphi\sqrt{g}\, dv^1 dv^2 \\
&\quad + \int_{B^+} f_{,K}\, \Box\varphi\, \sqrt{g}\, dv^1 dv^2 + \frac{1}{2}\int_{B^+} f_{,H}\, \Delta\varphi\, \sqrt{g}\, dv^1 dv^2\,.
\end{aligned}$$

By integrating by parts the last two integrals one gets

$$\begin{aligned}
\int_{B^+} f_{,K}\, \Box\varphi\, \sqrt{g}\, dv^1 dv^2 &= \int_{B^+} (\Box f_{,K})\, \varphi\, \sqrt{g}\, dv^1 dv^2 \\
&\quad - \int_\Gamma f_{,K}\, K b^{\alpha\beta}\, n^\alpha\, D_\beta\varphi\, \sqrt{g}\, dv^1 + \int_\Gamma D_\alpha f_{,K}\, K b^{\alpha\beta}\, n^\beta\, \varphi\, \sqrt{g}\, dv^1
\end{aligned}$$

and

$$\begin{aligned}
\int_{B^+} f_{,H}\, \Delta\varphi\, \sqrt{g}\, dv^1 dv^2 &= \int_{B^+} (\Delta f_{,H})\, \varphi\, \sqrt{g}\, dv^1 dv^2 \\
&\quad - \int_\Gamma f_{,H}\, g^{\alpha\beta}\, n^\alpha\, D_\beta\varphi\, \sqrt{g}\, dv^1 + \int_\Gamma D_\alpha f_{,H}\, g^{\alpha\beta}\, n^\beta\, \varphi\, \sqrt{g}\, dv^1,
\end{aligned}$$

then, finally,

$$\begin{aligned}
&\frac{d}{d\epsilon}\mathcal{F}(\mathfrak{b}_\epsilon, \varphi\nu)_{|\epsilon=0} \\
&\quad = \int_{\mathfrak{b}} \left\{ \Box f_{,K} + \frac{1}{2}\Delta f_{,H} + 2HK f_{,K} + (2H^2 - K) f_{,H} - 2fH \right\} \varphi\, d\mathcal{H}^2 \\
&\quad - \int_\Gamma \sqrt{g} \left\{ f_{,K}\, K\, b^{\alpha\beta} + \frac{1}{2}\, f_{,H}\, g^{\alpha\beta} \right\} n^\alpha\, D_\beta\varphi\, dv^1 \\
&\quad + \int_\Gamma \sqrt{g} \left\{ K\, b^{\alpha\beta}\, D_\beta(f_{,K}) + \frac{1}{2}\, g^{\alpha\beta}\, D_\beta(f_{,H}) \right\} n^\alpha\, \varphi\, dv^1\,,
\end{aligned}$$

where $n \equiv (0,1)$ is the inward unit normal to Γ in $\mathbb{R}^2$.

Finally, in the special case $f(H,K) = (2H^2 - K)$, compare (27) with $p_2 = 2$, by putting $\mathfrak{b} = \mathfrak{b}_2$ and $\mathfrak{b}_1 = \partial\mathfrak{b}_2 \cap \mathcal{B}$, and using the identity

$$\int_\Gamma \sqrt{g}\, g^{\alpha\beta}\, D_\beta\, H\, n^\alpha\, \varphi\, dv^1 = \int_{\mathfrak{b}_2} \nabla^{\mathfrak{b}_2} H \cdot m\, \varphi\, d\mathcal{H}^1\,,$$

the variational formula

$$\frac{d}{d\epsilon}\mathcal{F}(\mathfrak{b}_\epsilon,\varphi\nu)_{|\epsilon=0} = \int\limits_{\mathfrak{b}_2}\Big\{2\Delta H + 4H^3 - 4HK\Big\}\varphi\, d\mathcal{H}^2 + 2\int\limits_{\mathfrak{b}_2}\nabla^{\mathfrak{b}_2} H\cdot m\,\varphi\, d\mathcal{H}^1$$
$$-\int\limits_{\Gamma}\sqrt{g}\,(-K\,b^{\alpha\beta} + 2\,H\,g^{\alpha\beta})\,n^\alpha\, D_\beta\varphi\, dv^1 \tag{34}$$

follows.

