COMPACTNESS OF SPECIAL FUNCTIONS OF BOUNDED HIGHER VARIATION

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Abstract. Given an open set $\Omega \subset \mathbb{R}^m$ and $n > 1$, we introduce the new spaces $GB_nV(\Omega)$ of *Generalized functions of bounded higher variation* and $GSB_nV(\Omega)$ of *Generalized special functions of bounded higher variation* that generalize, respectively, the space B_nV introduced by Jerrard and Soner in [43] and the corresponding SB_nV space studied by De Lellis in [24]. In this class of spaces, which allow as in [43] the description of singularities of codimension n, the distributional jacobian Ju need not have finite mass: roughly speaking, finiteness of mass is not required for the $(m-n)$-dimensional part of Ju, but only finiteness of size. In the space GSB_nV we are able to provide compactness of sublevel sets and lower semicontinuity of Mumford-Shah type functionals, in the same spirit of the codimension 1 theory [5, 6].

1. Introduction

The space BV of functions of bounded variation, consisting of real-valued functions u defined in a domain of $\mathbb{R}^m$ whose distributional derivative Du is a finite Radon measure, may contain discontinuous functions and, precisely for this reason, can be used to model a variety of phenomena, while on the PDE side it plays an important role in the theory of conservation laws [20, 14]. In more recent times, De Giorgi and the first author introduced the distinguished subspace SBV of special functions of bounded variation, whose distributional derivative consists of an absolutely continuous part and a singular part concentrated on a $(m-1)$-dimensional set, called (approximate) discontinuity set S_u. See [7] for a full account of the theory, whose applications include the minimization of the Mumford-Shah functional [52] and variational models in fracture mechanics. In a vector-valued setting, also the spaces BD and SBD play an important role, in connection with problems involving linearized elasticity and fracture (see also the recent work by Dal Maso on the space $GSBD$ [21])

It is well know that $|Du|$ vanishes on $\mathscr{H}^{n-1}$-negligible sets, hence BV and all related spaces can't be used to describe singularities of higher codimension. For this reason, having in mind application to the Ginzburg-Landau theory (where typically singularities, e.g. line vortices in $\mathbb{R}^3$ have codimension 2) Jerrard and Soner introduced in [43] the space B_nV of functions of bounded higher variation, where n stands for the codimension: roughly speaking it consists of Sobolev maps $u : \Omega \to \mathbb{R}^n$ whose distributional Jacobian Ju (well defined, at least as a distribution, under appropriate integrability assumptions) is representable by a vector-valued measure: in this case the natural vector space is the space $\Lambda_{m-n}\mathbb{R}^m$ of $(m-n)$-vectors. Remarkable extensions of the BV theory have been discovered in [43], as the counterpart of the coarea formula and of De Giorgi's rectifiability theorem for sets of finite perimeter. Even before [43], the distributional jacobian has been studied in many fundamental works as [46, 13, 38, 55, 51] in connection with variational problems in nonlinear elasticity (where typically $m = n$ and u represents a deformation map), e.g. to model cavitation effects.

As a matter of fact, since Ju can be equivalently described as a flat $(m-n)$-dimensional current, an important tool in the study of Ju is the well-developed machinery of currents, both in the Euclidean and in metric spaces, see [31, 34, 9, 57]. The fine structure of the measure Ju has been investigates in subsequent papers: using precisely tools from the theory of metric currents [9], De Lellis in [24] characterized Ju in terms of slicing and proved rectifiability of the (measure theoretic) support S_u of the $(m-n)$-dimensional part of Ju, while in [26] the second author and De Lellis characterized the absolutely continuous part of Ju with respect to $\mathscr{L}^m$ in terms of the Sobolev gradient ∇u. Also, in [24] the analog of the space SBV has been introduced, denoted SB_nV: it consists of all functions $u \in B_nV$ such that $Ju = R + T$, with $\|R\| \ll \mathscr{L}^m$ and $\|T\|$ concentrated on a $(m-n)$-dimensional set.

The main goal of this paper is to study the compactness properties of SB_nV. Even in the standard SBV theory, a uniform control on the energy of Mumford-Shah type

$$\int (|u_h|^s + |\nabla u_h|^p) d\mathscr{L}^m + \mathscr{H}^{m-n}(S_{u_h})$$

(with $s > 0$, $p > 1$) along a sequence (u_h) does not provide a control on Du_h. Indeed, only the $\mathscr{H}^{m-1}$-dimensional measure of S_{u_h} does not provide a control on the width of the jump. This difficulty leads [22] to the space $GSBV$ of *generalized* special functions of bounded variation, i.e. the space of all real-valued maps u whose truncates $(-N) \vee u \wedge N$ are all SBV. Since both the approximate gradient ∇u and the approximate discontinuity set S_u behave well under truncation, it turns out that also the energy of $u_n^N := (-N) \vee u_n \wedge N$ is uniformly controlled, and now also $|Du_n^N|$; this is the very first step in the proof of the compactness-lower semicontinuity theorem in $GSBV$, which shows that the sequence (u_h) has limit points with respect to local convergence in measure, that any limit point u belongs to $GSBV$, and that

$$\int (|u|^s + |\nabla u|^p) d\mathscr{L}^m \le \liminf_h \int (|u_h|^s + |\nabla u_h|^p) d\mathscr{L}^m, \qquad \mathscr{H}^{m-1}(S_u) \le \liminf_h \mathscr{H}^{m-1}(S_{u_h}).$$

In the higher codimension case, if we look for energies of the form

$$\int (|u|^s + |\nabla u|^p + |M(\nabla u)|^\gamma) d\mathscr{L}^m + \mathscr{H}^{m-n}(S_u)$$

(with $s^{-1} + (n-1)/p < 1$, $\gamma > 1$) now involving also the minors $M(\nabla u)$ of ∇u, the same difficulty exists, but the truncation argument does not work anymore. Indeed, the absence of S_u, namely the absolute continuity of Ju, may be due to very precise cancellation effects that tend to be destroyed by a left composition, thus causing the appearance of new singular points (see Example 2.5.3 and the subsequent observation). Also, unlike the codimension 1 theory, no "pointwise" description of S_u is presently available.

For these reasons, when looking for compactness properties in SB_nV, we have been led to define the space GSB_nV of *generalized* special functions of bounded higher variation as the space of functions u such that Ju is representable in the form $R + T$, with R absolutely continuous with respect to $\mathscr{L}^m$ and T having finite *size* in an appropriate sense, made rigorous by the slicing theory of flat currents (in the same vein, one can also define GB_nV, but our main object of investigation will be GSB_nV). In particular, for $u \in GSB_nV$ the distribution Ju is not necessarily representable by a measure. The similarity between $GSBV(\Omega)$ and $GSB_nV(\Omega)$ is not coincidental, and in fact we prove that in the scalar case $n = 1$ these two spaces are essentially the same; on the contrary for $n \ge 2$ their properties are substantially different. In order to study the T part of Ju we use the notion of size of flat current with possibly infinite mass developed, even in metric spaces, in [8], see also [27] for the case of currents with finite mass.

The paper is organized as follows: after posing the proper definitions in the context of metric currents we briefly review the space B_nV studied in [43, 24, 25]. In section 3 we present the notion of size of a flat current, we relate it to the concept of distributional jacobian and define our new space of functions GSB_nV. The main result of the paper is presented in section 4, where with the help of the slicing theorem we will generalize to our setting the compactness theorem of $GSBV$, as well as the closure theorem in SB_nV due to De Lellis in [24, 25].

We finally apply the compactness theorem to show the existence of minimizers for a general class of energies that feature both a volume and a size term. The model problem is a new functional of Mumford-Shah type that we here introduce, in the spirit of [23]. We analyse its minimization together with suitable Dirichlet boundary conditions, both in the interior and in the closure of Ω. In particular we show that minimizers must be nontrivial (i.e.: $S_u \neq \emptyset$), at least for suitable boundary data; we also compare our choice of the energy with the classical p-energy of sphere-valued maps, see [38, 39, 15]. Regarding the problem in $\overline{\Omega}$ the higher codimension of the singular set allows concentration of the jacobian at the boundary, providing some interesting examples that we briefly include in subsection 5.3. Similar variational problems in the framework of cartesian currents have been considered in [47], where the author proves existence of minimizers in the set of maps whose graph is a normal current: the boundaries of these graphs enjoy a decomposition into vertical parts of integer dimension, inherited from general properties of integral currents, which relates to the space B_nV, see [48].

In a forthcoming paper [37] we show how the Mumford-Shah energy can be approximated, in the sense of Γ-convergence, by a family of functionals defined on maps with absolutely continuous jacobian:

$$F_\varepsilon(u,v;\Omega) := \int_\Omega (|u|^s + |\nabla u|^p + (v+k_\varepsilon)|M(\nabla u)|^\gamma) d\mathscr{L}^m + \beta \int_\Omega \varepsilon^{q-n}|\nabla v|^q + \frac{(1-v)^n}{\varepsilon^n}\, d\mathscr{L}^m,$$

(here $\beta, \gamma, q, k_\varepsilon$ are suitable parameters, $v : \Omega \to [0,1]$ is Borel). Following [11, 12, 3, 45] the control variable v dims the concentration of $M(\nabla u)$: the price of the transition between 0 and 1 is captured by the Modica-Mortola term which detects $(m-n)$-dimensional sets.

We occasionally appeal to the metric theory of currents because the main tool in the definition of size and in the proof of the rectifiability theorem is the slicing technique, a basic ingredient of the metric theory. For instance the argument in [8] proving the rectifiability of currents with finite size uses Lipschitz restrictions and maps of metric bounded variation taking values into an appropriate space of flat chains with a suitable hybrid metric. However, no significant simplification comes from the Euclidean theory, except the fact that suffices to consider linear instead of Lipschitz maps.

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2. Distributional Jacobians

We begin by fixing some basic notions about currents and recalling some properties of the distributional jacobian.

2.1. **Exterior algebra and projections.** Our ambient space will be $\mathbb{R}^m$ with the standard basis $e_1, \ldots, e_m$ and its dual $e^1, \ldots, e^m$. For every $1 \leq k \leq m$ we let

$$\mathbf{O}_k = \left\{\pi : \mathbb{R}^m \to \mathbb{R}^k : \pi \circ \pi^* = I_k\right\}$$

be the space of orthogonal projections of rank k. We will also need to fix coordinates according to some projection $\pi \in \mathbf{O}_k$: we agree that $\mathbb{R}^m \ni z = (x,y) \in \mathbb{R}^k \times \mathbb{R}^{m-k}$ are orthogonal coordinates with positive orientation such that $\pi(z) = x$. In particular we let $A^x = A \cap \pi^{-1}(x)$ be the restriction of any $A \subset \mathbb{R}^m$ to the fiber $\pi^{-1}(x)$ and $i^x = \mathbb{R}^{m-k} \to \mathbb{R}^m$ be the isometric injection $i^x(y) = (x,y)$.

As customary the symbols $\Lambda_k\mathbb{R}^m$ and $\Lambda^k\mathbb{R}^m$ will respectively denote the spaces of k-vectors and k-covectors in $\mathbb{R}^m$. The contraction operation $\llcorner : \Lambda_q\mathbb{R}^m \times \Lambda^p\mathbb{R}^m \to \Lambda_{q-p}\mathbb{R}^m$ between a q-vector ζ

and a p-covector α, with $q \geq p$, is defined as:

$$\langle \zeta \llcorner \alpha, \beta \rangle = \langle \zeta, \alpha \wedge \beta \rangle \qquad \text{whenever } \beta \in \Lambda^{q-p}\mathbb{R}^m. \tag{1}$$

If $L : \mathbb{R}^m \to \mathbb{R}^n$ is linear then

$$M_n L := e_1 \wedge \cdots \wedge e_m \llcorner L^1 \wedge \cdots \wedge L^n \in \Lambda_{m-n}\mathbb{R}^m \tag{2}$$

represents the collection of determinants of $n \times n$ minors of L. In fact, if $\underline{i} : \{1, \dots, m-n\} \to \{1, \dots, m\}$ is an increasing selection of indexes, and if $\bar{i} : \{1, \dots, n\} \to \{1, \dots, m\}$ is the complementary increasing selection, then the $e_{\underline{i}}$ component of $M_n L$ is

$$\langle M_n L, e^{\underline{i}} \rangle = \langle e_1 \wedge \cdots \wedge e_m, L^1 \wedge \cdots \wedge L^n \wedge e^{\underline{i}} \rangle = (-1)^{\sigma} \det([L]_{\bar{i}}), \tag{3}$$

where $[L]_{\bar{i}}$ is the $n \times n$ submatrix L^{ℓ}_j with $j = \bar{i}(1), \dots, \bar{i}(n)$, $\ell = 1, \dots, n$ and σ is the sign of the permutation

$$(1, \dots, m) \mapsto (\bar{i}(1), \dots, \bar{i}(n), \underline{i}(1), \dots, \underline{i}(m-n)).$$

When $\mathrm{rk}(L) = n$, choosing an orthonormal frame (e_i) so that $\ker(L) =< e_{n+1}, \dots, e_m >$ we have $L = (A, \mathbf{0})$ and by (3) $M_n L = \det(A) e_{n+1} \wedge \cdots \wedge e_m$. In particular $M_n L$ is a simple $(m-n)$-vector.

Recall that the spaces $\Lambda_k \mathbb{R}^m$ and $\Lambda^k \mathbb{R}^m$ can be endowed with two different pairs of dual norms. The first one is called norm, it is denoted by $|\cdot|$ and it comes from the scalar product where the multivectors

$$\{e_{i(1)} \wedge \cdots \wedge e_{i(k)}\}_i \quad \text{and} \quad \{e^{i(1)} \wedge \cdots \wedge e^{i(k)}\}_i \tag{4}$$

indexed by increasing maps $i : \{1, \dots, k\} \to \{1, \dots, m\}$ form a pair of dual orthonormal bases. The second one is called mass, comass for the space of covectors, and it is defined as follows: the comass of $\phi \in \Lambda^k \mathbb{R}^m$ is

$$\|\phi\| := \sup\big\{\langle \phi, v_1 \wedge \cdots \wedge v_k \rangle : v_i \in \mathbb{R}^m, |v_i| \leq 1\big\};$$

and the mass of $\xi \in \Lambda_k \mathbb{R}^m$ is defined, by duality, by

$$\|\xi\| := \sup\big\{\langle \xi, \phi \rangle : \|\phi\| \leq 1\big\}.$$

As described in [31, 1.8.1], in general $\|\xi\| \leq |\xi|$ and equality holds if and only if ξ is simple.

