PUBLISHED IN FORUM MATH. 23 (6): 1281-1322, 2011

REGULARITY IN PARABOLIC DINI CONTINUOUS SYSTEMS

PAOLO BARONI

ABSTRACT. We consider a weak solution to the non-linear, parabolic systems of the form

$$u_t - div\, A(x,t,u,Du) = 0,$$

where the vector field A satisfies a Dini-type continuity condition with respect to the variables (x,t,u), and we prove a partial regularity result for such a solution. Moreover, we give an estimate of the size of the singular set of a solution in terms of a generalized parabolic Hausdorff measure associated to an appropriate modulus of continuity naturally associated to the coefficients of the system.

1. INTRODUCTION

In this paper we consider a non-linear parabolic system of second order of the following type:

$$u_t - div\, A(x,t,u,Du) = 0 \qquad (x,t) \in \Omega \times (-T,0) =: \Omega_T. \tag{1.1}$$

Here Ω is a bounded open set in $\mathbb{R}^n$, $T > 0$ and u mapping Ω_T into $\mathbb{R}^N$. We suppose that the continuous vector field $A : \Omega_T \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ satisfies the standard hypotheses of growth and ellipticity; see (A1), (A2) below.

Duzaar and Mingione in [12] proved sharp partial regularity results for weak solutions of system as (1.1) under the hypothesis of α-Hölder continuity of the function

$$(z,u) \longmapsto \frac{A(z,u,p)}{1+|p|} \tag{1.2}$$

with respect the parabolic metric in $\mathbb{R}^{n+1}$ given by

$$d_{\mathcal{P}}(z,\tilde z) := \max\Big\{|x-\tilde x|, \sqrt{|t-\tilde t|}\Big\} \approx \Big(|x-\tilde x|^2 + |t-\tilde t|\Big)^{1/2}, \tag{1.3}$$

whenever $z = (x,t)$, $\tilde z = (\tilde x, \tilde t) \in \mathbb{R}^{n+1}$. The spatial gradient Du is found to be α-Hölder continuous with respect to the parabolic metric (1.3) outside a relatively closed negligible set. For similar results in the case of coefficients with polynomial growth we refer to [13]; see also [2] for systems with nonstandard growth and [3] for higher order systems.

The aim of this paper is to weaken the assumption (1.2) in the sense that the coefficients are merely Dini continuous. Note that Dini continuity is strictly weaker than α-Hölder continuity for any exponent $\alpha > 0$, but stronger than simply continuity. More precisely, a function f that maps some metric space into $\mathbb{R}^N$ is said to be

Date: October 6, 2012.

Dini continuous if its modulus of continuity μ fulfills the so-called *Dini condition*, i.e.

$$\mathcal{M}(r) := \int_0^r \frac{\mu(\rho)}{\rho}\, d\rho < \infty$$

for some $r > 0$. Dini is a in a certain sense the limit case of Hölder continuity, is that it provides the worst possible modulus of continuity for which certain dyadic decay estimates still hold true; we refer to the papers [4, 8, 9, 20, 26, 23, 29] for recent regularity results concerning problems involving Dini continuous coefficients.

As a first result we show the partial regularity of weak solutions of (1.1) with Dini continuous coefficients, also satisfying the standard hypotheses of growth and ellipticity. More precisely, if we set

$$Reg(u) := \Big\{z \in \Omega_T : Du \text{ is continuous in a neighborhood of } z\Big\}$$

and $Sing(u) := \Omega_T \setminus Reg(u)$, we can state our first result in the following way: suppose u is a weak solution of system (1.1) under the hypotheses (A1), (A2), the Dini continuity of (1.2) and with $\rho^{-\alpha}\mu(\rho)$ decreasing for some $\alpha \in (0,1)$. Then

$$Du \in \mathcal{C}^0_{loc}(Reg(u), \mathbb{R}^{nN}).$$

Moreover, for each $\beta \in [\alpha, 1)$, Du has modulus of continuity $r \mapsto r^\beta + \mathcal{M}(r)$.

To show this we use an approach similar to that in [12], combined with the technical tools from [8, 9], dealing with partial regularity for elliptic systems with Dini continuous coefficients. In particular, the most important tool we used in proving partial regularity is the *$\mathcal{A}$-caloric approximation lemma*. This Lemma, which was proved in [12], is the evolutionary version of the *$\mathcal{A}$-harmonic approximation lemma* (see [14, 10]) which is in turn a generalization of the harmonic approximation lemma De Giorgi used in the proof of the regularity of minimal surfaces (see [6] and also [28]). For a complete overview about approximation lemmata, see [11]. The *$\mathcal{A}$-caloric approximation lemma* exhibits a sort of rigidity property of approximately $\mathcal{A}$-caloric maps, and its application allows us to take over good estimates for weak solutions to linear parabolic system with constant coefficients, to our original weak solution and to prove regularity via Dini-Campanato spaces.

Furthermore, following the approach taken in [12] - see also [2] for a differentiable case - as a second result we give an estimate for the dimension of the singular set $Sing(u)$ in terms of a *generalized parabolic Hausdorff measure* associated to an appropriate modulus of continuity. If $\theta : [0,\infty) \to [0,\infty)$ is monotone, non-decreasing and continuous, with $\theta(0) = 0$, the *generalized parabolic Hausdorff measure* associated to θ is

$$\mathcal{P}^\theta(E) := \lim_{\delta \searrow 0}\bigg(\inf\bigg\{\sum_{j=0}^{\infty} \theta\big(diam_{\mathcal{P}}\,(S_j)\big);\ E \subset \bigcup_{j=0}^{\infty} S_j;\ diam_{\mathcal{P}}\,(S_j) < \delta\bigg\}\bigg),$$

for each $E \subset \mathbb{R}^{n+1}$; $diam_{\mathcal{P}}\,(S_j)$ is the diameter of S_j calculated with respect the parabolic metric. In the case of uniform α-Hölder continuity of the vectorial field A with respect u, in [12] the authors proved that there exists a number $\delta = \delta(\alpha, L/\lambda) > 0$ such that $dim_{\mathcal{P}}(Sing(u)) \le n + 2 - \delta$. $dim_{\mathcal{P}}(E)$ is the parabolic Hausdorff dimension of a subset $E \subset \mathbb{R}^{n+1}$, and it is the supremum of the real numbers $s \ge 0$ for which $\mathcal{P}^s(E)$ is infinite. The s-dimensional parabolic Hausdorff measure $\mathcal{P}^s$ is the *generalized parabolic Hausdorff measure* in the special case where

$\theta(\rho) = \rho^s$. Note that the limit parabolic Hausdorff dimension of a subset E of $\mathbb{R}^{n+1}$ is $n+2$, differently from the elliptic case.

In our general case here we prove that there exists an exponent $\gamma \in (0, 1/4)$ such that if

$$\Lambda(r) := \int_0^r \frac{\mu^\gamma(\rho)}{\rho}\, d\rho < \infty$$

and $\omega(r) := r^{n+2}\Lambda(r)^{\varepsilon-3}$, then $\mathcal{P}^\omega(Sing(u)) = 0$ for all $\varepsilon > 0$. Moreover the previous result can be improved in the case where the vector field A does not explicitly depend on u, but only on its spatial gradient Du and on (x,t):

$$u_t - div\, A(x,t,Du) = 0 \qquad (x,t) \in \Omega_T. \tag{1.4}$$

In this case we prove that an analogue result holds if we replace $\Lambda(r)$ with

$$\widetilde{\Lambda}(r) := \int_0^r \frac{\mu(\rho)^{\frac{2}{3}\gamma'}}{\rho}\, d\rho$$

and $\omega(r)$ with $\tilde{\omega}(r) := r^{n+2}\widetilde{\Lambda}(r)^{\varepsilon-3\gamma'}$, for some $\gamma' > 1$. We observe (see also [9]) that in the general case we have to assume the Dini continuity of the function μ^γ in order to obtain an estimate on the Hausdorff dimension of $Sing(u)$. This is a stronger assumption than the Dini continuity of the function μ, since γ (which could explicitly be estimated from below in terms of the higher integrability exponent δ_0 appearing in Giaquinta-Struwe Lemma [16], which in turn depends on n and L/λ) can be very small. On the other hand in the case $A = A(z, Du)$ we have to require the Dini continuity of μ^t (with $t > 2/3$) which could be even weaker than the assumption we made to obtain partial regularity. We finally remark that in the elliptic/stationary setting singular estimates via fractional differentiability have been obtained in [24, 25] for solutions to non-linear elliptic systems, and in [21] for minima of (non-differentiable) integral functionals.

Acknowledgements. We thank Prof. Giuseppe Mingione for several useful discussions and Dr. Verena Bögelein for reading a preliminary version of the paper.

2. Notations and statement of the results

2.1. **Notations.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain; in the following Ω_T will denote the parabolic cylinder $\Omega \times (-T, 0)$, where $T > 0$; if $z \in \Omega_T$, we denote $z = (x,t)$ with $x \in \Omega$ and $t \in (-T, 0)$. $|\cdot|$ will be the usual $(n+1)$-dimensional Lebesgue measure, and α_n will be the volume of the unit ball in $\mathbb{R}^n$. We denote with $Q_\rho(z_0)$ the cylinder $B_\rho(x_0) \times (t_0 - \rho^2, t_0)$. If $v \in L^1_{loc}(\Omega_T, \mathbb{R}^n)$, then, for each $Q_\rho(z_0) \subset\subset \Omega_T$,

$$(v)_{z_0,\rho} := \fint_{Q_\rho(z_0)} v(z)\, dz = \frac{1}{\alpha_n \rho^{n+2}} \int_{Q_\rho(z_0)} v(z)\, dz$$

will be its mean value on $Q_\rho(z_0)$. Points of $\mathbb{R}^{nN}$ will be thought as $n \times N$ matrices; in that way notations like Ap, with $A \in \mathbb{R}^{nN}$ and $p \in \mathbb{R}^n$, will be meaningful. We will often omit the brackets of the inner product. In all the paper the symbol c will denote a positive, finite constant that may vary from line to line; the relevant dependencies will be specified enclosed in parentheses. We will denote differently relevant constants that will reappear frequently.

2.2. **Assumptions on the vector field.** We assume that the vector field $A : \Omega_T \times \mathbb{R}^n \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ is differentiable in p, with continuous derivatives satisfying

$$\left|\frac{\partial A}{\partial p}(z,u,p)\right| \leq L, \tag{A1}$$

$$\left\langle \frac{\partial A}{\partial p}(z,u,p)\,\xi,\xi\right\rangle \geq \lambda|\xi|^2 \tag{A2}$$

for any $z \in \Omega_T$, $u \in \mathbb{R}^N$, $p, \xi \in \mathbb{R}^{nN}$, where $\lambda > 0$ and $1 \leq L < \infty$.

With regard to the regularity of A with respect the variables (z,u), we assume that there exists a bounded modulus of continuity $\mu : (0,\infty) \to [0,1]$ and a non-decreasing function $K : [0,\infty) \to [1,\infty)$ such that

$$\frac{\big|A(z,u,p) - A(\tilde z,\tilde u,p)|}{1+|p\,|} \leq K(|u|)\,\mu\Big(|x-\tilde x| + \sqrt{|t-\tilde t|} + |u-\tilde u|\Big) \tag{A3}$$

for all $z=(x,t), \tilde z = (\tilde x,\tilde t) \in \Omega_T$, $u,\tilde u \in \mathbb{R}^N$, $p \in \mathbb{R}^{nN}$. In order to avoid trivialities (case when no coefficients are considered) we shall always assume that $\mu(r) > 0$ for $r > 0$. We require that the modulus of continuity μ fulfills the Dini condition

$$\mathcal{M}(r) := \int_0^r \frac{\mu(\rho)}{\rho}\,d\rho < \infty \tag{μ0}$$

for some $r > 0$ and that there exists an exponent $\alpha \in (0,1)$ such that

$$r \longmapsto \frac{\mu(r)}{r^\alpha} \qquad \text{is non-increasing.} \tag{μ1}$$

In order to prove our singular set estimate, we shall be forced to strengthen assumption (A3) by requesting uniform Dini continuity of the vector field, i.e. demanding the function K to be constant: therefore, from Chapter 6 on we shall assume that

$$\frac{\big|A(z,u,p) - A(\tilde z,\tilde u,p)|}{1+|p\,|} \leq L\,\mu\Big(|x-\tilde x| + \sqrt{|t-\tilde t|} + |u-\tilde u|\Big), \tag{A4}$$

with μ satisfying (μ0) and (μ1). We note that in the case (1.4) of no explicit dependence of A on u, assumptions (A3) and (A4) coincide.

2.3. **Remarks.** Up to enlarging the constant L, from (A1) we may deduce that

$$|A(z,u,p)\,| \leq L\big(1+|p\,|\big), \tag{2.1}$$

and that

$$|A(z,u,p) - A(z,u,\tilde p)| \leq L\,|p-\tilde p| \tag{2.2}$$

for any $z \in \Omega_T$, $u \in \mathbb{R}^N$, $p, \tilde p \in \mathbb{R}^{nN}$. Using (A2) we may also deduce that

$$\big\langle A(z,u,p) - A(z,u,\tilde p),\, p-\tilde p\big\rangle \geq \lambda\,|p-\tilde p|^2, \tag{2.3}$$

for any $z \in \Omega_T$, $u \in \mathbb{R}^N$, $p, \tilde p \in \mathbb{R}^{nN}$.

Note that we may always suppose a modulus of continuity to be non-decreasing, continuous and concave. So we can assume that

$$\mu \text{ is non-decreasing and continuous with } \mu(0^+) = 0 \tag{2.4}$$

and

$$\mu^2 \text{ is concave.} \tag{2.5}$$

The hypothesis ($\mu 1$) ensures that

$$r \longmapsto \frac{\mu(r)}{r} \qquad \text{and} \qquad r \longmapsto \frac{\mu(r)}{r^2} \tag{2.6}$$

are non-increasing. Hence

$$\mu(rs) \le r\mu(s) \qquad \text{for all } r \ge 1,\ s > 0. \tag{2.7}$$

Changing K by a constant, but keeping $K \ge 1$, we may assume that

$$\mu(1) = 1;\ \text{this implies that } r \le \mu(r) \le 1 \text{ for all } r \in [0,1]. \tag{2.8}$$

Since μ^2 is non-decreasing, we deduce $s^2\mu^2(r) \le s^2\mu^2(s)$ for all $0 \le r \le s$. We also note that $s^2\mu^2(r) \le r^2\mu^2(s) \le r^2$ for $0 \le s \le r$ and for $0 \le s \le 1$, by the non-increasing property (2.6) of $r \longmapsto \mu(r)/r$ and by (2.8), $\mu^2(s) \le 1$. Combining both cases we infer that

$$s^2\mu^2(r) \le s^2\mu^2(s) + r^2 \quad \text{for all } s \in [0,1],\ r > 0, \tag{2.9}$$

and similarly, using the non-increasing property of $r \longmapsto \mu(r)/r^2$,

$$s^2\mu(r) \le s^2\mu(s) + r^2 \quad \text{for all } s \in [0,1],\ r > 0. \tag{2.10}$$

($\mu 1$) imply, for all $\theta \in (0,1)$, $r > 0$, $k \in \mathbb{N} \cup \{0\}$,

$$\frac{1}{\alpha}(1-\theta^\alpha)\,\mu(\theta^k r) = \frac{\mu(\theta^k r)}{(\theta^k r)^\alpha}\int_{\theta^{k+1}r}^{\theta^k r} \tau^{\alpha-1}\,d\tau \le \int_{\theta^{k+1}r}^{\theta^k r} \frac{\mu(\tau)}{\tau}\,d\tau,$$

so that, recalling ($\mu 0$),

$$\sum_{k=0}^{\infty} \mu(\theta^k r) \le \frac{\alpha}{1-\theta^\alpha}\int_0^r \frac{\mu(\tau)}{\tau}\,d\tau = \frac{\alpha}{1-\theta^\alpha}\mathcal{M}(r). \tag{2.11}$$

Hence

$$\mu^2(r) \le \alpha^2\mathcal{M}(r)^2\,; \tag{2.12}$$

moreover $r \mapsto \mathcal{M}(r)r^{-\alpha}$ is non-increasing. We remark, finally, that the continuity of $\partial A/\partial p$ implies the existence of a function $\omega : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$, with $\omega(r,0) = 0$ for all r, such that $r \mapsto \omega(r,s)$ is non-decreasing for fixed s, $s \mapsto \omega^2(r,s)$ is concave and non-decreasing for fixed r and such that, for all $(z,u,p), (\tilde{z},\tilde{u},\tilde{p}) \in \Omega_T \times \mathbb{R}^N \times \mathbb{R}^{nN}$ with $|u| + |p| + |u-\tilde{u}| + |p-\tilde{p}| \le M$, we have

$$\left|\frac{\partial A}{\partial p}(z,u,p) - \frac{\partial A}{\partial p}(\tilde{z},\tilde{u},\tilde{p})\right| \le \omega(M, |x-\tilde{x}|^2 + |t-\tilde{t}| + |u-\tilde{u}|^2 + |p-\tilde{p}|^2)\,. \tag{2.13}$$

2.4. **Statement of the results.** We recall that a weak solution to (1.1) is a function $u \in L^2(-T,0;W^{1,2}(\Omega,\mathbb{R}^N))$ satisfying

$$\int_{\Omega_T} \Big[u\,\varphi_t - A(z,u,Du)D\varphi\Big]\,dz = 0 \tag{2.14}$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega_T,\mathbb{R}^N)$. The first result we state in this paper regards partial regularity of weak solutions of system (1.1):

Theorem 2.1. *Let $u \in L^2(-T,0;W^{1,2}(\Omega,\mathbb{R}^N))$ be a weak solution to* (1.1) *under the assumptions* (A1), (A2), (A3), (μ0) *and* (μ1). *Then $Sing(u) \subset \Sigma_1 \cup \Sigma_2 =: \Sigma$, where*

$$\Sigma_1 := \left\{ z_0 \in \Omega_T : \liminf_{\rho \searrow 0} \fint_{Q_\rho(z_0)} \left| Du - (Du)_{z_0,\rho} \right|^2 dz > 0 \right\}$$
$$\cup \left\{ z_0 \in \Omega_T : \liminf_{\rho \searrow 0} \fint_{Q_\rho(z_0)} \left| u - (u)_{z_0,\rho} \right|^2 dz > 0 \right\} =: \Sigma_{1,1} \cup \Sigma_{1,2} \quad (2.15)$$

and

$$\Sigma_2 := \left\{ z_0 \in \Omega_T : \limsup_{\rho \searrow 0} \Big(\left|(u)_{z_0,\rho}\right| + \left|(Du)_{z_0,\rho}\right| \Big) = +\infty \right\}.$$

So $Du \in \mathcal{C}^0_{loc}(Reg(u), \mathbb{R}^{nN})$ and $|Sing(u)| = 0$. Moreover, for all $\beta \in [\alpha, 1)$ and $z_0 \in \Omega_T \setminus \Sigma$, the spatial gradient Du has modulus of continuity $\rho \mapsto \rho^\beta + \mathcal{M}(\rho)$ in a neighborhood of z_0 with respect to the standard parabolic metric (1.3).

