SKETCHES OF NONLINEAR CALDERÓN-ZYGMUND THEORY

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Abstract. I am going to survey a number of recent non-linear regularity results that, put together, outline what might be considered a nonlinear version of Calderón-Zygmund theory.

Contents

1. Introduction

Calderón-Zygmund (CZ) theory, in its classical formulation, is concerned with determining, possibly in a sharp way, the integrability and differentiability properties of solutions to linear elliptic and parabolic equations in terms of the regularity of the given data. In these notes we shall briefly outline some results aimed at forming a nonlinear version of the classical CZ theory [21], [22], where linear equations are replaced by quasilinear, possibly degenerate ones of the type

$$-\operatorname{div} a(x, Du) = \mu \tag{1.1}$$

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defined on a bounded domain $\Omega \subset \mathbb{R}^n$. Here μ is a Borel measure in $\mathbb{R}^n$ with finite total mass. More in general we shall also consider equations of the type

$$-\operatorname{div} a(x, Du) = H \tag{1.2}$$

where H is a distribution having a suitable structure. The chief model example we have in mind involves the p-Laplacean operator

$$-\operatorname{div}(|Du|^{p-2}Du) = \mu, H. \tag{1.3}$$

In some cases we shall also consider systems. The general assumptions considered here, and modeled on the structure properties of the operator in (1.3), prescribe that $a\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carátheodory vector field – and therefore *a priori only measurable with respect to* x – satisfying the following strong p-monotonicity and growth assumptions:

$$\begin{cases} \nu(s^2+|z_1|^2+|z_2|^2)^{(p-2)/2}|z_2-z_1|^2 \le \langle a(x,z_2)-a(x,z_1), z_2-z_1\rangle \\ |a(x,z)| \le L(s^2+|z|^2)^{(p-1)/2} \end{cases} \tag{1.4}$$

whenever $x \in \Omega$, $z, z_1, z_2 \in \mathbb{R}^n$ where $0 < \nu \le L$. Here we initially take $p > 1$, $s \ge 0$; further restrictions on the range of p will be eventually specified. While assumptions (1.4) are nearly minimal in order to obtain certain low order regularity results, when instead looking for higher regularity on-for instance differentiability estimates on Du-we shall need additional regularity on the vector field and we shall for instance consider the following:

$$\begin{cases} |a(x,z)| + (s^2+|z|^2)^{1/2}|a_z(x,z)| \le L(s^2+|z|^2)^{(p-1)/2}, \\ \nu(s^2+|z|^2)^{(p-2)/2}|\lambda|^2 \le \langle a_z(x,z)\lambda, \lambda\rangle, \\ |a(x,z)-a(x_0,z)| \le L\omega(|x-x_0|)(s^2+|z|^2)^{(p-1)/2}, \end{cases} \tag{1.5}$$

whenever $x, x_0 \in \Omega$, $z, \lambda \in \mathbb{R}^n$ where $0 < \nu \le L$ and $s \ge 0$. The symbol a_z denotes the partial derivative of $a(\cdot)$, and a_z is again to be assumed Carathéodory regular, while $\omega\colon (0,\infty) \to [0,1]$ is *a modulus of continuity*, i.e. a non-decreasing function such that

$$\lim_{\varrho \to 0} \omega(\varrho) = 0.$$

This means that $a(\cdot)$ depends on the "coefficients" x in a continuous way. The parameter $s \ge 0$ is used to distinguish the case of degenerate ellipticity ($s = 0$) of equations as (1.3), from the nondegenerate one ($s > 0$).

Basic notation. In the rest of the paper $\Omega \subset \mathbb{R}^n$ will denote a bounded, Lipschitz regular domain, and $n \geq 2$; by $B(x,R) \subset \mathbb{R}^n$ we denote the open ball with radius $R > 0$, centered at x, i.e. $B(x,R) := \{y \in \mathbb{R}^n : |x-y| < R\}$. When the center will not be relevant we shall simply denote $B_R \equiv B(x,R)$. In a similar way, we shall denote by Q_R the general Euclidean hypercube with sidelength equal to $2R$, and sides parallel to the coordinate axes. We shall denote by c, δ, ε etc. general positive constants; relevant functional dependence on the parameters will be emphasized by displaying them in parentheses; for example, to indicate a dependence of c on the real parameters n, p, ν, L we shall write $c \equiv c(n,p,\nu,L)$. Finally, according to a standard notation, given a set $A \subset \mathbb{R}^n$ with positive measure and a map $v \in L^1(A,\mathbb{R}^n)$, we shall denote by

$$(v)_A := \fint_A v(x)\,dx$$

its integral average over the set A. Moreover, in the following, when treating equations as in (1.1) and the measure μ is actually an integrable function, we shall denote also

$$|\mu|(A) = \int_A |\mu|\,dx.$$

2. The duality range and energy solutions

In this section we shall consider equations and systems of the type (1.2) where $H \in W^{-1,p'}$ and in particular

$$H = \operatorname{div}(|F|^{p-2}F), \quad F \in L^p(\Omega,\mathbb{R}^n).$$

Accordingly, we shall consider the following notion of energy solution:

Definition 2.1 (Energy solutions)**.** An energy solution u to

$$\operatorname{div} a(x,Du) = \operatorname{div}(|F|^{p-2}F), \tag{2.1}$$

under assumptions (1.4) or (1.5), is a function belonging to $W^{1,p}(\Omega)$ such that

$$\int_\Omega \langle a(x,Du), D\varphi\rangle\,dx = \int_\Omega \langle |F|^{p-2}F, D\varphi\rangle\,dx \tag{2.2}$$

holds for every $\varphi \in C_c^\infty(\Omega)$.

In the rest of this section we shall consider such a notion of solution and we shall abbreviate, as usual, *energy solution* by *solution*. Moreover, unless otherwise specified, in this Section 2 we shall always assume that $p > 1$.

2.1. Basic results. Let us consider the following non-homogeneous p-Laplacean equation, as a model problem:

$$\operatorname{div}(|Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F) \quad \text{for } p > 1. \tag{2.3}$$

Note that the right hand side of (2.3) is written in the peculiar form $\operatorname{div}(|F|^{p-2}F)$ in order to facilitate a more elegant presentation of the results, and also because such form naturally arises in the study of certain projections problems motivated by multi-dimensional quasi-conformal geometry [43]. Anyway, one could immediately consider a right hand side of the type $\operatorname{div} G$ by an obvious change of the vector field

$$G \equiv |F|^{p-1}\frac{F}{|F|} \iff F \equiv |G|^{\frac{1}{p-1}}\frac{G}{|G|}.$$

The following fundamental result in essentially due to Tadeusz Iwaniec:

Theorem 2.1 ([43])**.** *Let $u \in W^{1,p}(\mathbb{R}^n)$ be a weak solution to the equation* (2.3) *in $\mathbb{R}^n$. Then*

$$F \in L^{\gamma}(\mathbb{R}^n) \Longrightarrow Du \in L^{\gamma}(\mathbb{R}^n) \quad \textit{for every } \gamma \geq p.$$

The innovative method introduced by Iwaniec in [4] essentially replaces the classical use of singular integrals and explicit representation formulas (typical of the linear setting) with that of maximal operators and local regularity estimates for homogeneous equations; we shall come on this point later on. The local version of the previous result is

Theorem 2.2. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to the equation* (2.3) *in Ω, where Ω is a bounded domain in $\mathbb{R}^n$. Then*

$$F \in L^{\gamma}_{\mathrm{loc}}(\Omega) \Longrightarrow Du \in L^{\gamma}_{\mathrm{loc}}(\Omega) \quad \textit{for every } \gamma \geq p. \tag{2.4}$$

Moreover, there exists a constant $c \equiv c(n,p,\gamma)$ such that for every ball $B_R \Subset \Omega$ it holds that

$$\Big(\fint_{B_{R/2}} |Du|^{\gamma}\,dx\Big)^{\frac{1}{\gamma}} \leq c\Big(\fint_{B_R} |Du|^{p}\,dx\Big)^{\frac{1}{p}} + c\Big(\fint_{B_R} |F|^{\gamma}\,dx\Big)^{\frac{1}{\gamma}}. \tag{2.5}$$

See for instance [1] for a proof in a more general setting. From now on, for ease of presentation, we shall confine ourselves to treat local regularity results.

The non-trivial extension to the case when (2.3) is a system has been obtained by DiBenedetto & Manfredi, who caught a borderline case too; the case $1 < p < 2$ has been treated in [29].

Theorem 2.3 ([28], [29])**.** *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution to the system* (2.3)*, where Ω is a bounded domain in $\mathbb{R}^n$, and $N \geq 1$. Then* (2.4) *holds. Moreover*

$$F \in \mathrm{BMO}_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn}) \Longrightarrow Du \in \mathrm{BMO}_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn}).$$

2.2. BMO. To introduce BMO functions let us consider the quantity

$$[w]_{R_0} \equiv [w]_{R_0,\Omega} := \sup_{B_R \subset \Omega, R \leq R_0} \fint_{B_R} |w(x) - (w)_{B_R}| \, dx.$$

Then a measurable map w belongs to $\mathrm{BMO}(\Omega)$ iff $[w]_{R_0} < \infty$, for every $R_0 < \infty$. It turns out that $BMO \subset L^\gamma$ for every $\gamma < \infty$, while a deep and celebrated result of John & Nirenberg tells that every BMO function actually belongs to a suitable weak Orlicz space generated by an N-function with exponential growth [48], and depending on the BMO norm of w. Specifically, we have

$$\frac{\big|\{x \in Q_R : |w(x) - (w)_{Q_R}| > \lambda\}\big|}{|Q_R|} \leq c_1(n) \exp\Big(-\frac{c_2 \lambda}{[w]_{2R,Q_R}}\Big) \tag{2.6}$$

where Q_R is a cube whose sidelength equals R, and c_1, c_2 are absolute constants. Anyway BMO functions can be unbounded, as shown by $\log(1/|x|)$. For the proof of (2.6) a good reference is for instance [32, Theorem 6.11].

Related to BMO functions are functions with vanishing mean oscillation. These have been originally defined by Sarason [68] as those BMO functions w, such that

$$\lim_{R \to 0} [w]_{R,\Omega} = 0.$$

In this way one prescribes a way to allow only mild discontinuities, since the oscillations of w are measured in an integral, averaged way.

By now classical results due to Chiarenza, jmenoFrasca and Longo [23] assert the validity of linear CZ theory for those problems/operators involving VMO coefficients. This happens also in the non-linear case, as proved by Kinnunen & Zhou who considered a class of degenerate equations whose model is given by

$$\mathrm{div}(c(x)|Du|^{p-2}Du) = \mathrm{div}(|F|^{p-2}F) \quad \text{for } p > 1, \tag{2.7}$$

where the coefficient $c(\cdot)$ is a function satisfying

$$c(\cdot) \in \mathrm{VMO}(\Omega) \quad \text{and} \quad 0 < \nu \leq c(x) \leq L < \infty. \tag{2.8}$$

The outcome is now

Theorem 2.4 ([54]). *Let $u \in W^{1,p}(\Omega)$ be a weak solution to the equation* (2.7) *in Ω, where Ω is a bounded domain in $\mathbb{R}^n$, and the function $c(\cdot)$ satisfies* (2.8)*. Then assertions* (2.4)–(2.5) *hold for u, and the constant in estimate* (2.5) *also depends on the coefficient function $c(\cdot)$.*

2.3. More general operators. We will now turn to more general equations of the type in (2.1), where $a\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field satisfying (1.5). The regularity of solutions to homogeneous equations as

$$\operatorname{div} a(x, Dv) = 0, \tag{2.9}$$

is an important ingredient in the proof of the gradient estimates for non homogeneous equations, in that the regularity estimates for solutions to (2.9) are then used in a local comparison scheme to get proper size estimates for the gradient of solutions to (2.1). Therefore in order to state a theorem of the type 2.2 one has to consider operators such that solutions v to (2.9) enjoy the maximal regularity, which in our case is $Dv \in L^\gamma$ for every $\gamma < \infty$. On the other hand, an obvious a posteriori argumentation is that if an analogue of Theorem 2.2 would hold for equation (2.1), then applying it with the choice $F \equiv 0$ would in fact yield $Dv \in L^\gamma$ for every $\gamma < \infty$.