Therefore $|M_n L| = \|M_n L\|$. Moreover using the Pitagora's Theorem for the norm and Binet's formula we have the following relation:

$$\begin{aligned}\sup_{\pi \in \mathbf{O}_{m-n}} |M_n L \llcorner d\pi| &= \sup_{\pi \in \mathbf{O}_{m-n}} |d\pi(M_n L)| = \sup_{\pi \in \mathbf{O}_{m-n}} \Big|\sum_{\underline{i}} M_n L^{\underline{i}} \, d\pi(e_{\underline{i}})\Big| \\ &\leq \sup_{\pi \in \mathbf{O}_{m-n}} |M_n L| \Big(\sum_{\underline{i}} |d\pi(e_{\underline{i}})|^2\Big)^{\frac{1}{2}} = |M_n L| \sup_{\pi \in \mathbf{O}_{m-n}} |\det(\pi \circ \pi^*)| = |M_n L|\end{aligned} \tag{5}$$

where $d\pi$ stands for $d\pi^1 \wedge \cdots \wedge d\pi^{m-n}$, and the equality is realized by the orthogonal projection onto $\ker(L)$.

We adopt the convention of choosing the mass and comass norms to measure the length of k-vectors and k-covectors respectively.

2.2. **Currents in $\Omega \subset \mathbb{R}^m$.** We briefly recall the basic definitions and properties of classical currents in $\mathbb{R}^m$. This theory was introduced by De Rham in [29], along the lines of the previous work on distributions by Schwartz [53], and the subsequently put forward by Federer and Fleming in [33]; we refer to [31] for a complete account of it. The classical framework is best for treating the concepts of distributional jacobian in a subset of $\mathbb{R}^m$; however we will need to use the metric theory of Ambrosio and Kirchheim [9] to define the concept of size and of concentration measure. We will clearly outline the interplay between the two approaches.

We give for granted the concepts of derivative, exterior differentiation, pull-back and support of a test functions: they can all be defined by expressing the form in the coordinates given by the frame (4), see [31, 4.1.6].

We begin by defining the space of smooth, compactly supported test forms:

Definition 2.2.1 (Smooth test forms)**.** *We let $\mathscr{D}^k(\Omega)$ be the space of smooth, compactly supported k-differential forms:*

$$\mathscr{D}^k(\Omega) = \bigcup_{K \Subset \Omega} \mathscr{D}^k_K(\Omega), \qquad \mathscr{D}^k_K(\Omega) = \big\{\omega \in C^\infty(\Omega, \Lambda^k \mathbb{R}^m), \operatorname{spt}(\omega) \subset K\big\}. \tag{6}$$

Each space $\mathscr{D}^k_K(\Omega)$ is endowed with the topology given by the seminorms

$$p_{K,j}(\omega) = \sup\{\|D^\alpha \omega(x)\|, x \in K, |\alpha| \le j\},$$

and $\mathscr{D}^k(\Omega)$ is endowed with the finest topology making the inclusions $\mathscr{D}^k_K(\Omega) \hookrightarrow \mathscr{D}^k(\Omega)$ are continuous.

This topology is locally convex, translation invariant and Hausdorff; moreover a sequence $\omega_j \to \omega$ in $\mathscr{D}^k(\Omega)$ if and only if there exists $K \Subset \Omega$ such that $\operatorname{spt}(\omega_j) \subset K$ and $p_{K,j}(\omega_j - \omega) \to 0$ for every $j \ge 0$.

Definition 2.2.2 (Classical currents and weak* convergence)**.** *A current T is a continuous linear functional on $\mathscr{D}^k(\Omega)$. The space of k-currents is denoted by $\mathscr{D}_k(\Omega)$. We say that a sequence (T_h) weak* converges to T, $T_h \overset{*}{\rightharpoonup} T$, whenever*

$$T_h(\omega) \to T(\omega) \qquad \forall \omega \in \mathscr{D}^k(\Omega). \tag{7}$$

The support of a current is

$$\operatorname{spt}(T) := \bigcap \{C : T(\omega) = 0 \ \forall \omega \in \mathscr{D}^k(\Omega), \operatorname{spt}(\omega) \cap C = \emptyset\}.$$

The boundary operator is the adjoint of exterior differentiation:

$$\partial T(\omega) := T(d\omega);$$

let also $\phi : \Omega \to \mathbb{R}^p$ be a proper Lipschitz map: we let the push-forward of $T \in \mathscr{D}_k(\Omega)$ via Φ by duality:

$$(\Phi_\# T)(\omega) := T(\Phi^\# \omega) \qquad \forall \omega \in \mathscr{D}^k(\mathbb{R}^p).$$

According to (1) given $T \in \mathscr{D}_k(\Omega)$ and $\tau \in \mathscr{D}^\ell(\Omega)$ with $\ell \le k$ we set the restriction

$$(T \llcorner \tau)(\eta) = T(\tau \wedge \eta) \qquad \forall \eta \in \mathscr{D}^{k-\ell}(\Omega).$$

Definition 2.2.3 (Finite mass and Normal currents)**.** *We say that $T \in \mathscr{D}_k(\Omega)$ is a current of finite mass if there exists a finite Borel measure μ in Ω such that*

$$|T(\omega)| \le \int_\Omega \|\omega(x)\| d\mu(x) \qquad \forall \omega \in \mathscr{D}^k(\Omega). \tag{8}$$

The total variation of T is the minimal μ satisfying (8) and is denoted by $\|T\|$, and the mass $\mathbf{M}(T) := \|T\|(\Omega)$. As customary we let $\mathbf{M}_k(\Omega)$ be the space of finite mass k-dimensional currents and $\mathbf{N}_k(\Omega)$ be the subspace of normal currents:

$$\mathbf{N}_k(\Omega) = \mathbf{M}_k(\Omega) \cap \{T : \partial T \in \mathbf{M}_{k-1}(\Omega)\}.$$

Therefore every finite mass current can be represented as $T = \vec{T} \wedge \|T\|$, and admits and extension to k-forms with bounded Borel coefficients. In particular we can restrict every finite mass current T to every open set A. We denote

$$\mathbf{E}^m = e_1 \wedge \cdots \wedge e_m \wedge \mathscr{L}^m$$

the top dimensional m-current representing the Lebesgue integration on $\mathbb{R}^m$ with the standard orientation. As explained in [31, 4.1.7, 4.1.18], [9, 3.2], every function $f \in L^1(\Omega, \Lambda_{m-k}\mathbb{R}^m)$ induces a k-current of finite mass via the action

$$(\mathbf{E}^m \llcorner f)(\omega) = \int_\Omega \langle f \wedge \omega, e_1 \wedge \cdots \wedge e_m \rangle \, d\mathscr{L}^m \qquad \forall \omega \in \mathscr{D}^k(\Omega). \tag{9}$$

Note that $f_h \rightharpoonup f$ weakly in L^1 entails $\mathbf{E}^m \llcorner f_h \overset{*}{\rightharpoonup} \mathbf{E}^m \llcorner f$.

2.3. **Flat currents.** In order to treat objects with possibly infinite mass, the right subspace of $\mathscr{D}^k(\Omega)$ retaining some useful properties such as slicing and restrictions is the space of flat currents.

Definition 2.3.1 (Flat norm and flat currents, [31, 4.1.12])**.** *For every $\omega \in \mathscr{D}^k(\Omega)$ we let*

$$\mathbf{F}(\omega) = \max\big\{ \sup_{x \in \Omega} \|\omega(x)\|, \sup_{x \in \Omega} \|d\omega(x)\| \big\}.$$

The flat norm is defined as

$$\mathbf{F}(T) = \inf\{\mathbf{M}(T - \partial Y) + \mathbf{M}(Y) : \, Y \in \mathbf{M}_{k+1}(\Omega)\} \tag{10}$$

$$= \sup\{T(\omega) : \, \omega \in \mathscr{D}^k(\Omega), \, \mathbf{F}(\omega) \le 1\}. \tag{11}$$

The space $\mathbf{F}_k(\Omega)$ of flat k-dimensional currents in an open subset $\Omega \subset \mathbb{R}^m$ is the $\mathbf{F}$-completion of $\mathbf{N}_k(\Omega)$ (see [34] and [31, 4.1.12] for the equivalence of (10) and (11)).

It is straightforward to prove that $\mathbf{F}$ is a norm; furthermore if T is flat, so is ∂T and

$$\mathbf{F}(\partial T) \le \mathbf{F}(T) \le \mathbf{M}(T).$$

Throughout all the paper we will deal with three notions of convergence:

- the convergence w.r.t. the flat norm $\mathbf{F}$ defined in (10) above;
- the weak convergence in L^p, $1 \le p < \infty$, denoted by $\rightharpoonup$;
- the weak* convergence of currents (7).

The map (9) $f \mapsto \mathbf{E}^m \llcorner f$ embeds $L^1(\Omega, \Lambda_{m-k}\mathbb{R}^m)$ into $\mathbf{F}_k(\Omega)$ by [31, 4.1.18], and the three aforementioned topologies are ordered from the strongest to the weakest.

2.4. **Slicing.** As explained in [31, 4.2] and [8], every $T \in \mathbf{F}_k(\Omega)$ can be sliced via a Lipschitz map $\pi \in \mathrm{Lip}(\Omega, \mathbb{R}^\ell)$, $1 \le \ell \le k$: the result is a collection of currents

$$\langle T, \pi, x \rangle \in \mathbf{F}_{k-\ell}(\Omega) \qquad \text{uniquely determined up to } \mathscr{L}^\ell \text{ negligible sets}$$

expressing the action of T against tensor product forms $(\phi \circ \pi) d\pi \wedge \psi$, for $\phi \in \mathscr{D}^0(\mathbb{R}^\ell)$ and $\psi \in \mathscr{D}^{k-\ell}(\Omega)$:

$$T\big((\phi \circ \pi) d\pi \wedge \psi\big) = \int_{\mathbb{R}^\ell} \phi(x) \langle T, \pi, x \rangle (\psi) d\mathscr{L}^\ell(x).$$

The slices satisfy several properties: amongst them we recall

$$\langle T, \pi, x \rangle \text{ is concentrated on } \pi^{-1}(x) \text{ for } \mathscr{L}^\ell\text{-a.e. } x \in \pi(\Omega), \tag{12}$$

$$\int_{\pi(\Omega)} \mathbf{F}(\langle T, \pi, x \rangle) \, d\mathscr{L}^\ell(x) \le \mathrm{Lip}(\pi)^\ell \mathbf{F}(T), \tag{13}$$

and we refer to [31, 4.2.1] and to [9] for a general account in the Euclidean and general metric settings. We stress the following fact, which is a key tool to extend many properties like restrictions and slicing from normal to flat currents, and that will be used later on. Suppose $(T_h) \subset \mathbf{F}_k(\Omega)$ satisfy

$$\sum_h \mathbf{F}(T_{h+1} - T_h) < +\infty$$

and let $\pi \in \mathrm{Lip}(\Omega, \mathbb{R}^\ell)$ fixed. Then

$$\mathbf{F}\big(\langle T_h, \pi, x\rangle - \langle T, \pi, x\rangle\big) \to 0$$

for $\mathscr{L}^\ell$-almost every $x \in \mathbb{R}^\ell$. Recall that for the finite mass current $R = \rho\mathscr{L}^m$ with $\rho \in L^1(\Omega, \Lambda_k\mathbb{R}^m)$ Federer's coarea formula implies that at almost every $x \in \mathbb{R}^\ell$ it holds:

$$\langle R, \pi, x\rangle = (\rho(x, \cdot) \llcorner d\pi)\, \mathscr{H}^{m-\ell} \llcorner \pi^{-1}(x). \tag{14}$$

2.5. **Distributional jacobian.** We will assume throughout all the paper that $m \geq n$ are positive integers and that p and s are positive exponents satisfying

$$\frac{1}{s} + \frac{n-1}{p} \leq 1. \tag{15}$$

The definition of distributional jacobian takes advantage of the divergence structure of jacobians

$$d(u^1 du^2 \wedge \cdots \wedge du^n) = du^1 \wedge \cdots \wedge du^n \quad \forall u \in C^1(\mathbb{R}^n, \mathbb{R}^n),$$

which allows to pass the exterior derivative to the test form and hence weakens the minimal regularity assumptions on the map u.

Definition 2.5.1 (Distributional Jacobian)**.** *Let $u \in W^{1,p}(\Omega, \mathbb{R}^n) \cap L^s(\Omega, \mathbb{R}^n)$. We denote by $j(u)$ the $(m-n+1)$-dimensional flat current*

$$\langle j(u), \omega\rangle := (-1)^n \int_\Omega u^1 du^2 \wedge \cdots \wedge du^n \wedge \omega \qquad \forall \omega \in \mathscr{D}^{m-n+1}(\Omega); \tag{16}$$

we define the distributional Jacobian of u as the $(m-n)$-dimensional flat current

$$Ju := \partial j(u) \in \mathbf{F}_{m-n}(\Omega). \tag{17}$$

A few observations are in order: first of all the integrability assumption $u \in W^{1,p} \cap L^s$ and the exponent bound (15) ensure that $j(u)$ is a well-defined flat current of finite mass, since it acts on test forms as the integration against an $L^1(\Omega, \Lambda_{m-n+1}\mathbb{R}^m)$ function: $j(u) = (-1)^n \mathbf{E}^m \llcorner u^1 du^2 \wedge \cdots \wedge du^n$. As a consequence $Ju \in \mathbf{F}_{m-n}(\Omega)$ as declared in (17). Furthermore for $p \geq \frac{mn}{m+1}$ the constraint (15) is satisfied with the Sobolev exponent p^* in place of s, hence definition 2.5.1 makes sense for $u \in W^{1,p}$ in this range of summability.

In [16] the authors showed that Ju can be defined in the space $W^{1-\frac{1}{m},m}(\Omega)$, which contains $L^s \cap W^{1,p}$ for every s, p as in (15). This extension exploits the trace space nature of $W^{1-\frac{1}{m},m}$, expressing Ju as a boundary integral in $\mathbb{R}^m_+$.