7.4 Tangential variations of the surface curvature term

Consider tangential variations $x = x+\epsilon\lambda\,(x)$ to $\mathfrak{b} = \mathfrak{b}_2$, i.e., $\lambda\,(x) = \varphi\,(x)\,\tau\,(x)$, where $\tau : \mathfrak{b} \to \mathbb{R}^3$ is a tangential vector field to $\mathfrak{b}$, $\varphi\,(x)$ a scalar. In local coordinates, such variations read

$$\chi_\epsilon(v) := \chi(v) + \epsilon\,\xi^\alpha(v)\,\chi_{,\alpha}(v)\,,$$

where the vector field $\xi := (\xi^1,\xi^2)\in C^\infty(\overline{B}^+,\mathbb{R}^2)$, are now discussed for $\mathcal{F}(\mathfrak{b})$.

The first order term of compactly supported tangential variations is equivalent to a compactly supported internal variation. Roughly speaking, this is due to the fact that an infinitesimal tangential variation of $\mathfrak{b}$ can be decomposed into the sum of a re-parameterization plus a higher order normal variation, whereas $\mathcal{F}(\mathfrak{b})$ is a parametric functional. As a consequence, if φ is zero near $\partial\mathfrak{b}$, then

$$\frac{d}{d\epsilon}\mathcal{F}(\mathfrak{b}_\epsilon,\varphi\tau)\,|_{\epsilon=0} = 0\,. \tag{35}$$

Therefore, if φ is zero near $\partial\mathcal{B}$, *the sole contribution given by tangential variations has to be a boundary term.*

Proposition 3 *Let $\tau : \mathfrak{b} \to \mathbb{R}^3$ be a smooth function such that $\tau\,(x) \in T_x\mathfrak{b}$ for every $x \in \mathfrak{b}$, and $\varphi \in C_c^\infty\,(\mathcal{B},\mathbb{R})$. Then*

$$\frac{d}{d\epsilon}\mathcal{F}(\mathfrak{b}_\epsilon,\varphi\tau)\,|_{\epsilon=0} = -\int_{\mathfrak{b}} f\,(H,K)\,(\tau\cdot m)\,\varphi d\mathcal{H}^1, \tag{36}$$

where $m\,(x)$ is the inward unit normal to $\partial\mathfrak{b}$ in $T_x\mathfrak{b}$ at x.

Proof. Due to the invariance property, the treatment can be restricted to tangential variations of the type

$$\underline{\chi} := \chi + \epsilon\,\varphi\,\chi_{,1}\,, \qquad \varphi\in C^\infty(\overline{B}^+)$$

where $\epsilon > 0$ is small. Moreover, from the above mentioned properties, *the terms containing the test function φ can be neglected*, so they do not produce boundary terms. Only the terms involving the derivatives of φ have to be computed. To this purpose, denote by Ψ a generic rational function of the components of χ and of their derivatives.

Explicit computations lead to

$$\underline{\chi}_{,1} = \chi_{,1} + \epsilon\,(\varphi_{,1}\,\chi_{,1} + \varphi\,\chi_{,11})\,, \qquad \underline{\chi}_{,2} = \chi_{,2} + \epsilon\,(\varphi_{,2}\,\chi_{,1} + \varphi\,\chi_{,12})\,,$$

so that, by setting $\underline{g}_{\alpha\beta} := \underline{\chi}_{,\alpha} \cdot \underline{\chi}_{,\beta}$, one gets

$$\begin{aligned}
\underline{g}_{11} &= g_{11}\,(1 + 2\epsilon\,\varphi_{,1}) + \epsilon\,\varphi\,\Psi + \cdots \\
\underline{g}_{22} &= g_{22} + 2\epsilon\,\varphi_{,2}\,g_{12} + \epsilon\,\varphi\,\Psi + \cdots \\
\underline{g}_{12} &= g_{12} + \epsilon\,\Big(\varphi_{,1}\,g_{12} + \varphi_{,2}\,g_{11}\Big) + \epsilon\,\varphi\,\Psi + \cdots
\end{aligned}$$

