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NONLINEAR POTENTIAL ESTIMATES IN PARABOLIC PROBLEMS

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Abstract. We report on some recent result allowing to get pointwise bounds for the spatial gradient of solutions to degenerate and singular parabolic equations via linear and nonlinear potentials of the data.

1. Introduction

In this note we report on some of the results obtained in [18, 19], concerning the possibility of giving pointwise bounds for the gradient of solutions to possibly degenerate parabolic equations of the type

$$u_t - \operatorname{div} a(Du) = \mu \,, \tag{1.1}$$

defined in cylindrical domains $\Omega_T = \Omega \times (-T, 0)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $n \geq 2$, and $T > 0$. In the most general case μ is a Borel measure with finite total mass: $|\mu|(\Omega_T) < \infty$. From now on, without loss of generality, we shall assume that the measure is defined on $\mathbb{R}^{n+1}$ by letting $\mu\lfloor_{\mathbb{R}^{n+1}\setminus\Omega_T} = 0$; therefore we shall assume that $|\mu|(\mathbb{R}^{n+1}) < \infty$. A chief model example for the equations treated here is given by the familiar evolutionary p-Laplacean equation

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \mu \,, \tag{1.2}$$

and in fact, when considering (1.1), we shall consider the following growth and parabolicity assumptions on the C^1-vector field $a\colon \mathbb{R}^n \to \mathbb{R}^n$

$$\begin{cases} |a(z)| + |\partial a(z)|(|z|^2 + s^2)^{1/2} \leq L(|z|^2 + s^2)^{(p-1)/2} \\ \nu(|z|^2 + s^2)^{(p-2)/2}|\xi|^2 \leq \langle \partial a(z)\xi, \xi\rangle \end{cases} \tag{1.3}$$

whenever $z, \xi \in \mathbb{R}^n$, where $0 < \nu \leq L$ and $s \geq 0$. In the following λ will always denote a finite and positive real number.

1.1. Review of the elliptic background.

Let us consider the Poisson equation $-\triangle u = \mu$ in $\mathbb{R}^n$ - here we take $n \geq 3$, μ being an integrable function and u being the only solution decaying to zero at infinity. In this case the classical representation formula

$$u(x_0) = \frac{1}{n(n-2)|B_1|} \int_{\mathbb{R}^n} \frac{d\mu(x)}{|x_0 - x|^{n-2}} \,,$$

allows to derive the Riesz potential estimates

$$|u(x_0)| \leq c\mathbf{I}_2^{|\mu|}(x_0, \infty) \,, \qquad \text{and} \qquad |Du(x_0)| \leq c\mathbf{I}_1^{|\mu|}(x_0, \infty) \,, \tag{1.4}$$

where

$$\mathbf{I}_\beta^\mu(x_0, r) = \int_0^r \frac{|\mu|(B(x_0, \varrho))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho} \,, \qquad \beta > 0 \,,$$

is the (truncated) Riesz potentials of the measure $|\mu|$. Whilst such a result seems to be very much linked to the linearity of the equation considered, a breakthrough

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of Kilpeläinen & Malý [16] established a nonlinear analog of the first formula in (1.4) for solutions to general quasilinear equations

$$-\mathrm{div}\, a(Du) = \mu \tag{1.5}$$

by means of Wolff potentials

$$\mathbf{W}^{\mu}_{\beta,p}(x_0,r) := \int_0^r \left(\frac{|\mu|(B(x_0,\varrho))}{\varrho^{n-\beta p}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho}\,, \qquad \beta > 0\,. \tag{1.6}$$

These are natural objects in classical linear and nonlinear potential theory, and coincide with classical Riesz potentials when $p = 2$, as $\mathbf{W}^{\mu}_{\beta/2,2} = \mathbf{I}^{\mu}_{\beta}$ for nonnegative measures. The estimate of Kilpeläinen & Malý [16] is

$$|u(x_0)| \le c \fint_{B(x_0,r)} (|u| + rs)\, dx + c\mathbf{W}^{\mu}_{1,p}(x_0,2r)\,, \tag{1.7}$$

and holds whenever $B(x,2r) \subset \Omega$ is a ball centered at x_0 with radius $2r$, with x_0 being a Lebesgue point of u; the constant c depends only on n, p, ν, L. Another interesting approach to (1.7) was later given by Trudinger & Wang in [24, 25]. Estimate (1.7) has been upgraded to the gradient level in [22] for the case $p = 2$, and then in [10, 11] for $p \neq 2$, where the following estimates have been proved:

$$|Du(x_0)| \le c \fint_{B(x_0,r)} (|Du| + s)\, dx + c\mathbf{W}^{\mu}_{1/p,p}(x_0,2r) \tag{1.8}$$

for $p \ge 2$ and

$$|Du(x_0)| \le c \fint_{B(x_0,r)} (|Du| + s)\, dx + c\left[\mathbf{I}^{|\mu|}_{1}(x_0,2r)\right]^{1/(p-1)}\,, \tag{1.9}$$

for $2 - 1/n < p \le 2$. The importance of estimates as (1.7)-(1.9) mainly relies in the fact that they allow to deduce several basic properties of solutions to quasilinear equations by simply analyzing the behavior of related Wolff potentials, providing estimates in rearrangement invariant function spaces [21, 22]. We refer to [23] for an outline of the main issues. In turn, Wolff potentials are essential tools in order to study the fine properties of Sobolev functions and, more in general, to build a reasonable nonlinear potential theory [14].

2. Parabolic potential estimates

In [18, 19] we concentrate on the most delicate case of the higher order estimates (1.8)-(1.9) and give a natural analog of them in the case of possibly degenerate/singular parabolic equations of p-Laplacean type as those in (1.1) and (1.2). The nondegenerate case $p = 2$ has been already dealt within [10], and in that case the proof of the Wolff potential (spatial) gradient estimate is similar to the one for the elliptic case. The case $p \neq 2$ requires instead very different means as the equations considered become anisotropic (multiple of solutions no longer solve similar equations) and as a consequence all the a priori estimates available for solutions - starting from those concerning the homogeneous case $\mu = 0$ - are not scaling invariant. Ultimately, the iteration methods introduced in [16, 24, 25, 21, 22, 10, 11] and that are based on the use of homogeneous estimates, cannot be any longer applied. As a matter of fact, even the notion of potentials used must be revisited in a way that fits the local structure of the equations considered. This is not only a technical fact but is instead linked to behavior that the p-Laplacean type degeneracy exhibits in the parabolic case. Indeed, as we shall see in the next section, the so-called intrinsic geometry of the equations considered will appear [5, 6].

Remark 2.1 (Approximation and a priori estimates)**.** In view of the standard approximation theory available, it is not restrictive to consider in the following energy solutions, i.e. to start with solutions $u \in L^p(-T,0;W^{1,p}(\Omega))$ and such that Du is continuous in Ω_T, while the measure μ will be considered as being actually an integrable function, that is $\mu \in L^1(\mathbb{R}^{n+1})$. Indeed, solutions u to Cauchy-Dirichlet problems involving equations as (1.1) - with μ being now a general measure - are usually found via approximation arguments, and actually as limits of solutions to suitably regularized problems where both solutions $u_m \to u$ and and data $L^1 \ni \mu_m \to \mu$ are more regular. We refer to [18] for a comprehensive discussion and references.

2.1. **The intrinsic approach, and intrinsic potentials.** The anisotropic structure of the equations considered here naturally leads to the concept of *intrinsic geometry*, widely discussed in [5, 6]. This prescribes that, although the equations considered are anisotropic, they behave as isotropic equations when considered in space/time cylinders *whose size depend on the solution itself.* To outline how the *intrinsic approach* works, let us consider a domain, actually a cylinder Q, where, roughly speaking, the size of the gradient norm is approximately λ – possibly in some integral averaged sense – i.e.

$$|Du| \approx \lambda > 0\,. \tag{2.1}$$

In this case we shall consider *intrinsic cylinders*, i.e. cylinders of the type

$$Q = Q_r^\lambda(x_0,t_0) \equiv B(x_0,r) \times (t_0 - \lambda^{2-p}r^2, t_0)\,, \tag{2.2}$$

where $B(x_0,r) \subset \mathbb{R}^n$ is the usual Euclidean ball centered at x_0 and with radius $r > 0$. Note, when $\lambda \equiv 1$ or when $p = 2$, the cylinder in (2.2) reduces to the standard parabolic cylinder given by

$$Q_r(x_0,t_0) \equiv Q_r^1(x_0,t_0) \equiv B(x_0,r) \times (t_0 - r^2, t_0)\,.$$