Finally in the special situation $n = 1$ the minimal requirement to give meaning to (16) is $u \in L^1(\Omega)$, and the Jacobian reduces to the distributional derivative $Ju = -\partial(\mathbf{E}^m \llcorner u)$:

$$\Big\langle Ju, \sum_i (-1)^{i-1} \omega_i \widehat{dx^i}\Big\rangle = -\sum_i \int_\Omega u \frac{\partial \omega_i}{\partial x^i} dx = \sum_i \langle D_i u, \omega_i\rangle. \tag{18}$$

Regarding the convergence properties of these currents, we note the following:

Proposition 2.5.2. *Let $u_h, u \in W^{1,p}(\Omega, \mathbb{R}^n) \cap L^s(\Omega, \mathbb{R}^n)$ satisfy*

- *$u_h \to u$ in $L^s(\Omega, \mathbb{R}^n)$,*
- *$\nabla u_h \rightharpoonup \nabla u$ weakly in $L^p(\Omega, \mathbb{R}^{n\times m})$.*

Then $\mathbf{F}(Ju_h - Ju) \to 0$.

Proof. Let us rewrite the difference $u_h^1 du_h^2 \wedge \cdots \wedge du_h^n - u^1 du^2 \wedge \cdots \wedge du^n$ in the following way:

$$u_h^1 du_h^2 \wedge \cdots \wedge du_h^n - u^1 du^2 \wedge \cdots \wedge du^n =$$
$$= (u_h^1 - u^1) du_h^2 \wedge \cdots \wedge du_h^n + u^1 \sum_{k=2}^{n} du_h^2 \wedge du_h^{k-1} \wedge d(u_h^k - u^k) \wedge du^{k+1} \wedge \cdots \wedge du^n.$$

We can actually write each addendum in the last summation as

$$-(u_h^k - u^k) du_h^2 \wedge du_h^{k-1} \wedge du^1 \wedge du^{k+1} \wedge \cdots \wedge du^n + d\zeta_h^k,$$

where we set

$$\zeta_h^k = (-1)^{k-2} u^1 (u_h^k - u^k) du_h^2 \wedge \cdots \wedge du_h^{k-1} \wedge du^{k+1} \wedge \cdots \wedge du^n \in L^1(\Omega, \Lambda^{n-2}\mathbb{R}^m). \tag{19}$$

Notice that we can always assume $s \geq p$, hence $\zeta_h^k \in L^1$. To show (19) it is sufficient to approximate both u and u_h in the strong topology with regular functions and apply the Leibniz rule; the same approximation shows that $d\zeta_h^k \in L^1$ and hence $\int_\Omega d\zeta_h^k \wedge d\omega = 0$ for each $\omega \in \mathscr{D}^{m-n+1}(\Omega)$. By the calculations above we can estimate

$$\begin{aligned} |\langle Ju_h - Ju, \omega\rangle| &= |\langle j(u_h) - j(u), d\omega\rangle| \\ &\leq \|d\omega\|_{L^\infty} \sum_{k=1}^{n} \|u_h^k - u^k\|_{L^s} \|du_h^1\|_{L^p} \cdots \|du_h^{k-1}\|_{L^p} \|du_h^{k+1}\|_{L^p} \cdots \|du_h^n\|_{L^p} \\ &\leq C\,\mathbf{F}(\omega) \left(\sup_h \|\nabla u_h\|_{L^p}\right)^{n-1} \|u_h - u\|_{L^s}. \end{aligned}$$

Taking the supremum on test functions ω with $\mathbf{F}(\omega) \leq 1$ we immediately obtain the asserted convergence. $\square$

A natural question is the relation between the summability exponent p and the regularity of the distribution Ju. There is a main difference between $p \geq n$ and $p < n$: if the gradient ∇u has a sufficiently high summability, then Ju is an absolutely continuous measure. In fact let $u_h = u * \rho_h$, where ρ_h is a standard approximation of the identity: since $p \geq n$ the continuous embedding $W^{1,p} \hookrightarrow W^{1,n}_{\mathrm{loc}}$ implies that $u_h \to u$ both in $W^{1,p} \cap L^s$ and $W^{1,n}_{\mathrm{loc}}$. Taking a test form ψ with compact support we can use Proposition 2.5.2 to pass to the limit in the integration by parts formula

$$\langle Ju_h, \psi\rangle = (-1)^n \int_\Omega u_h^1 du_h^2 \wedge \cdots \wedge du_h^n \wedge d\psi = \int_\Omega du_h^1 \wedge du_h^2 \wedge \cdots \wedge du_h^n \wedge \psi,$$

yielding $Ju = \mathbf{E}^m \llcorner du^1 \wedge \cdots \wedge du^n$.

On the other hand when $p < n$ there are several examples of functions whose jacobian is not in L^1: for instance when $m = n$ the "monopole" function $u(x) := \frac{x}{|x|}$ satisfies $Ju = \mathscr{L}^n(B_1)[\![0]\!]$, where $[\![0]\!]$ is the Dirac's mass in the origin. More generally:

Example 2.5.3 (Zero homogeneous functions, $m = n$, [43, 3.2])**.** Let $\gamma : S^{n-1} \to \mathbb{R}^n$ be smooth and let $u(x) := \gamma(\frac{x}{|x|})$. Then

$$Ju = Area(\gamma)[\![0]\!] \tag{20}$$

where $Area(\gamma)$ is the signed area enclosed by γ.

Proof. Outside the origin u is smooth and takes values into the $(n-1)$-dimensional submanifold $\gamma(S^{n-1})$, hence $\mathrm{spt}(Ju) \subset \{0\}$. Set $t = |x|$ and $y = \frac{x}{|x|}$: then $d\gamma = \sum_i \frac{\partial u}{\partial x^k} t dy^k$, so

$$t^{n-1} du^2 \wedge \cdots \wedge du^n = d\gamma^2 \wedge \cdots \wedge d\gamma^n \in \Lambda^{n-1} \mathrm{Tan}\, S^{n-1}.$$

Hence the only term of $d\omega$ surviving in the wedge product is $\frac{\partial\omega}{\partial t}dt$. Therefore

$$
\begin{aligned}
(-1)^n \int_{\mathbb{R}^n} u^1 du^2 \wedge \cdots \wedge du^n \wedge d\omega &= (-1)^n \int_{\mathbb{R}^n} \frac{\partial\omega}{\partial t}\gamma^1(y) du^2 \wedge \cdots \wedge du^n \wedge dt \\
&= -\int_{\mathbb{R}^n} \frac{\partial\omega}{\partial t} u^1(y) dt \wedge du^2 \wedge \cdots \wedge du^n \\
&= -\int_{\partial B_t} \left(\int_0^{+\infty} \frac{\partial\omega}{\partial t} dt\right) u^1(x) du^2 \wedge \cdots \wedge du^n \\
&= -\int_{\partial B_1} \left(\int_0^{+\infty} \frac{\partial\omega}{\partial t} dt\right) \gamma^1(y) d\gamma^2 \wedge \cdots \wedge d\gamma^n \\
&= \omega(0) \int_{S^{n-1}} \gamma^1(y) d\gamma^2 \wedge \cdots \wedge d\gamma^n. \qquad (21)
\end{aligned}
$$

Setting $\Upsilon(t,y) := t\gamma(y)$ the Lipschitz extension to the unit ball $B_1 \subset \mathbb{R}^n$, by Stokes' Theorem (21) equals to

$$
\begin{aligned}
\omega(0) \int_{\partial B_1} \Upsilon^1(1,y) d\Upsilon^2 \wedge \cdots \wedge d\Upsilon^n &= \omega(0) \int_{B_1} d\Upsilon^1 \wedge \cdots \wedge d\Upsilon^n \\
&= \omega(0) \int_{B_1} \det(\nabla\Upsilon) dx = \omega(0) \int_{\mathbb{R}^n} \deg(\Upsilon, w, B_1) dw. \qquad (22)
\end{aligned}
$$

It is well known that (22) represents the signed area enclosed by the surface $\gamma(S^{n-1})$. $\square$

This example immediately outlines one of the biggest differences with the scalar case. Consider as in [43] the "eight-shaped" loop in $\mathbb{R}^2$:

$$
\gamma(\theta) = \begin{cases} (\cos(2\theta) - 1, \sin(2\theta)) & \text{for } \theta \in [0,\pi], \\ (1 - \cos(2\theta), \sin(2\theta)) & \text{for } \theta \in [\pi, 2\pi]. \end{cases} \qquad (23)
$$

and let u be the zero homogeneous extension. γ encloses the union $B_1(-e_1) \cup B_1(e_1)$ with degree $+1$ and -1 respectively: in light of (20) $Ju = 0$. However a left composition with a smooth map $F : \mathbb{R}^2 \to \mathbb{R}^2$ easily destroys the cancellation, causing the appearance of a Dirac's mass in 0. Hence the estimate

$$
\|J(F \circ u)\| \le \mathrm{Lip}(F)^2 \|Ju\| \qquad (24)
$$

doesn't hold anymore if u is not regular. Note that this phenomenon does not appear for $n = 1$ and $u \in BV(\Omega)$, as Vol'pert chain rule provides exactly the estimate (24) (see [7, Theorem 3.96]).

The failure of (24) is related to the validity of a strong coarea formula for jacobians of vector valued maps, namely equation (1.7) in [43]. The (weak) coarea amounts instead to decompose the current Ju of a B_nV map (see next paragraph for the definition) into the superposition of integral currents corresponding to the level sets of u: letting $u_y(x) := \frac{u(x)-y}{|u(x)-y|}$, it is proved in [43, Theorem 1.2] that

$$
Ju = \frac{1}{\mathscr{L}^n(B_1)} \int_{\mathbb{R}^n} Ju_y \, dy
$$

as currents. However, because of some cancellation phenomena like in (23), (24), the strong version of the coarea formula

$$
\|Ju\| = \frac{1}{\mathscr{L}^n(B_1)} \int_{\mathbb{R}^n} \|Ju_y\| \, dy \qquad (25)
$$

might well fail. Once again observe that for $n = 1$ the equality (25) has been proved by Fleming and Rishel to holds for every $u \in BV$, see [7, Theorem 3.40]. For a more detailed analysis we refer to [43, 25, 51, 28].

For later purposes we report the dipole construction, introduced by Brezis, Coron and Lieb in [15]: it consists of a map taking values into a sphere which is constant outside a prescribed compact set, its jacobian is the difference of two Dirac's masses and satisfies suitable $W^{1,p}$ estimate. We write $(y,z) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and denote by $\mathcal{N} = (0,1) \in S^{n-1}$ the north pole.

Example 2.5.4 (Dipole, [16, 2.2])**.** Let $n \geq 2$, $\nu \in \mathbb{Z}$, $\rho > 0$: there exists a map $f_{\nu,\rho} : \mathbb{R}^n \to S^{n-1}$ with the following properties:

- $f_{\nu,\rho} \equiv \mathcal{N}$ outside $\{|y| + |z| < \rho\}$;
- $f_{\nu,\rho} - \mathcal{N} \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$ for every $p < n$ with estimates
$$\|\nabla f_{\nu,\rho}\|_{L^p}^p \leq C_p \, \nu^{\frac{p}{n-1}} \rho^{n-p};$$
- $J f_{\nu,\rho} = \nu \mathscr{L}^n(B_1^n)\big([\![(0,-\rho)]\!] - [\![(0,\rho)]\!]\big)$.

The locality of the dipole construction allows to glue several copies of dipoles to produce interesting examples.

Example 2.5.5 (Finiteness of $\mathbf{F}(Jg)$ does not imply finiteness of $M(Jg)$)**.** We build a map g such that $\mathbf{F}(Jg)$ is finite and Jg has an infinite mass. The construction starts from $f_{\nu,\rho}(\cdot,0) : \mathbb{R}^{n-1} \to S^{n-1}$, which is a smooth map equal to $\mathcal{N}$ outside B_ρ^{n-1} and such that $\deg(f_{\nu,\rho}(\cdot,0)) = \nu$. For $|z| < \rho$ we extend by $f_{\nu,\rho}(y,z) = f_{\nu,\rho}(\frac{\rho y}{\rho - |z|}, 0)$ and we set $f_{\nu,\rho} \equiv \mathcal{N}$ at points $|z| \geq \rho$. Choosing a sequence of positive radii (ρ_k) we can glue an infinite number of dipoles along the z axis:
$$g(y,z) = f_{1,\rho_k}(y, z - z_k) \qquad \text{for} \qquad |z - z_k| \leq 2\rho_k,$$
where $z_0 = 0$ and $z_k = 2\sum_{j=0}^k \rho_j$. The function g belongs to $L^\infty \cap W^{1,p}$ provided $\sum_k \rho_k^{n-p} < \infty$: in this case note that
$$Jg = \mathscr{L}^n(B_1) \sum_k [\![(0, z_k - \rho_k)]\!] - [\![(0, z_k + \rho_k)]\!],$$
hence $\mathbf{F}(Jg) \leq 2\mathscr{L}^n(B_1) \sum_k \rho_k < \infty$ but $\mathbf{M}(Jg) = +\infty$.

More complicated examples, including maps such that Ju is not even a Radon measure, are presented in [43, 51, 3].

2.6. **Functions of bounded n-variation.** The space of functions of bounded n-variation has been introduced by Jerrard and Soner in the fundamental paper [43].

Definition 2.6.1. *$B_nV(\Omega, \mathbb{R}^n)$ is the space of functions $u \in W^{1,p}(\Omega, \mathbb{R}^n) \cap L^s(\Omega, \mathbb{R}^n)$ such that Ju is a current of finite mass.*

Notice that in this case the action of Ju can be represented as the integration against a $\Lambda_{m-n}\mathbb{R}^m$-valued Radon measure. Clearly the following statement is an easy improvement of 2.5.2, since every continuous function with compact support can be uniformly approximated by a Lipschitz function with the same L^∞ bound.

Corollary 2.6.2. *Assume the same hypotheses of Proposition 2.5.2. If in addition*
$$(u_h) \subset B_nV(\Omega, \mathbb{R}^n) \qquad \text{and} \qquad \|Ju_h\|(\Omega) \leq C < \infty$$
then $u \in B_nV(\Omega, \mathbb{R}^n)$ and $Ju_h \overset{*}{\rightharpoonup} Ju$ in the sense of measures.