This is the case for solutions to (2.9) under assumptions (1.5). Therefore it holds the following:

Theorem 2.5. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to the equation* (2.1) *under the assumptions* (1.5)*, where Ω is a bounded domain in $\mathbb{R}^n$. Then* (2.4) *holds for u and moreover*

$$\Big(\fint_{B_{R/2}} |Du|^\gamma \, dx \Big)^{\frac{1}{\gamma}} \le c \Big(\fint_{B_R} (|Du| + s)^p \, dx \Big)^{\frac{1}{p}} + c \Big(\fint_{B_R} |F|^\gamma \, dx \Big)^{\frac{1}{\gamma}}, \tag{2.10}$$

holds for every ball $B_R \Subset \Omega$, where $c \equiv c(n, p, \nu, L, \gamma)$.

See for instance [1], from which a proof of the previous result can be adapted. There is a number of possible variants of the previous result, and here we present one featuring VMO coefficients, a conditions weaker than the one of continuous coefficients considered in Theorem 2.5 above. More precisely we are dealing with equations of the type

$$\operatorname{div}[c(x) a(Du)] = \operatorname{div}(|F|^{p-2} F) \quad \text{for } p > 1, \tag{2.11}$$

where the vector field $a\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfies (1.5) – obviously recast for the case where there is no x-dependence, while the coefficient function $c(\cdot)$ satisfies (2.8). We have

Theorem 2.6. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to the equation* (2.11) *in Ω, where Ω is a bounded domain in $\mathbb{R}^n$, and such that assumptions* (2.8) *and* (1.5) *are satisfied. Then the assertions in* (2.4) *and* (2.10) *hold for u, and the constant c appearing in* (2.10) *depends also on the coefficient $c(\cdot)$.*

For a proof one could for instance adapt the arguments from [1], [67]. The previous result can be also extended to the boundary when considering the Dirichlet problem – as (2.14) below, under very mild assumptions on the regularity of the boundary $\partial\Omega$; we will not deal very much with boundary regularity, and for such issues we for instance refer to [18] and related references.

2.4. The case of systems. Theorem 2.3 tells us that CZ estimates extend to the case of systems when considering the specific p-Laplacean system. We now wonder up which extent the results of the previous section extend to general systems. The reason for Theorem 2.3 to hold is that, as first shown by Uraltseva [76] and Uhlenbeck [75], solutions to the homogeneous p-Laplacean system $\operatorname{div}(|Du|^{p-2}Du) = 0$ are actually of class $C^{1,\alpha}$ for some $\alpha > 0$. This makes the local comparison argument work, finally leading to Theorem 2.3. We also recall that, as pointed out in the previous section, the regularity of solutions to associated homogeneous problems is crucial to obtain the desired CZ estimates.

In the case of general systems as (2.1), and satisfying (1.5), we cannot expect a theorem like 2.3 to hold, and for a very simple reason. It is known that solutions to general homogeneous systems as

$$\operatorname{div} a(Dv) = 0, \tag{2.12}$$

are not everywhere regular; they are $C^{1,\alpha}$-regular only when considered outside a closed negligible subset of Ω, in fact called the singular set of the solution. Moreover, even for $p = 2$, and in the case of a smooth vector field $a(\cdot)$, Šverák & Yan [70] have shown that solutions to (2.12) may even be unbounded in the interior of Ω; for such issues see for instance the recent survey paper [62]. This rules out the validity of Theorem 2.3 for general systems in that, should it hold, when applied to the case (2.12) it would imply the everywhere Hölder continuity of v in Ω, clearly contradicting the existence of unbounded solutions proved in [70].

On the other hand an intermediate version of Theorem 2.3 which is valid for general systems, and designed to match the regularity suggested by the examples of Šverák & Yan [70], holds in the following form:

Theorem 2.7 ([55])**.** *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution to the system*

$$\operatorname{div} a(x, Du) = \operatorname{div}(|F|^{p-2}F),$$

for $N \geq 1$, where Ω is a bounded domain in $\mathbb{R}^n$ and the continuous vector field $a\colon \Omega \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ satisfies (1.5) *when suitably recast for the vectorial case. Then there exists $\delta \equiv \delta(n, N, p, \nu, L) > 0$ such that*

$$F \in L^{\gamma}_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn}) \Longrightarrow Du \in L^{\gamma}_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn}),$$

whenever

$$p \leq \gamma < p + \frac{2p}{n-2} + \delta \quad \text{when } n > 2, \tag{2.13}$$

while no upper bound is prescribed on γ in the two-dimensional case $n = 2$. Moreover, the local estimate (2.10) *holds.*

Note that the previous theorem does not contradict the counterexample in [70], since this does not apply when $n = 2$. The previous result comes along with a global one. For this we shall consider the Dirichlet problem

$$\begin{cases} \operatorname{div} a(x, Du) = 0 & \text{in } \Omega \\ u = v & \text{on } \partial\Omega \end{cases} \tag{2.14}$$

for some boundary datum $v \in W^{1,p}(\Omega, \mathbb{R}^N)$; here we assume for simplicity that $\partial\Omega \in C^{1,\alpha}$, but such an assumption can be relaxed. The main result for (2.14) is

Theorem 2.8 ([55])**.** *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be the solution to the Dirichlet problem* (2.14) *for $N \geq 1$, where Ω is a bounded domain in $\mathbb{R}^n$ and the continuous vector field $a\colon \Omega \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ satisfies* (1.5) *when suitably recast for the vectorial case. Then there exists $\delta \equiv \delta(n, N, p, \nu, L) > 0$ such that*

$$\int_{\Omega} |Du|^{\gamma}\, dx \leq c \int_{\Omega} (|Dv| + s)^{\gamma}\, dx,$$

holds whenever (2.13) *is satisfied for $n > 2$, while no upper bound is imposed on γ in the two-dimensional case $n = 2$; the constant c depends only on n, N, p, ν, L, γ, $\partial\Omega$.*

The previous theorem reveals to be crucial when deriving certain improved bounds for the Hausdorff dimension of the singular set of minima of integral functions – see [55] – and when proving the existence of regular boundary points for solutions to Dirichlet problems involving non-linear elliptic systems and vectorial functionals [33], [56]. Moreover, the peculiar upper bound on γ appearing in (2.13) perfectly fits with the parameters values in order to allow the convergence of certain technical iterations occurring in [33], [55], [56].

The proof of Theorems 2.7, 2.8 is based on an argument different from those in [44], but rather relying on some more recent methods used by Caffarelli & Peral [20] in order to prove higher integrability of solutions to some homogenization problems and relying, as those from [43], [28], on the use of maximal operators.

2.5. Parabolic problems. The extension to the parabolic case of the results of the previous sections is quite non-trivial, and in fact the validity of Theorem 2.2 for the parabolic p-Laplacean system

$$u_t - \mathrm{div}(|Du|^{p-2}Du) = \mathrm{div}(|F|^{p-2}F) \tag{2.15}$$

remained an open problem for a while in the case $p \neq 2$, even in the case of one scalar equation $N = 1$; it was settled only recently in [2]. All the parabolic problems in this section, starting by (2.15), will be considered in the cylindrical domain

$$\Omega_T := \Omega \times (0, T),$$

where, as usual, Ω is a bounded domain in $\mathbb{R}^n$, and $T > 0$.

Let us now explain where the additional difficulties are coming from. The proof of the higher integrability results in the elliptic case strongly relies on the use of maximal operators. This approach is completely ruled out in the case of (2.15). This is deeply linked to the fact that the homogeneous system

$$u_t - \mathrm{div}(|Du|^{p-2}Du) = 0 \tag{2.16}$$

locally follows an **intrinsic geometry** dictated by the solution itself. This is essentially DiBenedetto's approach to the regularity of parabolic problems [27] we are going to briefly streamline. The right cylinders on which the problem (2.16) enjoys good a priori estimates when $p \geq 2$ are of the type

$$Q_{z_0}(\lambda^{2-p}R^2, R) \equiv B_R(x_0) \times (t_0 - \lambda^{2-p}R^2, t_0 + \lambda^{2-p}R^2),$$

where $z_0 \equiv (x_0, t_0) \in \mathbb{R}^{n+1}$ and the main point is that λ must be such that

$$\int_{Q_{z_0}(\lambda^{2-p}R^2, R)} |Du|^p \approx \lambda^p. \tag{2.17}$$

The last line says that $Q_{z_0}(\lambda^{2-p}R^2, R)$ is defined in an intrinsic way. It is actually the main core of DiBenedetto's ideas to show that such cylinders can be constructed and used. Now the point is very simple: since the cylinders in (2.17) depend on the size of the solution itself, then it is not possible

to associate to them, and therefore to the problem (2.16), a universal family of cylinders-that is independent of the solution considered. In turn this rules out the possibility of using parabolic type maximal operators.

In the paper [2] we overcame this point by introducing a completely new technique *bypassing the use of maximal operators*, and giving the first *Harmonic Analysis free, purely pde proof*, of non-linear CZ estimates. The result is split in the case $p \geq 2$ and $p < 2$.

Theorem 2.9 ([2])**.** *Let $u \in C(0,T,L^2(\Omega,\mathbb{R}^N)) \cap L^p(0,T,W^{1,p}(\Omega,\mathbb{R}^N))$ be a weak solution to the parabolic system* (2.15)*, where Ω is a bounded domain in $\mathbb{R}^n$, and $p \geq 2$. Then*

$$F \in L^\gamma_{\mathrm{loc}}(\Omega_T,\mathbb{R}^{Nn}) \Longrightarrow Du \in L^\gamma_{\mathrm{loc}}(\Omega_T,\mathbb{R}^{Nn}) \quad \textit{for every } \gamma \geq p.$$

Moreover, there exists a constant $c \equiv c(n,N,p,\nu,L,\gamma)$ such that for every parabolic cylinder $Q_R \equiv B_R(x_0) \times (t_0 - R^2, t_0 + R^2) \Subset \Omega_T$ it holds that

$$\begin{aligned} &\Big(\fint_{Q_{R/2}} |Du|^\gamma \, dx \, dt \Big)^{\frac{1}{\gamma}} \\ &\qquad \leq c \Big[\Big(\fint_{Q_R} (|Du|^p + 1) \, dx \, dt \Big)^{\frac{1}{p}} + \Big(\fint_{Q_R} |F|^\gamma \, dx \, dt \Big)^{\frac{1}{\gamma}} \Big]^{\frac{p}{2}}. \end{aligned} \tag{2.18}$$

We note the peculiar form of the a priori estimate (2.18), which fails to be a reverse Hölder type inequality as (2.10) due to the presence of the exponent $p/2$, which is the *the scaling deficit of the system* (2.15). The presence of such exponent is natural, and can be explained as follows: in fact, let us consider the case $F \equiv 0$, that is (2.16). We note that if u is a solution, then, with $c \in \mathbb{R}$ being a fixed constant, the function cu fails to be a solution of a similar system, unless $p = 2$. Therefore, we cannot expect homogeneous a priori estimates of the type (2.10) to hold for solutions to (2.15), unless $p = 2$, when (2.18) becomes in fact homogeneous. Instead, the appearance of the scaling deficit exponent $p/2$ in (2.18) precisely reflects the lack of homogeneity. Another sign of the lack of scaling is the presence of the additive constant in the second integral, this is a purely parabolic fact, linked to the presence of a diffusive term – that is u_t – in the system.