We remark that the previous theorem provides a characterization of *regular points* of Du: if

$$\liminf_{\rho \searrow 0} \fint_{Q_\rho(z_0)} \left| u - (u)_{z_0,\rho} \right|^2 dz = 0, \quad \liminf_{\rho \searrow 0} \fint_{Q_\rho(z_0)} \left| Du - (Du)_{z_0,\rho} \right|^2 dz = 0$$

and both $|(u)_{z_0,\rho}|$ and $|(Du)_{z_0,\rho}|$ are bounded in the limit $\rho \searrow 0$, then z_0 is a regular point. So the estimate of the Lebesgue dimension of $Sing(u)$ is in fact an immediate consequence of the fact that the set of non-Lebesgue points has measure zero. This is a typical result in partial regularity; see for instance the elliptic result of Giaquinta and Modica [15].

With the notations introduced in the introductory chapter we can now state the second theorem of the paper, which gives a more precise estimate on the "dimension" of the singular set of u:

Theorem 2.2. *Let $u \in L^2(-T,0;W^{1,2}(\Omega,\mathbb{R}^N))$ be a weak solution of the parabolic system* (1.1) *under the assumptions* (A1) – (A2), (A4), (μ0) *and* (μ1). *Then there exists $\gamma = \gamma(n, L/\lambda) \in (0, 1/4)$ such that if*

$$\Lambda(r) := \int_0^r \frac{\mu(\rho)^{\frac{2}{3}\gamma}}{\rho}\, d\rho < +\infty$$

then, for any $\varepsilon > 0$, we have $\mathcal{P}^\omega(Sing(u)) = 0$ where $\omega(r) := r^{n+2}\Lambda^{\varepsilon-3}(r)$.

As we already stated, when the vector field A is independent of u, we can improve the previous theorem in the following way:

Theorem 2.3. *Let $u \in L^2(-T,0;W^{1,2}(\Omega,\mathbb{R}^N))$ be a weak solution of the parabolic system* (1.4) *under the assumptions* (A1) – (A2), (A4), (μ0) *and* (μ1) *conveniently modified. Then there exists $\gamma' = \gamma'(n, L/\lambda) > 1$ such that if*

$$\widetilde{\Lambda}(r) := \int_0^r \frac{\mu(\rho)^{\frac{2}{3}\gamma'}}{\rho}\, d\rho < +\infty$$

then, for any $\varepsilon > 0$, we have $\mathcal{P}^{\widetilde{\omega}}(Sing(u)) = 0$ where $\widetilde{\omega}(r) := r^{n+2}\widetilde{\Lambda}^{\varepsilon-3\gamma'}(r)$.

2.5. **Preliminary lemmata.** For $u \in L^2(Q_\rho(z_0), \mathbb{R}^N)$ let's denote with $\ell_{z_0,\rho}$ the unique affine function in space ($\ell(z) = \ell(x)$) minimizing $\ell \longmapsto \fint_{Q_\rho(z_0)} |u - \ell|^2 \, dz$ amongst all affine functions independent of t. We note that such a minimum point exists and a straightforward computation yields that, if $\ell_{z_0,\rho} = \xi_{z_0,\rho} + \nu_{z_0,\rho}(x - x_0)$, then

$$\xi_{z_0,\rho} = (u)_{z_0,\rho} \quad \text{and} \quad \nu_{z_0,\rho} = \frac{n+2}{\rho^2} \fint_{Q_\rho(z_0)} u \otimes (x - x_0) \, dz.$$

The following estimate may be found in [22]:

Lemma 2.4. *Let $u \in L^2(Q_\rho(z_0), \mathbb{R}^N)$, $\vartheta \in (0,1)$ and $\ell_{z_0,\rho}$ respectively $\ell_{z_0,\vartheta\rho}$ the unique affine functions in space minimizing $\ell \mapsto \fint_{Q_\rho(z_0)} |u - \ell|^2 \, dz$ respectively $\ell \mapsto \fint_{Q_{\vartheta\rho}(z_0)} |u - \ell|^2 \, dz$. Then*

$$\left|\nu_{z_0,\vartheta\rho} - \nu_{z_0,\rho}\right|^2 \leq \frac{n(n+2)}{(\vartheta\rho)^2} \fint_{Q_{\vartheta\rho}(z_0)} \left|u - (u)_{z_0,\rho} - \nu_{z_0,\rho}(x - x_0)\right|^2 dz.$$

Moreover, if $Du \in L^2(Q_\rho(z_0), \mathbb{R}^{nN})$, then

$$\left|\nu_{z_0,\rho} - (Du)_{z_0,\rho}\right|^2 \leq \frac{n(n+2)}{\rho^2} \fint_{Q_\rho(z_0)} \left|u - (u)_{z_0,\rho} - (Du)_{z_0,\rho}(x - x_0)\right|^2 dz.$$

A bilinear form $\mathcal{A}$ on $\mathbb{R}^{nN}$ is called strongly elliptic bilinear (with ellipticity constant $\lambda > 0$ and upper bound $L > 0$) if

$$\mathcal{A}(\tilde{p}, \tilde{p}) \geq \lambda \, |\tilde{p}|^2, \qquad \mathcal{A}(p, \tilde{p}) \leq L \, |p| \, |\tilde{p}| \qquad \text{for all } p, \tilde{p} \in \mathbb{R}^{nN}.$$

We call a function belonging to $L^2(-T, 0; W^{1,2}(B_\rho, \mathbb{R}^N))$ $\mathcal{A}$*-caloric* on $Q_\rho(z_0)$ if the equality

$$\int_{Q_\rho(z_0)} \Big[h \, \varphi_t - \mathcal{A}(Dh, D\varphi)\Big] \, dz = 0$$

holds for each $\varphi \in \mathcal{C}_0^\infty(Q_\rho(z_0), \mathbb{R}^N)$. The following lemma is a standard estimate for weak solutions to linear parabolic systems with constant coefficients and may be found in [5]:

Lemma 2.5. *Let $h \in L^2(-T, 0; W^{1,2}(B_\rho(x_0), \mathbb{R}^N))$ be $\mathcal{A}$-caloric on $Q_\rho(z_0)$, with $\mathcal{A}$ being a strongly elliptic bilinear form on $\mathbb{R}^{nN}$ with ellipticity constant λ and upper bound L. Then h is smooth in $Q_\rho(z_0)$ and there exists a constant $c_{Pa} = c_{Pa}(n, N, L/\lambda) \geq 1$ such that*

$$\begin{aligned} \frac{1}{(\theta\rho)^2} \fint_{Q_{\vartheta\rho}(z_0)} & \left|h - (h)_{z_0,\theta\rho} - (Dh)_{z_0,\theta\rho}(x - x_0)\right|^2 dz \\ & \leq c_{Pa} \, \theta^2 \frac{1}{\rho^2} \fint_{Q_\rho(z_0)} \left|h - (h)_{z_0,\rho} - (Dh)_{z_0,\rho}(x - x_0)\right|^2 dz \end{aligned}$$

for all $0 < \theta < 1$.

The next lemma, whose proof may be found in [12], is a fundamental tool to prove partial regularity, and it's the parabolic version of the classical De Giorgi's harmonic approximation lemma.

Lemma 2.6 ($\mathcal{A}$-caloric approximation). *Let $\varepsilon > 0$ and $\mathcal{A}$ a bilinear form on $\mathbb{R}^{nN}$ which is strongly elliptic with ellipticity constant λ and upper bound L. Then there exists $\delta = \delta(n, N, \lambda, \Lambda, \varepsilon) \in (0, 1]$ such that if $u \in L^2(t_0 - \rho^2, t_0; W^{1,2}(B_\rho(x_0), \mathbb{R}^N))$, with*

$$\frac{1}{\rho^2} \fint_{Q_\rho(z_0)} |u|^2 \, dz + \fint_{Q_\rho(z_0)} |Du|^2 \, dz \le 1, \tag{2.16}$$

is approximatively $\mathcal{A}$-caloric, in the sense that

$$\left| \fint_{Q_\rho(z_0)} \Big[u \, \varphi_t - \mathcal{A}(Du, D\varphi) \Big] \, dz \right| \le \delta \sup_{Q_\rho(z_0)} |D\varphi|$$

for all $\varphi \in \mathcal{C}_0^\infty(Q_\rho(z_0), \mathbb{R}^N)$, then there exists an $\mathcal{A}$-caloric function h such that

$$\frac{1}{\rho^2} \fint_{Q_\rho(z_0)} |h|^2 \, dz + \fint_{Q_\rho(z_0)} |Dh|^2 \, dz \le 1 \quad \textit{and} \quad \frac{1}{\rho^2} \fint_{Q_\rho(z_0)} |u - h|^2 \, dz \le \varepsilon.$$

3. A Caccioppoli inequality

We set $H(s) := K(s)(1 + s)$, where K is the function from (A3). We now derive the following Caccioppoli inequality:

Lemma 3.1. *Let $u \in L^2(-T, 0; W^{1,2}(\Omega, \mathbb{R}^N))$ be a weak solution to system (1.1) under the assumptions (A1) – (A3), ($\mu 0$) and ($\mu 1$). Then for any $M > 0$, any affine function $\ell(z) = \ell(x)$ with $|\ell(z_0)| + |D\ell| \le M$ and for any $Q_\rho(z_0) \subset\subset \Omega_T$, with $\rho \le \rho_1 := 1/H(M)$, we have*

$$\fint_{Q_{\rho/2}(z_0)} |D(u - \ell)|^2 \, dz \le c_{Cacc} \left[\fint_{Q_\rho(z_0)} \left| \frac{u - \ell}{\rho} \right|^2 dz + \mu^2(\rho) \right]$$

where the constant c_{Cacc} depends only on λ, L and $H(M)$.

Proof. We let $\varphi(x, t) := \eta^2(x)\zeta^2(t)(u - \ell)$, with $\eta \in \mathcal{C}_0^1(B_\rho(x_0))$ being a cut-off function in space such that $0 \le \eta \le 1$, $\eta \equiv 1$ in $B_{\rho/2}(x_0)$, $|D\eta| \le 4/\rho$ and $\zeta \in \mathcal{C}^1(\mathbb{R})$ a cut-off function in time with, being $0 < \varepsilon < \rho^2/4$ arbitrary, $\zeta \equiv 1$ on $(t_0 - \rho^2/4, t_0 - \varepsilon^2)$, $\zeta \equiv 0$ on $(-\infty, t_0 - \rho^2) \cup (t_0, +\infty)$ and $0 \le \zeta \le 1$. Moreover $\zeta_t \le 0$ on $(t_0 - \rho^2/4, +\infty)$ and $|\zeta_t| \le 3/\rho^2$ on $(t_0 - \rho^2, t_0 - \rho^2/4)$. Let's remark that φ has not the required regularity with respect time to be admissible as test function. Therefore, if we want to proceed in a rigorous way, we should smooth it in time for instance by the use of Steklov averages (see e.g. [7]). In the following we proceed formally, and use φ as a test function in the weak formulation of (1.1), letting to the reader the needed adjustments and the standard, technical changes. Differentiating $D\varphi$ and using the weak formulation (2.14) we obtain

$$\int_{Q_\rho(z_0)} A(z, u, Du) \, D(u - \ell) \, \zeta^2 \eta^2 \, dz$$
$$= \int_{Q_\rho(z_0)} \Big[-2A(z, u, Du) \, \zeta^2 \eta \, D\eta \otimes (u - \ell) + u \, \varphi_t \Big] \, dz.$$

Using $0 = \int_{Q_\rho(z_0)} A(z_0, \ell(z_0), D\ell) \, D\varphi \, dz$ and the fact that $\ell_t = 0$ we obtain

$$\int_{Q_\rho(z_0)} \Big[A(z, u, Du) - A(z, u, D\ell) \Big] \, D(u - \ell) \, \zeta^2 \eta^2 \, dz$$

$$
\begin{aligned}
&= -2\int_{Q_\rho(z_0)} \Big[A(z,u,Du)-A(z,u,D\ell)\Big]\,\zeta^2\eta\,D\eta\otimes(u-\ell)\,dz\\
&\quad -\int_{Q_\rho(z_0)} \Big[A(z,u,D\ell)-A(z,\ell(z),D\ell)\Big]\,D\varphi\,dz\\
&\quad -\int_{Q_\rho(z_0)} \Big[A(z,\ell(z),D\ell)-A(z_0,\ell(z_0),D\ell)\Big]\,D\varphi\,dz\\
&\quad +\int_{Q_\rho(z_0)} (u-\ell)\,\varphi_t\,dz =: I+II+III+IV. \qquad (3.1)
\end{aligned}
$$

In the following we shall in turn estimate separately these integrals. In order to estimate I we use the Lipschitz bound for A (2.2) and Young's inequality, yielding for $\gamma\in(0,1]$ that

$$
\begin{aligned}
|I| &\le 2L\int_{Q_\rho(z_0)} |Du-D\ell|\;\zeta^2\eta\;|D\eta|\,|u-\ell|\;dz\\
&\le \gamma\int_{Q_\rho(z_0)} |Du-D\ell|^2\,\zeta^2\eta^2\,dz+\frac{64L^2}{\gamma}\int_{Q_\rho(z_0)}\left|\frac{u-\ell}{\rho}\right|^2 dz. \qquad (3.2)
\end{aligned}
$$

For IV we proceed formally exactly as in [12], obtaining

$$
|IV|\le 3\int_{Q_\rho(z_0)}\left|\frac{u-\ell}{\rho}\right|^2 dz. \qquad (3.3)
$$

We remark that here the smoothing procedure mentioned above becomes necessary in order to obtain the estimate. To evaluate the second addendum, using (A3),

$$
|II|\le\int_{Q_\rho(z_0)} K\big(\ell(z)\big)\mu\big(|u-\ell|\big)\big(1+|D\ell|\big)|D\varphi|\,dz.
$$

By our assumptions $|D\ell|\le M$, and moreover $|\ell(z)|\le|\ell(z_0)|+|D\ell|\,|x-x_0|\le M$, since ℓ does not depend on time and $\rho\le 1$. We have

$$
\begin{aligned}
|II| &\le H(M)\int_{Q_\rho(z_0)}\mu(|u-\ell|)\,\zeta^2\eta^2|D(u-\ell)|\,dz\\
&\quad + 2H(M)\int_{Q_\rho(z_0)}\mu(|u-\ell|)\,\zeta^2\eta\,|D\eta||u-\ell|\,dz =: II_1+II_2.
\end{aligned}
$$