Furthermore it is a general result on normal currents, contained for example in [31, 4.1.21] and in [9, Theorem 3.9] for the metric spaces statement, that if T is a normal k-current then $\|T\| \ll \mathscr{H}^k$. In light of the Example 2.6.3 (with a trivial extension in case $m > n$) $\|Ju\| \ll \mathscr{H}^{m-n}$ is the only possible bound on the Hausdorff dimension of Ju.

As in the theory of BV functions Ju satisfies a canonical decomposition in three mutually singular parts according to the dimensions (see [24, 7, 43]):

$$Ju = \nu \cdot \mathscr{L}^m + J^c u + \theta \cdot \mathscr{H}^{m-n} \llcorner S_u \tag{26}$$

where the decomposition is uniquely determined by these three properties:

- $\nu = \frac{dJu}{d\mathscr{L}^m} \in L^1(\Omega, \Lambda_{m-n}\mathbb{R}^m)$ is the Radon-Nikodym derivative of Ju with respect to $\mathscr{L}^m$;
- $\|J^c u\|(F) = 0$ whenever $\mathscr{H}^{m-n}(F) < \infty$;
- $\theta \in L^1(\Omega, \Lambda_{m-n}\mathbb{R}^m, \mathscr{H}^{m-n})$ is a $\mathscr{H}^{m-n}$-measurable function and $S_u \subset \Omega$ is σ-finite w.r.t. $\mathscr{H}^{m-n}$.

The intermediate measure $J^c u$ is known as the Cantor part of Ju.

Example 2.6.3 (Summability exponent p versus $\dim_{\mathscr{H}} \operatorname{spt}(Ju)$, [50, Theorem 5.1])**.** For every $\alpha \in [0, n]$ there exists a continuous B_nV map

$$u_\alpha \in C^0(\mathbb{R}^n, \mathbb{R}^n) \cap \bigcap_{p<n} W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$$

such that Ju_α is a nonnegative Cantor measure satisfying

$$c\mathscr{H}^\alpha \llcorner \operatorname{spt}(Ju_\alpha) \leq Ju_\alpha \leq C\mathscr{H}^\alpha \llcorner \operatorname{spt}(Ju_\alpha)$$

for some $c, C > 0$. In particular $\operatorname{spt}(Ju)$ has Hausdorff dimension α.

Hence no bound on p is sufficient to constrain the singularity of Ju. Adding $m-n$ dummy variables to the domain the same examples show α can range in the interval $[m-n, m]$ regardless how close p is to n.

It has been proved in [49] and [26] that

$$\nu(x) = M_n \nabla u(x) = e_1 \wedge \cdots \wedge e_m \llcorner du^1(x) \wedge \cdots \wedge du^n(x) \in \Lambda_{m-n}\mathbb{R}^m \tag{27}$$

at $\mathscr{L}^m$-almost every point $x \in \Omega$. The set S_u is unique up to $\mathscr{H}^{m-n}$-negligible sets, and can be characterized by

$$S_u := \left\{ x \in \Omega : \limsup_{\rho \downarrow 0} \frac{\|Ju\|(B_\rho(x))}{\rho^{m-n}} > 0 \right\}.$$

Moreover S_u it has been shown in [24] using some general properties of normal and flat currents that S_u is countably $\mathscr{H}^{m-n}$-rectifiable and that for $\mathscr{H}^{m-n}$-a.e. $x \in S_u$ the multivector $\theta(x)$ is simple and it orients the approximate tangent space $\operatorname{Tan}^{(m-n)}(S_u, x)$.

Definition 2.6.4. *We denote, in analogy with the SBV theory, by SB_nV the set of B_nV functions such that $J^c u = 0$.*

The space SB_nV enjoys a closure property proved in [24]:

Theorem 2.6.5 (Closure Theorem for SB_nV)**.** *Let us consider u, $u_h \in B_nV(\Omega, \mathbb{R}^n)$ and suppose that*

(a) *$u_h \to u$ strongly in $L^s(\Omega, \mathbb{R}^n)$ and $\nabla u_h \rightharpoonup \nabla u$ weakly in $L^p(\Omega, \mathbb{R}^{n\times m})$,*
(b) *if we write*

$$Ju_h = \nu_h \cdot \mathscr{L}^m + \theta \cdot \mathscr{H}^{m-n} \llcorner S_{u_h}$$

then $|\nu_h|$ are equiintegrable in Ω and $\mathscr{H}^{m-n}(S_{u_h}) \leq C < \infty$.

Then $u \in SB_nV(\Omega, \mathbb{R}^n)$ and

$$\nu_h \rightharpoonup \nu \text{ weakly in } L^1(\Omega, \Lambda_{m-n}\mathbb{R}^m), \quad \mathscr{H}^{m-n}(S_u) \leq \liminf_h \mathscr{H}^{m-n}(S_{u_h}).$$

2.7. **Slicing Theorem.** We aim to apply the slicing operation to $Ju \in \mathbf{F}_{m-n}(\Omega)$ in the special case $\ell = m-n$, thus reducing ourselves to 0-dimensional slices; moreover we want to relate these slices to the Jacobian of the restriction $J(u|_{\pi^{-1}(x)})$. In [24], the author extended a classical result on restriction of BV functions (see [7, Section 3.11]) to Jacobians:

Theorem 2.7.1 (Slicing)**.** *Let $u \in W^{1,p} \cap L^s(\Omega, \mathbb{R}^n)$ be a function, and let $\pi \in \mathbf{O}_{m-n}$. Then for $\mathscr{L}^{m-n}$-almost every $x \in \mathbb{R}^{m-n}$*

$$\langle Ju, \pi, x\rangle = (-1)^{(m-n)n} i^x_\#(Ju^x), \tag{28}$$

where $u^x = u \circ i^x$. Moreover $u \in B_nV(\Omega, \mathbb{R}^n)$ if and only if for every $\pi \in \mathbf{O}_{m-n}$ the following two conditions hold:

$$\text{(i)} \qquad u^x \in B_nV(\Omega^x, \mathbb{R}^n) \quad \textit{for} \quad \mathscr{L}^{m-n}\textit{-almost every } x \in \mathbb{R}^{m-n},$$

$$\text{(ii)} \qquad \int_{\pi(\Omega)} |Ju^x|(\Omega^x)\, d\mathscr{L}^{m-n}(x) < \infty.$$

In this case the Distributional Jacobian of the restriction u^x is equal (up to sign) to the slice of Ju at x:

$$\langle Ju, \pi, x\rangle = (-1)^{(m-n)n} i^x_\#(Ju^x), \tag{29}$$

and this slicing property holds separately for the absolutely continuous part, the Cantor part and the Jump part of Ju, namely:

- $\langle J^a u, \pi, x\rangle = (-1)^{(m-n)n} i^x_\#(J^a u^x)$,
- $\langle J^c u, \pi, x\rangle = (-1)^{(m-n)n} i^x_\#(J^c u^x)$,
- $\langle J^s u, \pi, x\rangle = (-1)^{(m-n)n} i^x_\#(J^s u^x)$.

3. Size of a current and a new class of maps

As anticipated in the abstract, we are interested in broadening the class B_nV to include vector valued maps satisfying a weaker control than the mass bound: this lack of control on $\mathbf{M}(Ju)$ already appears in Theorem 2.6.5 when we require *a priori* the limit u to be in B_nV. We relax our energy by considering a mixed control of Ju, where we bound part of the current Ju with its size. In general it is possible to define $\mathbf{S}(T)$ for every flat current $T \in \mathbf{F}_k(\Omega)$, even if T has infinite mass: this size quantity was introduced in [8], borrowing some ideas already used by Hardt and Rivière in [40], Almgren [4], Federer [32], and agrees with the classical notion of size for finite mass currents. For example a polyhedral chain

$$P = \sum_{i=1}^n a_i [\![Q_i]\!]$$

where $a_i \in \mathbb{R}$ and $[\![Q_i]\!]$ are the integration currents over some pairwise disjoint k-polygons Q_i, has mass $\mathbf{M}(P) = \sum_i |a_i| \mathscr{H}^k(Q_i)$ and size $\mathbf{S}(P) = \sum_i \mathscr{H}^k(Q_i)$. The main idea behind the definition is to detect the support of the 0-dimensional slices of T via some $\pi \in \mathbf{O}_k$ and then to optimize the choice of projection π.

Definition 3.0.2 (Size of a flat current, [8, Definition 3.1])**.** *We say that $T \in \mathbf{F}_k(\Omega)$ has finite size if there exists a positive Borel measure μ such that*

$$\begin{aligned} &\mathscr{H}^0 \llcorner \operatorname{spt}(T) \le \mu && \textit{in the case } k = 0,\\ &\mu_{T,\pi} := \int_{\mathbb{R}^k} \mathscr{H}^0 \llcorner \operatorname{spt}\langle T, \pi, x\rangle\, d\mathscr{L}^k(x) \le \mu \quad \forall \pi \in \mathbf{O}_k && \textit{in the case } k \ge 1. \end{aligned} \tag{30}$$

The choice of μ can be optimized by choosing the least upper bound of the family $\{\mu_{T,\pi}\}$ in the lattice of nonnegative measures:

$$\mu_T := \bigvee_{\pi\in\mathbf{O}_k} \mu_{T,\pi} = \bigvee_{\pi\in\mathbf{O}_k} \int_{\mathbb{R}^k} \mathscr{H}^0 \llcorner \operatorname{spt}\langle T,\pi,x\rangle \, d\mathscr{L}^k(x). \tag{31}$$

We set $\mathbf{S}(T) := \mu_T(\Omega)$.

It can be proved (see [8]) that for every flat current with finite size there exists unique (up to null sets) countably $\mathscr{H}^{m-n}$-rectifiable set, denoted $set(T)$, such that

$$\mu_T = \mathscr{H}^{m-n} \llcorner set(T),$$

so that in particular $\mathscr{H}^{m-n}(set(T)) = \mathbf{S}(T)$. A pointwise constructions of $set(T)$ can also be given as follows

$$set(T) := \left\{ x\in\Omega : \limsup_{\rho\downarrow 0} \frac{\mu(B_\rho(x))}{\rho^{m-n}} > 0 \right\}.$$

The following result, which we will not use, holds for a fairly general class of metric spaces and fits naturally in the context of calculus of variations:

Theorem 3.0.3 (Lower semicontinuity of size, [8, Theorem 3.4])**.** *Let $(T_h) \subset \mathbf{F}_k(\Omega)$ be a sequence of currents with equibounded sizes and converging to T in the flat norm:*

$$\mathbf{S}(T_h) \le C < \infty, \qquad \lim_h \mathbf{F}(T_h - T) = 0.$$

Then T has finite size and

$$\mathbf{S}(T) \le \liminf_h \mathbf{S}(T_h). \tag{32}$$

We remark that the definition of size in the metric space contest of [8] is slightly different, since supremum (31) was taken among all 1-Lipschitz maps $\pi \in \mathrm{Lip}_1(\Omega,\mathbb{R}^k)$. However, when the ambient space is Euclidean, the rectifiability and lower semicontinuity results obtained there, as well as the characterization of μ_T in terms of $set(T)$ can be readily proved using only the subset of orthogonal projections.

The space of generalized functions of bounded higher variation is described in terms of the decomposition (26): we relax the requirement on the addendum of lower dimension and require only a size bound, retaining the mass bound on the diffuse part. Following the previous definitions we consider the Sobolev functions u whose jacobian can be split in the sum of two parts, R and T, such that:

- R has finite mass and $\|R\|(F) = 0$ whenever $\mathscr{H}^{m-n}(F) < \infty$;
- T is a flat chain of finite size.

In formulas:

Definition 3.0.4 (Special functions of bounded higher variation)**.** *The space of generalized functions of bounded higher variation is defined by*

$$\begin{aligned} GB_nV(\Omega) = \big\{ u \in W^{1,p}(\Omega,\mathbb{R}^n) \cap L^s(\Omega,\mathbb{R}^n) : Ju = R + T,\, \mathbf{M}(R) + \mathbf{S}(T) < \infty, \\ \|R\|(F) = 0 \,\forall F : \mathscr{H}^{m-n}(F) < \infty \big\}. \end{aligned} \tag{33}$$

Analogously, the space of generalized special functions of bounded higher variation is defined by

$$GSB_nV(\Omega) = \big\{ u \in W^{1,p}(\Omega,\mathbb{R}^n) \cap L^s(\Omega,\mathbb{R}^n) : Ju = R + T,\, \mathbf{M}(R) + \mathbf{S}(T) < \infty,\, \|R\| \ll \mathscr{L}^m \big\}. \tag{34}$$

In accordance with the classical BV theory we denote $S_u := set(T_u)$.

This space is clearly meant to mimic the aforementioned SB_nV class. In particular, thanks to the slicing properties of flat currents and the definition of size, the slicing theorem for $GSB_mV(\Omega)$ can be stated in the following way:

$$u \in GSB_nV(\Omega) \Longleftrightarrow \forall \pi \in \mathbf{O}_{m-n} \quad \begin{cases} u^x \in GSB_nV(\Omega^x), \\ \int_{\pi(\Omega)} \mathbf{M}(R_{u^x}) + \mathbf{S}(T_{u^x})\, d\mathscr{L}^{m-n}(x) < \infty. \end{cases} \tag{35}$$

In the following propositions we describe some useful properties of the class $GSB_nV(\Omega)$.

Lemma 3.0.5. *If $m = n$ then $GSB_nV(\Omega) = SB_nV(\Omega)$.*

Proof. The statement relies on the fact that a flat 0-current of finite size coincides with a finite sum of Dirac masses, and in particular it has finite mass. This property, reminiscent of Schwartz lemma for distributions, has been proved in [8], Theorem 3.3. Therefore the current

$$T = Ju - R$$

has finite mass, hence $\mathbf{M}(Ju) \le \mathbf{M}(R) + \mathbf{M}(T) < \infty$ which means $u \in B_nV(\Omega)$. □

Since the Radon-Nikodym decomposition of a measure into the sum of an absolutely continuous and a singular part is unique, by slicing also R and T are uniquely determined in the decomposition. Therefore we can write $Ju = R_u + T_u$, so that S_u is a well defined set.