We turn now to the case $p < 2$. This is the so called singular case since when $|Du|$ approaches zero, the quantity $|Du|^{p-2}$, which roughly speaking represents the lowest eigenvalue of the operator div $(|Du|^{p-2}Du)$, tends to infinity. Anyway, this interpretation is somewhat misleadins here: we are

interested in determining the integrability rate of Du, therefore we are interested in the large values of the gradient. Therefore, in a sense, this is the real degenerate case for us. Here a new phenomenon appears: we cannot consider values of p which are arbitrarily close to 1, as described in [27]. The right condition turns out to be

$$p > \frac{2n}{n+2}, \tag{2.19}$$

otherwise, as shown by counterexamples, solutions to (2.16) may be even unbounded. This can be explained by looking at (2.16) when $|Du|$ is very large: if $p < 2$, and it is far from 2, then the regularizing effect of the elliptic part – the diffusion – is too weak as $|Du|^{p-2}$ is very small, and the evolutionary part develops singularities like in odes, where no diffusion is involved. For the case $p < 2$ the result is now

Theorem 2.10 ([2]). *Let $u \in C(0,T,L^2(\Omega,\mathbb{R}^N)) \cap L^p(0,T,W^{1,p}(\Omega,\mathbb{R}^N))$ be a weak solution to the parabolic system* (2.15)*, where Ω is a bounded domain in $\mathbb{R}^n$, and $p < 2$ satisfies* (2.19)*. Then*

$$F \in L^\gamma_{\mathrm{loc}}(\Omega_T,\mathbb{R}^{Nn}) \Longrightarrow Du \in L^\gamma_{\mathrm{loc}}(\Omega_T,\mathbb{R}^{Nn}) \quad \textit{for every } \gamma \geq p.$$

Moreover, there exists a constant $c \equiv c(n,N,p,\nu,L,q)$ such that for every parabolic cylinder $Q_R \equiv B_R(x_0) \times (t_0 - R^2, t_0 + R^2) \Subset \Omega_T$ it holds that

$$\begin{aligned} &\Big(\fint_{Q_{R/2}} |Du|^\gamma \, dx\, dt \Big)^{\frac{1}{\gamma}} \\ &\quad \leq c \bigg[\Big(\fint_{Q_R} (|Du|^p + 1)\, dx\, dt \Big)^{\frac{1}{p}} + \Big(\fint_{Q_R} |F|^\gamma \, dx\, dt \Big)^{\frac{1}{\gamma}} \bigg]^{\frac{2p}{p(n+2)-2n}}. \end{aligned} \tag{2.20}$$

Note how in the previous estimate the scaling deficit exponent $p/2$ in (2.18) is replaced by $2p/(p(n+2)-2n)$, a quantity that stays finite as long as (2.19) is satisfied. Therefore estimate (implespar2) exhibits in quantitative way the role of assumption (2.19).

Theorems 2.9 and 2.10 admit of course several generalizations; for instance, a possible one concerns general parabolic *equations* of the type

$$u_t - \operatorname{div} a(Du) = \operatorname{div}(|F|^{p-2}F),$$

where the vector field $a(\cdot)$ satisfies (1.5). In this case Theorems 2.9 and 2.10 hold in the form described above, for a constant c depending also on ν, L.

Further extensions to the case of systems, in the spirit of Theorem 2.7, are available in [36].

We conclude with a further integrability result in the stationary case, recently obtained in [17], and concerning gradient estimates for obstacle problems. A point of interest here is that, differently from the usual results available in the literature, the obstacles considered here are just Sobolev functions, and therefore discontinuous, in general. We shall just report on the simplest model elliptic case, involving the minimization problem

$$\mathrm{Min} \int_{\Omega} |Dv|^p \, dx, \quad v \in K, \tag{2.21}$$

where

$$K := \{v \in W^{1,p}_0(\Omega) : v \geq \psi \text{ a.e.}\}, \quad \psi \in W^{1,p}_0(\Omega).$$

We now have

Theorem 2.11 ([17])**.** *Let $u \in W^{1,p}(\Omega)$ be the unique solution to the obstacle problem* (2.21)*, where Ω is a bounded domain in $\mathbb{R}^n$. Then*

$$\psi \in W^{1,\gamma}_{\mathrm{loc}}(\Omega) \Longrightarrow u \in W^{1,\gamma}_{\mathrm{loc}}(\Omega) \quad \textit{for every } \gamma \geq p.$$

Moreover, for every $B_R \Subset \Omega$ it holds that

$$\Big(\fint_{B_{R/2}} |Du|^\gamma \, dx \Big)^{\frac{1}{\gamma}} \leq c \Big(\fint_{B_R} |Du|^p \, dx \Big)^{\frac{1}{p}} + c \Big(\fint_{B_R} |D\psi|^\gamma \, dx \Big)^{\frac{1}{\gamma}},$$

where $c \equiv c(n, p, \nu, L, \gamma)$.

The previous theorem is obviously optimal, as it follows by considering the gradient integrability of u on the contact set $\{u \equiv \psi\}$, where Du and $D\psi$ coincide almost everywhere. For more general elliptic cases and parabolic extensions we refer to [17].

3. The subdual range and very weak solutions

In this section we shall deal with the case the right hand side does not belong to $W^{-1,p'}$, and, specifically, with the case considered in (1.1). Following a rather consolidated tradition we shall talk about measure data problems also in those cases when the datum involved is not genuinely a measure, but also a function with low integrability properties. For the sake of simplicity we shall

concentrate on Dirichlet problems, with homogeneous boundary datum, of the type

$$\begin{cases} -\operatorname{div} a(x, Du) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open subset with $n \geq 2$, while μ is a (signed) Borel measure with finite total mass $|\mu|(\Omega) < \infty$. Of course, it is always possible to assume that the measure μ is defined on the whole $\mathbb{R}^n$ by letting $|\mu|(\mathbb{R}^n \setminus \Omega) = 0$, therefore in the following we shall do so. As for the structure properties of the problem,we shall consider assumptions (1.4). Moreover, when considering (1.4) the structure constants will satisfy

$$2 - 1/n < p \leq n \quad 0 < \nu \leq 1 \leq L \quad s \geq 0. \tag{3.2}$$

This time the adopted notion of solution requires a larger discussion. We start by the following crude distributional definition, particularizing the one in (2.2).

Definition 3.1 (Very weak solutions)**.** A very weak solution u to the problem (3.1) under assumptions (1.4), is a function $u \in W^{1,1}_0(\Omega)$ such that $a(x, Du) \in L^1(\Omega, \mathbb{R}^n)$ and

$$\int_\Omega \langle a(x, Du), D\varphi \rangle \, dx = \int_\Omega \varphi \, d\mu \quad \text{holds for every } \varphi \in C^\infty_c(\Omega). \tag{3.3}$$

Very weak solutions exist beside usual energy solutions and non-uniqueness occurs. Even for simple linear homogeneous equations of the type

$$\operatorname{div}(A(x)Du) = 0,$$

as shown by a classical counterexample of Serrin [69], for a proper choice of the strongly elliptic and bounded, measurable matrix $A(x)$, two solutions show up: one of them belongs to the natural energy space $W^{1,2}$, and it is therefore an energy solution; the other one does not belong to $W^{1,2}$, and for this reason in a time where the concept of very weak solution was not very familiar, was conceived as a pathological solution. This situation immediately poses the problem of *uniqueness of solutions*. When $\mu \in W^{-1,p'}$ uniqueness takes place, of course. Two examples when this happens are given below.

Example 1 (Density type conditions)**.** In the case of a Borel measure μ in the right hand side, there is a classical trace type theorem due to Adams [3] stating that if the density condition

$$|\mu|(B_R) \lesssim R^{n-p+\varepsilon} \tag{3.4}$$

holds for some $\varepsilon > 0$, then it follows that $\mu \in (W_0^{1,p}(\Omega))' \equiv W^{-1,p'}(\Omega)$. Therefore for such measures we have the existence of a unique energy solution. We notice that the p-capacity of a ball B_R is comparable to R^{n-p}, therefore (3.4) implies that the measure in question is absolutely continuous with respect to the p-capacity. Indeed Sobolev functions are those that can be defined up to set of negligible p-capacity.

Example 2 (High integrable functions)**.** When the measure is a function and $\mu \in L^\gamma(\Omega)$, then for certain values of γ we have $\mu \in W^{-1,p'}(\Omega)$. In fact, Sobolev imbedding theorem yields, when $p < n$

$$W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega) \quad p^* := \frac{np}{n-p}.$$

Therefore $L^{(p^*)'}(\Omega) \subset W^{-1,p'}(\Omega)$. This means that if $\mu \in L^\gamma(\Omega)$ and

$$\gamma \geq (p^*)' = \frac{np}{np - n + p}$$

then there is a unique energy solution to (3.1). This argument can be refined up to Lorentz spaces – see Definition 4.4 below. In fact the improved Sobolev embedding theorem gives

$$W_0^{1,p}(\Omega) \subset L(p^*, p)(\Omega) \subsetneqq L^{p^*}(\Omega) = L(p^*, p^*)(\Omega).$$

For a proof see for instance [73]. But since $(L(p^*, p))' = L((p^*)', p')$, we have that if $\mu \in L((p^*)', q)(\Omega)$ with $q \leq p'$, then there exists a unique energy solution to (3.1).

Example 3 (Non-linear Green's functions)**.** By the fundamental solution to the p-Laplacean equation we mean the function

$$G_p(x) = \begin{cases} |x|^{\frac{p-n}{p-1}} & \text{if } 2 \leq p < n, \\ \log|x| & \text{if } p = n, \end{cases} \tag{3.5}$$

which is indeed *the unique solution* to the problem

$$\begin{cases} -\operatorname{div}(|Du|^{p-2}Du) = c(n,p)\delta & \text{in } \Omega, \\ u = 0 & \text{on } |\partial\Omega, \end{cases} \tag{3.6}$$

amongst those obtainable via approximation (see next section for the precise meaning). Here δ is Dirac mass charging the origin and $c(n,p)$ is a constant depending only on n and p.

The notion of solution used in this paper for treating measure data problems and equations as (1.1) is stronger than that simply given by the definition 3.1 of very weak solution. Indeed, in the following we shall deal with solutions obtained via approximation methods and this in completely natural in view of the existence theory available. Next section is dedicated to this.