To evaluate both the integrals the following estimate will be useful: using (2.9) (keeping in mind that $\rho\le\rho_1\le 1/H(M)$) and (2.7):

$$
\begin{aligned}
&H(M)^2\alpha_n\rho^{n+2}\mu^2\Big(\fint_{Q_\rho(z_0)}|u-\ell|\,dz\Big)\\
&\qquad\le \alpha_n\rho^n\Big[H(M)^2\rho^2\mu^2\big(H(M)\rho\big)+\Big(\fint_{Q_\rho(z_0)}|u-\ell|\,dz\Big)^2\Big]\\
&\qquad\le \alpha_n\rho^{n+2}H(M)^4\mu^2(\rho)+\alpha_n\int_{Q_\rho(z_0)}\left|\frac{u-\ell}{\rho}\right|^2 dz. \qquad (3.4)
\end{aligned}
$$

Estimate for II_1: using Young's inequality, $\gamma\in(0,1]$, the concavity of μ^2 (2.5), Jensen's inequality and (3.4) we get

$$
|II_1|\le\gamma\int_{Q_\rho(z_0)}\zeta^2\eta^2|D(u-\ell)|^2\,dz+\frac{H(M)^2\alpha_n\rho^{n+2}}{\gamma}\fint_{Q_\rho(z_0)}\mu^2(|u-\ell|)\,dz
$$

$$\leq \gamma\int_{Q_\rho(z_0)}\zeta^2\eta^2|D(u-\ell)|^2\,dz+\frac{H(M)^2\alpha_n\rho^{n+2}}{\gamma}\,\mu^2\bigg(\fint_{Q_\rho(z_0)}|u-\ell|\,dz\bigg)$$
$$\leq \gamma\int_{Q_\rho(z_0)}\zeta^2\eta^2|D(u-\ell)|^2\,dz+\frac{\alpha_n}{\gamma}\int_{Q_\rho(z_0)}\left|\frac{u-\ell}{\rho}\right|^2dz$$
$$+\frac{\alpha_n\rho^{n+2}H(M)^4}{\gamma}\,\mu^2(\rho). \quad (3.5)$$

Estimate for II_2: analogously, using again (3.4) and Young's inequality with $\gamma\in(0,1]$

$$|II_2|=2H(M)\int_{Q_\rho(z_0)}\mu(|u-\ell|)\,\zeta^2\eta\,|D\eta||u-\ell|\,dz$$
$$\leq 64\int_{Q_\rho(z_0)}\left|\frac{u-\ell}{\rho}\right|^2dz+H(M)^2\alpha_n\rho^{n+2}\fint_{Q_\rho(z_0)}\mu^2(|u-\ell|)\,dz$$
$$\leq 64\int_{Q_\rho(z_0)}\left|\frac{u-\ell}{\rho}\right|^2dz+H(M)^2\alpha_n\rho^{n+2}\mu^2\bigg(\fint_{Q_\rho(z_0)}|u-\ell|\,dz\bigg)$$
$$\leq c(n)\int_{Q_\rho(z_0)}\left|\frac{u-\ell}{\rho}\right|^2dz+\alpha_n\rho^{n+2}H(M)^4\mu^2(\rho). \quad (3.6)$$

In a very similar way we estimate

$$|III|\leq\int_{Q_\rho(z_0)}\big|A(z,\ell(z),D\ell)-A(z_0,\ell(z_0),D\ell)\big||D\varphi|\,dz$$
$$\leq H(M)\int_{Q_\rho(z_0)}\mu\big(|x-x_0|+\sqrt{|t-t_0|}+|\ell(z)-\ell(z_0)|\big)|D\varphi|\,dz.$$

Since $|x-x_0|+\sqrt{|t-t_0|}+|\ell(z)-\ell(z_0)|\leq 2\rho\,(M+1)$ and using Young's inequality, (2.7) and our assumption $|D\eta|\leq 4/\rho$ we obtain

$$|III|\leq 2H(M)^2\mu(\rho)\int_{Q_\rho(z_0)}\Big[2\zeta^2\eta\;|D\eta|\;|u-\ell|+\zeta^2\eta^2\;|D(u-\ell)|\Big]\,dz$$
$$\leq 16^2\int_{Q_\rho(z_0)}\left|\frac{u-\ell}{\rho}\right|^2dz+H(M)^4\mu^2(\rho)\,\alpha_n\rho^{n+2}$$
$$+\gamma\int_{Q_\rho(z_0)}\zeta^2\eta^2|D(u-\ell)|^2\,dz+\frac{4H(M)^4\mu^2(\rho)}{\gamma}\,\alpha_n\rho^{n+2}. \quad (3.7)$$

Finally, combining (3.1) with (3.2), (3.5), (3.6), (3.7) and (3.3) and using assumption (2.3) to estimate the left-hand side of that inequality from below we get

$$(\lambda-3\gamma)\int_{Q_\rho(z_0)}|D(u-\ell)|\,\zeta^2\eta^2\,dz$$
$$\leq\frac{c\,(L,H(M))}{\gamma}\bigg(\int_{Q_\rho(z_0)}\left|\frac{u-\ell}{\rho}\right|^2dz+\alpha_n\rho^{n+2}\mu^2(\rho)\bigg).$$

Now we choose $\gamma=\min\{1,\lambda/6\}$; recalling that $\zeta\equiv 1$ on $[t_0-\rho^2/4,t_0-\varepsilon^2]$ and $\eta\equiv 1$ in $B_{\rho/2}(x_0)$, we deduce

$$\int_{B_{\rho/2}(x_0)} \int_{t_0-\rho^2/4}^{t_0-\varepsilon^2} |D(u-\ell)|\,dz \\ \leq c\,(\lambda, L, H(M)) \Big(\int_{Q_\rho(z_0)} \Big| \frac{u-\ell}{\rho} \Big|^2 dz + \alpha_n \rho^{n+2} \mu^2(\rho) \Big).$$

We get the desired Caccioppoli inequality dividing by $|Q_\rho|$ and letting $\varepsilon \searrow 0$. □

4. Linearization

Now we want to deduce an inequality which will allow us to apply the $\mathcal{A}$*-caloric approximation lemma*. We write

$$\Phi(z_0, \rho, \nu) := ⨍_{Q_\rho(z_0)} |Du - \nu|^2\,dz \quad \text{and} \quad \Psi(z_0, \rho, \xi) := ⨍_{Q_\rho(z_0)} \Big| \frac{u-\xi}{\rho} \Big|^2 dz.$$

Lemma 4.1. *Let $u \in L^2(-T, 0; W^{1,2}(\Omega, \mathbb{R}^N))$ be a weak solution of (1.1) under the assumptions (A1), (A2), (A3). Then, for all $M > 0$*

$$\Big| ⨍_{Q_\rho(z_0)} \Big[(u-\ell)\,\varphi_t - \frac{\partial A}{\partial p}\big(z_0, \ell(z_0), D\ell\big)(Du - D\ell) D\varphi \Big]\,dz \Big| \\ \leq c_{lin} \Big[\omega(M+1, \Phi)\sqrt{\Phi} + \Phi + \Psi + \mu(\rho) \Big] \sup_{Q_\rho(z_0)} |D\varphi| \quad (4.1)$$

holds for all $Q_\rho(z_0) \subset\subset \Omega_T$, with $\rho \leq \rho_1(M) := 1/H(M)$, $\varphi \in \mathcal{C}_0^\infty(Q_\rho(z_0), \mathbb{R}^N)$ and for any affine function $\ell(z) = \ell(x)$ which does not depend on t and satisfies $|\ell(z_0)| + |D\ell| \leq M$. The constant $c_{lin} \geq 1$ appearing in (4.1) only depends on L and $H(M)$. Here we wrote $\Phi := \Phi(z_0, \rho, D\ell)$ and $\Psi := \Psi(z_0, \rho, \ell)$.

Proof. Without loss of generality we can suppose $\sup_{Q_\rho(z_0)} |D\varphi| \leq 1$. The general case will follow by a scaling argument. Since

$$⨍_{Q_\rho(z_0)} u\,\varphi_t\,dz = ⨍_{Q_\rho(z_0)} A(z, u, Du) D\varphi\,dz \quad \text{and} \quad ⨍_{Q_\rho(z_0)} \ell\,\varphi_t\,dz = 0,$$

we can write

$$⨍_{Q_\rho(z_0)} \Big[(u-\ell)\,\varphi_t - \frac{\partial A}{\partial p}\big(z_0, \ell(z_0), D\ell\big)(Du - D\ell) D\varphi \Big]\,dz \\ = ⨍_{Q_\rho(z_0)} \Big[A(z_0, \ell(z_0), Du) - \frac{\partial A}{\partial p}\big(z_0, \ell(z_0), D\ell\big)(Du - D\ell) \Big] D\varphi\,dz \\ + ⨍_{Q_\rho(z_0)} \Big[A(z, u, Du) - A(z, \ell, Du) \Big] D\varphi\,dz \\ + ⨍_{Q_\rho(z_0)} \Big[A(z, \ell, Du) - A(z_0, \ell(z_0), Du) \Big] D\varphi\,dz =: I + II + III.$$

The estimate of the first integral goes on exactly as in [12], so that

$$|I| \leq \omega\Big(M+1, ⨍_{Q_\rho(z_0)} |Du - D\ell|^2\,dz \Big) \Big(⨍_{Q_\rho(z_0)} |Du - D\ell|^2\,dz \Big)^{1/2} \\ + 4L(M+1) ⨍_{Q_\rho(z_0)} |Du - D\ell|^2\,dz. \quad (4.2)$$

In order to estimate the remaining integrals, the following auxiliary calculation will be useful: from (2.9) and (2.7), using that $1 \le K(M) \le H(M)$, $\rho \le 1/H(M)$ and $\mu^2(\rho) \le \mu(\rho)$,

$$\begin{aligned}\frac{1}{\rho^2}\Big[K(M)^2\rho^2\,\mu^2(|u-\ell|)\Big] &\le \frac{1}{\rho^2}\Big[K(M)^2\rho^2\mu^2\big(K(M)\rho\big)+|u-\ell|^2\Big]\\ &\le H(M)^4\mu(\rho)+\left|\frac{u-\ell}{\rho}\right|^2.\end{aligned}$$

Integrating over $Q_\rho(z_0)$ and dividing by $|Q_\rho|$ yields

$$K(M)^2 \fint_{Q_\rho(z_0)} \mu^2(|u-\ell|)\,dz \le H(M)^4\mu(\rho) + \fint_{Q_\rho(z_0)} \left|\frac{u-\ell}{\rho}\right|^2 dz.$$

Analogously, from (2.10), we get

$$H(M) \fint_{Q_\rho(z_0)} \mu(|u-\ell|)\,dz \le H(M)^2\mu(\rho) + \fint_{Q_\rho(z_0)} \left|\frac{u-\ell}{\rho}\right|^2 dz.$$

From (A3), using the previous estimates, we deduce

$$\begin{aligned}|II| &\le K(M) \fint_{Q_\rho(z_0)} \mu(|u-\ell|)\Big[1+|D\ell|+|Du-D\ell|\Big]\,dz\\ &\le H(M)\fint_{Q_\rho(z_0)} \mu(|u-\ell|)\,dz + K(M)\fint_{Q_\rho(z_0)} \mu(|u-\ell|)\,|Du-D\ell|\,dz\\ &\le H(M)\fint_{Q_\rho(z_0)} \mu(|u-\ell|)\,dz + K(M)^2\fint_{Q_\rho(z_0)} \mu^2(|u-\ell|)\,dz\\ &\qquad\qquad + \fint_{Q_\rho(z_0)} |Du-D\ell|^2\,dz\\ &\le 2H(M)^4\mu(\rho) + 2\fint_{Q_\rho(z_0)} \left|\frac{u-\ell}{\rho}\right|^2 dz + \fint_{Q_\rho(z_0)} |Du-D\ell|^2\,dz. \qquad (4.3)\end{aligned}$$

Finally, in order to to evaluate the third addendum, since $|x-x_0|+\sqrt{|t-t_0|}+|D\ell|\,|x-x_0| \le (M+2)\rho$, from (A3) and (2.7) we infer

$$\begin{aligned}|III| &\le K(M)\,\mu\big((M+2)\rho\big) \fint_{Q_\rho(z_0)} \Big[1+|D\ell|+|Du-D\ell|\Big]\,dz\\ &\le 2H(M)\,\mu(\rho)\Big[(1+M)+\fint_{Q_\rho(z_0)}|Du-D\ell|\,dz\Big]\\ &\le 2H(M)^2\mu(\rho) + \fint_{Q_\rho(z_0)}|Du-D\ell|^2\,dz + 4H(M)^2\mu^2(\rho)\\ &\le \fint_{Q_\rho(z_0)}|Du-D\ell|^2\,dz + 6H(M)^2\mu(\rho). \qquad (4.4)\end{aligned}$$

Combining (4.2), (4.3) and (4.4) we finally arrive at

$$\begin{aligned}&\left|\fint_{Q_\rho(z_0)}\Big[(u-\ell)\varphi_t - \frac{\partial A}{\partial p}\big(z_0,\ell(z_0),D\ell\big)(Du-D\ell)D\varphi\Big]\,dz\right|\\ &\le \omega\bigg(M+1, \fint_{Q_\rho(z_0)}|Du-D\ell|^2\,dz\bigg)\bigg(\fint_{Q_\rho(z_0)}|Du-D\ell|^2\,dz\bigg)^{1/2}\end{aligned}$$

$$+ c(L, H(M))\Big[\fint_{Q_\rho(z_0)} |Du - D\ell|^2\,dz + \fint_{Q_\rho(z_0)} \Big|\frac{u-\ell}{\rho}\Big|^2 dz + \mu(\rho)\Big].$$

Keeping in mind the definitions of Φ and Ψ this shows the statement of the Lemma. □

5. Regular points and partial regularity

In this section we will prove Theorem 2.1, using the inequalities proved in last sections, together with the *$\mathcal{A}$-caloric approximation lemma* and Lemma 2.5. We now fix $z_0 \in \Omega_T$. Throughout the whole Chapter $u \in L^2(-T, 0; W^{1,2}(\Omega, \mathbb{R}^N))$ denotes a weak solution of the non-linear parabolic system (1.1) on $Q_\rho(z_0) \subset\subset \Omega_T$, satisfying the hypotheses of Section 2.2. Moreover, we let $M > 0$ and $\rho_1(M) := 1/H(M)$. In the following we shall denote by $\widetilde{\Psi}(z, \rho, \ell)$ the quantity $\Psi(z, \rho, \ell) + \mu^2(\rho)$. With this notation we can state the following

Lemma 5.1. *For each $\beta \in [\alpha, 1)$ there exists $\vartheta \in (0, 1/2)$ and $\delta \in (0, 1]$, depending on n, N, λ, L, α, β and $H(M)$, such that if the smallness condition*

$$\omega^2\big(M + 1, \widetilde{\Psi}(z_0, \rho, \ell_{z_0,\rho})\big) + \widetilde{\Psi}(z_0, \rho, \ell_{z_0,\rho}) \le \frac{1}{2}\,\delta^2 \tag{5.1}$$

is satisfied on $Q_\rho(z_0) \subset\subset \Omega_T$ and such if $|\ell_{z_0,\rho}| + |D\ell_{z_0,\rho}| \le M$ then

$$\widetilde{\Psi}(z_0, \vartheta\rho, \ell_{z_0,\vartheta\rho}) \le \vartheta^{2\beta}\widetilde{\Psi}(z_0, \rho, \ell_{z_0,\rho}) + c_1\,\mu^2(\rho)$$

where $c_1 := 1 + \delta^{-2}$.