where $\cdots$ denotes higher order terms. Further computations lead to

$$\underline{g} := \det(\underline{g}_{\alpha\beta}) = g\,(1 + 2\,\epsilon\,\Phi) + \cdots\,, \qquad \text{where} \quad \Phi := \varphi_{,1} + \varphi\,g^{-1}\,\Psi\,,$$

and

$$\underline{\chi}_{,1} \wedge \underline{\chi}_{,2} = \chi_{,1} \wedge \chi_{,2}\,(1 + \epsilon\,\varphi_{,1}) + \epsilon\,\varphi\,\Psi + \cdots\,.$$

Since $\underline{g}^{-1/2} = g^{-1/2}\,(1 - \epsilon\,\Phi) + \cdots$, it follows that

$$\begin{aligned}
\underline{\nu} &:= \underline{g}^{-1/2}(\underline{\chi}_{,1} \wedge \underline{\chi}_{,2}) \\
&= g^{-1/2}\,(1 - \epsilon\,\Phi)\,(1 + \epsilon\,\varphi_{,1})\,\chi_{,1} \wedge \chi_{,2} + \epsilon\,g^{-3/2}\,\varphi\,\Psi + \cdots \\
&= \nu + \epsilon\,g^{-3/2}\,\varphi\,\Psi + \cdots\,.
\end{aligned}$$

Moreover, direct exploitation of the derivatives of χ implies

$$\begin{aligned}
\underline{\chi}_{,11} &= \chi_{,11} + \epsilon\,(\varphi_{,11}\,\chi_{,1} + 2\,\varphi_{,1}\,\chi_{,11}) + \epsilon\,\varphi\,\Psi + \cdots \\
\underline{\chi}_{,22} &= \chi_{,22} + \epsilon\,(\varphi_{,22}\,\chi_{,1} + 2\,\varphi_{,2}\,\chi_{,12}) + \epsilon\,\varphi\,\Psi + \cdots \\
\underline{\chi}_{,12} &= \chi_{,12} + \epsilon\,\Big(\varphi_{,12}\,\chi_{,1} + \varphi_{,1}\,\chi_{,12} + \varphi_{,2}\,\chi_{,11}\Big) + \epsilon\,\varphi\,\Psi + \cdots
\end{aligned}$$

whence, setting $\underline{b}_{\alpha\beta} := \underline{\nu} \cdot \chi_{,\alpha\beta}$, recalling that $\chi_{,1} \wedge \chi_{,2} \cdot \chi_{,\alpha} \equiv 0$ and $b_{\alpha\beta} = g^{-1/2}\,(\chi_{,1} \wedge \chi_{,2} \cdot \chi_{,\alpha\beta})$, one obtains

$$\begin{aligned}
\underline{b}_{11} &= b_{11}\,(1 + 2\,\epsilon\,\varphi_{,1}) + \epsilon\,\varphi\,g^{-1/2}\,\Psi + \cdots \\
\underline{b}_{22} &= b_{22} + 2\,\epsilon\,\varphi_{,2}\,b_{12} + \epsilon\,\varphi\,g^{-1/2}\,\Psi + \cdots \\
\underline{b}_{12} &= b_{12} + \epsilon\,(\varphi_{,1}\,b_{12} + \varphi_{,2}\,b_{11}) + \epsilon\,\varphi\,g^{-1/2}\,\Psi + \cdots
\end{aligned}$$

and then

$$\underline{b} := \det(\underline{b}_{\alpha\beta}) = b\,(1 + 2\,\epsilon\,\varphi_{,1}) + \epsilon\,\varphi\,g^{-1}\,\Psi + \cdots .$$

Therefore, since $\underline{g}^{-1} = g^{-1}\,(1 - 2\,\epsilon\,\Phi) + \cdots$, the Gauss curvature $\underline{K}$ of $\underline{\chi}$ is given by

$$\begin{aligned} \underline{K} &= \underline{b}\,\underline{g}^{-1} = g^{-1}(1 - 2\,\epsilon\,\Phi)\Big(b\,(1 + 2\,\epsilon\,\varphi_{,1}) + \epsilon\,\varphi\,g^{-1}\,\Psi\Big) + \cdots \\ &= K + \epsilon\,\varphi\,g^{-2}\,\Psi + \cdots . \end{aligned}$$