Indeed, the case $p = 2$ is the only one admitting a non-intrinsic scaling and local estimates have a natural homogeneous character. In this case the equations in question are automatically non-degenerate. The *heuristics of the intrinsic scaling method* can now be easily described as follows: assuming that in the cylinder Q as in (2.2), the size of the gradient is approximately λ as in (2.1). Then we have that the equation $u_t - \operatorname{div}(|Du|^{p-2}Du) = 0$ looks like $u_t = \operatorname{div}(\lambda^{p-2}Du) = \lambda^{p-2}\triangle u$ which, after a scaling, that is considering $v(x,t) := u(x_0 + \varrho x, t_0 + \lambda^{2-p}\varrho^2 t)$ in $B(0,1) \times (-1,0)$, reduces to the heat equation $v_t = \triangle v$ in $B(0,1) \times (-1,0)$. This equation, in fact, admits favorable a priori estimates for solutions. The success of this strategy is therefore linked to a rigorous construction of such cylinders in the context of intrinsic definitions. Indeed, the way to express a condition as (2.1) is typically in an averaged sense like for instance

$$\left(\frac{1}{|Q_r^\lambda|}\int_{Q_r^\lambda} |Du|^{p-1}\,dx\,dt\right)^{1/(p-1)} = \left(\fint_{Q_r^\lambda} |Du|^{p-1}\,dx\,dt\right)^{1/(p-1)} \approx \lambda\,. \tag{2.3}$$

A problematic aspect in (2.3) occurs as the value of the integral average must be comparable to a constant which is in turn involved in the construction of its support $Q_r^\lambda \equiv Q_r^\lambda(x_0,t_0)$, exactly according to (2.2). As a consequence of the use of such intrinsic geometry, all the a priori estimates for solutions to evolutionary equations of p-Laplacean type admit a formulation that becomes natural only when formulated in terms of intrinsic parameters and cylinders as Q_r^λ and λ.

We shall present the result distinguishing the usually called *degenerate case* $p \geq 2$ from *the singular one*, that is when $2 - 1/(n+1) < p \leq 2$.

2.2. **The case $p \geq 2$.** The first novelty of [18, 19] is in that *we adopt the intrinsic geometry approach in the context of nonlinear potential estimates.* This will naturally give raise to a class of *intrinsic Wolff and Riesz potentials* that reveal to be the natural objects to be considered, as their structure allows to recast the behavior of the Barenblatt solution - the so-called nonlinear fundamental solution - see Section 2.4 below.

Theorem 2.1 (Intrinsic potential bound)**.** *Let u be a solution to* (1.1) *such that Du is continuous in Ω_T and that $\mu \in L^1$; assume that* (1.3) *hold with $p \geq 2$. There exists a constant $c \geq 1$, depending only on n, p, ν, L, such that if $\lambda > 0$ is a generalized root of*

$$\lambda = c\beta + c\int_0^{2r} \left(\frac{|\mu|(Q_\varrho^\lambda(x_0,t_0))}{\lambda^{2-p}\varrho^{n+1}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho}\,, \tag{2.4}$$

and if

$$\left(\fint_{Q_r^\lambda} (|Du|+s)^{p-1}\,dx\,dt \right)^{1/(p-1)} \leq \beta\,,$$

where $Q_{2r}^\lambda \equiv Q_{2r}^\lambda(x_0,t_0) \equiv B(x_0,2r) \times (t_0 - \lambda^{2-p}4r^2, t_0) \subset \Omega_T$ is an intrinsic cylinder with vertex at (x_0,t_0), then

$$|Du(x_0,t_0)| \leq \lambda\,.$$

By saying that λ is a generalized root of equation (2.4), where $\beta > 0$ and $c \geq 1$ are given constants, we mean a positive solution of (2.4) (the smallest can be taken), with the word generalized referring to the possibility that no root exists in which case we simply set $\lambda = \infty$. The finiteness of the integral in the right hand side of (2.4) anyway rules this case out. Statements as the one of Theorem 2.1, i.e. involving intrinsic quantities and cylinders, are completely natural when describing the local properties of the evolutionary p-Laplacean equation (see for instance [6]). The last integral appearing in (2.4) is the natural intrinsic counterpart of the Wolff potential $\mathbf{W}^\mu_{1/p,p}$ intervening in (1.8). In fact, when considering the associated elliptic stationary problem and μ is time independent, Theorem 2.1 gives back (1.8); for this see also Theorem 2.3 below. Moreover, in the case $p = 2$ it is easy to see that Theorem 2.1 implies the bound

$$|Du(x_0,t_0)| \leq c \fint_{Q_r} |Du|\,dx\,dt + c\mathbf{I}_1^\mu(x_0,t_0;2r) \tag{2.5}$$

whenever $Q_{2r} \equiv Q_{2r}(x_0,t_0) \subset \Omega_T$ is a standard parabolic cylinder, where

$$\mathbf{I}_\beta^\mu(x_0,t_0;2r) := \int_0^{2r} \frac{|\mu|(Q_\varrho(x_0,t_0))}{\varrho^{N-\beta}}\,\frac{d\varrho}{\varrho}\,, \qquad \beta < N \tag{2.6}$$

is the parabolic Riesz potential of μ and $N = n+2$ is the parabolic dimension. Estimate (2.5) has been originally obtained in [10].