Proof. We only sketch the proof, since it is very similar to the one of the analogue Lemma 7.1 in [12]. For given $\varepsilon > 0$, let δ be the constant appearing in Lemma 2.6. If we write

$$w := \frac{u - \ell}{\gamma} \quad \text{with} \quad \gamma := 4\tilde{c}\sqrt{\Psi(z_0, \rho, \ell) + \frac{\mu^2(\rho)}{\delta^2}},$$

where ℓ is an affine function in space satisfying $|\ell(z_0)| + |D\ell| \le M$ and $\tilde{c}$ is a constant (depending on c_{Cacc}, c_{lin}) chosen large enough, we can deduce from Lemma 4.1 applied on $Q_{\rho/2}(z_0)$ and Lemma 3.1 that

$$\Big|\fint_{Q_{\rho/2}(z_0)} \Big[w\,\varphi_t - \frac{\partial A}{\partial p}\big(z_0, \ell(z_0), D\ell\big) Dw\,D\varphi\Big]\,dz\Big|$$
$$\le \Big[\omega^2\big((M + 1, \widetilde{\Psi}(z_0, \rho, \ell)\big) + \widetilde{\Psi}(z_0, \rho, \ell) + \frac{1}{2}\,\delta^2\Big]^{1/2} \sup_{Q_\rho(z_0)} |D\varphi|$$

holds for all $\varphi \in \mathcal{C}_0^\infty(Q_{\rho/2}(z_0), \mathbb{R}^N)$. Moreover, with the help of the Caccioppoli inequality Lemma 3.1 we can show that ω satisfies the hypothesis (2.16) of Lemma 2.6. We further set

$$\mathcal{A}(p, \widetilde{p}) := \Big\langle \frac{\partial A}{\partial p}(z_0, \ell(z_0), D\ell)\,p, \,\widetilde{p}\Big\rangle \qquad \text{for } p, \tilde{p} \in \mathbb{R}^{nN}.$$

We remark that, if A satisfies (A1) and (A2), then $\mathcal{A}$ is a strongly elliptic and bounded bilinear form on $\mathbb{R}^{nN}$. So, if the smallness condition (5.1) is satisfied, we can apply Lemma 2.6 to infer that there exists an $\mathcal{A}$-caloric function $h \in L^2(t_0 - \rho^2/4, t_0; W^{1,2}(B_{\rho/2}(x_0), \mathbb{R}^N))$ on the cylinder $Q_{\rho/2}(z_0)$ which is ε-close to w (in the sense of Lemma 2.6). Using Lemma 2.5 and the ε-closeness of w and h, recalling the definition of w and the minimizing property of $\ell_{z_0,\theta\rho/2}$, we can show that for each $\theta \in (0,1)$ there holds

$$\left(\frac{\theta\rho}{2}\right)^{-2} \fint_{Q_{\vartheta\rho/2}(z_0)} \left|u-\ell_{z_0,\theta\rho/2}\right|^2 dz \le c\,(\theta^{-n-4}\varepsilon+\theta^2)\left(\Psi(z_0,\rho,\ell)+\frac{\mu^2(\rho)}{\delta^2}\right).$$

With an appropriate choice of ε and θ, writing $\vartheta := \theta/2$, we get our assertion. □

Now we want to iterate the previous Lemma: for given $M > 0$ and $\beta \in (\alpha, 1)$, we determine $\delta = \delta(2M)$, $\vartheta = \vartheta(2M)$ and $c_1 = c_1(2M)$ accordingly to Lemma 5.1. Then, we can find $\widetilde{\Psi}_0(M) > 0$ sufficiently small, such that

$$\omega^2\big(2M+1, 2\widetilde{\Psi}_0(M)\big) + 2\,\widetilde{\Psi}_0(M) \le \frac{1}{2}\,\delta^2 \tag{5.2}$$

and

$$\widetilde{\Psi}_0(M) \le \frac{1}{4\,(n+2)^2}\, M^2 \vartheta^{n+4}(1-\vartheta^\beta)^2. \tag{5.3}$$

Given this we can also find $\rho_0(M) \in (0, \rho_1(2M)]$ so small that writing

$$c_2(M) := \frac{c_3(2M)}{\vartheta^{2\alpha}-\vartheta^{2\beta}},$$

we have, by (2.4),

$$c_2(M)\,\mu^2\big(\rho_0(M)\big) \le \min\left\{\delta^2, \widetilde{\Psi}_0(M), M^2\frac{\vartheta^{n+4}}{4(n+2)^2}\,(1-\vartheta^\alpha)^2\right\} \tag{5.4}$$

and, by ($\mu 0$),

$$c_2(M)\mathcal{M}(\rho_0(M))^2 \le M^2\frac{\vartheta^{n+4}}{4\alpha(n+2)^2}\,(1-\vartheta^\alpha)^2. \tag{5.5}$$

Now, suppose that the conditions

- (i) $|\ell_{z_0,\rho}(z_0)| + |D\ell_{z_0,\rho}| \le M$,
- (ii) $\rho \le \rho_0(M)$,
- (iii) $\widetilde{\Psi}(z_0,\rho,\ell_{z_0,\rho}) \le \widetilde{\Psi}_0(M)$

are satisfied on $Q_\rho(z_0)$. Then, for $j = 1, 2, \dots$ we shall inductively derive the following assertions:

- (I_j) $\widetilde{\Psi}(z_0,\vartheta^j\rho,\ell_{z_0,\vartheta^j\rho}) \le \vartheta^{2\beta j}\widetilde{\Psi}(z_0,\rho,\ell_{z_0,\rho}) + c_2(M)\mu^2(\vartheta^j\rho)$,
- (II_j) $|\ell_{z_0,\vartheta^j\rho}(z_0)| + |D\ell_{z_0,\vartheta^j\rho}| \le 2M$.

For the sake of brevity, in the following we shall omit in any notation the point z_0, unless when essential. We first note that (I_j), combined with (ii), (iii) and (5.4) yields

- (I'_j) $\widetilde{\Psi}(\vartheta^j\rho,\ell_{\vartheta^j\rho}) \le 2\,\widetilde{\Psi}_0(M)$.

Now we prove by induction (I_j) and (II_j). We first consider the case $j = 1$. We deduce, from the monotonicity of ω, (iii) and (5.2)

$$\omega^2\big(M+1, \widetilde{\Psi}(\rho,\ell_\rho)\big) + \widetilde{\Psi}(\rho,\ell_\rho) \le \omega^2\big(2M+1, \widetilde{\Psi}_0(M)\big) + 2\widetilde{\Psi}_0(M) \le \frac{1}{2}\,\delta^2.$$

Taking into account that $\rho \le \rho_0(M) \le \rho_1(M)$ and $|\ell_\rho(z_0)| + |D\ell_\rho| \le M$, we can apply Lemma 5.1 in order to conclude that (I_1) holds, using also ($\mu 1$):

$$\widetilde{\Psi}(\vartheta\rho,\ell_{\vartheta\rho}) \le \vartheta^{2\beta}\widetilde{\Psi}(\rho,\ell_\rho) + c_1(2M)\,\mu^2(\rho)$$

$$\begin{aligned}&\le \vartheta^{2\beta}\widetilde{\Psi}(\rho,\ell_\rho)+\frac{c_1(2M)}{\vartheta^{\alpha}}\,\mu^2(\vartheta\rho)\\&\le \vartheta^{2\beta}\widetilde{\Psi}(\rho,\ell_\rho)+c_2(M)\,\mu^2(\vartheta\rho).\end{aligned}$$

Furthermore, using Lemma 2.4, (iii) and (5.3) we infer

$$\begin{aligned}|\ell_{\vartheta\rho}(z_0)|+|D\ell_{\vartheta\rho}|&\le |\ell_\rho(z_0)|+|D\ell_\rho|+|\ell_{\vartheta\rho}(z_0)-\ell_\rho(z_0)|+|D\ell_{\vartheta\rho}-D\ell_\rho|\\&\le M+\left|\fint_{Q_{\vartheta\rho}(z_0)}\big[u-(u)_{z_0,\rho}\big]\,dz\right|+\left[\frac{n(n+2)}{\vartheta^2\rho^2}\fint_{Q_{\vartheta\rho}(z_0)}|u-\ell_\rho|^2\,dz\right]^{1/2}\\&\le M+\left[\fint_{Q_{\vartheta\rho}(z_0)}|u-\ell_\rho|^2\,dz\right]^{1/2}+\left[\frac{n(n+2)}{\vartheta^2\rho^2}\fint_{Q_{\vartheta\rho}(z_0)}|u-\ell_\rho|^2\,dz\right]^{1/2}\\&\le M+\left[\frac{\vartheta^2\rho^2}{\vartheta^{n+4}}\,\widetilde{\Psi}(\rho,\ell_\rho)\right]^{1/2}+\left[\frac{n(n+2)}{\vartheta^{n+4}}\,\widetilde{\Psi}(\rho,\ell_\rho)\right]^{1/2}\\&\le M+\frac{1+\sqrt{n(n+2)}}{\sqrt{\vartheta^{n+4}}}\sqrt{\widetilde{\Psi}_0(M)}\le M+\frac{n+2}{\sqrt{\vartheta^{n+4}}}\sqrt{\widetilde{\Psi}_0(M)}\le 2M.\end{aligned}$$

i.e. II_1 holds. We next want to show (I_j): we assume (I_m) and (II_m) (and therefore (I'_m)) hold for $m=1,\dots,j-1$. Then (I'_m), (II_m) and (5.2) imply that we can apply Lemma 5.1 for $m=1,\dots,j-1$. Recalling the definition of $c_2(M)$ and ($\mu1$) we obtain

$$\begin{aligned}\widetilde{\Psi}(\vartheta^j\rho,\ell_{\vartheta^j\rho})&\le \vartheta^{2\beta j}\widetilde{\Psi}(\rho,\ell_\rho)+c_1(2M)\sum_{m=0}^{j-1}\vartheta^{2\beta m}\mu^2(\vartheta^{j-m-1}\rho)\\&\le \vartheta^{2\beta j}\widetilde{\Psi}(\rho,\ell_\rho)+c_1(2M)\mu^2(\vartheta^j\rho)\,\vartheta^{-2\alpha}\sum_{m=0}^{j-1}\vartheta^{2(\beta-\alpha)m}\\&\le \vartheta^{2\beta j}\widetilde{\Psi}(\rho,\ell_\rho)+\frac{c_1(2M)}{\vartheta^{2\alpha}-\vartheta^{2\beta}}\mu^2(\vartheta^j\rho),\end{aligned}$$

showing (I_j). In order to show (II_j) we use Lemma 2.4, (I_m) for $m=1,\dots,j-1$, (2.11), (5.3) and (5.5) to infer

$$\begin{aligned}|\ell_{\vartheta^j\rho}(z_0)|+|D\ell_{\vartheta^j\rho}|&\le |\ell_\rho(z_0)|+|D\ell_\rho|\\&\quad+\sum_{m=1}^{j}|\ell_{\vartheta^m\rho}(z_0)-\ell_{\vartheta^{m-1}\rho}(z_0)|+\sum_{m=1}^{j}|D\ell_{\vartheta^m\rho}-D\ell_{\vartheta^{m-1}\rho}|\\&\le M+\sum_{m=1}^{j}\left[\fint_{Q_{\vartheta^m\rho/2}(z_0)}|u-\ell_{\vartheta^{m-1}\rho}|^2\,dz\right]^{1/2}\\&\qquad+\sum_{m=1}^{j}\left[\frac{n(n+2)}{(\vartheta^m\rho)^2}\fint_{Q_{\vartheta^m\rho/2}(z_0)}|u-\ell_{\vartheta^{m-1}\rho}|^2\,dz\right]^{1/2}\\&\le M+\frac{1+\sqrt{n(n+2)}}{\sqrt{\vartheta^{n+4}}}\sum_{m=1}^{j}\sqrt{\widetilde{\Psi}(\vartheta^{m-1}\rho,\ell_{\vartheta^{m-1}\rho})}\\&\le M+\frac{n+2}{\sqrt{\vartheta^{n+4}}}\sum_{m=0}^{j-1}\sqrt{\vartheta^{2m\beta}\widetilde{\Psi}(\rho,\ell_\rho)+c_2(M)\mu^2(\vartheta^m\rho)}\end{aligned}$$

$$\le M + \frac{n+2}{\sqrt{\vartheta^{n+4}}}\biggl(\frac{\sqrt{\widetilde{\Psi}(\rho,\ell_\rho)}}{1-\vartheta^\beta} + \frac{\alpha\sqrt{c_2(M)}}{1-\vartheta^\alpha}\mathcal{M}(\rho)\biggr)$$
$$\le M + \frac{M}{2} + \frac{n+2}{\sqrt{\vartheta^{n+4}}}\frac{\alpha\sqrt{c_2(M)}}{1-\vartheta^\alpha}\mathcal{M}(\rho_0(M)) \le 2M.$$

Since at this point the case $\beta = \alpha$ is trivial, the above reasoning proves the first assertion of the following:

Lemma 5.2. *For $M > 0$ and $Q_\rho(z_0) \Subset \Omega_T$, suppose that the conditions*

(*i*) $|\ell_{z_0,\rho}(z_0)| + |D\ell_{z_0,\rho}| \le M$,
(*ii*) $\rho \le \rho_0(M)$,
(*iii*) $\widetilde{\Psi}(z_0,\rho,\ell_{z_0,\rho}) \le \widetilde{\Psi}_0(M)$

are satisfied. Then, for all $j \in \mathbb{N}$, we have

$$\widetilde{\Psi}(z_0,\vartheta^j\rho,\ell_{z_0,\vartheta^j\rho}) \le \vartheta^{2\beta j}\widetilde{\Psi}(z_0,\rho,\ell_{z_0,\rho}) + c_2(M)\mu^2(\vartheta^j\rho) \tag{5.6}$$

and $|\ell_{z_0,\vartheta^j\rho}(z_0)| + |D\ell_{z_0,\vartheta^j\rho}| \le 2M$. Moreover, Du is continuous in a neighborhood of z_0 and its modulus of continuity, with respect the parabolic metric, is $r \longmapsto r^\beta + \mathcal{M}(r)$ for each $\beta \in [\alpha, 1)$.

Proof. It remains to prove the existence of a neighborhood of z_0 where Du is continuous. Since the functions $z \mapsto \ell_{z,\rho}$, $z \mapsto D\ell_{z,\rho}$ and $z \mapsto \widetilde{\Psi}(z,\rho,\ell_{z,\rho})$ are continuous, there exists a neighborhood $\mathcal{J}$ of z_0 such that for all $z \in \mathcal{J}$, $Q_\rho(z) \Subset \Omega_T$ and both (i) and (iii) hold, with M replaced by $M+1$ and $\widetilde{\Psi}_0(M)$ by $\widetilde{\Psi}_0(M)+1$.

We now fix $z \in \mathcal{J}$. We shall again omit in any notation the point z, unless when necessary. Since $|D\ell_{\vartheta^j\rho}| \le 2(M+1)$, we are in a position to apply Lemma 3.1 which, together with the minimizing property of $(Du)_{\vartheta^j\rho/2}$ and (5.6), yields

$$\Phi\bigl(\vartheta^j\rho/2,(Du)_{\vartheta^j\rho/2}\bigr) \le \Phi\bigl(\vartheta^j\rho/2, D\ell_{\vartheta^j\rho/2}\bigr) \le c_{Cacc}\widetilde{\Psi}(\vartheta^j\rho,\ell_{\vartheta^j\rho})$$
$$\le c(\lambda,L,M)\Bigl(\vartheta^{2\beta j}\widetilde{\Psi}(\rho,\ell_\rho) + \mu^2(\vartheta^j\rho)\Bigr). \tag{5.7}$$

Now we prove that $\{(Du)_{\vartheta^j\rho/2}\}_j$ is a Cauchy sequence: using (5.7) and arguing as in the proof of the inductive step in (II_j) we deduce, for $k > j$,

$$\bigl|(Du)_{\vartheta^j\rho/2} - (Du)_{\vartheta^k\rho/2}\bigr| \le \sum_{m=j+1}^{k}\bigl|(Du)_{\vartheta^m\rho/2} - (Du)_{\vartheta^{m-1}\rho/2}\bigr|$$
$$\le \sqrt{\vartheta^{-n-2}}\sum_{m=j}^{k-1}\biggl[\fint_{Q_{\vartheta^m\rho/2}(z_0)}\bigl|Du - (Du)_{\vartheta^m\rho/2}\bigr|^2\,dz\biggr]^{1/2}$$
$$= \sqrt{\vartheta^{-n-2}}\sum_{m=j}^{k-1}\sqrt{\Phi\bigl(\vartheta^m\rho/2,(Du)_{\vartheta^m\rho/2}\bigr)}$$
$$\le \sqrt{c(\lambda,L,M)\,\vartheta^{-n-2}}\biggl[\frac{\sqrt{\widetilde{\Psi}(\rho,\ell_\rho)}}{1-\vartheta^\beta}\,\vartheta^{\beta j} + \frac{\alpha}{1-\vartheta^\alpha}\,\mathcal{M}(\vartheta^j\rho)\biggr]. \tag{5.8}$$

Hence $\{(Du)_{z,\vartheta^j\rho/2}\}$ is a Cauchy sequence in $\mathbb{R}^{nN}$ and it converges to some $\widetilde{Du}(z) \in \mathbb{R}^{nN}$. Taking the limit $k \to \infty$ in (5.8) we infer that

$$\bigl|(Du)_{\vartheta^j\rho/2} - \widetilde{Du}(z)\bigr| \le c\Bigl[\vartheta^{\beta j} + \mathcal{M}(\vartheta^j\rho)\Bigr] \tag{5.9}$$

with $c = c(n, N, \lambda, L, \alpha, \beta, H(M))$ (note that $\widetilde{\Psi}(\rho) \le \widetilde{\Psi}_0(M) + 1$ by our choice of $\mathcal{J}$). Since $z \mapsto (Du)_{z,\vartheta^j\rho/2}$ is continuous in $\mathcal{J}$ for all j and estimate (5.9) is uniform in z, the limit function $\widetilde{Du}(z)$ is continuous, being the uniform limit of continuous functions. Let $z_1 = (x_1, t_1),\ z_2 = (x_2, t_2) \in \mathcal{J}$, with $t_1 \le t_2$. We could, if necessary, shrink $\mathcal{J}$ in order to have $diam_{\mathcal{P}}(\mathcal{J}) \le \rho/4$. We set $\delta := d_{\mathcal{P}}(z_1, z_2)$, and let $j \in \mathbb{N} \cup \{0\}$ such that $\vartheta^{j+1}\rho/4 < \delta \le \vartheta^j\rho/4$. We have

$$|\widetilde{Du}(z_1) - \widetilde{Du}(z_2)| \le |\widetilde{Du}(z_1) - (Du)_{z_1,\vartheta^j\rho/2}| + |\widetilde{Du}(z_2) - (Du)_{z_2,\vartheta^j\rho/2}| + |(Du)_{z_1,\vartheta^j\rho/2} - (Du)_{z_2,\vartheta^j\rho/2}|. \tag{5.10}$$