The ensuing relations hold obviously:

$$\underline{g}^{11} = \frac{1}{\underline{g}}\,\underline{g}_{22}\,, \qquad \underline{g}^{22} = \frac{1}{\underline{g}}\,\underline{g}_{11}\,, \qquad \underline{g}^{12} = -\frac{1}{\underline{g}}\,\underline{g}_{12}\,.$$

Consequently, by recalling that $\Phi = \varphi_{,1} + \varphi\,g^{-1}\,\Psi$, one gets

$$\begin{aligned} \underline{g}^{11} &= g^{-1}\,(1 - 2\,\epsilon\,\Phi)\,\Big[g_{22} + 2\,\epsilon\,\varphi_{,2}\,g_{12} + \epsilon\,\varphi\,\Psi\Big] + \cdots \\ &= g^{-1}\,\Big(g_{22}\,(1 - 2\,\epsilon\,\varphi_{,1}) + 2\,\epsilon\,\varphi_{,2}\,g_{12}\Big) + \epsilon\,g^{-1}\,\varphi\,\Psi + \cdots \\ \underline{g}^{22} &= g^{-1}\,(1 - 2\epsilon\,\Phi)\,\Big[g_{11}\,(1 + 2\epsilon\,\varphi_{,1}) + \epsilon\,\varphi\,\Psi\Big] + \cdots \\ &= g^{-1}\,g_{11} + \epsilon\,g^{-1}\,\varphi\,\Psi + \cdots \\ \underline{g}^{12} &= -g^{-1}\,(1 - 2\epsilon\,\Phi)\,\Big[g_{12} + \epsilon\,\Big(\varphi_{,1}\,g_{12} + \varphi_{,2}\,g_{11}\Big) + \epsilon\,\varphi\,\Psi\Big] + \cdots \\ &= -g^{-1}\,\Big(g_{12}\,(1 - \epsilon\,\varphi_{,1}) + \epsilon\,\varphi_{,2}\,g_{11}\Big) + \epsilon\,g^{-1}\,\varphi\,\Psi + \cdots . \end{aligned}$$

Such a result implies

$$\begin{aligned} \underline{b}_{11}\,\underline{g}^{11} &= \Big[b_{11}\,(1 + 2\,\epsilon\,\varphi_{,1}) + \epsilon\,\varphi\,g^{-1/2}\,\Psi\Big] \\ &\qquad \times \Big[g^{-1}\,\Big(g_{22}\,(1 - 2\,\epsilon\,\varphi_{,1}) + 2\,\epsilon\,\varphi_{,2}\,g_{12}\Big) + \epsilon\,g^{-1}\,\varphi\,\Psi\Big] + \cdots \\ &= g^{-1}\,\Big[b_{11}\,g_{22} + 2\,\epsilon\,\varphi_{,2}\,b_{11}\,g_{12}\Big] + \epsilon\,g^{-3/2}\,\varphi\,\Psi + \cdots \end{aligned}$$

$$\begin{aligned} \underline{b}_{22}\,\underline{g}^{22} &= \Big[b_{22} + 2\,\epsilon\,\varphi_{,2}\,b_{12} + \epsilon\,\varphi\,g^{-1/2}\,\Psi\Big]\,\Big[g^{-1}\,g_{11} + \epsilon\,g^{-1}\,\varphi\,\Psi\Big] + \cdots \\ &= g^{-1}\,\Big[b_{22}\,g_{11} + 2\,\epsilon\,\varphi_{,2}\,b_{12}\,g_{11}\Big] + \epsilon\,g^{-3/2}\,\varphi\,\Psi + \cdots \end{aligned}$$