3.1. Solvability and SOLA. Here we discuss the basic solvability of (3.1) and, accordingly, the notion of solution adopted here. The main point is that although uniqueness is still lacking, it is always possible to solve (3.1) in the plain sense of Definition 3.1. Since distributional solutions are not unique, at this point there are in the literature several definitions of solution adopted towards the settlement of the uniqueness problem – see for instance [12] for the definition of entropy solutions, and [26] for the definition of re-normalized ones. Here we shall adopt the notion of *solution obtained by limits of approximations* (SOLA); these are solutions obtained via an approximation scheme using solutions of regularized problems. The approximation procedure and its convergence has been settled down by Boccardo & Gallöuet [15], [16]; see also [25]. The idea is to approximate the measure μ via a sequence of smooth functions $\{f_k\} \subset L^\infty(\Omega)$, such that $f_k \to \mu$ weakly in the sense of measures, or $f_k \to \mu$ strongly in $L^1(\Omega)$ in the case μ is a function. At this point, by standard monotonicity methods, one finds a unique solution $u_k \in W^{1,p}_0(\Omega)$ to

$$\begin{cases} -\operatorname{div} a(x, Du_k) = f_k & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

The arguments in [15] lead to establish that there exists $u \in W^{1,\max\{1,p-1\}}_0(\Omega)$ such that, up to a not relabeled subsequence,

$$u_k \to u \quad \text{and} \quad Du_k \to Du \ \text{ strongly in } \ L^{\max\{1,p-1\}}(\Omega), \ \text{ and a.e.}$$

and (3.1) is solved by u in the sense of (3.3). We have therefore found a distributional solution having the remarkable additional feature of having been selected via an approximation argument through regular energy solutions.

It is important to notice that, as described for instance in [25], in the case μ is an L^1-function, *by considering a different approximating sequence* $\{\bar{f}_k\}$ *strongly converging to* f *in* $L^1(\Omega)$*, we still get the same limiting solution* u. As a consequence the described approximation process allows to build a class of solutions, those in fact obtained by approximation, in which the unique solvability of (3.1) is possible. For this reason, from now on, *when dealing with the case the measure* μ *is a actually an* L^1*-function, we shall talk about*

the solution to (3.1), meaning by this the unique solution found by the above settled approximation scheme.

3.2. Basic regularity results. It is interesting to compute the degree of integrability of $G_p(\cdot)$ introduced in (3.5), since, by the unique solvability of (3.6), it is possible to test the optimality of the regularity results for general measure data problems by comparing them with the properties of $G_p(\cdot)$. Note that $|DG_p| \approx |x|^{(1-n)/(p-1)}$ and therefore it follows that

$$|G_p|^{p-1} \in \mathcal{M}_{\mathrm{loc}}^{\frac{n}{n-p}}(\mathbb{R}^n) \quad \text{and} \quad |DG_p|^{p-1} \in \mathcal{M}_{\mathrm{loc}}^{\frac{n}{n-1}}(\mathbb{R}^n),$$

the first being meaningful of course when $p < n$. As shown later in Theorem 4.4 and Remark 4.2, $G_p(\cdot)$ exhibits the worst behavior amongst the solutions with measure data problems, according to the rough principle stating that "the more the measure concentrates, the worse solutions behave".

There is a vast literature on the regularity of solutions to measure data problems; here we report a few basic ones.

Theorem 3.1 ([12], [31])**.** *Under the assumptions* (1.4) *there exists a solution* $u \in W_0^{1,p-1}(\Omega)$ *to* (3.1) *such that*

$$|u|^{p-1} \in \mathcal{M}^{\frac{n}{n-p}}(\Omega) \quad \textit{for } p < n,$$

and

$$|Du|^{p-1} \in \mathcal{M}^{\frac{n}{n-1}}(\Omega). \tag{3.7}$$

The result of the previous theorem has been obtained in some preliminary forms in [15], [72]; the form above has been obtained in [12] for the case $p < n$, while the case $p = n$, with the consequent $\mathcal{M}^n$ estimate, is treated in [31]. Results for systems have been obtained in [30].

We now switch to the case when the measure is actually a function

$$\mu \in L^\gamma(\Omega), \quad \gamma \geq 1. \tag{3.8}$$

For this we premise the following:

Remark 3.1 (Maximal regularity)**.** The equations we are considering have measurable coefficients, and this means that $x \mapsto a(x, z)$ is a measurable map. The maximal regularity in terms of gradient integrability we may expect, even for energy solutions to the homogeneous equation $\operatorname{div} a(x, Du) = 0$, is at most $Du \in L^q_{\mathrm{loc}}$, for some q which is in general only slightly larger than p, and depends in a critical way on n, p, ν, L. This

is basically a consequence of Gehring's lemma [40], [44], which indeed allows to prove that in the case $\mu = 0$ we have $Du \in L^q$. Therefore we are not expecting to get much more that $Du \in L^q$ in general for solutions to the measure data problems considered in the following. Therefore, with abuse of terminology, we shall consider $Du \in L^p$ as the maximal regularity for the gradient of solutions u.

The previous remark allows to restrict the range of parameters of γ, the exponent appearing in (3.8). We first look for values of γ such that $L^\gamma \not\subset W^{-1,p'}$, otherwise the existence of an energy solution such $Du \in L^p$ follows. We are in fact almost at the maximal regularity. As a matter of fact when considering measure data problems one is mainly interested in those solutions which are not energy ones. By Example 2 we see that the right condition is

$$1 < \gamma < \frac{np}{np - n + p} = (p^*)' \quad \text{for } p < n. \tag{3.9}$$

Theorem 3.2 ([16])**.** *Under the assumptions* (1.4) *and* (3.8)–(3.9)*, the solution* $u \in W_0^{1,1}(\Omega)$ *to* (3.1) *is such that*

$$|Du|^{p-1} \in L^{\frac{n\gamma}{n-\gamma}}(\Omega).$$

Finally, a borderline case

Theorem 3.3 ([16])**.** *Assume that* (1.4) *hold and that the measure* μ *is a function belonging to* $L \log L(\Omega)$*. The solution* $u \in W_0^{1,1}(\Omega)$ *to* (3.1) *is such that*

$$|Du|^{p-1} \in L^{\frac{n}{n-1}}(\Omega).$$

Theorems 3.1–3.3 establish a low order CZ-theory for elliptic problems with measure data which is completely analogous to that available in the linear case for the Poisson equation $-\triangle u = \mu$ and therefore optimal in the scale of Lebesgue's spaces.

Remark 3.2. The lower bound $2 - 1/n < p$ assumed in (3.2) is linked to the fact that the nonlinear Green's function $G_p(\cdot)$, and SOLA in general, stop belonging to $W^{1,1}$ when $1 < p \leq 2 - 1/n$. We refer to [12] for more on this last case.

4. Nonlinear Adams theorems

The results in this section, taken from [63], [64], extend in different directions the regularity results available for measure data problems presented in

Section 3, showing CZ estimates in new types of function spaces. We shall in fact present optimal non-linear extensions of classical results of Adams [4] and Adams & Lewis [7]. Moreover, we shall present a localization of the classical Lorentz spaces estimates obtained by Talenti [72]. Some of these local estimates have been later obtained and extended in [24]. A few extensions of the results presented here to the parabolic setting are contained in [11].

We recall the reader that in the following by solution we mean a SOLA, with the meaning given in Section 3.1. Therefore, the results below hold in general for *any* SOLA to (3.1); moreover, a unique SOLA exists when the measure right hand side belongs to L^1. Moreover, for the sake of simplicity we shall confine ourselves to the case $p \geq 2$; results for the case $2 - 1/n < p < 2$ can be achieved combining the methods in [66] with those from [64].

4.1. Morrey spaces. Morrey spaces provide a way of measuring the size of functions which is in some sense orthogonal to that of Lebesgue spaces. In fact, while these read the size of the super-level sets of functions-as rearrangement invariant spaces-Morrey spaces use instead density conditions in their formulation. Specifically, the condition is

$$\int_{B_R} |w|^\gamma \, dx \leq M^\gamma R^{n-\theta} \quad \text{and} \quad 0 \leq \theta \leq n, \tag{4.1}$$

to be satisfied for all balls $B_R \subseteq \Omega$ with radius R.

Definition 4.1. A measurable map $w \colon \Omega \to \mathbb{R}^k$, belongs to the Morrey space $L^{\gamma,\theta}(\Omega, \mathbb{R}^k) \equiv L^{\gamma,\theta}(\Omega)$ iff satisfies (4.1), and moreover one sets

$$\|w\|^\gamma_{L^{\gamma,\theta}(\Omega)} := \inf\{M^\gamma \geq 0 : (4.1) \text{ holds}\} = \sup_{B_R \subseteq \Omega} R^{\theta-n} \int_{B_R} |w|^\gamma \, dx.$$

Obviously $L^{\gamma,n} \equiv L^\gamma$, and $L^{\gamma,0} \equiv L^\infty$. The Morrey scale is orthogonal to the one provided by Lebesgue spaces in fact

$$L^{\gamma,\theta} \not\subset L^\gamma \log L \quad \text{for every } \theta > n,$$

that is, no matter how close θ is to zero, therefore no matter how close $L^{\gamma,\theta}$ is to L^∞ in the Morrey scale. We recall that the space $L^\gamma \log L(\Omega)$ is that of those functions w satisfying

$$\int_\Omega |w|^\gamma \log(\mathrm{e} + |w|) \, dx < \infty,$$

so that $L^{\gamma'} \subset L^\gamma \log L \subset L^\gamma$ for every $\gamma' > \gamma \geq 1$. This space can be structured as a Banach space, and a norm (actually equivalent to the standard Luxemburg one usually adopted when dealing with Orlicz spaces) is given by the quantity

$$\|w\|^\gamma_{L^\gamma \log L(\Omega)} := \fint_\Omega |w|^\gamma \log\Big(\mathrm{e} + \frac{w}{\fint_\Omega |w(y)|\,dy}\Big)\,dx. \tag{4.2}$$

This astonishing fact has been first observed in [47]. In a similar way the classical Marcinkiewicz-Morrey spaces [7], [5], [71] are naturally defined.

Definition 4.2. A measurable map $w\colon \Omega \to \mathbb{R}^k$ belongs to the Marcinkiewicz-Morrey space $\mathcal{M}^{\gamma,\theta}(\Omega, \mathbb{R}^k) \equiv \mathcal{M}^{\gamma,\theta}(\Omega)$ iff

$$\begin{aligned}\sup_{B_R \subset \Omega} R^{\theta-n} \|w\|^\gamma_{\mathcal{M}^\gamma(B_R)} &= \sup_{B_R \subset \Omega} \sup_{\lambda > 0} \lambda^\gamma R^{\theta-n} |\{x \in B_R : |w(x)| > \lambda\}| \\ &=: \|w\|^\gamma_{\mathcal{M}^{\gamma,\theta}(\Omega)} < \infty.\end{aligned} \tag{4.3}$$

Extending in an endpoint way previous results of Stampacchia [71], Adams considered Riesz potentials

$$I_\beta(\mu)(x) := \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\beta}}, \quad \beta \in (0, n], \tag{4.4}$$

and proved the following:

Theorem 4.1 ([4])**.** *Let $\beta \in [0, \theta)$; for every $\gamma > 1$ such that $\beta\gamma < \theta$ we have*

$$\mu \in L^{\gamma,\theta}(\mathbb{R}^n) \Longrightarrow I_\beta(\mu) \in L^{\frac{\theta\gamma}{\theta-\beta\gamma},\theta}(\mathbb{R}^n). \tag{4.5}$$

When $\gamma = 1$ and $\beta < \theta$ we have

$$\mu \in L^{1,\theta}(\mathbb{R}^n) \Longrightarrow I_\beta(\mu) \in \mathcal{M}^{\frac{\theta}{\theta-\beta},\theta}(\mathbb{R}^n), \tag{4.6}$$

and

$$\mu \in L^{1,\theta}(\mathbb{R}^n) \cap L\log L(\mathbb{R}^n) \Longrightarrow I_\beta(\mu) \in L^{\frac{\theta}{\theta-\beta},\theta}(\mathbb{R}^n). \tag{4.7}$$

Note that in the standard case $n = \theta$ (no genuine Morrey spaces are considered) the previous theorem extends classical results about regularizing properties of Riesz potentials, where θ replaces n everywhere. Theorem 4.1 has a few immediate consequences for the regularity of solutions to Poisson equation $-\triangle u = \mu$ (for simplicity considered in $\mathbb{R}^n$ when μ has compact support).

$$\mu \in L^{\gamma,\theta} \Longrightarrow Du \in L^{\frac{\theta\gamma}{\theta-\gamma},\theta} \quad \text{provided } 1 < \gamma < \theta. \tag{4.8}$$

The previous implication is obvious since estimates via fundamental solutions give $|Du(x)| \lesssim I_1(|\mu|)(x)$.