In order to estimate the third term we write, for any $w \in Q_{\vartheta^j\rho/4}(z_2)$,

$$\left|(Du)_{z_1,\vartheta^j\rho/2} - (Du)_{z_2,\vartheta^j\rho/2}\right| \le \left|(Du)_{z_1,\vartheta^j\rho/2} - Du(w)\right| + \left|(Du)_{z_2,\vartheta^j\rho/2} - Du(w)\right|;$$

now we observe that $Q_\delta(z_2) \subset Q_{2\delta}(z_1) \cap Q_{2\delta}(z_2)$, and hence

$$Q_{\vartheta^j\rho/4}(z_2) \subset Q_{\vartheta^j\rho/2}(z_1) \cap Q_{\vartheta^j\rho/2}(z_2). \tag{5.11}$$

Integrating over $Q_{\vartheta^j\rho/4}(z_2)$ and dividing by $|Q_{\vartheta^j\rho/4}|$ we get, using (5.11), (5.7) and (2.12)

$$\begin{aligned}\left|(Du)_{z_1,\vartheta^j\rho/2} - (Du)_{z_2,\vartheta^j\rho/2}\right| &\le c(n) \Big[\fint_{Q_{\vartheta^j\rho/2}(z_1)} \left|Du - (Du)_{z_1,\vartheta^j\rho/2}\right|^2 dw \\ &\quad + \fint_{Q_{\vartheta^j\rho/2}(z_2)} \left|Du - (Du)_{z_2,\vartheta^j\rho/2}\right|^2 dw \Big]^{1/2} \\ &\le c\left(\vartheta^{\beta j}\sqrt{\widetilde{\Psi}(\rho)} + \mu(\vartheta^j\rho)\right) \le c\left(\vartheta^{\beta j}\sqrt{\widetilde{\Psi}(\rho)} + \mathcal{M}(\vartheta^j\rho)\right),\end{aligned}$$

with $c = c(n, N, \lambda, L, M, \alpha, \beta)$. Next, we estimate the first two terms in (5.10) using (5.9) and the third one using the previous inequality, obtaining

$$\begin{aligned}|\widetilde{Du}(z_1) - \widetilde{Du}(z_2)| &\le c\left(\vartheta^{\beta j}\sqrt{\widetilde{\Psi}(\rho)} + \mathcal{M}(\vartheta^j\rho)\right) \\ &\le c\left[\left(\frac{2\delta}{\rho}\right)^\beta \sqrt{\widetilde{\Psi}(\rho)} + \mathcal{M}(\delta)\left(\frac{\vartheta}{2}\right)^\alpha\right] \\ &\le c\left[d_{\mathcal{P}}(z_1, z_2)^\beta + \mathcal{M}(d_{\mathcal{P}}(z_1, z_2))\right],\end{aligned}$$

and this shows that the modulus of continuity of $\tilde{u}$ is $r \mapsto r^\beta + \mathcal{M}(r)$ for each $\beta \in [\alpha, 1)$. Since by Lebesgue differentiation theorem Du and $\widetilde{Du}$ coincide almost everywhere, we now have proved also the second part of the statement. □

Up to now, in Lemma 5.2 and recalling also Lemma 2.4 and the definition of $\ell_{z_0,\rho}$, we have proved that if u is a weak solution of the parabolic system (1.1) under the assumption (A1), (A2), (A3), ($\mu 0$) and ($\mu 1$), then $Sing(u) \subset \Sigma_0 \cup \Sigma_2$ with Σ_0 being the subset of the points $z_0 \Omega_T$ such that

$$\liminf_{\rho \searrow 0} \frac{1}{\rho^2} \fint_{Q_\rho(z_0)} \left|u - (u)_{z_0,\rho} - (Du)_{z_0,\rho}(x - x_0)\right|^2 dz > 0 \tag{5.12}$$

and Σ_2 as in the statement of Theorem 2.1. Moreover, for each $z_0 \in \Omega_T \setminus \Sigma$ and for each $\beta \in [\alpha, 1)$, Du has the modulus of continuity $\rho \mapsto \rho^\beta + \mathcal{M}(\rho)$ with respect the standard parabolic metric (1.3) in a neighborhood of z_0.

This property of singular points does not still allow us to deduce that the singular set is Lebesgue-negligible. We can however easily deduce the property stated in Theorem 2.1 arguing as in Section 8 of [12], substituting the Hölder modulus of continuity with a generic Dini one.

6. Fractional estimates

The aim of this Chapter is to derive L^2-estimates for the finite differences of the spatial gradient Du, both in space and in time, in terms of our modulus of continuity μ. The classical parabolic higher integrability lemma (see [16]) tells us that a weak solution $v \in L^2(-T,0;W^{1,2}(\Omega,\mathbb{R}^N))$ to our parabolic system (1.1) is higher integrable and satisfies a Reverse-Hölder inequality: more precisely, there exists $\delta_0 = \delta_0(n, L/\lambda) > 0$ such that for all open sets $\mathcal{O} \subset\subset \Omega_T$

$$\left(\fint_{\mathcal{O}} |Dv|^{2(1+\delta_0)}\,dz\right)^{1/(1+\delta_0)} \le c_{RH} \fint_{\Omega_T} \big(1+|Dv|\big)^2\,dz, \tag{6.1}$$

where $c_{RH} = c_{RH}(n, L/\lambda, dist(\partial\Omega,\mathcal{O}))$. Let us remark that for such an estimate the only properties of the vector field we request are the linear growth (A1) and the ellipticity (A2). For more recent results on gradient estimates in the parabolic setting we refer to [19, 1]. We define, for e unit vector in $\mathbb{R}^n$ and both $(x,t),(x+h\,e,t) \in \Omega_T$, the finite difference $\tau_{h,e} \equiv \tau_{h,e}(x,t) := u(x+h\,e,t) - u(x,t)$. Since $u(t,\cdot) \in W^{1,2}(\Omega,\mathbb{R}^N)$ for $a.e.$ $t \in [-T,0]$, we immediately have

$$\int_{-T}^{0}\!\!\int_{\widetilde{\Omega}} |u(x+he,t)-u(x,t)|^2\,dx\,dt \le |h|^2 \int_{\Omega_T} |Du|^2\,dz, \tag{6.2}$$

with $\widetilde{\Omega} \subset\subset \Omega$ and $|h| \le \min\{dist(\partial\Omega,\widetilde{\Omega}),1\}/2$ and also the L^1-version:

$$\int_{-T}^{0}\!\!\int_{\widetilde{\Omega}} |u(x+he,t)-u(x,t)|\,dx\,dt \le |h| \int_{\Omega_T} |Du|\,dz. \tag{6.3}$$

Let us fix $-T < \tilde{t} < t_1 < 0$, arbitrarily, and let $t_0 := (-T+\tilde{t})/2 < \tilde{t}$. In the following we shall always take $|h| \le \min\big\{dist(\partial\Omega,\widetilde{\Omega}),1\big\}/2$ and $t \in (t_0,t_1)$. We choose $\zeta(t) \in \mathcal{C}^\infty((-T,0))$ such that $\zeta \equiv 0$ on $(-T,t_0)$, $\zeta \equiv 1$ on $(\tilde{t},0)$, $0 \le \zeta' \le 4/(T+t_0)$ and $\eta \in \mathcal{C}^\infty_0(\Omega)$ a cut-off function in space, $0 \le \eta \le 1$ and $\eta \equiv 1$ on $\widetilde{\Omega}$. Using the formulation of (2.14) in terms of Steklov averages, one can prove (see, e.g., [12]) that for $a.e.$ t_1 as before

$$\begin{aligned}\frac{1}{2}\int_{\widetilde{\Omega}} \zeta^2(t_1)\,\eta^2 |\tau_{h,e}u(\cdot,t_1)|^2\,dx - \int_{-T}^{t_1}\!\!\int_{\widetilde{\Omega}} \zeta\zeta'\eta^2|\tau_{h,e}u|^2\,dx\,dt\\ + \int_{t_0}^{t_1}\!\!\int_{\widetilde{\Omega}} \zeta^2\tau_{h,e}\big[A(x,t,u,Du)\big]\Big[2\eta D\eta\otimes(\tau_{h,e}u) + \eta^2 D(\tau_{h,e}u)\Big]\,dx\,dt = 0.\end{aligned}$$

To obtain an estimate of $\tau_{h,e}Du$ we shall ignore the contribution of the first integral and decompose $\tau_{h,e}[A]$ as follows:

$$\begin{aligned}&\tau_{h,e}\Big[A\big(\cdot,\cdot\cdot,u(\cdot,\cdot\cdot),Du(\cdot,\cdot\cdot)\big)\Big](x,t)\\ &= A\big(x+he,t,u(x+he,t),Du(x+he,t)\big) - A\big(x+he,t,u(x+he,t),Du(x,t)\big)\\ &\quad + A\big(x+he,t,u(x+he,t),Du(x,t)\big) - A\big(x+he,t,u(x,t),Du(x,t)\big)\\ &\quad + A\big(x+he,t,u(x,t),Du(x,t)\big) - A\big(x,t,u(x,t),Du(x,t)\big)\end{aligned}$$

$$=: \mathscr{A}(h) + \mathscr{B}(h) + \mathscr{C}(h). \tag{6.4}$$

We estimate separately the integrals: we write $\mathscr{A}(h)(x,t) = \widetilde{\mathscr{A}(h)}\,[\tau_{h,e} Du](x,t)$, with

$$\widetilde{\mathscr{A}(h)} := \int_0^1 \frac{\partial A}{\partial p}\big(x+he, t, u(x+he,t), Du(x,t) + s\tau_{h,e}Du(x,t)\big)\, ds, \tag{6.5}$$

and we infer from the ellipticity (A2) of A

$$\int_{t_0}^{t_1}\int_{\widetilde{\Omega}} \zeta^2\eta^2 \mathscr{A}(h) D(\tau_{h,e}u)\,dx\,dt \ge \lambda \int_{t_0}^{t_1}\int_{\widetilde{\Omega}} \zeta^2\eta^2 |\tau_{h,e}Du|^2\,dx\,dt.$$

In order to evaluate the remaining integral involving $\mathscr{A}(h)$ we use (2.2), Young's inequality with $\varepsilon \in (0,1)$ to be chosen, (6.2) and the fact that $s \le \mu(s)$ for $s \le 1$:

$$\begin{aligned}
&\int_{t_0}^{t_1}\int_{\widetilde{\Omega}} \zeta^2\eta\,|D\eta|\,|\mathscr{A}(h)|\,|\tau_{h,e}u|\;dx\,dt\\
&\qquad \le \varepsilon \int_{t_0}^{t_1}\int_{\widetilde{\Omega}} \zeta^2\eta^2|\tau_{h,e}Du|^2\,dx\,dt + \frac{L^2}{\varepsilon}\,||D\eta||^2_{L^\infty} \int_{t_0}^{t_1}\int_{\widetilde{\Omega}} |\tau_{h,e}u|^2\,dx\,dt\\
&\qquad \le \varepsilon \int_{t_0}^{t_1}\int_{\widetilde{\Omega}} \zeta^2\eta^2|\tau_{h,e}Du|^2\,dx\,dt + \frac{L^2}{\varepsilon}\,||D\eta||^2_{L^\infty}\,\mu^2(|h|)\int_{\Omega_T} |Du|^2\,dz.
\end{aligned}$$

Similarly, using $\zeta' \le 4/(T+t_0)$, (6.2) and $s \le \mu(s)$ for $s \le 1$, we obtain

$$\int_{-T}^{t_1}\int_{\widetilde{\Omega}} \eta^2\zeta\zeta'|\tau_{h,e}u|^2\,dx\,dt \le c\,(t_0)\,\mu^2(|h|)\int_{\Omega_T}|Du|^2\,dz.$$

Next we estimate the integrals involving $\mathscr{C}(h)$: using the uniform Dini continuity (A4), Hölder's inequality, (6.2) and the fact that $|h| \le \mu(|h|)$: we get

$$\begin{aligned}
&\int_{t_0}^{t_1}\int_{\widetilde{\Omega}} \zeta^2\eta\,|D\eta|\,|\mathscr{C}(h)|\,|\tau_{h,e}u|\;dx\,dt\\
&\le L\,\mu(|h|)\int_{t_0}^{t_1}\int_{\widetilde{\Omega}} \zeta^2\eta\,|D\eta|\,\big(1+|Du|\big)\,|\tau_{h,e}u|\;dx\,dt\\
&\le L\,||D\eta||_{L^\infty}\mu(|h|)\bigg(\int_{\Omega_T}\big(1+|Du|^2\big)\,dz\bigg)^{1/2}\bigg(\int_{t_0}^{t_1}\int_{\widetilde{\Omega}}|\tau_{h,e}u|^2\,dx\,dt\bigg)^{1/2}\\
&\le c\,\big(L, |\Omega_T|, ||D\eta||_{L^\infty}, ||Du\,||_{L^2}\big)\,\mu^2(|h|).
\end{aligned}$$

(A4) and Young's inequality allow us to estimate

$$\begin{aligned}
&\int_{t_0}^{t_1}\int_{\widetilde{\Omega}} \zeta^2\eta^2|\mathscr{C}(h)|\;|\tau_{h,e}Du|\;dx\,dt\\
&\qquad \le \mu(|h|)\int_{t_0}^{t_1}\int_{\widetilde{\Omega}} \zeta^2\eta^2 L\big(1+|Du|\big)\,|\tau_{h,e}Du|\;dx\,dt\\
&\qquad \le \varepsilon\int_{t_0}^{t_1}\int_{\widetilde{\Omega}} \zeta^2\eta^2|\tau_{h,e}Du|^2\,dx\,dt + \frac{2L^2\mu^2(|h|)}{\varepsilon}\int_{\Omega_T}\big(1+|Du|^2\big)\,dz.
\end{aligned}$$

In order to evaluate terms containing $\mathscr{B}(h)$ we need an auxiliary estimate: using Hölder inequality, the fact that $\mu \le 1$, Jensen's inequality, (6.1) and (6.3), we get

$$\bigg(\int_{t_0}^{t_1}\int_{\widetilde{\Omega}} \big(1+|Du|\big)^2\mu^2\big(|\tau_{h,e}u|\big)\,dx\,dt\bigg)^{1+\delta_0}$$

$$
\begin{aligned}
&\le \left(\int_{t_0}^{t_1}\int_{\widetilde\Omega} \big(1+|Du|\big)^{2(1+\delta_0)}\,dx\,dt\right)\left(\int_{t_0}^{t_1}\int_{\widetilde\Omega} \mu^{2(1+\delta_0)/\delta_0}\big(|\tau_{h,e}u|\big)\,dx\,dt\right)^{\delta_0}\\
&\le c\left(\fint_{(t_0,t_1)\times\widetilde\Omega} \big(1+|Du|\big)^{2(1+\delta_0)}\,dx\,dt\right)\left(\fint_{(t_0,t_1)\times\widetilde\Omega} \mu\big(|\tau_{h,e}u|\big)\,dx\,dt\right)^{\delta_0}\\
&\le c\left(\fint_{\Omega_T} \big(1+|Du|\big)^{2}\,dx\,dt\right)^{1+\delta_0}\mu\left(\fint_{(t_0,t_1)\times\widetilde\Omega} |\tau_{h,e}u|\,dx\,dt\right)^{\delta_0}\\
&\le c\,\mu\left(\fint_{(t_0,t_1)\times\widetilde\Omega} |\tau_{h,e}u|\,dx\,dt\right)^{\delta_0} \le c\,\mu\left(|h|\int_{\Omega_T}|Du|\,dz\right)^{\delta_0}\\
&\le c\,\mu(|h|)^{\delta_0},
\end{aligned}
\tag{6.6}
$$

so that

$$
\int_{t_0}^{t_1}\int_{\widetilde\Omega} \big(1+|Du|\big)^2\mu^2\big(|\tau_{h,e}u|\big)\,dx\,dt \le c\,\mu(|h|)^{\delta_0/(1+\delta_0)},
\tag{6.7}
$$

with $c = c\,(n, L/\lambda, |\Omega_T|, \|Du\|_{L^2}, dist(\partial\Omega,\widetilde\Omega))$. In the last line in (6.6) we used (2.7), if $\int_{\Omega_T}|Du|\,dz \ge 1$, or the non-decreasing property of μ, otherwise. Using (6.7) we have

$$
\begin{aligned}
&\int_{t_0}^{t_1}\int_{\widetilde\Omega}\zeta^2\eta\,|D\eta|\,|\mathscr{B}(h)|\,|\tau_{h,e}u|\,dx\,dt\\
&\qquad\le ||D\eta||_{L^\infty}\int_{t_0}^{t_1}\int_{\widetilde\Omega}\big(1+|Du|\big)\mu\big(|\tau_{h,e}u|\big)\,|\tau_{h,e}u|\,dx\,dt\\
&\qquad\le c\left(\int_{t_0}^{t_1}\int_{\widetilde\Omega}\big(1+|Du|^2\big)\mu^2\big(|\tau_{h,e}u|\big)\,dx\,dt\right)^{1/2}\\
&\qquad\qquad\qquad\qquad\left(\int_{t_0}^{t_1}\int_{\widetilde\Omega}|\tau_{h,e}u|^2\,dx\,dt\right)^{1/2}\\
&\qquad\le c\,\mu(|h|)^{\delta_0/(2+2\delta_0)}\,|h|\int_{\Omega_T}|Du|^2\,dz\\
&\qquad\le c\,\mu(|h|)^{\delta_0/(2+2\delta_0)+1},
\end{aligned}
$$

where $c = c\,(n, L/\lambda, |\Omega_T|, \|D\eta\|_{L^\infty}, \|Du\|_{L^2}, dist(\partial\Omega,\widetilde\Omega))$. Similarly we get, using also Young's inequality:

$$
\begin{aligned}
&\int_{t_0}^{t_1}\int_{\widetilde\Omega}\zeta^2\eta^2|\mathscr{B}(h)|\,|\tau_{h,e}Du|\,dx\,dt\\
&\qquad\qquad\le \varepsilon\int_{t_0}^{t_1}\int_{\widetilde\Omega}\zeta^2\eta^2|\tau_{h,e}Du|^2\,dx\,dt + \frac{c}{\varepsilon}\,\mu(|h|)^{\delta_0/(1+\delta_0)}.
\end{aligned}
$$