A very well known space of functions implemented in the calculus of variations is $GSBV$. The main idea behind this space, introduced in [22] (see also [7, Section 4.5]), is to consider functions u whose derivative Du loses any kind of local integrability, but nevertheless retains some of the structure of SBV functions. Setting $u^N := (-N) \vee u \wedge N$ for every $N > 0$ we define

$$GSBV(\Omega) = \{u : \Omega \to \mathbb{R} \text{ Borel} : \ u^N \in SBV(\Omega) \text{ for all integers } N > 0\}.$$

The countable set of truncation given by $N \in \mathbf{N}$ is enough to provide the existence of an approximate differential $\nabla^* u$ and of a countably $\mathscr{H}^{m-1}$-rectifiable singular set S^*_u such that for every N

$$\|Du^N\| \le |\nabla^* u| \chi_{\{|u| \le N\}} \mathscr{L}^m + 2N \mathscr{H}^{m-1} \llcorner S^*_u.$$

Moreover an analog of the slicing theorem for BV function is available also in $GSBV$, see [7, Proposition 4.35].

Proposition 3.0.6 (Comparison between GSB_1V and $GSBV$)**.** *A function $u \in GSB_1V(\Omega)$ if and only if $u \in GSBV(\Omega)$, $u \in L^1(\Omega)$, $\nabla^* u \in L^1(\Omega, \mathbb{R}^n)$ and $\mathscr{H}^{m-1}(S^*_u) < \infty$.*

Proof. With abuse of notation, motivated by (18) we identify for scalar functions the action of Ju on $\mathscr{D}^{m-1}(\Omega)$ with the action of the distributional derivative Du on $C^\infty_c(\Omega, \mathbb{R}^n)$ (see the map $\mathbf{D}^{m-1}$ in [31, 1.5.2]). Consider first the case $m = 1$. Let $u \in GSB_1V(\Omega)$: writing $R_u = \rho \mathscr{L}^1$ and $T_u = \sum_{k=1}^{\mathbf{S}(T_u)} a_k [\![x_k]\!]$, thanks to (18) we know that for $\omega \in \mathscr{D}^0(\Omega)$

$$\langle Du, \omega \rangle = \int_\Omega \rho \omega \, dx + \sum_{k=1}^{\mathbf{S}(T_u)} a_k \omega(x_k).$$

This proves that $u \in SBV(\Omega)$, $S_u \subset set(T_u)$ and $u'(x) = \rho(x)$ almost everywhere. In particular for $N > 0$ fixed

$$\|Du^N\| \le |\rho| \mathscr{L}^1 + 2N \mathscr{H}^{m-1} \llcorner set(T_u). \tag{36}$$

For $m \ge 2$ the slicing Theorem 2.7.1 applied to a coordinate projection onto a hyperspace implies that almost every slice u^x is in $GSB_1V(\Omega^x)$, hence for every $N > 0$ the estimate (36) holds for u^x. Integrating back we have $\|Du^N\|(\Omega) < \infty$, hence $u \in GSBV(\Omega)$.

On the other if $u \in GSBV(\Omega)\cap L^1(\Omega)$ we know that $Du^N \overset{*}{\rightharpoonup} Du$ in the sense of distributions, and also in the flat norm, since the weak derivative is a distribution of order 1. Moreover $\nabla u^N \to \nabla^* u$ strongly in L^1, hence also in the flat norm. Therefore the jump parts also converge to some flat T_u:

$$D^j u^N \xrightarrow{\mathbf{F}} T_u \in \mathbf{F}_{m-1}(\Omega).$$

Recall that for $v \in BV$ the jump part of the derivative Dv can be expressed in terms of the approximate upper and lower limits $v_\pm$ and of the approximate tangent $(m-1)$-vector τ in the following way:

$$D^j v = (v_+ - v_-)\tau\, \mathscr{H}^{m-1} \llcorner S_v. \tag{37}$$

Hence if $m = 1$ then $\mathscr{H}^0(\mathrm{spt}T_u) \le \lim_N \mathscr{H}^0(S_{u^N}) \le \mathscr{H}^0(S_u^*)$; in the general case can be achieved using the slicing Theorem 2.7.1 and Proposition [7, 4.35]. □

3.1. **Some examples.** The following observation shows that when $n \ge 2$ it is hopeless to rely on truncation to get mass bounds for Ju.

Example 3.1.1 (L^∞ bound for $n \ge 2$)**.** For $n \ge 2$ let $\gamma_k : S^{n-1} \to S^{n-1}$ be a smooth map with degree k, and call u_k its zero homogeneous extension to $\mathbb{R}^n$. Then $\|u_k\|_{L^\infty} \le 1$ but by Example 2.5.3 $Ju = k\mathscr{L}^n(B_1)[\![0]\!]$.

On the contrary for $n = 1$ and $u \in BV(\Omega)$ the approximate upper and lower limits $u_\pm$ of u characterize the singular set: $S_u = \{x \in \Omega : u_+(x) > u_-(x)\}$. Equation (37) implies that an L^∞ bound on u together with a size bound $\mathscr{H}^{m-1}(S_u) < \infty$ gives a mass bound on Du.

We now adapt the construction in 2.5.5, building a map whose jacobian has infinite mass but finite size:

Example 3.1.2 ($\mathbf{S}(Ju) < \infty$ but $\mathbf{M}(Ju) = \infty$)**.** Set $m = n+1$ and let us write (x, y, z) the coordinates of $\mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$. Besides $\nu \in \mathbb{Z}$ and $\rho > 0$ fix an extra parameter $R \ge \rho$. We extend the function $f_{\nu,\rho}(y, z)$ of Example 2.5.4 to $\mathbb{R}^{n+1}$ by

$$h_{\nu,\rho,R}(x, y, z) = \begin{cases} f_{\nu,\rho}\left(\frac{Ry}{R-|x|}, \frac{Rz}{R-|x|}\right) & \text{for} \quad |x| < R, \\ \mathcal{N} & \text{for} \quad |x| \ge R. \end{cases}$$

Clearly $h_{\nu,\rho,R} \ne \mathcal{N}$ in the set $\{|x|/R + |y|/\rho + |z|/\rho < 1\}$; by simmetry we can do the computations in $\{x < 0\}$. Let us estimate the partial derivatives:

$$\begin{aligned} &\left|\frac{\partial h_{\nu,\rho,R}}{\partial x}(x, y, z)\right| \le \tfrac{R(|y|+|z|)}{(R+x)^2}|\nabla f_{\nu,\rho}|\left(\tfrac{Ry}{x+R}, \tfrac{Rz}{x+R}\right) \le \tfrac{\rho}{x+R}|\nabla f_{\nu,\rho}|\left(\tfrac{Ry}{x+R}, \tfrac{Rz}{x+R}\right), \\ &|\nabla_{y,z} h_{\nu,\rho,R}| \le \tfrac{R}{x+R}|\nabla f_{\nu,\rho}|\left(\tfrac{Ry}{x+R}, \tfrac{Rz}{x+R}\right). \end{aligned}$$

Since $\rho \le R$

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} |\nabla h_{\nu,\rho,R}|^p \, dxdydz &\le 2(n+1)\int_{-R}^0 \int_{\{|y|/\rho+|z|/\rho<(x+R)/R\}} \left(\tfrac{R}{x+R}\right)^p |\nabla f_{\nu,\rho}|^p \left(\tfrac{Ry}{x+R}, \tfrac{Rz}{x+R}\right) dydz\, dx \\ &\le 2(n+1)\int_0^R \left(\tfrac{R}{x}\right)^{p-n} \int_{\{|y|+|z|<\rho\}} |\nabla f_{\nu,\rho}(y,z)|^p \, dydz\, dx \\ &\le C_p\, \nu^{\frac{p}{n-1}} \rho^{n-p} R. \end{aligned} \tag{38}$$

Moreover $Jh_{\nu,\rho,R}$ is the integral cycle $\nu \cdot \zeta_\#[\![[0,1]]\!]$, where $\zeta : [0,1] \to \mathbb{R}^{n+1}$ is the following closed curve:

$$\zeta(t) = \begin{cases} (4Rt - R, 0, -4\rho t) & \text{for} \quad t \in [0, \frac{1}{4}], \\ (4Rt - R, 0, 4\rho t - 2\rho) & \text{for} \quad t \in [\frac{1}{4}, \frac{1}{2}], \\ (3R - 4Rt, 0, 4\rho t - 2\rho) & \text{for} \quad t \in [\frac{1}{2}, \frac{3}{4}], \\ (3R - 4Rt, 0, 4\rho - 4\rho t) & \text{for} \quad t \in [\frac{3}{4}, 1]. \end{cases}$$

Since $\mathrm{Lip}(\zeta) \le CR$ we have $\mathbf{M}(Jh_{\nu,\rho,R}) \le C\nu R$ and $\mathbf{S}(Jh_{\nu,\rho,R}) \le CR$. Like in 2.5.4 we glue infinite copies of h_{ν_k,ρ_k,R_k} along the z axis and obtain a map g: the Sobolev norm of g can be estimated by (38):

$$\|\nabla g\|^p_{L^p} \le C \sum_k \nu_k^{\frac{p}{n-1}} \rho_k^{n-p} R_k$$

and

$$\mathbf{M}(Jg) \le C \textstyle\sum_k \nu_k R_k, \qquad \mathbf{S}(Jg) \le C \textstyle\sum_k R_k.$$

Choosing $\nu_k = k$, $R_k = \frac{1}{k^2}$ and $\rho_k = e^{-k}$ we obtain a S^{n-1}-valued $W^{1,p}$ function constant outside a compact set and whose Jacobian has infinite mass but finite size.

3.2. **Ju and approximate differentiability.** We now extend to GSB_nV the pointwise characterization of the absolutely continuous part of Ju.

Proposition 3.2.1 ($Det = det$ in the GSB_nV class)**.** *Let $u \in GSB_nV(\Omega)$ and write $Ju = R + T$ as in Definition 3.0.4. Let ∇u be the approximate differential of u. Then*

$$\frac{dR}{d\mathscr{L}^m} = M_n \nabla u \quad \mathscr{L}^m\text{-almost everywhere in } \Omega. \tag{39}$$

Proof. For the ease of notation let $\nu := \frac{dR}{d\mathscr{L}^m}$. Fix a projection $\pi \in \mathbf{O}_{m-n}$ and let us write the coordinates $z = (x, y)$ accordingly. For a fixed $x \in \mathbb{R}^{m-n}$ we note that the injection i^x and the complementary projection $\pi^\perp|_{\pi^{-1}(x)}$ are one the inverse of the other. Recall the slicing Theorem for general Sobolev functions gives

$$\langle Ju, \pi, x\rangle = (-1)^{(m-n)n} i^x_\#(Ju^x). \tag{40}$$

Taking Lemma 3.0.5 into account, for almost every $x \in \mathbb{R}^{m-n}$ it holds $u^x \in B_nV(\Omega^x)$ and $\mathbf{M}(\langle R, \pi, x\rangle)) + \mathbf{S}(\langle T, \pi, x\rangle) < \infty$, hence (14) gives

$$\langle Ju, \pi, x\rangle = \nu(x, \cdot) \llcorner d\pi \mathscr{H}^n \llcorner \pi^{-1}(x) + \langle T, \pi, x\rangle. \tag{41}$$

Pushing forward (41) via $\pi^\perp$ by (40) it follows that $(-1)^{(m-n)n} Ju^x = \nu(x, \cdot) \llcorner d\pi \mathscr{L}^n + \tilde{T}^x$, with $\tilde{T}^x = \pi^\perp_\# \langle T, \pi, x\rangle$. But the finiteness of the size of $\langle T, \pi, x\rangle$ implies that $\tilde{T}^x$ is a sum of $\mathbf{S}(\langle T, \pi, x\rangle)$ Dirac masses. In particular by equation (27) in the case $m = n$ we know that

$$(-1)^{(m-n)n} \nu(x, \cdot) \llcorner d\pi = \det \nabla_y u(x, \cdot).$$

Using (3) we obtain $\nu(x, \cdot) \llcorner d\pi = M_n \nabla u(x, \cdot) \llcorner d\pi$ for almost every x. We recover the equality (39) by taking orthogonal projections π onto every $(m-n)$-dimensional coordinate subspace. $\square$

It will be useful to extend the result of Proposition 3.2.1 to the lower order determinants: let $u \in GSB_nV(\Omega)$ and $w \in \mathrm{Lip}(\Omega, \mathbb{R}^n)$. We denote by $\Gamma(u, w)$ the sum of the jacobians of the functions obtained by replacing at least one component of u with the respective component of w, but not all of them. More precisely for every $I \subset \{1, \ldots, n\}$ such that $0 < |I| < n$ we construct the function u_I whose components are

$$u_I^k = \begin{cases} u^k \text{ if } k \notin I, \\ w^k \text{ if } k \in I. \end{cases}$$

Then we let $\Gamma(u, w) = \sum_{0<|I|<n} Ju_I$. By the multilinearity of jacobians, it is easy to check that if u is Lipschitz the identity

$$J(u + w) = Ju + \Gamma(u, w) + Jw \tag{42}$$

holds pointwise $\mathscr{L}^m$-a.e. in Ω.