4.2. Nonlinear versions. Here we shall introduce the non-linear potential theory versions of Theorem 4.1 and of (4.8) in the context of measure data problems: this means that images of Riesz potentials are replaced by solutions to non-linear equations with p-growth, for example p-harmonic functions. Specifically, we are dealing with solutions to problems of the type (3.1); due to the fact that $x \to a(x,z)$ is just measurable, the maximal regularity expected in terms of gradient integrability is essentially $Du \in L^p$; see also Remark 3.1.

Since after Theorem 4.1 we expect the Morrey space parameter θ to play the role of the dimension n, in formal accordance with (3.9), we start assuming that

$$1 < \gamma \leq \frac{\theta p}{\theta p - \theta + p} \quad \text{and} \quad p < \theta \leq n, \tag{4.9}$$

a condition whose actual role will be discussed in Remark 4.1 below. We have the following non-linear potential theory version of (4.5):

Theorem 4.2 ([64])**.** *Assume* (1.4)*, and that the measure μ is a function belonging to $L^{\gamma,\theta}(\Omega)$ with* (4.9)*. Then the solution $u \in W^{1,p-1}_0(\Omega)$ to the problem* (3.1) *is such that*

$$|Du|^{p-1} \in L^{\frac{\theta\gamma}{\theta-\gamma},\theta}_{\mathrm{loc}}(\Omega). \tag{4.10}$$

Moreover, the local estimate

$$\||Du|^{p-1}\|_{L^{\frac{\theta\gamma}{\theta-\gamma},\theta}(B_{R/2})} \leq cR^{\frac{\theta-\gamma}{\gamma}-n}\|(|Du|+s)^{p-1}\|_{L^1(B_R)} + c\|\mu\|_{L^{\gamma,\theta}(B_R)}$$

holds for every ball $B_R \subseteq \Omega$, with a constant c only depending on n, p, ν, L, γ.

Observe that, on one hand for $p = 2$ inclusion (4.10) locally gives back (4.8), while on the other (4.10) is also the natural Morrey space extension of the non-linear result of Theorem 3.2, to which it locally reduces for $n = \theta$. In the relevant borderline case $\gamma = \theta p/(\theta p - \theta + p)$ the "energy solution" regularity $Du \in L^{p,\theta}_{\mathrm{loc}}(\Omega) \subset L^p_{\mathrm{loc}}(\Omega)$ holds.

Remark 4.1 (Sharpness of condition (4.9))**.** The parameters choice in (4.9) is optimal for the gradient integrability in the sense that the upper bound for γ is the minimal one allowing for the maximal regularity $Du \in L^p$. In fact we have

$$\gamma \leq \frac{\theta p}{\theta p - \theta + p} \quad \text{iff} \quad \frac{\theta\gamma(p-1)}{\theta-\gamma} \leq p. \tag{4.11}$$

Related to this fact is Theorem 4.3 below. Moreover, by Example 1 in Section 3, a Borel measure μ satisfying (3.4) for some $\varepsilon > 0$ and for every ball $B_R \subset \mathbb{R}^n$ belongs to the dual space $W^{-1,p'}$, and therefore (3.1) is uniquely solvable in $W_0^{1,p}(\Omega)$. This is the reason for assuming $p < \theta$ in Theorem 4.2 and Theorem 4.4 below. The case $p = \theta$ forces $\gamma = 1$ and therefore falls in the realm of measure data problems: it will be treated in Theorem 4.5 below. Note that Hölder's inequality and (4.1) imply that $|\mu|(B_R) = [|\mu|\,dx](B_R) \lesssim MR^{n-\theta/\gamma}$, and therefore, again by the mentioned Adams' result, in order to avoid trivialities we should also impose that $p\gamma \le \theta$. But keep in mind (4.9) and note that

$$\frac{\theta p}{\theta p - \theta + p} \le \frac{\theta}{p} \quad \text{iff} \quad 1 \le p \le \theta.$$

Therefore, assuming the first inequality in (4.11) together with $p \le \theta$ implies $p\gamma \le \theta$.

Theorem 4.3 ([64]). *Assume* (1.4)*, and that the measure μ is a function belonging to $L^{\gamma,\theta}(\Omega)$ with $\gamma > \theta p/(\theta p - \theta + p)$ and $p \le \theta \le n$. Then the solution $u \in W_0^{1,p-1}(\Omega)$ to the problem* (3.1) *is such that*

$$Du \in L^{h,\theta}_{\mathrm{loc}}(\Omega), \quad \textit{for some } h \equiv h(n,p,\nu,L,\gamma,\theta) > p.$$

Moreover, for every ball $B_R \subseteq \Omega$ with $R \le 1$ the local estimate

$$\|Du\|_{L^{h,\theta}(B_{R/2})} \le cR^{\frac{\theta}{h}-\frac{n}{p-1}}\|(|Du|+s)\|_{L^{p-1}(B_R)} + c\|\mu\|^{\frac{1}{p-1}}_{L^{\gamma,\theta}(B_R)}$$

holds for a constant c only depending on n, p, ν, L.

The previous result tells that, provided μ is regular enough, the solutions enjoys the maximal regularity allowed by the operator.

In the case $\gamma = 1$ we cannot obviously expect Theorem 4.2 to hold; instead, imposing an $L\log L$ type integrability condition on μ allows to deal with the case $\gamma = 1$ too, obtaining the natural analogue of (4.7).

Theorem 4.4 ([64]). *Assume that* (1.4) *holds, and that the measure μ is a function belonging to $L^{1,\theta}(\Omega) \cap L\log L(\Omega)$ with $p \le \theta \le n$. Then the solution $u \in W_0^{1,p-1}(\Omega)$ to the problem* (3.1) *satisfies*

$$|Du|^{p-1} \in L^{\frac{\theta}{\theta-1}}_{\mathrm{loc}}(\Omega). \tag{4.12}$$

Moreover, the reverse-Hölder type inequality

$$\begin{aligned}\Big(\fint_{B_{R/2}} |Du|^{\frac{(p-1)\theta}{\theta-1}}\,dx\Big)^{\frac{\theta-1}{\theta}} &\le c\fint_{B_R}(|Du|+s)^{p-1}\,dx \\ &+ c\|\mu\|^{\frac{1}{\theta}}_{L^{1,\theta}(B_R)}\Big[\fint_{B_R}|\mu|\log\Big(e+\frac{\mu}{\fint_{B_R}|\mu(y)|\,dy}\Big)\,dx\Big]^{\frac{\theta-1}{\theta}}\end{aligned} \tag{4.13}$$

holds for every ball $B_R \subseteq \Omega$, with a constant c only depending on n, p, ν, L.

The previous theorem is the natural extension to Morrey spaces of Theorem 3.3. We note that the appearance of the $L\log L$-type functional in the right hand side of the last estimate is exactly what we expect in a reverse Hölder type inequality as (4.13), since the last quantity defines a norm in $L\log L$, as observed in (4.2).

In order to conclude the non-linear extension of Theorem 4.1, giving the analogue of (4.6), we introduce Morrey spaces of measures.

Definition 4.3. We say that a Borel measure, defined on Ω, belongs to the Morrey space $L^{1,\theta}(\Omega)$ iff

$$\|\mu\|_{L^{1,\theta}(\Omega)} := \sup_{B_R} R^{\theta-n}|\mu|(B_R) < \infty. \tag{4.14}$$

The non-linear analogue of (4.6) is now

Theorem 4.5 ([63])**.** *Under the assumptions* (1.4)*, and $\mu \in L^{1,\theta}(\Omega)$ with $p \le \theta \le n$, there exists a solution $u \in W^{1,p-1}_0(\Omega)$ to the problem* (3.1) *such that*

$$|Du|^{p-1} \in \mathcal{M}^{\frac{\theta}{\theta-1},\theta}_{\mathrm{loc}}(\Omega). \tag{4.15}$$

Moreover, the local estimate

$$\||Du|^{p-1}\|_{\mathcal{M}^{\frac{\theta}{\theta-1},\theta}(B_{R/2})} \le cR^{\theta-1-n}\|(|Du|+s)^{p-1}\|_{L^1(B_R)} + c\|\mu\|_{L^{1,\theta}(B_R)}$$

holds for every ball $B_R \Subset \Omega$, with a constant c only depending on n, p, ν, L.

Just a few comments: in order to avoid trivialities we assumed $p \le \theta \le n$, otherwise, in the case $\theta < p$, the measure μ satisfies (3.4) and therefore belongs to the dual space $W^{-1,p'}(\Omega)$ by a result of Adams [3] – compare with Example 2 – and there exists a unique solution $u \in W^{1,p}_0(\Omega)$, found via usual monotonicity methods – note that, obviously $\theta(p-1)/(\theta-1) \le p$.

Theorem 4.5 incorporates in a local way-being nevertheless extendable up to the boundary – all the main integrability results previously obtained for

equations with measure data in the literature. When $\theta = n$, and in particular no density information on the measure is assumed, we find back the result in (3.7). When $p = n$, a case forcing $\theta = p = n$, we find back for the case of n-Laplacean equation the $\mathcal{M}^n$-regularity results obtained in [31], and in particular the explicit $\mathcal{M}^n$ local estimates subsequently obtained in [52]. In the borderline case $\theta = p$ we have in particular $Du \in \mathcal{M}^p$, locally, and therefore this is in perfect accordance with what happens when $\theta < p$: the L^p-regularity of the gradient is here just missed by a natural Marcinkiewicz factor. Finally let us mention that original Adams' theorem [4] of course applies when f in (4.6) is a measure rather than a function-of course Riesz potentials naturally act on measures too.

Remark 4.2 (Dimensional remark)**.** The qualitative information yielded by Theorem 4.5 is somehow interesting: let us recall that a measure μ which belongs to $L^{1,\theta}$, and that therefore satisfies $|\mu|(B_R) \lesssim R^{n-\theta}$, cannot concentrate on sets with Hausdorff dimension larger than $n - \theta$. Therefore Theorem 4.5 tells that the less the measure μ concentrates, the better solutions behave, confirming that the Dirac measure case is in some sense the worst one when analyzing the qualitative properties of solutions to measure data problems.

4.3. Lorentz spaces, and finer regularity. Lorentz spaces are a two-parameter scale of spaces which refine Lebesgue spaces in a sense that will be clear in a few lines. Lorentz spaces can be actually released as interpolation spaces using the K-functional interpolation theory of Gagliardo, Lions and Peetre, or using trace theory; we shall not pursue this abstract approach in the following rather pointing at a very straight presentation.