$$\begin{aligned} 2\,\underline{b}_{12}\,\underline{g}^{12} &= 2\,\Big[b_{12}\,(1 + \epsilon\,\varphi_{,1}) + \epsilon\,\varphi_{,2}\,b_{11} + \epsilon\,\varphi\,g^{-1/2}\,\Psi\Big] \\ &\qquad \Big[-g^{-1}\,\Big(g_{12}\,(1 - \epsilon\,\varphi_{,1}) + \epsilon\,\varphi_{,2}\,g_{11}\Big) + \epsilon\,g^{-1}\,\varphi\,\Psi\Big] + \cdots \\ &= -2\,g^{-1}\,\Big[b_{12}\,g_{12} + \epsilon\,\varphi_{,2}\,(b_{11}\,g_{12} + b_{12}\,g_{11})\Big] + \epsilon\,g^{-3/2}\,\varphi\,\Psi + \cdots . \end{aligned}$$

Since $2H = g^{-1}\,(b_{11}\,g_{22} + b_{22}\,g_{11} - 2\,b_{12}\,g_{22})$, we get

$$\underline{H} = \frac{1}{2}\underline{b}_{\alpha\beta}\,\underline{g}^{\alpha\beta} = H + \epsilon\,g^{-3/2}\,\varphi\,\Psi + \cdots .$$

In conclusion, from the above computations we infer that

$$\delta\sqrt{g} = \sqrt{g}\,\Phi = \sqrt{g}\,(\varphi_{,1} + \varphi\, g^{-1}\,\Psi_1)\,, \quad \delta K = \varphi\, g^{-2}\,\Psi_2\,, \quad \delta H = g^{-3/2}\,\varphi\,\Psi_3$$

where the Ψ_i's are rational functions of the components of χ and of their derivatives. Therefore,

$$\frac{d}{d\epsilon}\mathcal{F}(\mathfrak{b}_\epsilon, \varphi\chi_{,1})\,|_{\epsilon=0} = \int_{B^+} f\,(H,K)\,D_1\varphi\sqrt{g}dx + \int_{B^+} L_1\,(\varphi)\,dv^1dv^2,$$

where $\varphi \longmapsto L_1\,(\varphi)$ is linear. As a consequence, due to the invariance property, we infer that for every $\xi \in C^\infty(\overline{B^+}, \mathbb{R}^2)$

$$\frac{d}{d\epsilon}\mathcal{F}(\mathfrak{b}_\epsilon, V_\xi)_{|\epsilon=0} = \int_{\partial B^+} f\,\mathrm{div}\xi\,\sqrt{g} + \int_{B^+} L(\xi)\,dv^1dv^2\,,$$

where $L(\xi)$ is some linear operator of ξ and $V_\xi := \sum_\alpha \xi^\alpha \chi_{,\alpha}$. Then, necessarily,

$$\frac{d}{d\epsilon}\mathcal{F}(\mathfrak{b}_\epsilon, V_\xi)_{|\epsilon=0} = \int_{\partial\mathfrak{b}} \mathrm{div}^{\mathfrak{b}}(fV_\xi)\,d\mathcal{H}^2 + \int_{\mathfrak{b}} \widetilde{L}(\xi)\,d\mathcal{H}^2.$$

By choosing $V_\xi = \varphi\tau$ and integrating by parts on the first term, the claim is proved on account of (35), and the remark thereafter. ■

7.5 Variation of the term associated with the term of the tip

By using the integration by parts formula (8), with $\mathfrak{b} = \mathfrak{b}_1$ and $k = 1$, and using that $\mathfrak{b}_1 = \partial\mathfrak{b}_2 \cap \mathcal{B}$, one obtains for every $\lambda \in C_c^\infty(\mathcal{B}, \mathbb{R}^3)$

$$\frac{d}{d\epsilon}\mathbf{M}(V_{(1)\epsilon})_{|\epsilon=0} = -\theta_1 \int_{\mathfrak{b}_1} \mathrm{div}^{\mathfrak{b}_1}\lambda\,d\mathcal{H}^1 = \theta_1 \int_{\mathfrak{b}_1} \mathbf{k}^\alpha\lambda^\alpha d\mathcal{H}^1, \tag{37}$$

where $\mathbf{k} = (\mathbf{k}^1, \mathbf{k}^2, \mathbf{k}^3)$ is the *curvature vector* of $\mathfrak{b}_1$.