Corollary 3.2.2. *Let $\Omega \subset \mathbb{R}^m$, $w \in \mathrm{Lip}(\Omega, \mathbb{R}^n)$ and $u \in GSB_nV(\Omega)$. Then, in the sense of distributions, it holds*

$$J(u + w) = Ju + \Gamma(u, w) + Jw. \tag{43}$$

Proof. The proof uses the following observation: if $u_h \to u$ in L^s and $\nabla u_h \rightharpoonup \nabla u$ in L^p, then by Reshetnyak's Theorem and the inequality $p > n - 1$ every minor of ∇u of order $k < n$ is weakly continuous in $L^{\frac{p}{k}}$. It follows that $\Gamma(u_h, w) \to \Gamma(u, w)$, so that we can pass to the limit in (42) to obtain (43). □

4. Compactness

Theorem 4.0.3 (Compactness for the class GSB_nV)**.** *Let $s > 0$, $p > 1$ be exponents with $\frac{1}{s} + \frac{n-1}{p} \leq 1$ and let $\Psi : [0, \infty) \to [0, \infty)$ be a convex increasing function satisfying $\lim_{t\to\infty} \Psi(t)/t = \infty$.*
Let $(u_h) \subset GSB_nV(\Omega)$ be such that $u_h \to u$ in $L^s(\Omega, \mathbb{R}^n)$ and $\nabla u_h \rightharpoonup \nabla u$ weakly in $L^p(\Omega, \mathbb{R}^{n\times m})$. Suppose that $Ju_h = R_{u_h} + T_{u_h}$ fulfil

$$K := \sup_h \int_\Omega \Psi\left(\left|\frac{dR_{u_h}}{d\mathscr{L}^m}\right|\right) d\mathscr{L}^m + \mathbf{S}(T_{u_h}) < \infty. \tag{44}$$

Then $u \in GSB_nV(\Omega)$ and, writing $Ju = R_u + T_u$,

$$\frac{dR_{u_h}}{d\mathscr{L}^m} \rightharpoonup \frac{dR_u}{d\mathscr{L}^m} \qquad \text{weakly in } L^1(\Omega, \Lambda_{m-n}\mathbb{R}^m), \tag{45}$$

$$\mathbf{S}(T_u) \leq \liminf_h \mathbf{S}(T_{u_h}). \tag{46}$$

Proof. Without loss of generality we can assume Ψ to have at most a polynomial growth at infinity, for otherwise it is sufficient to take $\tilde{\Psi}(t) := \min\{\Psi(t), t^2\}$. In particular we will use the inequality

$$\Psi(2t) \leq C\Psi(t) \qquad \forall t > 0 \tag{47}$$

(this inequality is known as Δ_2 condition in the literature, see for instance [2, 8.6]). We shorten T_h, R_h in place of T_{u_h} and R_{u_h} respectively and denote by $\rho_h = M_n\nabla u_h$ the densities of R_h with respect to $\mathscr{L}^m$. We know from Proposition 2.5.2 that $Ju_h \to Ju$ in the flat norm. Possibly extracting a subsequence we can assume with no loss of generality that:

(a) the limit $\lim_h \mathbf{S}(T_h)$ exists,
(b) $\rho_h \rightharpoonup \rho$ weakly in $L^1(\Omega, \Lambda_{m-n}\mathbb{R}^m)$,
(c) (u_h) rapidly converges to u in L^s: as a consequence of Proposition 2.5.2 we have also

$$\sum_h \mathbf{F}(Ju_h - Ju) < \infty. \tag{48}$$

Indeed, if we prove the result under these additional assumptions, then we can use the weak compactness of ρ_h in L^1 provided by the Dunford-Pettis Theorem, and the fact that any subsequence admits a further subsequence satisfying (a), (b), (c) to obtain the general statement.

We shall let $R := \rho\mathscr{L}^m$ be the limit current: since the flat and weak convergences in (b) and (c) are stronger than the weak* convergence for currents, putting them together we obtain a flat current $T := Ju - R$ such that

$$T_h \stackrel{*}{\rightharpoonup} T. \tag{49}$$

The proof is divided in three steps: we first address the special case $m = n$, then we use this case and the slicing Theorem (35) to show the lower semicontinuity of size in the second step. The main difficulty is in the third step, where we prove (45), because weak convergence behaves badly under the slicing operation.

Step 1: $m = n$**.** We can apply a very particular case of Blaschke's compactness Theorem [10, 4.4.15] to the sets $\mathrm{spt}(T_h)$, which have equibounded cardinality, to obtain a finite set $N \subset \overline{\Omega}$ and a subsequence $(T_{h'})$ such that $\mathrm{spt}(T_{h'}) \longrightarrow N$ in the sense of Hausdorff convergence. By (49) we immediately obtain that $\mathrm{spt}(T) \subset N \cap \Omega$, hence $\mathbf{S}(T) < \infty$ and $u \in GSB_nV(\Omega)$. In addition, since any point in $\mathrm{spt}T$ is the limit of points in $\mathrm{spt}T_{h'}$ it follows that

$$\mathbf{S}(T) \le \liminf_{h'} \mathbf{S}(T_{h'}) = \lim_h \mathbf{S}(T_h).$$

Finally, since $Ju = R + T$ it must be $T = T_u$, which yields (46), and $R = R_u$, which together with (b) yields (45).

Step 2: $m \ge n$**.** Let us fix $A \subset \Omega$ open, $\pi \in \mathbf{O}_{m-n}$ and $\varepsilon \in (0,1)$: the bound (44), (14) and Fatou's lemma imply that

$$\begin{aligned}
+\infty > K &\ge \liminf_h \left\{ \mu_{T_h}(A) + \varepsilon \int_A \Psi(|\rho_h|) d\mathscr{L}^m \right\} && (50)\\
&\ge \liminf_h \left\{ \mu_{T_h,\pi}(A) + \varepsilon \int_A \Psi(|\rho_h \llcorner d\pi|) d\mathscr{L}^m \right\} && (51)\\
&= \int_{\mathbb{R}^{m-n}} \liminf_h \left[\mathscr{H}^0(A^x \cap \mathrm{spt}(\langle T_h, \pi, x\rangle)) + \varepsilon \int_{A^x} \Psi(|\rho_h \llcorner d\pi|) dy \right] dx \\
&= \int_{\mathbb{R}^{m-n}} \liminf_h \left[\mathscr{H}^0(A^x \cap \mathrm{spt}(T_{u_h^x})) + \varepsilon \int_{A^x} \Psi(|\rho_h^x|) dy \right] dx, && (52)
\end{aligned}$$

with $\rho_h^x := M\nabla u_h(x,\cdot) \llcorner d\pi$. By (50) we can choose for almost every $x \in \mathbb{R}^{m-n}$ a subsequence $h' = h'(x, A)$, possibly depending on x and on the set A, realizing the finite lower limit:

$$\liminf_h \mathscr{H}^0(A^x \cap \mathrm{spt}(T_{u_h^x})) + \varepsilon \int_{A^x} \Psi(|\rho_h^x|) dy.$$

Recall that thanks to (c) $Ju_h^x \xrightarrow{\mathbf{F}} Ju^x$ for almost every x. We can therefore apply step 1 to the sequence $u_h^x \in GSB_nV(\Omega^x)$, which converges rapidly to u^x, to conclude that $u^x \in GSB_nV(\Omega^x)$ and that

$$\begin{aligned}
\mathscr{H}^0(A^x \cap \mathrm{spt}(T_{u^x})) &\le \liminf_{h'} \mathscr{H}^0(A^x \cap \mathrm{spt}(T_{u_{h'}^x})) \\
&\le \liminf_{h'} \mathscr{H}^0(A^x \cap \mathrm{spt}(T_{u_{h'}^x})) + \varepsilon \int_{A^x} \Psi(|\rho_{h'}^x|) dy \\
&= \liminf_h \mathscr{H}^0(A^x \cap \mathrm{spt}(T_{u_h^x})) + \varepsilon \int_{A^x} \Psi(|\rho_h^x|) dy. && (53)
\end{aligned}$$

Integrating in x and applying (51) as well as the monotonicity of Ψ we entail

$$\mu_{T,\pi}(A) \le \liminf_h \left\{ \mu_{T_h,\pi}(A) + \varepsilon \int_A \Psi(|\rho_h|) d\mathscr{L}^m \right\} =: \eta_\varepsilon(A). \tag{54}$$

The map $A \mapsto \eta_\varepsilon(A)$ is a finitely superadditive set-function, with $\eta_\varepsilon(\Omega) \le \liminf_h \mathbf{S}(T_{u_h}) + K\varepsilon$. Therefore if $B_1, \dots, B_N$ are pairwise disjoint Borel sets and $K_i \subset B_i$ are compact, we can find pairwise disjoint open sets A_i containing K_i and apply the superadditivity to get

$$\sum_{i=1}^N \mu_{T,\pi_i}(K_i) \le \sum_{i=1}^N \eta_\varepsilon(A_i) \le \eta_\varepsilon(\Omega).$$

Since K_i are arbitrary, the same inequality holds with B_i in place of K_i; since also B_i, π_i and N are arbitrary, it follows that μ_T is a finite Borel measure and $\mu_T(\Omega) \le \eta_\varepsilon(\Omega)$. Hence $u \in GSB_nV(\Omega)$ because $Ju = R + T$, $\mathbf{S}(T) < \infty$ and R is an absolutely continuous measure. Letting $\varepsilon \downarrow 0$ we also prove (46). For later purposes we notice that we proved

$$\mu_{T_u}(A) \le \liminf_h \mu_{T_{u_h}}(A). \tag{55}$$

Step 3: proof of (45). In order to prove (45), since the space $\Lambda_{m-n}\mathbb{R}^m$ is finite dimensional, we will prove that

$$\rho_h \llcorner d\pi \rightharpoonup M_n \nabla u \llcorner d\pi \qquad \text{weakly in } L^1(\Omega, \Lambda_{m-n}\mathbb{R}^m) \tag{56}$$

for every orthogonal projection π onto a coordinate subspace. We fix an open $A \subset \Omega$ and $a \in \mathbb{R}$. From now on $w : A \to \mathbb{R}^n$ will be an affine map such that

$$\nabla_x w = 0, \qquad \det(\nabla_y w) = a.$$

Let us compute $J(u_h + w)$: thanks to Corollary 3.2.2 we get

$$J(u_h + w) = Ju_h + \Gamma(u_h, w) + a\mathbf{E}^m \llcorner d\pi.$$

We are now ready to prove the last part of the Theorem. We argue as in step 2, but this time we change the form of the energy and we analyse the convergence of a perturbed sequence of maps. First of all we note that the sequence

$$\int_A \Psi\left(\left|\rho_h \llcorner d\pi + a\right|\right) d\mathscr{L}^m + \varepsilon\mu_{T_h,\pi}(A) + \varepsilon \int_A |\nabla u_h|^p d\mathscr{L}^m \tag{57}$$

is still bounded from above, because $|\alpha + \beta|^p \le 2^{p-1}(|\alpha|^p + |\beta|^p)$, (47) and the convexity of Ψ imply that

$$\int_A \Psi\left(\left|\rho_h \llcorner d\pi + a\right|\right) d\mathscr{L}^m \le \frac{C}{2} \int_A \Psi\left(\left|\rho_h\right|\right) d\mathscr{L}^m + \frac{C}{2}\Psi(|a|)\mathscr{L}^m(A) \le \frac{C}{2}\big(K + \Psi(|a|)\mathscr{L}^m(A)\big).$$

We consider the sequence $(u_h + w) \subset GSB_nV(A)$ and the perturbed energy (57): arguing as in the chain of inequalities (50)-(52) for almost every x we can find a suitable subsequence $h' = h'(x, A)$ realizing the finite lower limit of the sliced energies

$$\int_{A^x} \Psi\left(\left|\rho_h^x + a\right|\right) dy + \varepsilon\mathscr{H}^0(A^x \cap \mathrm{spt}(T_{u_h^x})) + \varepsilon \int_{A^x} |\nabla u_h^x|^p dy. \tag{58}$$

Since Ψ is superlinear at infinity, up to subsequences the densities $\rho_{h'}^x + a$ weakly converge to some function r^x in $L^1(A^x)$: in particular the associated currents weak* converge

$$(\rho_{h'}^x + a)\mathbf{E}^n \llcorner A^x \stackrel{*}{\rightharpoonup} r^x \mathbf{E}^n \llcorner A^x. \tag{59}$$

Thanks to the fast convergence (c) we also know that $u^x \to u$ in $L^s(A^x)$; moreover the boundedness of the Dirichlet term in (58) implies also that $\nabla_y u_{h'}^x \rightharpoonup \nabla u^x$ in $L^p(A^x, \mathbb{R}^n)$, hence by step 2 we get

$$u^x \in GSB_nV(A^x) \qquad \text{and} \qquad T_{u_{h'}^x} \stackrel{*}{\rightharpoonup} T_{u^x}.$$

The weak convergence of the gradients in L^p also allows to use the continuity property of $\Gamma(\cdot, w^x)$ along the sequence of restrictions $(u^x_{h'})$ and deduce that

$$(\rho^x_{h'} + a)\mathbf{E}^n \llcorner A^x = J(u^x_{h'} + w^x) - \Gamma(u^x_{h'}, w^x) - T_{u^x_{h'}} \overset{*}{\rightharpoonup} J(u^x + w^x) - \Gamma(u^x, w^x) - T_{u^x}$$

in the sense of distributions. By Corollary 3.2.2 and Proposition 3.2.1 we are able to identify the weak limit in (59)

$$r^x = \det \nabla_y u^x + a = M_n \nabla u(x, \cdot) \llcorner d\pi + a. \tag{60}$$

We fix a a convex increasing function with superlinear growth φ satisfying

$$\lim_{t\to+\infty} \frac{\Psi(t)}{\varphi(t)} = +\infty. \tag{61}$$

Using the previous convergence (59), (60) on almost every slice and integrating with respect to x we deduce by the convexity of φ that

$$\int_A \varphi\left(\left|M_n \nabla u \llcorner d\pi + a\right|\right) d\mathscr{L}^m \le \liminf_h \int_A \varphi(|\rho_h \llcorner d\pi + a|) d\mathscr{L}^m + \varepsilon \mu_{T_h,\pi}(A) + \varepsilon \int_A |\nabla u_h|^p d\mathscr{L}^m.$$

Adding this inequality on a finite number of disjoint open subsets A_j, with arbitrary choices of $a_i \in \mathbb{R}$, we obtain

$$\int_\Omega \varphi\left(\left|M_n \nabla u \llcorner d\pi + \xi\right|\right) d\mathscr{L}^m \le \liminf_h \int_\Omega \varphi(|\rho_h \llcorner d\pi + \xi|) d\mathscr{L}^m + \varepsilon \mathbf{S}(T_h) + \varepsilon \int_\Omega |\nabla u_h|^p d\mathscr{L}^m,$$

where $\xi := \sum_j a_j \chi_{A_j}$. Letting $\varepsilon \downarrow 0$ we can disregard the size and Dirichlet terms in the last inequality to get

$$\int_\Omega \varphi\left(\left|M_n \nabla u \llcorner d\pi + \xi\right|\right) d\mathscr{L}^m \le \liminf_h \int_\Omega \varphi(|\rho_h \llcorner d\pi + \xi|) d\mathscr{L}^m. \tag{62}$$