Definition 4.4. The Lorentz space

$$L(t,q)(\Omega), \quad \text{with } 1 \le t < \infty \text{ and } 0 < q \le \infty,$$

is defined prescribing that a measurable map w belongs to $L(t,q)(\Omega)$ iff

$$\|w\|^q_{L(t,q)(\Omega)} := q \int_0^\infty \left(\lambda^t |\{x \in \Omega : |w(x)| > \lambda\}|\right)^{\frac{q}{t}} \frac{d\lambda}{\lambda} < \infty, \tag{4.16}$$

when $q < \infty$; for $q = \infty$ we set $L(t,\infty)(\Omega) := \mathcal{M}^t(\Omega)$, and this means finding Marcinkiewicz spaces back.

The quantity in (4.16) is only a quasinorm, i.e., satisfies the triangle inequality only up to a multiplicative factor larger than one, and we remark that *in the following, when writing $L(t,q)$ without further specifications, we*

shall mean that t and q vary in the range specified in (4.4). We nevertheless remark that there is a canonical way to equip Lorentz spaces with a norm when $t > 1$, equivalent to the quantity introduced in (4.16). Good references for Lorentz spaces are for instance [13], [41].

Recalling that in this paper Ω has always finite measure, we remark that the spaces $L(t,q)(\Omega)$ "decrease" in the first parameter t, while increasing in q; moreover, they "interpolate" Lebesgue spaces as the second parameter q "tunes" t in the following sense: for $0 < q < t < r \leq \infty$ and we have, with continuous embeddings, that

$$L^r \equiv L(r,r) \subset L(t,q) \subset L(t,t) \subset L(t,r) \subset L(q,q) \equiv L^q.$$

Remark 4.3 (Lorentz spaces are not bizzarre)**.** In fact Lorentz spaces serve to describe finer scales of singularities, not achievable neither via the use of Lorentz spaces nor of Marcikiewicz ones. We have seen that Marcinkiewicz spaces describe in a sharp way potentials. For instance, with the ambient spaces being $\mathbb{R}^n$, we have

$$\frac{1}{|x|^{\frac{n}{\gamma}}} \in \mathcal{M}^{\gamma}(B_1) \setminus L^{\gamma}(B_1).$$

The perturbation of a potential via a logarithmic singularity is then described via Lorentz spaces

$$\frac{1}{|x|^{\frac{n}{\gamma}} \log^{\beta} |x|} \in L(\gamma,q)(B_1) \quad \text{iff} \quad q > \frac{1}{\beta}.$$

The last strict inequality tells us that Lorentz spaces are even less fine that one would wish! Note how the inverse relation between β and q demonstrates the fact that Lorentz spaces increase in the second index.

The "morreyzation" of the Lorentz norm leads to consider the so called Lorentz-Morrey spaces [7], [5]; this means coupling definition (4.16) with a density condition.

Definition 4.5. A measurable map w belongs to $L^{\theta}(t,q)(\Omega)$, for $1 \leq t < \infty$, $0 < q < \infty$ and $\theta \in [0,n]$, iff

$$\begin{aligned} \|w\|_{L^{\theta}(t,q)(\Omega)} &:= \sup_{B_R \subseteq \Omega} R^{\frac{\theta-n}{t}} \|w\|_{L(t,q)(B_R)} \\ &= \sup_{B_R \subseteq \Omega} \left(q \int_0^{\infty} \left(\lambda^t R^{\theta-n} |\{x \in B_R : |w(x)| > \lambda\}| \right)^{\frac{q}{t}} \frac{d\lambda}{\lambda} \right)^{\frac{1}{q}} < \infty. \end{aligned}$$

Moreover one sets $L^{\theta}(t,\infty)(\Omega) := \mathcal{M}^{t,\theta}(\Omega)$ and again we find back Marcinkiewicz-Morrey spaces defined in (4.3).

Remark 4.4. As already mentioned above, it follows from the definition in (4.16) that $L(t,t) \equiv L^t$, in fact by Fubini's theorem we have

$$\|w\|^t_{L^t(A)} = t\int_0^\infty \lambda^t \big|\{x \in A : |g(x)| > \lambda\}\big| \, \frac{d\lambda}{\lambda},$$

so that $\|w\|_{L^t(A)} = \|w\|_{L(t,t)(A)}$. As a consequence we also have that $L^{t,\theta} \equiv L^\theta(t,t)$, and $\|w\|_{L^{t,\theta}(A)} = \|w\|_{L^\theta(t,t)(A)}$ hold. Of course, $L^n(t,q) \equiv L(t,q)$.

We finally proceed with the "morreyzation" of $L\log L$, using the fact that the quantity in (4.2) defines a norm in $L\log L$.

Definition 4.6. The Morrey-Orlicz space $L\log L^\theta(\Omega)$ for $\theta \in [0,n]$ is defined by saying that a map w belongs to $L\log L^\theta(\Omega)$ iff

$$\begin{aligned}\|w\|_{L\log L^\theta(\Omega)} &:= \sup_{B_R \subseteq \Omega} R^\theta \|w\|_{L\log L(B_R)} \\ &\approx \sup_{B_R \subseteq \Omega} R^{\theta-n} \int_{B_R} |w| \log\Big(\mathrm{e} + \frac{w}{\fint B_R |w(y)|\, dy}\Big)\, dx < \infty.\end{aligned}$$

We are now ready for the results in such finer scales of spaces. Here is a classical result on Riesz potentials, which is actually a rather easy consequence of the off-diagonal version of Marcinkiewicz interpolation theorem and of the classical theorems on the boundedness of Riesz potentials.

Theorem 4.6. *Let $\beta \in [0,n]$; let $\gamma > 1$ be such that $\beta\gamma < n$, and let $q > 0$. We have*

$$\mu \in L(\gamma,q)(\mathbb{R}^n) \Longrightarrow I_\beta(\mu) \in L\Big(\frac{n\gamma}{n-\beta\gamma}, q\Big)(\mathbb{R}^n).$$

See [64] for additional details. A result of Adams & Lewis is instead

Theorem 4.7 ([7])**.** *Let $\beta \in [0,\theta)$; let $\gamma > 1$ be such that $\beta\gamma < \theta$, and let $q > 0$. We have*

$$\mu \in L^\theta(\gamma,q)(\mathbb{R}^n) \Longrightarrow I_\beta(\mu) \in L^\theta\Big(\frac{\theta\gamma}{\theta-\beta\gamma}, \frac{\theta q}{\theta-\beta\gamma}\Big)(\mathbb{R}^n).$$

We note two important points. In the case $\gamma = q$, when $L^\theta(\gamma,q) \equiv L^{\gamma,\theta}$, Theorem 4.7 reduces to Theorem 4.1, part (imb1mo), as obviously expected. On the other hand, when $\theta = n$, and therefore no Morrey space condition comes into the play, we have $L^\theta(\gamma,q) \equiv L(\gamma,q)$ but nevertheless Theorem 4.7 does not reduce to Theorem 4.6. This is not a gap in the theory, but a *genuine discontinuity phenomenon* discussed at length, and by mean of counterexamples, in [7], [5]. No surprise that a similar discontinuity phenomenon will pop-up in the non-linear case too.

The non-linear analogue of Theorem 4.7 is now

Theorem 4.8 ([64]). *Assume* (1.4), *and that the measure* μ *is a function belonging to* $L^\theta(\gamma,q)(\Omega)$ *with* γ,θ *as in* (4.9) *and* $0<q\le\infty$. *Then the solution* $u\in W^{1,p-1}_0(\Omega)$ *to* (3.1) *satisfies*

$$|Du|^{p-1}\in L^\theta\Big(\frac{\theta\gamma}{\theta-\gamma},\frac{\theta q}{\theta-\gamma}\Big)\quad \textit{locally in } \Omega.$$

Moreover, the local estimate

$$\begin{aligned}&\||Du|^{p-1}\|_{L^\theta(\frac{\theta\gamma}{\theta-\gamma},\frac{\theta q}{\theta-\gamma})(B_{R/2})}\\ &\qquad\le cR^{\frac{\theta-\gamma}{\gamma}-n}\|(|Du|+s)^{p-1}\|_{L^1(B_R)}+c\|\mu\|_{L^\theta(\gamma,q)(B_R)}\end{aligned}$$

holds for every ball $B_R\subseteq\Omega$, *where* c *depends only on* n, p, ν, L, γ, q.

We just notice that applying the previous result with the choice $\gamma=q$, therefore dropping the Lorentz space scale, we obtain Theorem 4.2 as a particular case. Taking instead $\theta=n$, therefore dropping the Morrey scale, does not yield the sharp result for the case of Lorentz spaces, due to the discontinuity phenomenon described after Theorem 4.7; for this we refer again to [5], [7]. The sharp version is anyway in Theorem 4.10 below.

Along with the higher integrability of Du comes a result about u.

Theorem 4.9 ([64]). *Assume* (1.4), *and that*

$$1<\gamma<\theta/p\quad\textit{and}\quad p<\theta\le n.$$

Assume that the measure μ *is a function belonging to* $L^\theta(\gamma,q)(\Omega)$ *with* $0<q\le\infty$. *Then the solution* $u\in W^{1,p-1}_0(\Omega)$ *to the problem* (3.1) *is such that*

$$|u|^{p-1}\in L^\theta\Big(\frac{\theta\gamma}{\theta-\gamma p},\frac{\theta q}{\theta-\gamma p}\Big)\quad \textit{locally in } \Omega.$$

Moreover, the local estimate

$$\begin{aligned}&\||u|^{p-1}\|_{L^\theta(\frac{\theta\gamma}{\theta-\gamma p},\frac{\theta q}{\theta-\gamma p})(B_{R/2})}\\ &\qquad\le cR^{\frac{\theta-\gamma}{\gamma}-n}\|(|u|+sR)^{p-1}\|_{L^1(B_R)}+c\|\mu\|_{L^\theta(\gamma,q)(B_R)}\end{aligned}$$

holds for every ball $B_R\subseteq\Omega$, *with a constant* c *only depending on* n, p, ν, L, γ, θ, q.

We finally conclude with the natural completion of Theorem 4.4. The point here is that in Theorem 4.4 the information $\mu\in L\log L$ is added to reach the full integrability (4.12), rather that the weak one (4.15), and acts only at this level. To have the proper analogue of (4.10), that is $|Du|^{p-1}\in L^{\theta/(\theta-1),\theta}$, necessitates to *transfer the Morrey density information, which is available only at* L^1*-scale in Theorem 4.4, at a full* $L\log L$*-level*, that is we have to assume $\mu\in L\log L^\theta$.

Theorem 4.10 ([64]). *Assume* (1.4) *and that the measure* μ *is a function belonging to* $L\log L^{\theta}(\Omega)$ *with* $p \le \theta \le n$. *Then the solution* $u \in W_0^{1,p-1}(\Omega)$ *to* (3.1) *is such that*

$$|Du|^{p-1} \in L^{\frac{\theta}{\theta-1},\theta}_{\mathrm{loc}}(\Omega).$$

Moreover, the local estimate

$$\||Du|^{p-1}\|_{L^{\frac{\theta}{\theta-1},\theta}(B_{R/2})} \le cR^{\theta-1-n}\||Du|^{p-1}\|_{L^1(B_R)} + c\|\mu\|_{L\log L^{\theta}(B_R)}$$

holds for every ball $B_R \subseteq \Omega$, *with a constant* c *only depending on* n, p, ν, L.