The formulation of Theorem 2.1 involves intrinsic quantities and conditions, and appears therefore at the first sight to be problematic. This is not actually the case as shown in the next Theorem, which in fact follows as a corollary. In other words, Theorem 2.1 always implies local a priori estimates *via parabolic Wolff potentials*, on arbitrary parabolic cylinders $Q_r \subset \Omega_T$.

Theorem 2.2 (Parabolic Wolff potential bound)**.** *Let u be a solution to* (1.1) *such that Du is continuous in Ω_T and $\mu \in L^1$; assume that* (1.3) *hold with $p \geq 2$. There exists a constant c, depending only on n, p, ν, L, such that*

$$|Du(x_0,t_0)| \quad \leq \quad c \fint_{Q_r} (|Du|+s+1)^{p-1}\,dx\,dt$$

$$+c\left[\int_0^{2r}\left(\frac{|\mu|(Q_\varrho(x_0,t_0))}{\varrho^{n+1}}\right)^{1/(p-1)}\frac{d\varrho}{\varrho}\right]^{p-1} \tag{2.7}$$

holds whenever $Q_{2r}\equiv Q_{2r}(x_0,t_0)\equiv B(x_0,2r)\times(t_0-4r^2,t_0)\subset\Omega_T$ *is a standard parabolic cylinder with vertex at* (x_0,t_0).

To check the consistency of estimate (2.7) with the ones already present in the literature we observe that when $\mu\equiv 0$, estimate (2.7) reduces the classical L^∞-gradient bound available for solutions to the evolutionary p-Laplacean equation; see [6, Chapter 8, Theorem 5.1']. It is interesting to see that when switching to a non-intrisic formulation, local estimates immediately show anisotropicity under the form of a deficit scaling exponent, which in this case is $p-1$ and precisely reflects the lack of homogeneity of the equations. This is typical when considering anisotropic problems, and similar deficit scaling exponents typically appear in the a priori estimates from [1, 2].

Finally, when μ is time independent, or admits a favorable decomposition, it is possible to get rid of intrinsic geometry effect in the potential terms and we go back to the elliptic regime.

Theorem 2.3 (Elliptic-Parabolic Wolff potential bound)**.** *Let u be a solution to* (1.1) *such that Du is continuous in Ω_T and $\mu\in L^1$; assume that* (1.3) *hold with $p\geq 2$. Assume that the measure μ satisfies*

$$|\mu|\leq \mu_0\otimes f\,, \tag{2.8}$$

where $f\in L^\infty(-T,0)$ and μ_0 is a Borel measure on Ω with finite total mass. Then there exists a constant c, depending only on n,p,ν,L, such that

$$|Du(x_0,t_0)|\leq c\fint_{Q_r}(|Du|+s+1)^{p-1}\,dx\,dt+c\|f\|_{L^\infty}^{1/(p-1)}\mathbf{W}^{\mu_0}_{1/p,p}(x_0,2r)$$

whenever $Q_{2r}(x_0,t_0)\equiv B(x_0,2r)\times(t_0-4r^2,t_0)\subset\Omega_T$ is a standard parabolic cylinder having (x_0,t_0) as vertex. The (elliptic) Wolff potential $\mathbf{W}^{\mu}_{1/p,p}$ is defined in (1.6).

2.3. **The case $2-1/(n+1)<p\leq 2$.** Let us immediately remark that the assumed lower bound on p is the standard one allowing for the existence of Sobolev solutions to measure data problems, as for instance proved in [3, 4]. In the subquadratic case the intrinsic geometry associated to the problem changes in the sense that the relevant intrinsic cylinders are of the type

$$\tilde{Q}^\lambda_r(x_0,t_0)=Q^\lambda_{r_\lambda}(x_0,t_0)=B(x_0,\lambda^{(p-2)/2}r)\times(t_0-r^2,t_0)\,,\qquad r_\lambda=\lambda^{(p-2)/2}r\,.$$

Moreover, we shall pass from nonlinear intrinsic potentials of Wolff type to intrinsic Riesz potentials. The main result is the following:

Theorem 2.4 (Intrinsic linear potential bound)**.** *Let u be a solution to* (1.1) *such that Du is continuous in Ω_T and $\mu\in L^1$; assume that* (1.3) *hold with $2-1/(n+1)<p\leq 2$. There exists a constant $c\geq 1$, depending only on n,p,ν,L, such that if $\lambda>0$ is a generalized root of*

$$\lambda=c\beta+c\int_0^{2r_\lambda}\left(\frac{|\mu|(Q^\lambda_\varrho(x_0,t_0))}{\varrho^{n+1}}\right)\frac{d\varrho}{\varrho}\,,\qquad r_\lambda=\lambda^{(p-2)/2}r\,,$$

and if

$$\fint_{Q^\lambda_{r_\lambda}}(|Du|+s)\,dx\,dt\leq\beta\,,$$

where $Q^{\lambda}_{2r_\lambda} \equiv Q^{\lambda}_{2r_\lambda}(x_0,t_0) \equiv B(x_0, 2\lambda^{(p-2)/2}r) \times (t_0 - 4r^2, t_0) \subset \Omega_T$ *is an intrinsic cylinder with vertex at* (x_0,t_0)*, then*

$$|Du(x_0,t_0)| \le \lambda\,.$$

As for the case $p \ge 2$, Theorem 2.4 implies a priori estimates on non-intrinsic cylinders, and involving the parabolic Riesz potentials defined in (2.6).

Theorem 2.5 (Parabolic Riesz potential bound)**.** *Let* u *be a solution to* (1.1) *such that* Du *is continuous in* Ω_T *and* $\mu \in L^1$*; assume that* (1.3) *hold with* $2-1/(n+1) < p \le 2$*. There exists a constant* c*, depending only on* n,p,ν,L*, such that*

$$\begin{aligned} |Du(x_0,t_0)| \quad &\le \quad c\left(\fint_{Q_r} (|Du|+s+1)\,dx\,dt\right)^{2/[2-n(2-p)]} \\ &\qquad + c\,[\mathbf{I}_1^{\mu}(x_0,t_0;2r)]^{2/[(n+1)p-2n]} \end{aligned} \tag{2.9}$$

holds whenever $Q_{2r} \equiv Q_{2r}(x_0,t_0) \equiv B(x_0,2r) \times (t_0-4r^2,t_0) \subset \Omega_T$ *is a standard parabolic cylinder with vertex at* (x_0,t_0)*.*

Finally, also in this case we have that when μ is time independent or admits a favorable decomposition, the elliptic Riesz potentials come back exactly as in (1.9).

Theorem 2.6 (Elliptic Riesz potential bound)**.** *Let* u *be a solution to* (1.1) *such that* Du *is continuous in* Ω_T *and* $\mu \in L^1$*; assume that* (1.3) *hold with* $2-1/(n+1) < p \le 2$*. Assume that the measure* μ *satisfies* (2.8)*, where* $f \in L^\infty(-T,0)$ *and* μ_0 *is a Borel measure on* Ω *with finite total mass. Then there exists a constant* c*, depending only on* n,p,ν,L*, such that*

$$\begin{aligned} |Du(x_0,t_0)| \quad &\le \quad c\left(\fint_{Q_r} (|Du|+s+1)\,dx\,dt\right)^{2/[2-n(2-p)]} \\ &\qquad + c\|f\|_{L^\infty}^{1/(p-1)}\,[\mathbf{I}_1^{\mu_0}(x_0,2r)]^{1/(p-1)} \end{aligned}$$

whenever $Q_{2r}(x_0,t_0) \equiv B(x_0,2r) \times (t_0-4r^2,t_0) \subset \Omega_T$ *is a standard parabolic cylinder having* (x_0,t_0) *as vertex. The (elliptic) Riesz potential* $\mathbf{I}_1^{\mu_0}$ *is defined in* (1.6)*.*

Remark 2.2. It is interesting to analyze the exponents appearing in (2.9). The exponent $2/[2-n(2-p)]$ is the same one appearing in the typical gradient estimates for homogeneous equations and reflects the gradient nature of the estimate in question. Indeed, when $\mu \equiv 0$ estimate (2.9) reduces to the classical one obtained in [6, Chapter 8, Theorem 5.2']. The exponent $2/[(n+1)p-2n]$ instead blows-up when $p \to 2n/(n+1)$ and reflects the non-homogeneity of the equation studied, as well as the structure of the fundamental solution. Such exponent indeed intervenes in those estimates related to the Barenblatt solution, as for instance the Harnack inequalities in [8, 9].