7.6 Variations of the tip curvature term

We now compute the first variation of a curvature integral of a space curve $\mathcal{C}$. If $\mathcal{C}$ is a smooth curve immersed in $\mathbb{R}^3$, described by a parametric representation $c(t)$, $t \in [a,b]$, $\dot{c}(t) \neq 0$, the curvature $k(t)$ and the torsion $\tau(t)$ with the representation $c(t)$ are given by

$$k_c := \frac{|\dot{c} \wedge \ddot{c}|}{|\dot{c}|^3}\,, \qquad {}_c := \frac{[c, \ddot{c}, \dddot{c}]}{|c \wedge \ddot{c}|^2}\,,$$

where $[\alpha_1, \alpha_2, \alpha_3] := (\alpha_1 \wedge \alpha_2) \cdot \alpha_3$ is the spatial product of vectors $\alpha_i \in \mathbb{R}^3$. On account of (28), we restrict to consider curvature integrals of the type

$$\mathcal{F}(\mathcal{C}) := \int_{\mathcal{C}} f(k)\, ds = \int_a^b f(k_c)\, |\dot{c}|\, dt\,,$$

where $ds = |\dot{c}(t)|\, dt$. Since we have assumed $\mathfrak{b}_1 = \partial \mathfrak{b}_2 \cap \mathcal{B}$, it suffices to consider the variational formula

$$\delta \mathcal{F}(\mathcal{C}, \phi) := \frac{d}{d\epsilon} \mathcal{F}(\mathcal{C}_\epsilon)_{|\epsilon=0} = 0$$

for an extremal $\mathcal{C}$, where the curve $\mathcal{C}_\epsilon$ is represented by

$$c_\epsilon(t) := c(t) + \epsilon\, \phi(t), \qquad t \in [a, b]$$

and $\phi : [a, b] \to \mathbb{R}^2$ is a compactly supported smooth vector field.

Consider $\zeta \in C_c^\infty((a, b))$, and the *moving frame* $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ along $\mathcal{C}$ satisfying the Frenet's formulas

$$\dot{\mathbf{t}} = k\, \mathbf{n}\,, \qquad \dot{\mathbf{n}} = -k\, \mathbf{t} + \tau\, \mathbf{b}\,, \qquad \dot{\mathbf{b}} = -\tau\, \mathbf{n}\,.$$

Tangential variations $\phi = \zeta\, \mathbf{t}$ do not furnish equations. By following e.g. [31, p. 76], normal variations $\phi = \zeta\, \mathbf{n}$ lead to

$$\frac{d}{d\epsilon} \mathcal{F}(\mathcal{C}_\epsilon, \varphi \mathbf{n})_{|\epsilon=0} = \int_{\mathcal{C}} \Big\{ f'''(k)\, \dot{k}^2 + f''(k)\, \ddot{k} + f'(k)\, (k^2 - \tau^2) - k\, f(k) \Big\} \varphi\, d\mathcal{H}^1,$$

where $\dot{k}$ denotes the derivative of k with respect to the *arc length parameter*.

Finally, binormal variations $\phi = \zeta\, \mathbf{b}$ lead to

$$\frac{d}{d\epsilon} \mathcal{F}(\mathcal{C}_\epsilon, \varphi \mathbf{b})_{|\epsilon=0} = \int_{\mathcal{C}} \Big\{ 2 f''(k)\, \dot{k}\, \tau + f'(k)\, \dot{\tau} \Big\} \varphi\, d\mathcal{H}^1.$$

In particular, if $\mathcal{C}$ is an extremal of $f(k) = k^2$ (compare (28) with $p_1 = 2$), the formulas

$$\frac{d}{d\epsilon} \mathcal{F}(\mathcal{C}_\epsilon, \varphi \mathbf{n})_{|\epsilon=0} = \int_{\mathcal{C}} q_1\, \varphi\, d\mathcal{H}^1\,, \qquad \frac{d}{d\epsilon} \mathcal{F}(\mathcal{C}_\epsilon, \varphi \mathbf{b})_{|\epsilon=0} = \int_{\mathcal{C}} q_2\, \varphi\, d\mathcal{H}^1\,,$$

hold, where

$$q_1 := (2\ddot{k} + k^3 - 2\, k\, \tau^2) \qquad q_2 := (4\, \dot{k}\, \tau + 2\, k\, \dot{\tau})\,. \tag{38}$$