4.4. Pure Lorentz spaces regularity. In this section we abandon the case of Morrey regularity, turning our attention to regularity results in classical Lorentz spaces; in other words we are considering here the case $\theta = n$. There is a rather large literature on the topic, basically going back to the original works of Talenti [72]. Here we shall present a few results, again from [64], which feature an alternative approach to the known estimates, and in particular to Talenti's one based on symmetrization. The differences are in two respects: the theorems presented here involve explicit local estimates which previously employed methods – symmetrization, truncation – do not immediately yield; second: here the borderline cases of gradient estimates are covered. Of course, being our approach aimed at obtaining local estimates, the problem of deriving the best constants in the a priori estimates, which is solved in [72], becomes here immaterial. This problem can be anyway faced by symmetrization methods.

The next results is the non-linear version of Theorem 4.6. As described after Theorem 4.7, we note a difference with respect to Theorem 4.8, in that we have a higher gain in the second Lorentz exponent here.

Theorem 4.11 ([64]). *Assume that* (1.4) *holds with* $p < n$, *and that the measure* μ *is a function belonging to* $L(\gamma, q)(\Omega)$, *with* $1 < \gamma \le np/(np-n+p)$ *and* $0 < q \le \infty$; *then the solution* $u \in W_0^{1,p-1}(\Omega)$ *to the problem* (3.1) *is such that*

$$|Du|^{p-1} \in L\Big(\frac{n\gamma}{n-\gamma}, q\Big) \quad \textit{locally in } \Omega. \tag{4.18}$$

Moreover, the local estimate

$$\||Du|^{p-1}\|_{L(\frac{n\gamma}{n-\gamma},q)(B_{R/2})} \le cR^{\frac{n-\gamma}{\gamma}-n}\|(|Du|+s)^{p-1}\|_{L^1(B_R)} + c\|\mu\|_{L(\gamma,q)(B_R)}$$

holds for every ball $B_R \subseteq \Omega$, *with a constant* c *only depending on* n, p, ν, L, γ, q.

Inclusion (4.18) was already known in the literature [9], [50], except for the borderline case $\gamma = np/(np-n+p) = (p^*)'$ which was left uncovered; in this case we are "around the maximal regularity": $Du \in L(p, q(p-1))$, locally, whenever $q \in (0, \infty]$. The problem for the borderline case was raised for the first time in [50] for $q = \infty$, and eventually in [9] where inclusion (4.18) is conjectured for $q < \infty$. An attempt in the case $q = \infty$ has been given by Zhong [78, Theorem 2.30]. Inclusion (4.18) is actually straightforward in the case $(\gamma, q) = ((p^*)', p')$, compare with Example 2.

5. Higher differentiability for measure data problems

Let us concentrate for the moment on the case $p = 2$. It is clear that, in general, even for solutions to the basic linear equations $\triangle u = \mu \in L^1$ we cannot assert $u \in W^{2,1}$, and even locally. We see that lies exactly in the initial lack of integrability of the solutions, *and not in an additional lack of full differentiability.* Indeed we notice that the lack of integrability starts at a very primitive level in that, looking at the fundamental solution $G_2(\cdot)$, defined in (3.5) when $p = 2$, we have that (for simplicity we restrict to the case $n > 2$)

$$G_2 \in \mathcal{M}_{\mathrm{loc}}^{\frac{n}{n-2}}(\mathbb{R}^n) \setminus L_{\mathrm{loc}}^{\frac{n}{n-2}}(\mathbb{R}^n) \quad \text{and} \quad DG_2 \in \mathcal{M}_{\mathrm{loc}}^{\frac{n}{n-1}}(\mathbb{R}^n) \setminus L_{\mathrm{loc}}^{\frac{n}{n-1}}(\mathbb{R}^n)$$

while, after differentiating twice we have

$$|D^2 G_2(x)| \lesssim \frac{1}{|x|^n}, \quad x \neq 0,$$

and so this lack of integrability propagates up to second order derivatives

$$D^2 G_2 \in \mathcal{M}_{\mathrm{loc}}^1(\mathbb{R}^n, \mathbb{R}^{n^2}) \setminus L_{\mathrm{loc}}^1(\mathbb{R}^n, \mathbb{R}^{n^2}). \tag{5.1}$$

We may therefore suspect, as stated above, that the absence of second derivatives for solutions does not depend on a lack in the differentiability scale, but *rather on a lack on the integrability scale.* We shall try to make this concept rigorous now in order to recover at least some part of the missing differentiability.

There is now a problem in assertion (5.1): what kind of derivatives are those in (5.1)? Certainly, not of distributional type. At the moment they are just the usual, good old *pointwise derivatives*, which exist, in the case of the fundamental solution, at every point but the origin. Therefore, in order to look for a way of formulating a corresponding result for general SOLA to (3.1) we shall propose a different approach, via fractional derivatives.

Definition 5.1. For a bounded open set $A \subset \mathbb{R}^n$ and $k \in \mathbb{R}^n$, parameters $\sigma \in (0,1)$ and $q \in [1,\infty)$, the fractional Sobolev space $W^{\sigma,q}(A,\mathbb{R}^n)$ is defined requiring that $w \in W^{\sigma,q}(A,\mathbb{R}^n)$ iff the following Gagliardo-type norm is finite:

$$\|w\|_{W^{\sigma,q}(A)} := \Big(\int_A |w(x)|^q\,dx\Big)^{1/q} + \Big(\int_A\int_A \frac{|w(x)-w(y)|^q}{|x-y|^{n+\sigma q}}\,dx\,dy\Big)^{1/q}.$$

To view the previous definition in a more intuitive way, the reader may think of $W^{\sigma,q}$-functions as those function having "derivatives of order σ", in turn integrable with exponent q. Roughly writing, this means

$$[w]^q_{\sigma,q;A} := \int_A\int_A \frac{|w(x)-w(y)|^q}{|x-y|^{n+\sigma q}}\,dx\,dy \approx \text{“}\int_A |D^\sigma w|^q\,dz\text{”}, \quad 0<\sigma<1.$$

In order to get higher regularity for Du it is therefore natural to require more regularity on the vector field $a(\cdot)$, and we shall therefore consider assumption (1.5) rather than (1.4); moreover we shall consider an unavoidable-for the type of results eventually derived-Lipschitz regularity assumption on $x \mapsto a(x,z)$:

$$\omega(R) \le R. \tag{5.2}$$

Theorem 5.1 ([63], [66])**.** *Under the assumptions* (1.5) *with* $2-1/n < p \le n$ *and* (5.2)*, let* $u \in W^{1,1}_0(\Omega)$ *be a SOLA to the problem* (3.1)*.*

- *If* $p \ge 2$ *then*

$$Du \in W^{\frac{1-\varepsilon}{p-1},p-1}_{\mathrm{loc}}(\Omega,\mathbb{R}^n) \quad \textit{for every } \varepsilon>0. \tag{5.3}$$

- *If* $2-1/n < p \le 2$ *then*

$$Du \in W^{\frac{p-np(2-p)-\varepsilon}{2p-2},1}_{\mathrm{loc}}(\Omega,\mathbb{R}^n) \quad \textit{for every } \varepsilon>0. \tag{5.4}$$

- *In particular, when* $p=2$ *it holds that*

$$Du \in W^{1-\varepsilon,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^n) \quad \textit{for every } \varepsilon>0. \tag{5.5}$$

Remark 5.1. The result in (5.5) positively answers to an old conjecture of Philippe Benilán formulated in the case $p=2$ [77]; when $p \ne 2$ the theorem above extends the validity of this conjecture to the p-Laplacean type problems. In the case $p>n$ different fractional differentiability results are available; for this we refer to [63].

We obviously observe the change from (5.3) to (5.4) when p becomes smaller than 2. Note the space in (5.3) makes sense only when $p \geq 2$ as $p-1$ must be larger or equal than 1. The two results match in the case $p = 2$. The bifurcation of differentiability statements is typical already when dealing with the p-Laplacean equation

$$\operatorname{div}(|Dv|^{p-2}Dv) = 0. \tag{5.6}$$

Indeed in this case it is known (see also [62]) that $W^{1,p}$-solutions v are such that

$$v \in \begin{cases} W^{2/p,p}_{\mathrm{loc}}(\Omega) & \text{if } p \geq 2, \\ W^{1,p}_{\mathrm{loc}}(\Omega) & \text{if } 1 < p \leq 2. \end{cases} \tag{5.7}$$

The analogy with the regularity results available for p-harmonic mappings goes anyway further. Indeed, a so called "*nonlinear uniformization of singularities principle*" holds – we are here adopting a terminology introduced by Tadeusz Iwaniec – in that although the gradient Dv is in general not differentiable, certain nonlinear expressions involving it, turn in fact out to be. Such quantities are devised to incorporate the degeneracy features of the equation. Specifically, with $s \geq 0$ as in (1.5) let us introduce the mapping

$$V(z) = V_s(z) := (s^2 + |z|^2)^{(p-2)/4} z, \quad z \in \mathbb{R}^n,$$

which is easily seen to be a locally bi-Lipschitz bijection of $\mathbb{R}^n$. Notice that

$$|V(Du)| \approx |Du|^{p/2} \tag{5.8}$$

for $|Du|$ large, and therefore by composing the map $V(\cdot)$ with Du we expect to diminish the integrability of $V(Du)$ and to gain in smoothness. The situation we are going to describe is typical in Complex Analysis, when there exists functions which turn out to be analytic only after having been raised to high enough powers. In fact for solutions to (5.6) (which satisfies assumptions (1.5) with $s = 0$) satisfy

$$V_0(Du) = |Du|^{(p-2)/2} Du \in W^{1,2}_{\mathrm{loc}} \quad \text{for every } p > 1.$$

A similar situation reproduces in the case of measure data problems as stated in the next

Theorem 5.2 ([63], [66])**.** *Under the assumptions* (1.5) *with* $2-1/n < p \leq n$ *and* (5.2)*, let* $u \in W^{1,1}_0(\Omega)$ *be a SOLA to the problem* (3.1)*.*

- *If* $p \geq 2$ *then*

$$V(Du) \in W^{\frac{p-\varepsilon}{2(p-1)},\frac{2(p-1)}{p}}_{\mathrm{loc}}(\Omega,\mathbb{R}^n) \quad \textit{for every } \varepsilon > 0.$$

- *If* $2-1/n < p \leq 2$ *then*

$$V(Du) \in W^{\frac{p-np(2-p)-\varepsilon}{2(p-1)},\frac{2}{p}}_{\mathrm{loc}}(\Omega,\mathbb{R}^n) \quad \textit{for every } \varepsilon > 0.$$

Remark 5.2 (Sharpness)**.** We observe that the previous result is sharp, in that we cannot allow for $\varepsilon = 0$. This can be tested directly looking at the nonlinear fundamental solution in (3.5) and using fractional Sobolev embedding theorem, that states (see for instance [8]) that

$$W^{\sigma,q}_{\mathrm{loc}} \hookrightarrow L^{nq/(n-\sigma q)}_{\mathrm{loc}} \quad \text{provided } \sigma q < n.$$

By the previous embedding and (5.8), in both the cases $p \geq 2$ and $2-1/n < p \leq 2$, assuming (5.3) and (5.4) with $\varepsilon = 0$ would give $|Du|^{p-1} \in L^{n/(n-1)}_{\mathrm{loc}}$, which is on the other hand impossible as the nonlinear Green's function G_p in (3.5) – i.e., the unique SOLA to (3.1) with $\mu = \delta$ – does not enjoy such an integrability property. We also note that in the case $p < 2$ the lower bound $p > 2 - 1/n$ serves to ensure that the differentiability exponent remains positive, i.e. some fractional differentiability holds:

$$0 < \frac{p - np(2-p)}{2(p-1)} \Longleftrightarrow 2 - \frac{1}{n} < p.$$

This is in turn clearly related to the fact that below the threshold $p = 2-1/n$ SOLA do not longer belong to $W^{1,1}$, in general. Observe also that it is easy to obtain the results in (5.3) and (5.4) for the case $u = G_p$, by direct computation.