Taking $\varphi_n(t) := \frac{\varphi(t)}{n} \vee t$, we have that φ_n are still convex, increasing, superlinear at infinity and satisfy (61), therefore (62) is applicable with $\varphi = \varphi_n$. Given $\delta > 0$ fix C_δ such that $\varphi_1(t) \le \delta \Psi(t)$ for $t > C_\delta$; we also let $\Omega_{h,\delta} = \left\{|\rho_h \llcorner d\pi + \xi| > C_\delta\right\}$. By applying (62) with $\varphi = \varphi_n$ we have therefore

$$\begin{aligned}
\int_\Omega \left|M_n \nabla u \llcorner d\pi + \xi\right| d\mathscr{L}^m &\le \int_\Omega \varphi_n\left(\left|M_n \nabla u \llcorner d\pi + \xi\right|\right) d\mathscr{L}^m \le \liminf_h \int_\Omega \varphi_n(|\rho_h \llcorner d\pi + \xi|) d\mathscr{L}^m \\
&\le \liminf_h \int_\Omega \left|\rho_h \llcorner d\pi + \xi\right| + \limsup_h \int_{\Omega_{h,\delta}} \varphi_1\left(\left|\rho_h \llcorner d\pi + \xi\right|\right) + \sup_{0\le t\le C_\delta} \{\varphi_n(t) - t\} \mathscr{L}^m(\Omega^c_{h,\delta}) \\
&\le \liminf_h \int_\Omega \left|\rho_h \llcorner d\pi + \xi\right| + \limsup_h \delta \int_{\Omega_{h,\delta}} \Psi\left(\left|\rho_h \llcorner d\pi + \xi\right|\right) + \sup_{0\le t\le C_\delta} \{\varphi_n(t) - t\} \mathscr{L}^m(\Omega).
\end{aligned}$$

Letting $n \to \infty$ the third term vanishes because $\varphi_n(t) \downarrow t$ uniformly on compact sets. Eventually, sending $\delta \downarrow 0$ we obtain

$$\int_\Omega \left|M_n \nabla u \llcorner d\pi + \xi\right| \le \liminf_h \int_\Omega \left|\rho_h \llcorner d\pi + \xi\right|. \tag{63}$$

Inequality (63) is actually valid for every $\xi \in L^1(\Omega)$ by approximation, since the set of functions of type $\sum_j a_j \chi_{A_j}$ is dense in L^1. We therefore address the last point (56) thanks to Lemma 4.0.4 below: the weak limit ρ must be the equal to $M_n \nabla u$, the density of R_u with respect to $\mathscr{L}^m$. $\square$

Lemma 4.0.4. *Let $(z_h) \subset L^1(\Omega)$ be a weakly compact sequence and suppose that, for some $z \in L^1(\Omega)$, it holds*

$$\int_\Omega |z + \xi| \, d\mathscr{L}^m \le \liminf_h \int_\Omega |z_h + \xi| \, d\mathscr{L}^m \qquad \forall \xi \in L^1(\Omega).$$

Then $z_h \rightharpoonup z$ weakly in $L^1(\Omega)$.

We refer to [5] for the proof.

5. Applications

We now present an application of Theorem 4.0.3 to a minimization problem. The choice of our Lagrangian is motivated by the introduction of a new functional of the calculus of variation, presented in 5.2, aiming to generalize the classical Mumford-Shah energy [52, 23, 7] to vector valued maps with singular set of codimension at least 2. The discussion in the introduction already mentioned the central role of the distributional jacobian in relation to low dimensional singularities: in this model we replace the singularities of the derivative by the singularities of the jacobian and we measure them with the size functional of section 3.

5.1. **Existence result for general Lagrangians.** We fix an open, regular and bounded subset Ω of $\mathbb{R}^m$. For approximately differentiable maps $u:\Omega\to\mathbb{R}^n$ we let $M\nabla u$ be the vector of all minors of $k\times k$ submatrices of ∇u, with k ranging from 1 to n and we let $\kappa=\sum_{k=1}^n\binom{m}{k}\binom{n}{k}$ be its dimension. Given $w\in\mathbb{R}^\kappa$ we let w_ℓ the variables relative to the $\ell\times\ell$ minors. We also denote $\mathscr{L}_m$ the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^m$ and $\mathscr{B}(\mathbb{R}^{n+\kappa})$ the σ-algebra of Borel subsets of $\mathbb{R}^{n+\kappa}$. For the bulk part of the energy it is natural to treat polyconvex Lagrangians: the lower semicontinuity properties of such energies with respect to the weak $W^{1,p}$ convergence for $p<n$ has been thoroughly studied, see [18, 36, 35, 44].

Theorem 5.1.1 (Existence of minimizers for polyconvex Lagrangians)**.** *Assume r,p satisfy $r<\infty$, $\frac{1}{r}+\frac{n-1}{p}<1$ and let $c>0$ be a positive constant. Let $f:\Omega\times\mathbb{R}^n\times\mathbb{R}^\kappa\to[0,+\infty)$ satisfy the following hypotheses:*

(a) *f is $\mathscr{L}_m\times\mathscr{B}(\mathbb{R}^{n+\kappa})$-measurable;*
(b) *for $\mathscr{L}^m$-a.e. $x\in\Omega$, $(u,w)\mapsto f(x,u,w)$ is lower semicontinuous;*
(c) *for $\mathscr{L}^m$-a.e. $x\in\Omega$, $w\mapsto f(x,u,w)$ is convex in $\mathbb{R}^\kappa$ for every $u\in\mathbb{R}^n$;*
(d) *$f(x,u,w)\ge c\big(|u|^r+|w_1|^p+\Psi(|w_n|)\big)$ for some function Ψ satisfying the hypotheses of Theorem 4.0.3.*

Let also $g:\Omega\to[c,+\infty)$ be a lower semicontinuous function.
Then, for every $u_0\in W^{1-\frac{1}{p},p}(\partial\Omega,\mathbb{R}^n)$ there exists a solution to the problem

$$\min_{u\in GSB_nV(\Omega),\,u=u_0\ \text{on}\ \partial\Omega}\left\{\int_\Omega f\big(x,u(x),M\nabla u(x)\big)\,dx+\int_{\Omega\cap S_u}g(x)\,d\mathscr{H}^{m-n}(x)\right\}. \tag{P}$$

Proof. Suppose the energy (P) is finite for some function in $GSB_nV(\Omega)$ with trace u_0, otherwise there is nothing to prove. Pick a minimizing sequence (u_h): by the growth assumption (d) there exist $u\in L^r\cap W^{1,p}$ and a subsequence (not relabeled) such that

$$u_h\to u \text{ in } L^1 \tag{64}$$

and $\nabla u_h\rightharpoonup\nabla u$ in L^p. Since $\frac{1}{r}+\frac{n-1}{p}<1$ we can choose $s<r$ such that $\frac{1}{s}+\frac{n-1}{p}\le 1$: by Chebycheff's inequality we know that $u_h\to u$ in L^s. Moreover we know that the absolutely continuous parts of Ju_h satisfy

$$\int_\Omega\Psi\left(\big|M_n\nabla u_h\big|\right)\,dx\le C,$$

and that by the lower bound $g\ge c$ we also have:

$$\sup_h\mathscr{H}^{m-n}(S_{u_h}\cap\Omega)<\infty.$$

Hence by the compactness Theorem 4.0.3, together with the classical Reshetnyak's Theorem for the minors of order less than n, we know that

$$M\nabla u_h\rightharpoonup M\nabla u\quad\text{ weakly in } L^1. \tag{65}$$

By (64) and (65) the lower semicontinuity result of Ioffe [41, 42] (see also [7, Theorem 5.8]) implies

$$\liminf_h \int_\Omega f\big(x, u_h(x), M\nabla u_h(x)\big)\,dx \geq \int_\Omega f\big(x, u(x), M\nabla u(x)\big)\,dx.$$

Finally, g being lower semicontinuous, the superlevel sets $\{g > t\}$ are open, hence

$$\begin{aligned}
\liminf_h \int_{\Omega\cap S_{u_h}} g(x)\,d\mathscr{H}^{m-n}(x) &= \liminf_h \int_0^{+\infty} \mathscr{H}^{m-n}(S_{u_h}\cap\{g>t\})\,dt \\
&\geq \int_0^{+\infty} \liminf_h \mathscr{H}^{m-n}(S_{u_h}\cap\{g>t\})\,dt \\
&\geq \int_0^{+\infty} \mathscr{H}^{m-n}(S_u\cap\{g>t\})\,dt \\
&= \int_{\Omega\cap S_u} g(x)\,d\mathscr{H}^{m-n}(x),
\end{aligned}$$

because the size is lower semicontinuous on open sets, see (55). $\square$

Recall that by the Sobolev embedding we can drop the growth condition on u provided $p > \frac{mn}{m+1}$. Notice also that we can formulate problem (P) and the corresponding boundary value condition in a slightly different way, in order to include in the energy the possible appearance of singularities at the boundary. Let $U \Supset \Omega$ be a bounded open subset of $\mathbb{R}^m$: we formulate the minimization problem in the following way:

$$\min_{u\in GSB_nV(U),\, u=u_0 \text{ in } U\setminus\Omega} \left\{ \int_U f\big(x, u(x), M\nabla u(x)\big)\,dx + \int_{U\cap S_u} g(x)\,d\mathscr{H}^{m-n}(x) \right\} \tag{P'}$$

Every competitor being equal to u_0 in $U \setminus \overline{\Omega}$, problem (P') accounts for variations of Ju in the closure $\overline{\Omega}$. Moreover Theorem 5.1.1 readily applies to this case, as the condition $u = u_0$ in $U \setminus \Omega$ is closed for the strong L^1 convergence. To explicit the dependence on the energy on the datum u_0 and on the domain U we adopt in the sequel the notation

$$F(u, \Omega; u_0, U)$$

for the energy in (P').

5.2. **A Mumford-Shah functional of codimension higher than one.** As anticipated in the beginning of the section the study of general functionals of the form (P) was modeled on the Mumford-Shah type functional

$$MS(u,\Omega) := \int_\Omega |u|^r + |\nabla u|^p + |M_n\nabla u|^\gamma\,dx + \mathscr{H}^{m-n}(S_u\cap\Omega) \tag{66}$$

defined on $GSB_nV(\Omega)$, with r, p satisfying (15), $\gamma > 1$, together with suitable boundary data. Theorem 5.1.1 shows the existence of minimizers of (66) for both Dirichlet problems (P) and (P'): it is however desirable that at least for some boundary datum u_0 the minimizer presents some singularity. In the next proposition we show that this is the case:

Proposition 5.2.1 (Nontrivial minimizers for MS, formulation (P'))**.** *Let $m = n$ and $u_0 : B_2 \to \mathbb{R}^n$ be the identity: $u_0(x) = x$. Then for ε sufficiently small every minimizer $u \in GSB_nV(B_2)$ of*

$$MS_\varepsilon(u, B_1; x, B_2) := \int_{B_2} \varepsilon\big(|u|^r + |\nabla u|^p\big) + |\det\nabla u|^\gamma\,dx + \varepsilon\mathscr{H}^0(S_u\cap B_2)$$

such that $u(x) = x$ in $B_2 \setminus B_1$ must satisfy

$$S_u \cap \overline{B_1} \neq \emptyset.$$

Proof. We show that for every competitor v with $\|Jv\| \ll \mathscr{L}^n$ and for ε small enough it holds:

$$MS_\varepsilon(v, B_1; x, B_2) > MS_\varepsilon(w, B_1; x, B_2),$$

where

$$w(x) = \begin{cases} \frac{x}{|x|} & \text{in } B_1, \\ x & \text{in } B_2 \setminus B_1. \end{cases}$$

For the rest of the proof c will denote a generic positive constant we do not keep track of. Let us compute the energy of $\frac{x}{|x|}$: the Dirichlet and L^r parts are simply constants. Moreover

$$\det \nabla w = \chi_{B_2 \setminus B_1} \quad \text{and} \quad S_w = \{0\}.$$

Hence $MS_\varepsilon(w, B_1; x, B_2) = c\varepsilon + \mathscr{L}^n(B_2 \setminus B_1)$. On the contrary for almost every radius ρ it holds:

$$\int_{B_\rho} \det \nabla v \, dx = \int_{\partial B_\rho} v^1 dv^2 \wedge \cdots \wedge dv^n$$

(see [26, Lemma 2.1] for a simple proof of this fact). Since $u(x) = x$ outside B_1 for almost every $\rho \in (1, 2)$ we have $\int_{B_\rho} \det \nabla v \, dx = \mathscr{L}^n(B_\rho)$, hence by Jensen's inequality

$$\int_{B_\rho} |\det \nabla v|^\gamma \, dx \geq \mathscr{L}^n(B_\rho).$$

Summing up:

$$MS_\varepsilon(v, B_1; x, B_2) \geq \int_{B_\rho} |\det \nabla v|^\gamma \, dx \geq \mathscr{L}^n(B_\rho) > c\varepsilon + \mathscr{L}^n(B_2 \setminus B_1) = MS_\varepsilon(w, B_1; x, B_2)$$

choosing first ρ sufficiently close to 2 and then ε sufficiently small. Therefore the minimizer u must have a nonempty singular set S_u, and since u is linear in the open set $B_2 \setminus B_1$, the singularity must be in $\overline{B_1}$. □

It is easy to generalize the same proposition to the case $m \geq n$, by simply taking the trivial extension in the extra variables and showing that every minimizer has a nontrivial singular set. In analogy with [39], we expect however the singularities to appear in the interior.

The argument in Proposition 5.2.1 essentially exploits the presence of the jacobian term: this is not coincidental, as the next proposition shows. Recall that the sum of a GSB_nV function and a C^1 function is again in GSB_nV.

Proposition 5.2.2. *Every local minimizer of*

$$u \in GSB_nV(\Omega) \mapsto \int_\Omega |\nabla u|^p \, dx + \mathscr{H}^{m-n}(S_u \cap \Omega)$$

is locally of class $C^{1,\alpha}$ in Ω.

Proof. It is sufficient to perform an outer variation of the minimizer u along a $\phi \in C^1_c(\Omega, \mathbb{R}^n)$ map: $\varepsilon \mapsto u + \varepsilon\phi$ and apply Corollary 3.2.2 to obtain that

$$S_{u+\varepsilon\phi} = S_u.$$

Hence the size term is constant and u satisfies:

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi \, dx = 0 \quad \forall \phi \in C^1_c(\Omega, \mathbb{R}^n).$$

Therefore u is a p-harmonic $W^{1,p}$ function, hence $u \in C^{1,\alpha}_{\text{loc}}$ by [56, 30]. □

5.3. **Traces.** In the spirit of solving (P'), the nonuniqueness Example 5.3.1 below raises the problem of the dependence of the energy on the extension $u_0 : U \setminus \Omega \to \mathbb{R}^n$ to a given Sobolev trace $u|_{\partial\Omega}$. The example was communicated to us by C. De Lellis, see also Examples 1 and 2 in Section 3.2.5 of [38], and the discussion on weak and strong anchorage condition therein. It shows that if we want to detect the presence of singularities of Ju at the boundary of Ω, the Sobolev trace is not sufficient to characterize it.