Moreover, at any point $x \in \mathfrak{b}_1$ the unit vectors $\mathbf{n}$ and $\mathbf{b}$ are orthogonal and belong to the 2-space generated by ν and m. In terms of $\mathbf{n}$ and $\mathbf{b}$, the vectors ν and m can be expressed by

$$\nu = r_1 \mathbf{n} + r_2 \mathbf{b}, \qquad m = r_3 \mathbf{n} + r_4 \mathbf{b}$$

where, of course, $r_1 = \nu \cdot \mathbf{n}$, etc. In conclusion, one finds

$$
\begin{aligned}
\frac{d}{d\epsilon}\mathcal{F}(\mathcal{C}_\epsilon, \varphi\nu)_{|\epsilon=0} &= \int_{\mathcal{C}} (r_1 q_1 + r_2 q_2)\, \varphi \, d\mathcal{H}^1 \,, \\
\frac{d}{d\epsilon}\mathcal{F}(\mathcal{C}_\epsilon, \varphi m)_{|\epsilon=0} &= \int_{\mathcal{C}} (r_3 q_1 + r_4 q_2)\, \varphi \, d\mathcal{H}^1 \,.
\end{aligned}
\tag{39}
$$

7.7 Pointwise balances

For the sake of simplicity, assume that both the exponents p_1 and p_2 in (4) are equal to 2. Thus, the energy functional (4) becomes

$$
\begin{aligned}
\int_{\mathcal{B}} e(x,u,Du)\, dx + \tilde{\alpha}_2\, \theta_2 \int_{\mathfrak{b}_2} (2H^2 - K)\, d\mathcal{H}^2 + \beta_2\, \theta_2\, \mathcal{H}^2(\mathfrak{b}_2) \\
+ \tilde{\alpha}_1\, \theta_1 \int_{\mathfrak{b}_1} k^2\, d\mathcal{H}^1 + \beta_1\, \theta_1\, \mathcal{H}^1(\mathfrak{b}_1)\,,
\end{aligned}
$$

where

$$
\tilde{\alpha}_2 := 4\alpha_2\,, \qquad \tilde{\alpha}_2 := 4\alpha_2.
$$

On the basis of (31), balance equations in the bulk are given by

$$
D_\beta \mathbb{P}^\beta_\alpha(x,u,Du) + e_{x^\alpha}(x,u,Du) = 0 \tag{40}
$$

for every $\alpha = 1,2,3$.

Balance equations on the 2D-crack $\mathfrak{b}_2$, corresponding to normal and tangential variations, respectively, are given by

$$
\begin{cases}
[[\mathbb{P}^\beta_\alpha]]\, \nu^\beta\, \nu^\alpha + \tilde{\alpha}_2\, \theta_2 \{2\Delta H + 4H^3 - 4HK\} - 2\,\beta_2\, \theta_2\, H = 0 & \text{on } \mathfrak{b}_2 \\
([[\mathbb{P}^\beta_1]]\, \nu^\beta, [[\mathbb{P}^\beta_2]]\, \nu^\beta, [[\mathbb{P}^\beta_3]]\, \nu^\beta) \perp T_x\mathfrak{b}_2 \qquad \text{for every} \quad x \in \mathfrak{b}_2 &
\end{cases}
\tag{41}
$$

where $[[\cdot]]$ denotes the jump of its argument, that is the difference between the outer and inner trace to $\mathfrak{b}_2$. The first equation above corresponds to normal variations, i.e. $\lambda = -\varphi\, \nu$, and is obtained from (31), (32), and (34) by taking into account that

$$
\mathbf{H} \cdot \nu = 2\, H\,.
$$

The orthogonality condition given by the second equation above, which corresponds to tangential variations, follows from the formula

$$
[[\mathbb{P}^\beta_\alpha]]\, \nu^\beta\, \lambda^\alpha = 0 \qquad \text{on} \quad \mathfrak{b}_2
$$

for every vector field $\lambda(x)$ that is tangent to $T_x\mathfrak{b}_2$ at every point $x \in \mathfrak{b}_2$. It is obtained from (31), (32), and (35) by using the relation $\mathbf{H} \cdot \lambda = 0$.