Combining the previous estimates we obtain

$$
\begin{aligned}
&(\lambda-3\varepsilon)\int_{t_0}^{t_1}\int_{\widetilde\Omega}\zeta^2\eta^2|\tau_{h,e}Du|^2\,dx\,dt\\
&\quad\le c\Big[\mu^2(|h|)+\mu(|h|)^{\delta_0/(2+\delta_0)+1}+\mu(|h|)^{\delta_0/(1+\delta_0)}\Big] \le C\,\mu(|h|)^{\delta_0/(2+2\delta_0)},
\end{aligned}
$$

with $C = C(n, \lambda, L, |\Omega_T|, ||D\eta||_{L^\infty}, ||Du||_{L^2}, dist(\partial\Omega, \widetilde{\Omega}), t_0, \varepsilon)$. Now if we choose $\varepsilon := \lambda/6$ and we write $\gamma := \delta_0/4(1+\delta_0)$, we get the following:

Lemma 6.1. *Let $u \in L^2(-T, 0; W^{1,2}(\Omega, \mathbb{R}^N))$ be a weak solution to the parabolic system (1.1) under the assumptions (A1), (A2), (A4), (μ0) and (μ1). For any $\widetilde{\Omega} \subset\subset \Omega$, $-T < t_0 < t_1 < 0$, $|h| \le \min\{dist(\partial\Omega, \widetilde{\Omega}), 1\}/2$ and e unit vector in $\mathbb{R}^n$, there exists $\gamma = \gamma(n, L/\lambda) \in (0, 1/4)$ and $C = C(n, L, \lambda, |\Omega_T|, t_0, dist(\partial\Omega, \widetilde{\Omega}), ||Du||_{L^2})$ such that*

$$\int_{t_0}^{t_1} \int_{\widetilde{\Omega}} |Du(x+he, t) - Du(x, t)|^2 \, dx \, dt \le C \, \mu^{2\gamma}(|h|). \tag{6.8}$$

Proceeding in a similar way we can prove the temporal analogue of Lemma 6.1. Thereby we have to keep in mind that a weak solution to (2.14) is not a priori weakly differentiable with respect the temporal variable, the analogue of (6.2) is not immediate. However, using the Steklov averages formulation, it can be easily proved (see for instance [12]) that a weak solution to (1.1) with the only assumption (2.1) satisfies

$$\int_{t_0}^{t_1} int_{\widetilde{\Omega}} |u(x, t+h) - u(x, t)|^2 \, dx \, dt \le c \, |h| \int_{\Omega_T} \left(1 + |Du|^2\right) dz, \tag{6.9}$$

whenever $\widetilde{\Omega} \subset\subset \Omega$ and $-T < t_0 < t_1 < 0$, for all $0 < |h| < \min\{|t_1|, T - |t_0|, 1\}/2$, with $c = c\,(L, dist(\widetilde{\Omega}, \partial\Omega))$. Using this estimate we can deduce with similar arguments as before the following

Lemma 6.2. *Let $u \in L^2(-T, 0; W^{1,2}(\Omega, \mathbb{R}^N))$ be a weak solution to the parabolic system (1.1) under the assumptions (A1), (A2), (A4), (μ0) and (μ1). For any $\widetilde{\Omega} \subset\subset \Omega$, $-T < t_0 < t_1 < 0$, $|h| \le \min\{|t_1|, T - |t_0|, 1\}/2$, there exists $\gamma \in (0, 1/4)$ and C, having both the same dependencies as the constants appearing in (6.8), such that*

$$\int_{t_0}^{t_1} \int_{\widetilde{\Omega}} |Du(x, t+h) - Du(x, t)|^2 \, dx \, dt \le C \mu^{2\gamma}(|h|^{1/2}).$$

We remark that the constant γ appearing in this lemma is exactly the same as in Lemma 6.1. Moreover this constant, which depends directly on the constant of higher integrability of Giaquinta-Struwe Lemma, has a critical dependence on the ellipticity ratio L/λ in the sense that $\lim_{L/\lambda \nearrow \infty} \gamma = 0$ this arises from the analogous behavior of δ_0.

7. Generalized Sobolev spaces

In the following, by a modulus of continuity we understand a continuous, increasing function $\widetilde{\omega} : [0, \infty) \to [0, \infty)$ with $\widetilde{\omega}(0) = 0$. Let $\widetilde{\omega}_1$, $\widetilde{\omega}_2$ be two moduli of continuity. For $q \in [1, \infty)$ we say that a function $u \in L^q(\Omega_T, \mathbb{R}^N)$ belongs to the Sobolev space $W^{\widetilde{\omega}_1, \widetilde{\omega}_2; q}(\Omega_T, \mathbb{R}^N)$ if its seminorm $[u]_{\widetilde{\omega}_1, \widetilde{\omega}_2; q}$ is finite:

$$[u]^q_{\widetilde{\omega}_1, \widetilde{\omega}_2; q} := \int_{-T}^{0} \int_\Omega \int_\Omega \frac{|u(x,t) - u(y,t)|^q}{|x-y|^n \widetilde{\omega}_1(|x-y|)^q} \, dx \, dy \, dt + \int_\Omega \int_{-T}^{0} \int_{-T}^{0} \frac{|u(x,t) - u(x,s)|^q}{|t-s| \, \widetilde{\omega}_2(|t-s|)^q} \, dt \, ds \, dx < \infty.$$

The local variant $W^{\widetilde{\omega}_1,\widetilde{\omega}_2;q}_{loc}(\Omega_T,\mathbb{R}^N)$ can be defined in the usual way. We remark that the choice of $\widetilde{\omega}_1(\rho):=\rho^\alpha$ e $\widetilde{\omega}_2(\rho):=\rho^\beta$, $\alpha,\beta\in(0,1)$, as moduli of continuity leads to the parabolic fractional Sobolev spaces $W^{\alpha,\beta;q}$. For fixed $u\in W^{\widetilde{\omega}_1,\widetilde{\omega}_2;q}(\Omega_T,\mathbb{R}^N)$ we define the following function:

$$\begin{aligned}\lambda_{\widetilde{\omega}_1,\widetilde{\omega}_2;q}(u,\cdot): I\times\mathcal{O}\longmapsto &\int_I\int_{\mathcal{O}}\int_{\mathcal{O}}\frac{|u(x,t)-u(y,t)|^q}{|x-y|^n\widetilde{\omega}_1(|x-y|)^q}\,dx\,dy\,dt\\ &+\int_{\mathcal{O}}\int_I\int_I\frac{|u(x,t)-u(x,s)|^q}{|t-s|\,\widetilde{\omega}_2(|t-s|)^q}\,dt\,ds\,dx \qquad (7.1)\end{aligned}$$

with $I\subset(-T,0)$ and $\mathcal{O}\subset\Omega$ Borel sets. Observe that $\lambda_{\widetilde{\omega}_1,\widetilde{\omega}_2;q}(u,\Omega_T)=[u]^q_{\widetilde{\omega}_1,\widetilde{\omega}_2;q}$.

Actually we shall need no function space theoretic property of such space, and we could have even avoided introducing them, but we did so in order to have a bit of notation at our disposal. The following is a Poincaré type inequality for functions belonging to $W^{\widetilde{\omega}_1,\widetilde{\omega}_2;q}(\Omega_T,\mathbb{R}^N)$:

Proposition 7.1. *Let $u\in W^{\widetilde{\omega}_1,\widetilde{\omega}_2;\,q}_{loc}(\Omega_T,\mathbb{R}^N)$, with $\widetilde{\omega}_1(2\rho)\le A\,\widetilde{\omega}_1(\rho)$ for some $A>0$ and $Q_\rho(z_0)\subset\subset\Omega_T$. Then*

$$\int_{Q_\rho(z_0)}\big|u(z)-(u)_{z_0,\rho}\big|^q\,dz\le c(n,q,A)\Big[\widetilde{\omega}_1^q(\rho)+\widetilde{\omega}_2^q(\rho^2)\Big]\lambda_{\widetilde{\omega}_1,\widetilde{\omega}_2;q}(u,Q_\rho(z_0)). \tag{7.2}$$

Proof. Let $z=(x,t)\in Q_\rho(z_0)=B_\rho(x_0)\times(t_0-\rho^2,t_0)$. Using Jensen's inequality we get

$$\begin{aligned}\big|u(z)-(u)_{z_0,\rho}\big|^q&\le \fint_{Q_\rho(z_0)}|u(x,t)-u(y,s)|^q\,dy\,ds\\ &\le c(q)\fint_{Q_\rho(z_0)}|u(x,t)-u(y,t)|^q\,K_{1,\varepsilon}(|x-y|)\,dy\,ds\\ &\quad+c(q)\fint_{Q_\rho(z_0)}|u(y,t)-u(y,s)|^q\,K_{2,\varepsilon}(|t-s|)\,dy\,ds,\end{aligned}$$

where, for $\varepsilon\in(0,1)$, we set $K_{1,\varepsilon}(\theta):=\min\{1/\varepsilon,(2\rho/\theta)^n\big(\widetilde{\omega}_1(2\rho)/\widetilde{\omega}_1(\theta)\big)^q\}$ and $K_{2,\varepsilon}(\theta):=\min\{1/\varepsilon,(\rho^2/\theta)\big(\widetilde{\omega}_2(\rho^2)/\widetilde{\omega}_2(\theta)\big)^q\}$. Observe that $K_{1,\varepsilon}(\theta)\ge 1$ whenever $\theta\le 2\rho$ and that $K_{2,\varepsilon}(\theta)\ge1$ whenever $\theta\le\rho^2$. Integrating with respect to z over $Q_\rho(z_0)$ and using Fubini's theorem, we get

$$\begin{aligned}&\int_{Q_\rho(z_0)}\big|u(z)-(u)_{z_0,\rho}\big|^q\,dz\\ &\qquad\le c(q)\int_{Q_\rho(z_0)}\fint_{Q_\rho(z_0)}|u(x,t)-u(y,t)|^qK_{1,\varepsilon}(|x-y|)\,dy\,ds\,dx\,dt\\ &\qquad\quad+c(q)\int_{Q_\rho(z_0)}\fint_{Q_\rho(z_0)}|u(y,t)-u(y,s)|^qK_{2,\varepsilon}(|t-s|)\,dy\,ds\,dx\,dt\\ &\qquad=\frac{c(n,q)}{\rho^n}\int_{Q_\rho(z_0)}\int_{B_\rho(x_0)}|u(x,t)-u(y,t)|^qK_{1,\varepsilon}(|x-y|)\,dy\,dx\,dt\\ &\qquad\quad+\frac{c(n,q)}{\rho^2}\int_{t_0-\rho^2}^{t_0}\int_{Q_\rho(z_0)}|u(y,t)-u(y,s)|^qK_{2,\varepsilon}(|t-s|)\,dy\,ds\,dt.\end{aligned}$$

Letting $\varepsilon \searrow 0$ we get

$$\begin{aligned}
&\int_{Q_\rho(z_0)} \left|u(z) - (u)_{z_0,\rho}\right|^q dz \\
&\quad \le \frac{c}{\rho^n}\rho^n \widetilde{\omega}_1^q(\rho) \int_{t_0-\rho^2}^{t_0}\int_{B_\rho(x_0)}\int_{B_\rho(x_0)} \frac{|u(x,t)-u(x,s)|^q}{|x-y|^n\,\widetilde{\omega}_1^q(|x-y|)}\,dy\,dx\,dt \\
&\qquad + \frac{c}{\rho^2}\rho^2\widetilde{\omega}_2^q(\rho^2)\int_{B_\rho(x_0)}\int_{t_0-\rho^2}^{t_0}\int_{t_0-\rho^2}^{t_0} \frac{|u(x,s)-u(y,s)|^q}{|t-s|\,\widetilde{\omega}_2^q(|t-s|)}\,ds\,dt\,dy \\
&\quad \le c(n,q,A)\left[\widetilde{\omega}_1^q(\rho) + \widetilde{\omega}_2^q(\rho^2)\right]\lambda_{\widetilde{\omega}_1,\widetilde{\omega}_2;q}(u, Q_\rho(z_0)).
\end{aligned}$$

□

We remark that if $u \in W^{\widetilde{\omega}_1,\widetilde{\omega}_2;p}(\Omega_T,\mathbb{R}^N)$ then the Poincaré inequality (7.2) holds for all $Q_\rho(z_0) \subset \Omega_T$. The proof of the following Proposition is similar to the one of its elliptic analogue in [9, Proposition 3.3].

Proposition 7.2. *Let $v \in L^q(\Omega_T,\mathbb{R}^k)$ and assume that there exists $M \ge 0$, $A \ge 1$ and a modulus of continuity $\widetilde{\omega}$ satisfying*

$$\int_0^1 \frac{\widetilde{\omega}(\rho)^q}{\rho}\,d\rho < \infty$$

such that for $\widetilde{\Omega} \subset\subset \Omega$ and $d \in (0, T/8)$

$$\int_{S^{n-1}}\int_{-T+d}^{d}\int_{\widetilde{\Omega}} |v(x+he,t)-v(x,t)|^q\,dx\,dt\,d\mathcal{H}^{n-1}(e) \le M\,\widetilde{\omega}^q(|h|),$$

for all $|h| < \min\{dist(\partial\Omega,\widetilde{\Omega}), A\}$. Then for every $\mathcal{O} \subset\subset \widetilde{\Omega}$ and for any modulus of continuity θ satisfying the relative Dini condition

$$\int_0^1 \frac{\widetilde{\omega}^q(\rho)}{\rho\,\theta^q(\rho)}\,d\rho < \infty$$

there exists a constant $c_3 > 0$ depending on n, A, M, $||v||_{L^q}$, $\widetilde{\omega}(\cdot), \theta(\cdot)$, $diam(\Omega)$, $dist(\partial\Omega,\widetilde{\Omega})$, $dist(\partial\widetilde{\Omega},\mathcal{O})$ such that

$$\int_{-T+d}^{-d}\int_{\mathcal{O}}\int_{\mathcal{O}} \frac{|v(x,t)-v(y,t)|^q}{|x-y|^n\theta(|x-y|)^q}\,dx\,dy\,dt \le c_3.$$

Similarly the following proposition can be proved

Proposition 7.3. *Let $v \in L^2(\Omega_T,\mathbb{R}^k)$ and suppose there exists $M' \ge 0$, $A' \ge 1$ and a modulus of continuity $\widetilde{\omega}'$ satisfying*

$$\int_0^1 \frac{\widetilde{\omega}'^q(\rho)}{\rho}\,d\rho < \infty$$

such that for $\widetilde{\Omega} \subset\subset \Omega$ and $d \in (0, T/8)$

$$\int_{-T+d}^{d}\int_{\widetilde{\Omega}} |v(x,t+h)-v(x,t)|^q\,dx\,dt \le M'\,\widetilde{\omega}'^q(h),$$

for all $|h| < \min\{d, A'\}$, with $d \in (0, T/8)$.