Remark 5.3. Comparing the change of exponents occurring when p crosses the value $p = 2$ it is easy to see a perfect analogy between what happens in the homogeneous case $\mu = 0$ and displayed in (5.7), and the differentiability results of Theorems 5.1 and 5.2.

Finally, we may of course wonder what happens with respect to gradient (fractional) derivatives when one considers the density condition decay as (4.14). In this case the density information transfers to the fractional derivatives as follows, and once again the parameter θ appearing in (4.14) plays the role of n:

Theorem 5.3 ([63], [66]). *Under the assumptions* (1.5) *with* $2-1/n < p \le n$ *and* (4.14) *in force with* $p \le \theta \le n$, *let* $u \in W^{1,1}_0(\Omega)$ *be a SOLA to the problem* (3.1). *Then*

- *when*
$$p \ge 2 \quad \text{and } 0 < \sigma < 1,$$
the local estimate
$$\int_{B_R}\int_{B_R} \frac{|Du(x)-Du(y)|^{p-1}}{|x-y|^{n+\sigma}}\,dx\,dy \le cR^{n-\theta+(1-\sigma)}$$
holds whenever $B_R \subset \Omega$, *with* c *depending only on* n, p, ν, L, $|\mu|(\Omega)$, $\|\mu\|_{L^{1,\theta}}$, σ, $\mathrm{dist}(B_R, \partial\Omega)$,
- *instead, when*
$$2-1/n < p \le 2 \quad \text{and } 0 < \sigma < \frac{p-\theta p(2-p)}{2p-2}$$
the local estimate
$$\int_{B_R}\int_{B_R} \frac{|Du(x)-Du(y)|}{|x-y|^{n+\sigma}}\,dx\,dy \le cR^{n-\theta+\left(\frac{1-\theta(2-p)}{p-1}-\sigma\right)}$$
holds whenever $B_R \subset \Omega$, *with* c *depending only on* n, p, ν, L, $|\mu|(\Omega)$, $\|\mu\|_{L^{1,\theta}}$, $\sigma, \mathrm{dist}(B_R, \partial\Omega)$

6. Wolff potential estimates

In this final section we shall briefly describe some very recent development from non-linear potential theory made in [65], [34], [35]. Again we shall for simplicity restrict to the case $p \ge 2$, referring to [35] for the case $2-1/n < p < 2$.

The point here is to give a complete non-linear analogue of the classical pointwise gradient estimates valid for the Poisson equation

$$-\triangle u = \mu \quad \text{in } \mathbb{R}^n, \tag{6.1}$$

via Riesz potentials. At the same time the results give a somehow unexpected but natural maximal order-and parabolic-version of a by now classical result due to Kilpeläinen & Malý [51] and later extended, by mean of a different approach, by Trudinger & Wang [74].

To better frame our setting, let us recall a few basic linear results concerning the basic example (6.1) – here for simplicity considered in the whole $\mathbb{R}^n$ – for which, due to the use of classical representation formulas, it is possible to get pointwise bounds for solutions via the use of Riesz potentials (4.4) such as

$$|u(x)| \leq cI_2(|\mu|)(x) \quad \text{and} \quad |Du(x)| \leq cI_1(|\mu|)(x). \tag{6.2}$$

We recall that the equivalent, localized version of the Riesz potential $I_\beta(\mu)(x)$ is given by the linear potential

$$\mathbf{I}^\mu_\beta(x,R) := \int_0^R \frac{\mu(B(x,\varrho))}{\varrho^{n-\beta}}\,\frac{d\varrho}{\varrho}, \quad \beta \in (0,n],$$

with $B(x_0,\varrho)$ being the open ball centered at x_0, with radius ϱ. In fact, it is not difficult to see that

$$\mathbf{I}^\mu_\beta(x,R) \lesssim \int_{B(x,R)} \frac{d\mu(y)}{|x-y|^{n-\beta}} = I_\beta(\mu \llcorner B(x,R))(x) \leq I_\beta(\mu)(x)$$

holds provided μ is a non-negative measure. A question is now, *is it possible to give an analogue of estimates* (6.2) *in the case of general quasilinear equations such as for instance, the degenerate p-Laplacean equation*? A first answer has been given in the papers [51], [74], where – for suitably defined solutions – the authors prove the following pointwise *zero order estimate* – i.e. for u-when $p \leq n$, via non – linear Wolff potentials:

$$|u(x)| \leq c\Big(\fint_{B(x,R)} (|u|+s)^{p-1}\,dy\Big)^{\frac{1}{p-1}} + c\mathbf{W}^\mu_{1,p}(x,R). \tag{6.3}$$

where the constant c depends on the quantities n,p, and

$$\mathbf{W}^\mu_{\beta,p}(x,R) := \int_0^R \left(\frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta p}}\right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho}, \quad \beta \in (0,n/p], \tag{6.4}$$

is the non-linear Wolff potential of μ. Estimate (6.3), which extends to a whole family of general quasi-linear equations, and which is commonly considered as a basic result in the theory of quasi-linear equations, is the natural non-linear analogue of the first linear estimate appearing in (6.2). *Here we present the non-linear analogue of the second estimate in* (6.2)*, thereby giving a pointwise gradient estimate via non-linear potentials which upgrades* (6.4) *up to the gradient/maximal level.* Specifically, we shall consider general non-linear, possibly degenerate equations with p-growth of the

type (1.1) under the assumptions (1.5). Moreover, on the modulus of continuity $\omega\colon [0,\infty) \to [0,\infty)$ we impose a natural decay property, which is essentially optimal for the result we are going to have, and prescribes a *Dini continuous dependence of the partial map* $x \mapsto a(x,z)/(|z|+s)^{p-1}$:

$$\int_0^R [\omega(\varrho)]^{\frac{2}{p}} \, \frac{d\varrho}{\varrho} =: d(R) < \infty. \tag{6.5}$$

The relevant results will be here presented in the form of a priori estimates – i.e. when solutions and data are taken to be more regular than needed, for instance $u \in C^1(\Omega)$ and $\mu \in L^1(\Omega)$ – but they actually hold, via a standard approximation argument, for general weak and very weak solutions – i.e., distributional solutions which are not in the natural space $W^{1,p}(\Omega)$ – to measure data problems such as (3.1), where, as usual, μ is a general Radon measure with finite total mass, defined on Ω. See for instance Section 3.1. The reason for such choice is that the approximation argument in question leads to different notions of solutions, according to the regularity/integrability properties of the right hand side μ, therefore, proving a priori estimates leads to results applying to several different notions of solutions.

The first result we present is now

Theorem 6.1 ([34])**.** *Let $u \in C^1(\Omega)$ be a weak solution to* (1.1) *with $\mu \in L^1(\Omega)$, under the assumptions* (1.5) *and* (6.5)*. Then there exists a positive constant $c \equiv c(n,p,\nu,L)$, and a positive radius $R_0 \equiv R_0(n,p,\nu,L,\omega(\cdot))$ such that the pointwise estimate*

$$|Du(x)| \le c\Big(\fint_{B(x,R)} (|Du|+s)^{\frac{p}{2}} \, dy \Big)^{\frac{2}{p}} + c\mathbf{W}^{\mu}_{1/p,p}(x,R) \tag{6.6}$$

holds whenever $B(x,R) \subseteq \Omega$, and $R \le R_0$. Moreover, when the vector field $a(\cdot)$ is independent of x, estimate (6.6) *holds with no restriction on R.*

The potential $\mathbf{W}^{\mu}_{1/p,p}$ appearing in (6.6) is the natural one since its shape respects the scaling properties of the equation with respect to the estimate in question; compare with the linear estimates (6.2). When extended to general weak solutions *estimate* (6.6) *tells us the remarkable fact that the boundedness of Du at a point x_0 is independent of the solution u, and of the vector field $a(\cdot)$ considered, but only depends on the behavior of $|\mu|$ in a neighborhood of x.*

A particularly interesting situation occurs in the case $p = 2$, when we have a pointwise potential estimate which is completely similar to the second one in (6.2), that is

Theorem 6.2 ([65]). *Let $u \in C^1(\Omega)$, be a weak solution to* (1.1) *with $\mu \in L^1(\Omega)$, under the assumptions* (1.5) *and* (6.5)*, considered with $p = 2$. Then there exists a positive constant $c \equiv c(n, p, \nu, L)$, and a positive radius $R_0 \equiv R_0(n, p, \nu, L, \omega(\cdot))$ such that the pointwise estimate*

$$|Du(x)| \leq c \fint_{B(x,R)} (|Du| + s)\, dy + c\mathbf{I}_1^{|\mu|}(x, R) \tag{6.7}$$

holds whenever $B(x, R) \subseteq \Omega$, and $R \leq R_0$. Moreover, when the vector field $a(\cdot)$ is independent of the variable x, estimate (6.7) *holds with no restriction on R.*

Beside their intrinsic theoretical interest, the point in estimates (6.6), (6.7) is that they allow to unify and recast essentially all the gradient L^q-estimates for quasilinear equations in divergence form; moreover they allow for an immediate derivation of estimates in intermediate spaces such as interpolation spaces. Indeed, by (6.6) it is clear that the behavior of Du can be controlled by that $\mathbf{W}^{\mu}_{1/p,p}$, which is in turn known via the behavior of Riesz potentials. In fact, this is a consequence of the pointwise bound of the Wolff potential via the Havin–Maz'ja non-linear potential [42], [6], that is

$$\mathbf{W}^{\mu}_{1/p,p}(x, \infty) = \int_0^{\infty} \left(\frac{|\mu|(B(x, \varrho))}{\varrho^{n-1}} \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \leq cI_1\big\{[I_1(|\mu|)]^{\frac{1}{p-1}}\big\}(x).$$

Ultimately, we have

$$\mu \in L^q \Longrightarrow \mathbf{W}^{\mu}_{1/p,p} \in L^{\frac{nq(p-1)}{n-q}}, \quad q \in (1, n), \tag{6.8}$$

while Marcikiewicz spaces must be introduced for the borderline case $q = 1$. Inequality (6.8) immediately allows to recast the classical gradient estimates for solutions to (3.1) such as those due to Boccardo & Gallöuet [15], [16] – when q is "small" – and Iwaniec [43] and DiBenedetto & Manfredi [28] – when q is "large" – that is, for solutions to (3.1) it holds that

$$\mu \in L^q \Longrightarrow Du \in L^{\frac{nq(p-1)}{n-q}}, \quad q \in (1, n).$$

Moreover, since the operator $\mu \mapsto \mathbf{W}^{\mu}_{1/p,p}$ is obviously sub-linear, using the estimates related to (6.8) and classical interpolation theorems for sub-linear operators one immediately gets estimates in refined scales of spaces such as Lorentz or Orlicz spaces, recovering some estimates of Talenti [72], but

directly for the gradient of solutions, rather than for solutions themselves. For several consequences of Theorem 6.1 we again refer to [34] and [24]. Notice that the results of Section 4 cannot be obtained as a corollary from Theorems 6.1 and 6.2 as, while the regularity with respect to x of $a(\cdot)$ considered there is the Dini continuity, in Section 4 a measurable dependence is considered. Finally, we mention that a parabolic analogue of the Wolff potential estimate of Theorem 6.1 has been recently obtained in [57], [58], [59].

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