Example 5.3.1 (Singularity at the boundary)**.** Let $u : \mathbb{R}^2 \to S^1$ be defined by

$$u(x,y) = \left(\frac{y^2 - (x-1)^2}{(x-1)^2 + y^2}, \frac{2(1-x)y}{(x-1)^2 + y^2} \right). \tag{67}$$

This map represents the normal unit vectorfield of the family of circles centered on the real axis and tangent to S^1 in the point $(1,0)$. If θ is the angle that the vector $(x-1,y)$ makes with the real axis, we can write $u(x,y) = (-\cos(2\theta), -\sin(2\theta))$, hence by Example (2.5.3) $Ju = 2\pi[\![(1,0)]\!]$. Note that u is the identity map when restricted to S^1. Nonetheless we can construct another map $\tilde{u}$

$$\tilde{u}(x,y) = \begin{cases} u(x,y) & \text{for} \quad |x| < 1, \\ \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right) & \text{for} \quad |x| \geq 1. \end{cases} \tag{68}$$

In this case, by Example 2.5.3, $J\tilde{u} = \pi[\![(1,0)]\!]$. Hence $u|_{B_1}$ admits two different Sobolev extensions u and $\tilde{u}$ sharing the same trace at the boundary but whose jacobians are different in $\overline{\Omega}$: the trace of a Sobolev function does not characterize the jacobian $Jv \llcorner \partial\Omega$ of all the possible extensions v.

It is interesting to know when part of the distributional jacobian can be represented as a boundary integral. Recall that the slicing Theorem 2.7.1 already provides an answer to this question, because if $u : \Omega \to \mathbb{R}^n$ then $\partial(j(u) \llcorner \{\pi > t\}) = Ju \llcorner \{\pi > t\} + \langle j(u), \pi, t\rangle$, where π is the distance from $\partial\Omega$. However, as Example 5.3.1 shows, this statement holds only for $\mathscr{L}^1$-a.e. t. The following proposition improves the general result by slicing, under additional hypotheses on the summability of u and of its trace. This result is already present in the literature, see [38, Vol. I, p. 274] and [3, Lemma 6.1]: we report the proof for the reader's convenience. Denote for simplicity $g(u) := u^1 du^2 \wedge \cdots \wedge du^n$.

Proposition 5.3.2 (Stokes' Theorem)**.** *If $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ and $u|_{\partial\Omega} \in W^{1,n-1}(\partial\Omega, \mathbb{R}^n) \cap L^\infty(\partial\Omega, \mathbb{R}^n)$ then Stokes' theorem holds:*

$$\partial(j(u) \llcorner \Omega) = Ju \llcorner \Omega + \langle j(u), \partial\Omega\rangle$$

with the representation

$$\langle j(u), \partial\Omega\rangle(\omega) = \int_{\partial\Omega} \langle g(u), \tau_{\partial\Omega}\rangle \omega \, d\mathscr{H}^{n-1}$$

where $\tau_{\partial\Omega}$ orients $\partial\Omega$ as the boundary of Ω. In particular $\langle j(u), \partial\Omega\rangle$ depends only on the trace $u|_{\partial\Omega}$.

Proof. Suppose for simplicity that $\Omega = \mathbb{R}^n_+ = \mathbb{R}^n \cap \{x^n > 0\}$, $\mathrm{spt}(u) \subset B_1$ and let $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a positive convolution kernel with compact support in $\mathbb{R}^{n-1}$. Set

$$u_\varepsilon(x', x^n) = \frac{1}{\varepsilon^{n-1}} \int_{\mathbb{R}^{n-1}} u(x' - y', x^n) \phi\left(\frac{x' - y'}{\varepsilon} \right) dy' :$$

since the convolution in the x' variables commutes with the trace operator we still have $u_\varepsilon|_{\mathbb{R}^{n-1}}(x') = u_\varepsilon(x', 0)$; moreover $u_\varepsilon(\cdot, 0) \in C^1(\mathbb{R}^{n-1}, \mathbb{R}^n)$ and the following estimates hold:

$$\|u_\varepsilon\|_{W^{1,n}(\mathbb{R}^n_+, \mathbb{R}^n)} \leq \|u\|_{W^{1,n}(\mathbb{R}^n_+, \mathbb{R}^n)}, \qquad \|u_\varepsilon(\cdot, 0)\|_{W^{1,n-1}(\mathbb{R}^{n-1}, \mathbb{R}^n)} \leq \|u(\cdot, 0)\|_{W^{1,n-1}(\mathbb{R}^{n-1}, \mathbb{R}^n)} \tag{69}$$

and since the translations are strongly continuous in L^p,

$$\|u_\varepsilon - u\|_{W^{1,n}(\mathbb{R}^n_+, \mathbb{R}^n)} + \|u_\varepsilon(\cdot, 0) - u(\cdot, 0)\|_{W^{1,n-1}(\mathbb{R}^{n-1}, \mathbb{R}^n)} \to 0. \tag{70}$$

We claim that Stokes' Theorem holds for u_ε: for every $\omega \in \mathscr{D}^0(\mathbb{R}^n)$

$$\partial(j(u_\varepsilon) \llcorner \mathbb{R}^n_+)(\omega) = \int_{\mathbb{R}^n_+} \omega \det \nabla u_\varepsilon \, dx + \int_{\mathbb{R}^{n-1}\times\{0\}} \omega g(u_\varepsilon(\cdot, 0)). \tag{71}$$

In fact extending $u_\varepsilon(x', x^n) := u_\varepsilon(x', 0)$ for $x^n \in [-1, 0]$ and then convolving with a smooth kernel ρ_δ supported in $B_\delta(0)$ we obtain a smooth $u_{\varepsilon,\delta} \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{spt}(u_{\varepsilon,\delta}) \subset B_2 \times [-2, 2]$,

$$\begin{aligned} u_{\varepsilon,\delta}(x', x^n) &\to u_\varepsilon(x', 0) && \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^n_-, \mathbb{R}^n), \\ u_{\varepsilon,\delta} &\to u_\varepsilon && \text{in } W^{1,n}_{\mathrm{loc}}(\mathbb{R}^{n-1} \times (-1, +\infty), \mathbb{R}^n). \end{aligned} \tag{72}$$

More precisely it holds: $u_{\varepsilon,\delta}(x', -\delta) \to u_\varepsilon(x', 0)$ in $C^1(\mathbb{R}^{n-1}, \mathbb{R}^n)$. Hence

$$\partial(j(u_{\varepsilon,\delta}) \llcorner \{x^n > -\delta\})(\omega) = \int_{\{x^n > -\delta\}} \omega \det \nabla u_{\varepsilon,\delta} \, dx + \int_{\mathbb{R}^{n-1}\times\{-\delta\}} \omega g(u_{\varepsilon,\delta}(\cdot, -\delta)) :$$

letting $\delta \downarrow 0$ the left hand side converges to $\partial(j(u_\varepsilon) \llcorner \mathbb{R}^n_+)(\omega)$ by (72). The boundary term in right hand side tends to

$$\int_{\mathbb{R}^{n-1}\times\{0\}} \omega(\cdot, 0) g(u_\varepsilon(\cdot, 0))$$

because the convergence is C^1 and ω is smooth. Regarding the volume integral we can estimate

$$|\nabla u_{\varepsilon,\delta}(x)| = |(\rho_\delta * \nabla u_\varepsilon)(x)| \le \|u_\varepsilon(\cdot, 0)\|_{C^1} + \fint_{B_\delta(x)\cap\{y^n > 0\}} |\nabla u_\varepsilon(y)| \, dy$$

hence

$$\begin{aligned} \left| \int_{\{|x^n| < \delta\}} \omega \det \nabla u_{\varepsilon,\delta} \, dx \right| &\le \|\omega\|_{C^0} \int_{\{|x^n| < \delta\}} |\nabla u_{\varepsilon,\delta}|^n \, dx \\ &\le c_n \|\omega\|_{C^0} \left(\|u_\varepsilon(\cdot, 0)\|^n_{C^1} \delta + \int_{\{|x^n| < \delta\}} \fint_{B_\delta(x)\cap\{y^n > 0\}} |\nabla u_\varepsilon(y)|^n \, dy \, dx \right) \\ &\le c_n \|\omega\|_{C^0} \left(\|u_\varepsilon(\cdot, 0)\|^n_{C^1} \delta + \int_{\{0 < x^n < 2\delta\}} |\nabla u_\varepsilon(x)|^n \, dx \right) \to 0. \end{aligned}$$

Clearly $\int_{\{x^n > \delta\}} \omega \det \nabla u_{\varepsilon,\delta} \, dx \to \int_{\{x^n > 0\}} \omega \det \nabla u_\varepsilon \, dx$, therefore (71) is true.

We now want to pass to the limit for $\varepsilon \downarrow 0$ in (71). The left hand side goes to $\partial(j(u) \llcorner \mathbb{R}^n_+)(\omega)$ because of (70); similarly for the volume term. Regarding the boundary term the convergence of the minors on the slice needs to be improved. The estimates (69) and the classical result [19] gives a uniform bound of the Hardy norm [54, Chapter IV] of the minors of order $n-1$:

$$\|du^2_\varepsilon(\cdot, 0) \wedge \cdots \wedge du^n_\varepsilon(\cdot, 0)\|_{\mathcal{H}^1(\mathbb{R}^{n-1}, \mathbb{R}^n)} \le \|u(\cdot, 0)\|^{n-1}_{W^{1,n-1}(\mathbb{R}^{n-1}, \mathbb{R}^n)}.$$

We already know from Reshetnyak's Theorem that $du^2_\varepsilon(\cdot, 0) \wedge \cdots \wedge du^n_\varepsilon(\cdot, 0) \overset{*}{\rightharpoonup} du^2(\cdot, 0) \wedge \cdots \wedge du^n(\cdot, 0)$ in the sense of distributions; moreover smooth functions are dense in $VMO(\mathbb{R}^{n-1}, \mathbb{R}^n)$ and $VMO^* = \mathcal{H}^1$, so

$$du^2_\varepsilon(\cdot, 0) \wedge \cdots \wedge du^n_\varepsilon(\cdot, 0) \overset{*}{\rightharpoonup} du^2(\cdot, 0) \wedge \cdots \wedge du^n(\cdot, 0) \qquad \text{in} \quad \sigma(\mathcal{H}^1, VMO).$$

Finally the trace $u_\varepsilon(\cdot, 0)$ belongs to

$$W^{1-\frac{1}{n}, n}(\mathbb{R}^{n-1}, \mathbb{R}^n) \subset VMO(\mathbb{R}^{n-1}, \mathbb{R}^n)$$

(see [1, Theorem 7.58], [17, Example 2] for the inclusions). Hence $\|u_\varepsilon(\cdot,0)-u(\cdot,0)\|_{VMO}\to 0$ strongly and we can pass to the limit in (71)

$$\int_{\mathbb{R}^{n-1}}\omega g(u_\varepsilon(\cdot,0))\to\int_{\mathbb{R}^{n-1}}\omega g(u(\cdot,0)).$$

By (70) also the left hand side of (71) converges to $\int_{\mathbb{R}^n_+}\omega\det\nabla u\,dx$. □

In the example above the smooth extension $\tilde{u}$ is certainly preferable to u, where an "extra" singularity comes from the outside. A partial answer to this problem can be given if we assume a better differentiability of the outer extension, up to $\partial\Omega$:

Proposition 5.3.3 (See [38, Vol. I, p.266]). *Let $v,w\in L^s\cap W^{1,p}(U,\mathbb{R}^n)$ satisfy the following conditions:*

- $v\big|_\Omega=w\big|_\Omega$;
- $v\big|_{U\setminus\Omega},w\big|_{U\setminus\Omega}\in W^{1,n}(U\setminus\Omega,\mathbb{R}^n)$;
- $v\big|_{\partial\Omega}=w\big|_{\partial\Omega}\in W^{1,n-1}(\partial\Omega,\mathbb{R}^n)$.

Then:

$$Jv-Jw=\big(\det\nabla v-\det\nabla w\big)\mathbf{E}^n\llcorner(U\setminus\Omega).$$

Proof. We can write

$$Jv=\partial j(v)=\partial(j(v)\llcorner\Omega)+\partial(j(v)\llcorner(U\setminus\Omega))=\partial(j(v)\llcorner\Omega)+Jv\llcorner(U\setminus\Omega)-\langle j(v),\partial\Omega\rangle. \tag{73}$$

Subtracting the analogous expression for Jw we obtain

$$Jv-Jw=(Jv-Jw)\llcorner(U\setminus\Omega)-\langle j(v)-j(w),\partial\Omega\rangle=\big(\det\nabla v-\det\nabla w\big)\mathbf{E}^n\llcorner(U\setminus\Omega)$$

because Proposition 5.3.2 applied to the open set $U\setminus\Omega$ implies that $v\big|_{\partial\Omega}=w\big|_{\partial\Omega}$, hence $\langle g(v)-g(w),\tau_{\partial\Omega}\rangle=0$. □

Therefore, if we aim at formulating problem (P') in a local way, that is depending only on the values of u in $\overline{\Omega}$, at least when the trace is sufficiently "nice", we can proceed as follows. If $u\big|_{\partial\Omega}$ belongs to $W^{1,n-1}$ and admits a $W^{1,n}$ extensions outside Ω, we can conventionally agree to pick one of such extensions to $U\setminus\Omega$: the result of Proposition 5.3.3 implies that the jacobian in $\overline{\Omega}$ of every competitor does not depend on the particular choice we made. Note however that the smoothness of the trace does not imply membership of the extension to $GSB_nV(U)$. In fact, it is sufficient to place the infinite dipoles of the function g in Example 2.5.4 so that the singularities lie on ∂B_1 and do not overlap. The constant extension outside the ball provides a map whose jacobian has both infinite mass and size.

In conclusion, in order to solve Problem (P') it seems necessary to impose membership of the competitors to $GSB_nV(U)$, while for a fairly broad class of boundary data the energy in $\overline{\Omega}$ shall not depend on the particular extension.

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