Then for every modulus of continuity θ satisfying the relative Dini condition

$$\int_0^1 \frac{\widetilde{\omega}'^q(\rho)}{\rho\,\theta^q(\rho)}\,d\rho < \infty$$

there exists a constant $c_4 = c_4\big(n, A', M', \widetilde{\omega}(\cdot), \theta(\cdot), d, diam\,(\Omega), ||v||_{L^q}\big) > 0$ *such that*

$$\int_{\widetilde{\Omega}} \int_{-T+d}^{-d} \int_{-T+d}^{-d} \frac{|v(x,t) - v(x,s)|^q}{|t-s|\,\theta(|t-s|)^q}\, dt\, ds\, dx \le c_4.$$

8. Generalized Hausdorff measures

We recall the definition of the generalized Hausdorff measure (see [18, Definition 2]). Let $\theta : [0,\infty) \to [0,\infty)$ a continuous and increasing function such that $\theta(0) = 0$ and let $E \subset \mathbb{R}^n$. The *generalized Hausdorff measure* on $\mathbb{R}^n$ associated to θ is

$$\mathcal{H}^\theta(E) := \lim_{\delta \searrow 0} \left(\inf \left\{ \sum_{j=0}^{\infty} \theta(diam\, S_j);\ E \subset \bigcup_{j=0}^{\infty} S_j;\ diam\, S_j < \delta \right\} \right).$$

Since we are dealing with the parabolic case, we will be interested in the *generalized parabolic Hausdorff measure*, hence the generalized Hausdorff measure on $\mathbb{R}^{n+1}$ coupled with the parabolic metric, and since we are interested only in negligible sets, we shall confine ourselves to coverings made of cylinders. So, if θ is as before, for $E \subset \mathbb{R}^{n+1}$ we define

$$\mathcal{P}^\theta(E) := \lim_{\delta \searrow 0} \left(\inf \left\{ \sum_{j=0}^{\infty} \theta(\rho_j);\ E \subset \bigcup_{j=0}^{\infty} Q_{\rho_j}(z_j);\ \rho_j < \delta \right\} \right).$$

The following parabolic generalization of *Giusti Lemma* [17] will be a fundamental tool in proving Theorem 2.2. Its proof is very similar to that of its elliptic version [9, Lemma 5.1], and therefore we shall omit it.

Lemma 8.1. *Let* $\omega : [0,\infty) \to [0,\infty)$ *be non-decreasing and continuous with* $\omega(0) = 0$ *and* $\mathcal{P}^\omega$ *the generalized parabolic Haudorff measure associated to* ω. *Let* $\mathcal{B}_n$ *and* $\mathcal{B}_1$ *respectively be the families of Borel subsets of* Ω *and* $(-T, 0)$ *and* $\lambda : \mathcal{B}_n \times \mathcal{B}_1 \to [0, +\infty)$ *a finite, non negative, increasing set function, which is countably superadditive in the sense that*

$$\sum_{i=1}^{\infty} \lambda\big(Q_{r_i}(z_i)\big) \le \lambda\left(\bigcup_{i=1}^{\infty} Q_{\rho_i}(z_i) \right)$$

whenever $\{Q_{r_i}(z_i)\}$ *is a family of pairwise disjoint parabolic cylinders in* Ω_T. *We assume also that* λ *satisfies* $\lim_{|\mathcal{O}| \to 0} \lambda\big(\mathcal{O}\big) = 0$, *with* $\mathcal{O} \subset \Omega_T$ *open set, that*

$$\rho \longmapsto \frac{\rho^{n+2}}{\omega(\rho)} \quad \text{is non-decreasing and} \quad \lim_{\rho \searrow 0} \frac{\rho^{n+2}}{\omega(\rho)} = 0. \tag{8.1}$$

Then

$$\mathcal{P}^\omega(A) = 0, \quad \text{where} \quad A := \left\{ z_0 \in \Omega_T :\ \liminf_{\rho \searrow 0} \frac{\lambda\big(Q_\rho(z_0)\big)}{\omega(\rho)} > 0 \right\}.$$

For a proof of the following comparison theorem we refer to Rogers [27, Theorem 40].

Proposition 8.2. *Let* $f, g : [0,\infty) \to [0,\infty)$ *be increasing continuous functions with* $f(0) = g(0) = 0$ *and such that*

$$\lim_{r \searrow 0} \frac{f(r)}{g(r)} = 0.$$

Then for any $E \subset \mathbb{R}^n$, *if* $\mathcal{P}^g(E) = 0$ *then* $\mathcal{P}^f(E) = 0$.

9. Proof of the singular set estimate

Let $u \in L^2(-T,0;W^{1,2}(\Omega,\mathbb{R}^N))$ be a weak solution to the parabolic system (1.1) under the assumptions (A1), (A2), (A4), ($\mu0$) and ($\mu1$). We start from the property of singular points stated in Theorem 2.1:

$$\Sigma \subset \Sigma_{1,1} \cup \Sigma_{1,2} \cup \Sigma_{2,1} \cup \Sigma_{2,2},$$

where the first two terms are the ones defined in (2.15) and the last two are

$$\Sigma_{2,1} := \left\{ z_0 \in \Omega_T : \limsup_{\rho \searrow 0} \left|(Du)_{z_0,\rho}\right| = +\infty \right\}$$

and

$$\Sigma_{2,2} := \left\{ z_0 \in \Omega_T : \limsup_{\rho \searrow 0} \left|(u)_{z_0,\rho}\right| = +\infty \right\}.$$

9.1. **Estimate for $\Sigma_{1,1}$.** In the following $\gamma = \gamma(n, L/\lambda) \in (0,1/4)$ will denote the constant appearing in Lemmata 6.1 and 6.2. Combining Lemma 6.1 and Proposition 7.2, with $q=2$ and $\widetilde{\omega}(\cdot) = \mu^\gamma(\cdot)$, and also Lemma 6.2 and Proposition 7.3, with $q = 2$ but $\widetilde{\omega}'(\cdot) = \mu^\gamma(|\cdot|^{1/2})$, we infer that the spatial gradient Du belongs to $W^{\theta,\theta';2}_{loc}(\Omega,\mathbb{R}^{nN})$ whenever θ and θ' are two moduli of continuity satisfying the relative Dini conditions

$$\int_0^1 \frac{\mu^{2\gamma}(\rho)}{\rho\,\theta^2(\rho)}\,d\rho < \infty \quad \text{and} \quad \int_0^1 \frac{\mu^{2\gamma}(\rho^{1/2})}{\rho\,\theta'^2(\rho)}\,d\rho < \infty. \tag{9.1}$$

Now we observe that the function $\lambda_{\theta,\theta';2}(Du,\cdot)$ defined in (7.1) satisfies the hypotheses of Lemma 8.1. In order to apply Proposition 7.1 we have additionally to require that

$$\theta(2\rho) \le A\,\theta(\rho) \tag{9.2}$$

for some $A > 0$. Under this assumption Proposition 7.1 ensures that

$$⨍_{Q_\rho(z_0)} \left|Du(z) - (Du)_{\rho,z_0}\right|^2 dz \le c(n,A)\frac{\theta^2(\rho) + \theta'^2(\rho^2)}{\rho^{n+2}}\lambda_{\theta,\theta';2}(Du,Q_\rho(z_0)) \tag{9.3}$$

for all $Q_\rho(z_0) \subset\subset \Omega_T$. Therefore $z_0 \in \Sigma_{1,1}$ implies that

$$\liminf_{\rho \searrow 0} \frac{\theta^2(\rho) + \theta'^2(\rho^2)}{\rho^{n+2}}\lambda_{\theta,\theta';2}(Du,Q_\rho(z_0)) > 0.$$

The function

$$(0,\infty) \ni \rho \longmapsto \frac{\rho^{n+2}}{\theta^2(\rho) + \theta'^2(\rho^2)} =: \nu(\rho) \tag{9.4}$$

satisfies the hypothesis (8.1). Hence, letting

$$\mathcal{S}_{1,1} := \left\{ z_0 \in \Omega_T : \ \liminf_{\rho \searrow 0} \frac{\lambda_{\theta,\theta';2}(Du,Q_\rho(z_0))}{\nu(\rho)} > 0 \right\},$$

we have $\Sigma_{1,1} \subset \mathcal{S}_{1,1}$ and we are in a position to apply Lemma 8.1 to deduce that $\mathcal{P}^\nu(\mathcal{S}_{1,1}) = 0$. So we obtain that $\mathcal{P}^\nu(\Sigma_{1,1}) = 0$. Observe that the moduli θ e θ' are still to be chosen.

9.2. **Estimate for $\Sigma_{1,2}$.** Estimate (6.9), the fact that $u(\cdot,t) \in W^{1,2}(\Omega,\mathbb{R}^N)$ and Propositions 7.2 and 7.3 assure us that

$$u \in W^{\theta,\theta';2}_{loc}(\Omega,\mathbb{R}^N) \qquad \text{if (9.1) holds.} \tag{9.5}$$

Arguing as in the estimate of the dimension of $\Sigma_{1,1}$ we get that $\mathcal{P}^{\nu}(\Sigma_{1,2}) = 0$, being ν the function defined in (9.4).

9.3. **Estimate for $\Sigma_{2,2}$.** We consider

$$\mathcal{S}_2 := \left\{ z \in \Omega_T : \liminf_{\rho\searrow 0} \frac{\lambda_{\theta,\theta';2}(Du, Q_\rho(z_0))}{\omega(\rho)} > 0 \right\},$$

with a modulus of continuity ω satisfying the assumptions of Lemma 8.1, i.e (8.1). For $z_0 \in \Omega_T \setminus \mathcal{S}_2$ the sequence $\{\omega(\rho_k)^{-1}\lambda_{\theta,\theta';2}(Du, Q_\rho(z_0))\}$, with $\rho_k := 2^{-k}\rho$, is bounded. Using (9.3) we obtain

$$\begin{aligned} |(Du)_{z_0,\rho_{k+1}} - (Du)_{z_0,\rho_k}|^2 &\le 2^{n+2} \fint_{Q_{\rho_k}(z_0)} \left|Du - (Du)_{z_0,\rho_k}\right|^2 dz \\ &\le c(n,A)\frac{\theta^2(\rho_k) + \theta'^2(\rho_k^2)}{\rho_k^{n+2}}\, \lambda_{\theta,\theta';2}(Du, Q_{\rho_k}(z_0)) \\ &= c(n,A)\frac{\lambda_{\theta,\theta';2}(Du, Q_{\rho_k}(z_0))}{\omega(\rho_k)}\, \sigma^2(\rho_k), \end{aligned}$$

where we introduced

$$\omega(\rho) := \frac{\rho^{n+2}}{\theta^2(\rho) + \theta'^2(\rho^2)}\, \sigma^2(\rho) \tag{9.6}$$

with $\sigma : [0,\infty) \to [0,\infty)$, $\sigma(0) = 0$, to be chosen. In order to apply Lemma 8.1, ω must satisfy the hypotheses (8.1). This can be achieved if we assume that

$$\rho \to \frac{\theta^2(\rho) + \theta'^2(\rho^2)}{\sigma^2(\rho)} \text{ is non-decreasing with } \lim_{\rho\searrow 0} \frac{\theta^2(\rho) + \theta'^2(\rho^2)}{\sigma^2(\rho)} = 0. \tag{9.7}$$

Hence $|(Du)_{z_0,\rho_{k+1}} - (Du)_{z_0,\rho_k}| \le c\,\sigma(\rho_k)$. In particular, if we impose the Dini condition

$$\int_0^1 \frac{\sigma(r)}{r}\, dr < \infty \tag{9.8}$$

we see that

$$|(Du)_{z_0,\rho_k}| \le c\sum_{k=0}^{\infty} \sigma(\rho_k) \le \frac{c}{\log 2}\int_0^\rho \frac{\sigma(r)}{r}\, dr < \infty.$$

Having estimated $|(Du)_{z_0,\rho_k}|$, we can also estimate $|(Du)_{z_0,r}|$ for all $r \in (0,\rho)$: given such r, there is a unique $k \in \mathbb{N}\cup\{0\}$ with $\rho_{k+1} < r \le \rho_k$; for this k we have

$$\begin{aligned} \frac{1}{2}\left|(Du)_{z_0,r}\right|^2 &\le \left|(Du)_{z_0,\rho_k}\right|^2 + \left|(Du)_{z_0,r} - (Du)_{z_0,\rho_k}\right|^2 \\ &\le c + \fint_{Q_r(z_0)} \left|Du - (Du)_{z_0,\rho_k}\right|^2 dz \\ &\le c + c(n) \fint_{Q_{\rho_k}(z_0)} \left|Du - (Du)_{z_0,\rho_k}\right|^2 dz, \end{aligned}$$

the latter integral being bounded by the argument above. Hence $z_0 \in \Omega_T \setminus \mathcal{S}_2$ implies $z_0 \in \Omega_T \setminus \Sigma_{2,2}$, i.e. $\Sigma_{2,2} \subset \mathcal{S}_2$. Lemma 8.1 then yields $\mathcal{P}^{\omega}(\Sigma_{2,2}) = 0$.

9.4. **Estimate for $\Sigma_{2,1}$.** In a similar way, since also u satisfies the estimate (9.3) with u in place of Du ($u \in W^{\theta,\theta';2}_{loc}(\Omega,\mathbb{R}^N)$, see (9.5)), we have $\Sigma_{2,1} \subset \mathcal{S}_2$, showing that $\mathcal{P}^\omega(\Sigma_{2,1}) = 0$.

We remark that we still have to make some choices: we have to fix θ and θ' satisfying (9.1) and (9.2). Moreover we have to select σ satisfying (9.7) and (9.8).

9.5. **Effective choice of θ, θ' and σ.** From the definition of ω in (9.6) and that of ν in (9.4) we see that

$$\lim_{\rho\searrow 0} \frac{\omega(\rho)}{\nu(\rho)} = 0.$$

Therefore proposition 8.2 implies also that $\mathcal{P}^\omega(\Sigma_1) = 0$. Hence, up to now we have proved that $\mathcal{P}^\omega(Sing\, u) = 0$. The following elementary lemma [9, Lemma 5.2] will be useful:

Lemma 9.1. *Whenever a modulus of continuity κ fulfills the Dini condition*

$$K(r) := \int_0^r \frac{\kappa(\rho)}{\rho}\, d\rho < \infty$$

for some $r > 0$, then also κK^{-s} satisfies the Dini condition for every $s \in (-\infty, 1)$.

We choose, for any $\varepsilon \in (0,1)$,

$$\Lambda(r) := \int_0^r \frac{\mu^\alpha(\rho)}{\rho}\, d\rho, \qquad \theta(r) := \mu^\alpha(r)\, \Lambda(r)^{1/2-\varepsilon/4},$$

$$\theta'(r) := \theta(r^{1/2}) = \mu^\alpha(r^{1/2})\, \Lambda(r^{1/2})^{1/2-\varepsilon/4}, \quad \sigma(r) := \sqrt{2}\,\mu^\alpha(r)\,\Lambda(r)^{\varepsilon/4-1}$$

with a parameter $\alpha > 0$ to be chosen. We want to check that, with an opportune choice of such a parameter, these functions satisfy all the imposed conditions.

σ fulfills the Dini condition if (9.8) if

$$\int_0^1 \frac{\mu^\alpha(r)\Lambda(r)^{-(1-\varepsilon/4)}}{r}\, dr < \infty,$$

and this happens, because of Lemma 9.1 with $s = 1 - \varepsilon/4$, if μ^α fulfills a Dini condition:

$$\int_0^1 \frac{\mu^\alpha(r)}{r}\, dr < \infty. \tag{9.9}$$

θ fulfills the first condition in (9.1) if

$$\int_0^1 \frac{\mu^{2\gamma}(r)}{r\,\mu^{2\alpha}(r)\Lambda(r)^{1-\varepsilon/2}}\, dr = \int_0^1 \frac{\mu^{2(\gamma-\alpha)}(r)\Lambda(r)^{-(1-\varepsilon/2)}}{r}\, dr < \infty;$$

in order to apply Lemma 9.1 with $s = 1 - \varepsilon/2$ we need $2(\gamma - \alpha) = \gamma$, that is $\alpha = 2\gamma/3$.

Analogously θ' satisfies the second one if

$$\int_0^1 \frac{\mu^{2(\gamma-\alpha)}(r^{1/2})\Lambda(r^{1/2})^{-(1-\varepsilon/2)}}{r}\, dr < \infty;$$

which, using a simple change of variable, is equivalent to condition imposed on θ.

Let's test the condition (9.7):

$$\frac{\theta^2(r)+\theta'^2(r^2)}{\sigma^2(r)}=\frac{2\theta^2(r)}{\sigma^2(r)}=\frac{\Lambda(r)^{1-\varepsilon/2}}{\Lambda(r)^{\varepsilon/2-2}}=\Lambda(r)^{3-\varepsilon};$$

we see that the imposed bind is satisfied if and only if (9.9) is satisfied.