At the tip, a first balance of configurational actions comes from the tangential variations to $\mathfrak{b}_1$. Since $\mathbf{t} \cdot m = 0$ in (32) and (36), the sole contribution comes from the term involving the transplacement field, so that, by (30) and (31), we find

$$\int_{\mathfrak{b}_1} \mathbf{J}^\alpha \, \mathbf{t}^\alpha \, \varphi \, d\mathcal{H}^1 = 0 \qquad \forall \varphi$$

i.e.

$$\mathbf{J} \cdot \mathbf{t} = 0 \qquad \text{on} \quad \mathfrak{b}_1 \tag{42}$$

where $\mathbf{J}$ is given in components by (29).

Tangential variations to $\mathfrak{b}_2$ in the normal direction to $\mathfrak{b}_1$, i.e. $\lambda = \varphi \, m$, yields from (30), (31), (32), (36), (37), and (39) to

$$\mathbf{J} \cdot m + \tilde{\alpha}_2 \, \theta_2 \, (2H^2 - K) = \beta_2 \, \theta_2 + \tilde{\alpha}_1 \, \theta_1 \, (r_3 \, q_1 + r_4 \, q_2) + \beta_1 \, \theta_1 \, \mathbf{k} \cdot m \text{ on } \quad \mathfrak{b}_1 \, , \tag{43}$$

where q_1, q_2 are given by (38).

Finally, consider variations in the normal direction to $\mathfrak{b}_2$, i.e. $\lambda = \varphi \nu$. From (30) and (31), (32), (34), (36), (37), and (39), and by taking into account that $m \cdot \nu = 0$, we find

$$\int_{\mathfrak{b}_1} \Big\{ -\mathbf{J} \cdot \nu + 2 \, \tilde{\alpha}_2 \, \theta_2 \nabla^{\mathfrak{b}_2} H \cdot m + \tilde{\alpha}_1 \, \theta_1 \, (r_1 \, q_1 + r_2 \, q_2) + \beta_1 \, \theta_1 \, \mathbf{k} \cdot \nu \Big\} \, \varphi \, d\mathcal{H}^1$$
$$- \, \tilde{\alpha}_2 \, \theta_2 \int_{\Gamma} \sqrt{g} \, (-K \, b^{\alpha\beta} + 2 \, H \, g^{\alpha\beta}) \, n^\alpha \, D_\beta \varphi \, dv^1 = 0 \, . \tag{44}$$

Therefore, by integrating by parts the last term, where $n \equiv (0, 1)$, and using the arbitrariness of φ, the equation (in coordinates)

$$|\chi_{,1}| \Big\{ -\mathbf{J} \cdot \nu + 2 \, \tilde{\alpha}_2 \, \theta_2 \nabla^{\mathfrak{b}_2} H \cdot m + \tilde{\alpha}_1 \, \theta_1 \, (r_1 \, q_1 + r_2 \, q_2) + \beta_1 \, \theta_1 \, \mathbf{k} \cdot \nu \Big\}$$
$$+ \, \tilde{\alpha}_2 \, \theta_2 \left(\sqrt{g} \, (-K \, b^{21} + 2 \, H \, g^{21}) \right)' = 0 \tag{45}$$

follows and is equipped with the compatibility condition

$$-K \, b^{22} + 2 \, H \, g^{22} = 0 \qquad \text{on} \quad \mathfrak{b}_1 \, .$$

Equation (40) is the standard bulk balance of configurational forces for elastic bodies. The other balances, namely (41), (42), (43), (44) – or (45) with the last compatibility condition – are not standard (see, for example, resuilts and discussions in [5]).

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