2.4. **Comparison with the Barenblatt solution.** For the sake of brevity we shall concentrate here on the case $p > 2$. The standard quality test for potential estimates as for instance those in (1.7)-(1.9) consists of measuring the extent they allow to reproduce the behavior of fundamental solutions, i.e. the behavior of those special solutions obtained by taking $\mu \equiv \delta$, where δ is the Dirac measure charging one point. In the case of the evolutionary p-Laplacean equations with Dirac datum δ charging the origin, the equation

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \delta \qquad \text{in } \mathbb{R}^{n+1}$$

has an explicit solution - so-called Barenblatt solution - given by

$$\mathcal{B}_p(x,t) = \begin{cases} t^{-n/\theta}\left(c_b - \theta^{1/(1-p)}\dfrac{p-2}{p}\left(\dfrac{|x|}{t^{1/\theta}}\right)^{p/(p-1)}\right)_+^{(p-1)/(p-2)} & t>0 \\ 0 & t \le 0\,. \end{cases}$$

Here $\theta := n(p-2)+p$ and c_b is a constant normalizing the solution so that $\int_{\mathbb{R}^n}\mathcal{B}_p(x,t)\,dx = 1$ for all $t>0$. A direct computation reveals that the gradient of $\mathcal{B}_p(x,t)$ satisfies the estimate

$$|D\mathcal{B}_p(x_0,t_0)| \le ct_0^{-(n+1)/\theta} \tag{2.10}$$

whenever $(x_0,t_0)\in\mathbb{R}^n\times(0,\infty)$; in turn this prescribes the blow-up behaviour at the origin of the fundamental solution, which is typical of a situation where a Dirac measure appears. *The crucial point is now that the bound appearing in* (2.10) *is directly implied by Theorem* 2.1. Moreover, as Theorem 2.1 holds for general equations, the same bound also holds for solutions to general equations of the type

$$u_t - \operatorname{div} a(Du) = \delta \qquad \text{in } \mathbb{R}^{n+1}\,, \tag{2.11}$$

under assumptions (1.3); see also [20]. This result should be anyway compared to the one in [6, Chapter 11, Theorem 2.1, (2.4)]. Of course, when considering equations with genuine measure data as (2.11), we have to consider those solutions considered in [3, 4], and obtained by approximation processes, as limits of solutions with more regular data. As our estimates are stable under such approximation methods, Theorem 2.1 applies to solutions of (2.11) modulo considering Lebesgue points (x_0,t_0) of Du rather than any point.

3. Techniques employed - extensions

The proof of the potential estimates of the Section 2 employs and extends virtually all the known aspects of the gradient regularity theory for evolutionary p-Laplacean type equations. A preliminary part consists of deriving suitable a priori estimate for homogeneous equations of the type

$$w_t - \operatorname{div} a(Dw) = 0\,. \tag{3.1}$$

The Hölder regularity of the (spatial) gradient of w has been established in the papers of DiBenedetto & Friedman in [7] for equations and systems of the type

$$w_t - \operatorname{div}(g(|Dw|)Dw) = 0\,, \qquad g(|Dw|)\approx |Dw|^{p-2} \tag{3.2}$$

via suitable linearization methods, that in general do not apply for general cases as (3.1). Moreover, even in the model case (3.2) the estimates available in [6] are not immediately usable in problems involving measure data, as these typically need *estimates below the natural growth exponent*. This means we need estimates formulated in terms of decay of suitable excess functionals in terms of L^q-spaces, for low values of q (see also (5.2) below). These aspects are briefly discusses in the next Section, and lead to a new treatment of the Hölder regularity theory of the gradient of solutions. The central result in this respect is Theorem 5.1 below.

The proof of the potential estimates of Section 2 is now based on a delicate iteration procedure where the size of the gradient is uniformly controlled

$$\fint_{Q^\lambda_{r_i}} |Du|^{p-1}\,dx\,dt \le \lambda$$

over a chain of shrinking intrinsic cylinders with vertex (x_0,t_0)

$$\cdots \subset Q^\lambda_{r_{i+1}} \subset Q^\lambda_{r_i} \subset Q^\lambda_{r_{i-1}} \subset \cdots \tag{3.3}$$

where λ is the one from Theorem 2.1 (we confine to this case for brevity). The essence is the following: either the gradient Du stays bounded (in mean) from above by some fraction of λ on infinitely many scales i, say for instance