Finally, using the concavity of μ, (9.2) is satisfied with $A:=2^{\alpha(3/2-\varepsilon/4)}$. With these choices ω assumes the form

$$\omega(r)=r^{n+2}\biggl(\int_0^r\frac{\mu(\rho)^{\frac{2}{3}\gamma}}{\rho}\,d\rho\biggr)^{\varepsilon-3},$$

and we have $\mathcal{P}^\omega(Sing(u))=0$; this proves the Theorem in the case $\varepsilon\in(0,1)$. The case $\varepsilon\geq 1$ is a simple application of Proposition 8.2.

10. The case with no dependence on u

In the case where direct dependence of the vector field A on u does not occur, Theorem 2.2 can be improved in the following way: we consider the system

$$u_t-div\,A(z,Du)=0\quad\text{in }\Omega_T \tag{10.1}$$

where $A:\Omega_t\times\mathbb{R}^{nN}\to\mathbb{R}^{nN}$ is as in Theorem 2.3. First of all, we will show a stronger result of partial regularity for such a system. Stronger means that the singular set of u is "smaller". We will show that $Sing(u)\subset\widetilde{\Sigma}_1\cup\widetilde{\Sigma}_2$, where

$$\widetilde{\Sigma}_1:=\biggl\{z_0\in\Omega_T:\liminf_{\rho\searrow 0}\fint_{Q_\rho(z_0)}\bigl|Du-(Du)_{z_0,\rho}\bigr|^2\,dz>0\biggr\}$$

and

$$\widetilde{\Sigma}_2:=\biggl\{z_0\in\Omega_T:\limsup_{\rho\searrow 0}\bigl|(Du)_{z_0,\rho}\bigr|=+\infty\biggr\}.$$

We shall give only an outline about the way to prove all these results, since the proofs are in many points very similar to those we already gave. The techniques of Chapters 3 and 4 need only slight modifications: a careful inspection of the proofs reveals that we can weaken the condition we have to take on the affine function $\ell=\ell(x)$ appearing in Lemmata 3.1 and 4.1. It is in fact enough to assume that $|D\ell|\leq M$. This will lead us to weaken assumption (i) in Lemma 5.2 to $|D\ell_\rho|\leq M$. So we will not need to suppose $(u)_{z_0,\rho}$ to be bounded in order to have continuity in z_0. We refer again to [12], Theorem 8.2, for the proof that the set Σ_0, defined in (5.12), is contained in $\widetilde{\Sigma}_1$. When estimating the fractional space (and time) derivative (decomposing $\tau_{h,e}\bigl[A(z,Du)\bigr]$ as in (6.4)), we see that the term $\mathscr{B}(h)$ does not appear anymore. So we can state Lemmata 6.1 and 6.2 with $\gamma\equiv 1$: with the same assumption as in these Lemmata, we obtain

$$\int_{t_0}^{t_1}\int_{\widetilde{\Omega}}|Du(x+he,t)-Du(x,t)|^2\,dx\,dt\leq C\,\mu^2(|h|) \tag{10.2}$$

and the temporal version

$$\int_{t_0}^{t_1}\int_{\widetilde{\Omega}}|Du(x,t+h)-Du(x,t)|^2\,dx\,dt\leq C\,\mu^2(|h|^{1/2}).$$

Now we consider the weak formulation of problem (10.1) and we choose as test function $\tau_{-h,e}\varphi$, with $\varphi \in \mathcal{C}_0^\infty(\widetilde{\Omega}_{\widetilde{T}}, \mathbb{R}^N)$, $\widetilde{\Omega} \subset\subset \Omega$, $d \in (0, T/4)$, $\widetilde{\Omega}_{\widetilde{T}} := \widetilde{\Omega} \times (-T + d, -d)$, $|h| \in (0, \min\{dist(\widetilde{\Omega}, \partial\Omega)/2, 1\})$. Using "integration by parts" we obtain

$$\int_{\Omega_T} \Big[\tau_{h,e} u \partial_t \varphi - \tau_{h,e}\big[A(x,t,Du)\big] D\varphi\Big]\, dz = 0.$$

Now if we use the decomposition of $\tau_{h,e}\big[A(z,Du)\big]$ described before and we write $\mathscr{A}(h)(x,t) =: \widetilde{\mathscr{A}(h)}\big[\tau_{h,e}Du\big](x,t)$, with $\widetilde{\mathscr{A}(h)}$ similar to the quantity defined in (6.5), then we can rewrite

$$\int_{\widetilde{\Omega}_{\widetilde{T}}} \Big[\tau_{h,e} u\, \partial_t \varphi - \widetilde{\mathscr{A}(h)} \tau_{h,e}\big[Du\big]\, D\varphi\Big]\, dz = -\int_{\widetilde{\Omega}_{\widetilde{T}}} \mathscr{C}(h) D\varphi\, dz \tag{10.3}$$

for all $\varphi \in \mathcal{C}_0^\infty(\widetilde{\Omega}_{\widetilde{T}}, \mathbb{R}^N)$. Hence, dividing (10.3) by $\mu(|h|)$ and writing $v_h := \tau_{h,e}u/\mu(|h|)$ and $\widetilde{\mathscr{C}}(h) := -\mathscr{C}(h)/\mu(|h|)$, we have that v_h solves a parabolic linear problem with measurable and bounded coefficients. We highlight that $\mathscr{A}(h)$, thanks to our assumptions (A1) and (A2), represents a strongly elliptic and bounded bilinear form on $\mathbb{R}^{nN}$ independent of h. Moreover, since we have

$$|\mathscr{C}(h)| \le L\, \mu(|h|)\big(1 + |Du|\big) \tag{10.4}$$

and since $Du \in L^{2(1+\delta)}_{loc}(\Omega_T)$ for some $\delta = \delta(n, L/\lambda) > 0$, then $\mathscr{C}(h)/\mu(|h|)$ belongs to $L^{2(1+\delta)}_{loc}(\Omega_T)$. So we are in position to apply Giaquinta-Struwe Lemma [16] to deduce that there exists $0 < \delta_0 < \delta$ regardless of h such that (here we used (10.4))

$$\left(\fint_{Q_{\rho/2}} |Dv_h|^{2(1+\delta_0)}\, dz\right)^{1/(1+\delta_0)} \le c \fint_{Q_\rho} |Dv_h|^2\, dz + c\left(\fint_{Q_\rho} \big(1 + |Du|\big)^{2(1+\delta)}\, dz\right)^{1/(1+\delta)}$$

for all $Q_{2\rho} \subset\subset \widetilde{\Omega}$. Using (10.2) and (6.1) we obtain

$$\left(\fint_{Q_{\rho/2}} |Dv_h|^{\,2(1+\delta_0)}\, dz\right)^{1/(1+\delta_0)} \le \frac{c}{\rho^{n+2}}\Big[1 + \int_{Q_\rho} \big(1 + |Du|\big)^2\, dz\Big],$$

$c = c(dist(Q_\rho, \partial\Omega_T))$. So, recalling the definition of v_h, we get

$$\int_{Q_\rho} |\tau_{h,e}Du|^{2(1+\delta_0)}\, dz \le c\, \mu^{2(1+\delta_0)}(|h|),$$

where c exhibits the same dependencies as the constant in (6.8) and depends also on ρ and $dist(Q_{2\rho}, \partial\Omega_T)$. By a standard covering argument we can infer that for all $\mathcal{O} \subset\subset \widetilde{\Omega}_{\widetilde{T}}$ there exists a constant c essentially depending on L/λ and $dist(\mathcal{O}, \widetilde{\Omega}_{\widetilde{T}})$ such that

$$\int_{\mathcal{O}} |\tau_{h,e}Du|^{2(1+\delta_0)}\, dz \le c\, \mu^{2(1+\delta_0)}(|h|),$$

with e unit vector in $\mathbb{R}^n$. In a similar way we can improve the estimate appearing in Lemma 6.2 up to

$$\int_{\mathcal{O}} |\tau_h Du|^{2(1+\delta_0)}\, dz \le c\, \mu^{2(1+\delta_0)}(|h|^{1/2}).$$

We use now Proposition 7.2 with $q = 2(1+\delta_0)$, $\omega(\cdot) = \mu(\cdot)$ and $\omega'(\cdot) = \mu\big((\cdot)^{1/2}\big)$ and we deduce that $Du \in W^{\theta,\theta';2(1+\delta_0)}_{loc}(\Omega, \mathbb{R}^{nN})$ for θ, θ' moduli of continuity such that

$$\int_0^1 \frac{\mu^{2(1+\delta_0)}(\rho)}{\rho\,\theta^{2(1+\delta_0)}(\rho)}\,d\rho < \infty \quad \text{and} \quad \int_0^1 \frac{\mu^{2(1+\delta_0)}(\rho^{1/2})}{\rho\,\theta'^{2(1+\delta_0)}(\rho)}\,d\rho < \infty.$$

Now, considering the function $\lambda_{\theta,\theta';2(1+\delta_0)}(Du,\cdot)$ as in (7.1) and following the proof of the general case, we have that $\mathcal{P}^{\widetilde{\theta}}(Sing(u)) = 0$, with

$$\widetilde{\theta}(\rho) := \frac{\rho^{n+2}}{\theta^{2(1+\delta_0)}(\rho) + \theta'^{2(1+\delta_0)}(\rho^2)}\,\sigma^{2(1+\delta_0)}(\rho)$$

provided that $\rho \longmapsto (\theta(\rho)+\theta'(\rho^2))/\sigma(\rho)$ is non-decreasing and infinitesimal in zero, $\theta(2\rho) \le A\,\theta(\rho)$ for some $A > 0$ and σ fulfills a Dini condition. We choose for any $\varepsilon \in (0,1)$

$$\Lambda(r) := \int_0^r \frac{\mu^\alpha(\rho)}{\rho}\,d\rho, \qquad \theta(r) := \mu^\alpha(r)\,\Lambda(r)^{1/(2+2\delta_0)-\varepsilon/(4+4\delta_0)},$$
$$\theta'(r) := \theta(r^{1/2}), \qquad \sigma(r) := 2^{1/(2+2\delta_0)}\,\mu^\alpha(r)\,\Lambda(r)^{\varepsilon/(4+4\delta_0)-1}$$

with $\alpha > 0$ to be chosen. All the requests will be satisfied if μ^α satisfies a Dini condition, with

$$\alpha = \frac{2+2\delta_0}{3+2\delta_0} = \frac{2}{3}\,\gamma_1, \quad \text{for some } \gamma_1 > 1.$$

These choices lead to

$$\widetilde{\theta}(\rho) := \rho^{n+2}\bigg(\int_0^r \frac{\mu^\alpha(\rho)}{\rho}\,dr\bigg)^{\varepsilon-3-2\delta_0} = \rho^{n+2}\Lambda(\rho)^{\varepsilon-3\gamma_2}$$

with $\gamma_2 := 1 + \frac{2}{3}\,\delta_0$. Now we choose $\gamma' := \min\{\gamma_1, \gamma_2\} > 1$ and write

$$\widetilde{\omega}(r) := r^{n+2}\bigg(\int_0^r \frac{\mu^{\frac{2}{3}\gamma'}(\rho)}{\rho}\,d\rho\bigg)^{\varepsilon-3\gamma'} =: r^{n+2}\widetilde{\Lambda}(r)^{\varepsilon-3\gamma'}.$$

In the case $\gamma' = \gamma_2 < \gamma_1$ one can verify that

$$\lim_{r\searrow 0} \frac{\widetilde{\omega}(r)}{\widetilde{\theta}(r)} = 0,$$

provided

$$\int_0^1 \frac{\mu^{\frac{2}{3}\gamma_2}(\rho)}{\rho}\,d\rho < \infty;$$

whereas in the case $\gamma' = \gamma_1 < \gamma_2$ the same conclusion holds, provided

$$\int_0^1 \frac{\mu^{\frac{2}{3}\gamma_1}(\rho)}{\rho}\,d\rho < \infty.$$

In the remaining case, we have $\widetilde{\omega} = \widetilde{\theta}$. Hence, by Proposition 8.2, we conclude the assertion of Theorem in the case $\varepsilon \in (0,1)$. The case $\varepsilon \ge 1$ trivially follows, again, by an application of Proposition 8.2.

References

[1] E. Acerbi and G. Mingione, Gradient estimates for a class of parabolic systems, *Duke Math. J.* **136 (2)** (2007), 285-320.

[2] E. Acerbi, G. Mingione and G. A. Seregin, Regularity results for parabolic systems related to a class of non-Newtonian fluids, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **21 (1)** (2004), 25-60.

[3] V. Bögelein, Partial regularity and singular sets of solutions of higher order parabolic systems, *Ann. Mat. Pura Appl. (4)* **188 (1)** (2009), 61-122.

[4] L. A. Caffarelli and Q. Huang, Estimates in the generalized Campanato-John-Nirenberg spaces for fully nonlinear elliptic equations, *Duke Math. J.* **118** (2003), 1–17.

[5] S. Campanato, Equazioni paraboliche del secondo ordine e spazi $\mathfrak{L}^{2,\theta}(\Omega,\delta)$, *Ann. Mat. Pura Appl.* **73 (4)** (1996), 55-102.

[6] E. De Giorgi, Frontiere orientate di misura minima, *Sem. Scuola Normale Superiore Pisa* 1960-1961.

[7] E. DiBenedetto, *Degenerate Parabolic Equations*, Universtext, vol. XV, Springer-Verlag, New York, 1993.

[8] F. Duzaar and A. Gastel, Dini continuous nonlinear elliptic systems, *Arch. Math.* **78** (2002), 58-73.

[9] F. Duzaar, A. Gastel and G. Mingione, Elliptic systems, singular sets and Dini continuity, *Comm. Partial Differential Equations* **29** (2004), 1215-1240.

[10] F. Duzaar and J. F. Grotowski, Optimal interior partial regularity for nonlinear elliptic systems: the method of A-harmonic approximation, *Manuscripta Math.* **103** (2000), 267-298.

[11] F. Duzaar and G. Mingione, Harmonic type approximation lemmas, *J. Math. Anal. Appl.* **352 (1)** (2009), 301-335.

[12] F. Duzaar and G. Mingione, Second order parabolic systems, optimal regularity, and singular sets of solutions, *Ann. I. H. Poincaré Anal. Non Linéaire* **22** (2005), 705-751.

[13] F. Duzaar, G. Mingione and K. Steffen, Parabolic systems with polynomial growth and regularity, *Memoirs of the Amer. Math. Soc.*, to appear.

[14] F. Duzaar and K. Steffen, Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals, *J. Reine Angew. Math.* **546** (2002), 73-138.

[15] M. Giaquinta and G. Modica, Almost-everywhere regularity for solutions of nonlinear elliptic systems, *Manuscr. Math.* **28** (1979), 109-158.

[16] M. Giaquinta and M. Struwe, On the partial regularity of weak solutions of nonlinear parabolic systems, *Math. Z.* **179** (1982), 437-451.

[17] E. Giusti, *Direct methods in the Calculus of Variations*, World Scientific, Singapore.

[18] F. Hausdorff, Dimension und äußeres Maß, *Math. Ann.* **79** (1918), 157-179.

[19] J. Kinnunen and J. L. Lewis, Higher integrability for parabolic systems of p-Laplacian type, *Duke Math. J.* **102 (2)** (2000), 253-271.

[20] J. Kovats, Fully nonlinear elliptic equations and the Dini condition, *Comm. Partial Differential Equations* **22** (1997), no. 11-12.

[21] J. Kristensen and G. Mingione, The singular set of minima of integral functionals, *Arch. Rat. Mech. Anal.* **180** (2006) 331-398.

[22] M. Kronz, Partial regularity results for minimizers of quasiconvex functionals of higher order, *Ann. Inst. H. Ponicaré Anal. Non Linéaire* **19** (2002), 81-112.

[23] G. M. Lieberman, Higher regularity for nonlinear oblique derivative problems in Lipschitz domains, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **1** (2002), 111-151.

[24] G. Mingione, The singular set of solutions to non-differentiable elliptic systems, *Arch. Rat. Mech. Anal.* **166** (2003), 287-301.

[25] G. Mingione, Bounds for the singular set of solutions to non linear elliptic systems, *Calc. Var. Partial Differ. Equ.* **18** (2003), 373-400.

[26] R. Monneau, Pointwise estimates for Laplace equation. Applications to the free boundary of the obstacle problem with Dini coefficients, *J. Fourier Anal. Appl.* **15** (2009), 279-335.

[27] C. A. Rogers, *Hausdorff Measures*, London-New York: Cambridge University Press.

[28] L. Simon, *Lectures on regularity and singularities of harmonic maps*, Birkhuser-Verlag, Basel-Boston-Berlin, 1996.

[29] J. Wolf, Partial regularity of weak solutions to nonlinear elliptic systems satisfying a Dini condition, *Z. Anal. Anwendungen* **20** (2001), 315-330.

Paolo Baroni, Scuola Normale Superiore, p.za dei Cavalieri 7, I-56126 Pisa, Italy
E-mail address: paolo.baroni@sns.it