$$\fint_{Q^\lambda_{r_i}} |Du|^{p-1}\,dx\,dt \le \frac{\lambda^{p-1}}{100}$$

and in this case we immediately arrive at $|Du(x_0,t_0)| \le \lambda$; or this doesn't happen and therefore we argue after the exit time with respect to the previous condition, i.e. the first index i_e such that

$$\fint_{Q^\lambda_{r_i}} |Du|^{p-1}\,dx\,dt \ge \frac{\lambda^{p-1}}{100}\,.$$

holds for every $i > i_e$. This alternative condition in turn allows to argue by induction using a two-sided inequality given by

$$\frac{\lambda^{p-1}}{100} \le \fint_{Q^\lambda_{r_i}} |Du|^{p-1}\,dx\,dt \le \lambda^{p-1}\,. \tag{3.4}$$

The point is that via suitable comparison estimates with solutions to homogeneous equations as (3.1), it is possible to prove that if (3.4) holds for a certain index $i > i_e$, then it also holds at stage $i+1$, and therefore we conclude with (3.3) for every $i > i_e$. In turn, this ultimately gives $|Du(x_0,t_0)| \le \lambda$. A main point in the proof of the inductive step is that the conditions in (3.4) allow in turn to apply Theorem 5.1 (via (5.1)) to solutions $w \equiv w_i$ of (3.1) in $Q^\lambda_{r_i}$, sharing the same Cauchy-Dirichlet data of u on $\partial_{\rm par} Q^\lambda_{r_i}$. In turn the decay estimate of Theorem 5.1 can be transferred to u modulo a remainder term - giving raise to the potential term - and allows to prove the induction step.

In this note we are concentrating on the basic case in (1.1). Moreover, more general cases as

$$u_t - \operatorname{div} a(x,t,Du) = \mu$$

can be treated, assuming suitable forms of Hölder continuity on the partial map $x \mapsto a(x,t,z)$ (while just measurability is needed on $t \mapsto a(x,t,z)$).

Another extension in the case $p \ge 2$ concerns an alternative exponent allowable in Theorem 2.1, where the potential in (2.4) can be replaced by

$$\left[\int_0^{2r} \left(\frac{|\mu|(Q^\lambda_\varrho(x_0,t_0))}{\lambda^{2-p}\varrho^{n+1}}\right)^{p/[2(p-1)]} \frac{d\varrho}{\varrho}\right]^{2/p}\,. \tag{3.5}$$

This is no longer a Wolff potential, but it allows for slightly better conditions when looking for corollaries formulated in terms of rearrangement invariant function spaces.

4. Gradient continuity estimates

The techniques leading to the pointwise gradient bounds of Section 2 also lead to establish continuity criteria for Du, thereby extending those recently obtained in [12]. The results we are summarizing here are contained in [19] and for the sake of brevity we shall confine ourselves to report those for the case $p \ge 2$. We again refer to [19] for the subquadratic case.

Theorem 4.1 (Gradient continuity criterium)**.** *Let u be a weak solution to* (1.1) *in Ω_T; assume that* (1.3) *hold with $p \ge 2$. If*

$$\lim_{r\to 0} \int_0^r \left(\frac{|\mu|(Q_\varrho(x,t))}{\varrho^{n+1}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho} = 0$$

holds locally uniformly with respect to (x,t) *in* Ω_T*, then* Du *is continuous in* Ω_T*.*

This criterium in particular extends the analogous elliptic one obtained in [11]. Weakening the condition of the previous result by requiring that the convergence occurs for the integrand leads to a weaker form of continuity.

Theorem 4.2 (Gradient VMO-regularity)**.** *Let* u *be a weak solution to* (1.1) *in* Ω_T*; assume that* (1.3) *hold with* $p \geq 2$*. If the function*

$$(x,t) \mapsto \int_0^r \left(\frac{|\mu|(Q_\varrho(x,t))}{\varrho^{n+1}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho}$$

is locally bounded in Ω_T *and if furthermore*

$$\lim_{r \to 0} \frac{|\mu|(Q_r(x,t))}{r^{n+1}} = 0$$

holds locally uniformly with respect to (x,t) *in* Ω_T*, then* Du *is locally VMO-regular in* Ω_T*.*

It is worthwhile remarking that Theorem 4.1 extends the results obtained by Kilpeläinen and Lieberman to the parabolic case, actually catching a borderline case. In fact, for solutions to elliptic equations as (1.5), in [15, 17] it has been proved that Du is Hölder continuous provided the density condition $|\mu|(B_\varrho) \leq c\varrho^{n-1+\alpha}$ is satisfied. Theorem 4.1 extends this type of result to the parabolic case, furthermore catching a borderline case. Another corollary of Theorem 4.1 worth of stating concerns a formulation via Lorentz spaces.

Corollary 4.1 (Lorentz spaces regularity)**.** *Let* u *be a weak solution to* (1.1) *in* Ω_T*; assume that* (1.3) *hold with* $p \geq 2$*. If* $\mu \in L(n+2, 1/(p-1))$*, then* Du *is continuous.*

Observe that the previous result is a considerable improvement of those in [6], where the author proves the boundedness of the gradient provided $\mu \in L^q$ for some $q > n+2$. Similar statements, using the nonlinear quantities as in (3.5), can be used as well to get an alternative and slightly stronger form of the results in this section. In this case which on the other hand do not make use of Wolff potentials, but rather of the alternative non-linear quantities in (3.5).

5. Revisiting the gradient Hölder continuity theory

In this section we illustrate a few a priori regularity estimates for solutions to homogeneous equations of the type (3.1). They play an essential role in the proof of the nonlinear potentials estimates of Sections 2.2 and 2.3.

Theorem 5.1. *Suppose that* w *is a weak solution to* (3.1) *in* Q_r^λ *under assumptions* (1.3) *with* $p > 1$*, and consider numbers*

$$A, B, q \geq 1 \qquad \text{and} \qquad \varepsilon \in (0,1)\,.$$

Then there exists a constant $\delta_\varepsilon \in (0, 1/2)$ *depending only on* $n, p, \nu, L, A, B, \varepsilon$ *but otherwise independent of* s, q*, of the solution* w *considered and of the vector field* $a(\cdot)$*, such that if*

$$\lambda/B \leq \sup_{Q^\lambda_{\delta_\varepsilon r}} \|Dw\| \leq s + \sup_{Q^\lambda_r} \|Dw\| \leq A\lambda \tag{5.1}$$

holds, then

$$E_q(Dw, \delta_\varepsilon Q_r^\lambda) \leq \varepsilon E_q(Dw, Q_r^\lambda)$$

holds too, where E_q denote the excess functional

$$E_q(Dw, Q_\varrho^\lambda) := \left(\fint_{Q_\varrho^\lambda} |Dw - (Dw)_{Q_\varrho^\lambda}|^q \, dx \, dt \right)^{1/q}, \qquad \varrho \le r \,. \tag{5.2}$$

The main novelty of Theorem 5.1 is that it allows to get a precise *homogeneous decay estimate* for the excess functional of the gradient. This point is really crucial as it allows to employ Theorem 5.1 in *iteration processes*, where homogeneity estimates are necessary. This is typical when for instance performing perturbation arguments. Indeed, one of the main missing point in the parabolic regularity theory is a suitable machinery to perform perturbation arguments, eventually leading to nonlinear Schauder estimates. Theorem 5.1 is a suitable to tackle such issue, as it is shown in [19]. We remark that the main assumption (5.1) is typically satisfied in various iteration process, where the parameters A, B depend on the kind of regularity considered.

As a matter of fact, as the exponent q considered is arbitrary, Theorem 5.1 encodes all the information about the gradient Hölder continuity and, indeed, the following result, which is essentially contained in [6, Chapter 9], can be also directly proved using Theorem 5.1 (actually a small variant of it stated in [19]). This novel approach to the gradient Hölder continuity is presented in [19].

Theorem 5.2. *Let w be a weak solution to* (3.1) *in a given cylinder Q, under assumptions* (1.3) *with $p > 1$. Then Dw is locally Hölder continuous in Q. Moreover, let $Q_r^\lambda \subset Q$ be an intrinsic cylinder such that*

$$s + \sup_{Q_r^\lambda} ||Dw|| \le A\lambda$$

holds for a certain constant $A \ge 1$. Then

$$|Dw(x,t) - Dw(x_1,t_1)| \le c_h \lambda \left(\frac{\varrho}{r} \right)^\alpha$$

holds whenever $(x,t), (x_1,t_1) \in Q_\varrho^\lambda$ for a constants $c_h \equiv c_h(n,p,\nu,L,A) \ge 1$ and $\alpha \equiv \alpha(n,p,\nu,L,A) \in (0,1)$ which is independent of s, of the solution w considered and of the vector field $a(\cdot)$. Here $Q_\varrho^\lambda \subset Q_r^\lambda$ are intrinsic cylinders sharing the same vertex.

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