A SURPRISING LINEAR TYPE ESTIMATE FOR NONLINEAR
ELLiptic EQUATIONS

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Abstract. Pointwise gradient bounds via Riesz potentials like those available
for the Poisson equation actually hold for $p$-Laplacean type equations.

1. Results

The aim of this note is to report on a surprising estimate, together with its
corollaries and related results, which has been recently obtained in [10]; this is in

Theorem 1.1. Let $u \in W^{1,p}(\Omega)$ be a weak solution to the equation

$$-\text{div} (|Du|^{p-2} Du) = \mu$$

with $\mu$ being a Borel measure with finite total mass and $p \geq 2$, where $\Omega \subset \mathbb{R}^n$ is an
open set and $n \geq 2$. Then there exists a constant $c$, depending only on $n, p$, such
that the pointwise estimate

$$|Du(x)|^{p-1} \leq c I_1^{[\mu]}(x, R) + c \left( \frac{1}{B(x, R)} \int_{B(x, R)} |Du| \, dy \right)^{p-1}$$

holds whenever $B(x, R) \subseteq \Omega$ is a ball centered at $x$ and with radius $R$, and $x$ is a
Lebesgue point of $Du$.

In (1.2), $I_1$ denotes the classical, linear (truncated) Riesz potential of $|\mu|$, which
is suited to problems defined in bounded domains, and it is defined by

$$I_1^{[\mu]}(x, R) := \int_0^R \frac{|\mu|(B(x, \rho)) \, d\rho}{\rho^{n-1}} \leq \int_{\mathbb{R}^n} \frac{|\mu|(y) \, dy}{|x-y|^{n-1}}.$$

Observe that we may always assume that $\mu$ is defined on $\mathbb{R}^n$.

Remark 1.1. Theorem 1.1, proposed above in the form of an a priori estimate
for energy solutions, actually extends to the case when $u$ is a so-called very weak
solution not necessarily belonging to $W^{1,p}(\Omega)$. This type of low integrability is
typical when dealing with measure data problems; the extension goes via a stan-
dard approximation argument; for these issues we refer for instance to [1, 4, 11].

Theorem 1.1 also extends to solutions to more general quasilinear, possibly degener-
ate equations of the type $-\text{div} a(Du) = \mu$, under the usual growth and ellipticity
assumptions of Ladyzhenskaya & Ural’tseva type

$$\left\{ \begin{array}{l}
|a(z)| + |a_z(z)||(z|^2 + s^2)^{1/2} \leq L(|z|^2 + s^2)^{(p-1)/2} \\
\nu(|z|^2 + s^2)^{(p-2)/2}|\lambda|^2 \leq \langle a_z(x, z) \lambda, \lambda \rangle
\end{array} \right.$$
to hold whenever \( z, \lambda \in \mathbb{R}^n \). Here \( 0 < \nu \leq L \) and \( s \geq 0 \) are fixed parameters. Moreover, the vector field \( a: \mathbb{R}^n \to \mathbb{R}^n \) is supposed to be \( C^1 \). Further extensions are possible for equations with coefficients of the type \(-\text{div} a(x, Du) = \mu\), where the dependence on \( x \) is supposed to be Dini-continuous.

The primary effect of Theorem 1.1 is that in some sense it makes the gradient integrability theory of quasilinear equations completely analogous to that of the basic Poisson equation \(-\Delta u = \mu\). For example, we have

**Corollary 1.1.** Let \( u \in W^{1,p}(\mathbb{R}^n) \) be a weak solution to the equation (1.1) in \( \mathbb{R}^n \), with \( \mu \) being a Borel measure with locally finite mass and \( p \geq 2 \). Then there exists a constant \( c \), depending only on \( n, p \), such that the following estimate holds for every Lebesgue point \( x \in \mathbb{R}^n \) of \( Du \):

\[
|Du(x)|^{p-1} \leq c \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}}.
\]

As a matter of fact, by using the basic results about Riesz potentials, all the classical integrability results available for the solutions to (1.1) are now recovered (for instance, those in [1, 2, 3, 4, 6, 11, 12]). Moreover, delicate and open borderline cases can be fixed. For instance, Theorem 1.1 also provides us with a sharp Lipschitz regularity criterium for solutions to non-homogeneous equations:

**Corollary 1.2.** Let \( u \in W^{1,p}(\Omega) \) be as in Theorem 1.1. If \( I_{1,p}^{(\cdot, R)} \in L^{\infty}(\Omega) \) for some \( R > 0 \), then \( Du \in L^{\infty}(\Omega, \mathbb{R}^n) \).

The other thing making Theorem 1.1 somehow unexpected is that, starting from the seminal paper of Kilpeläinen & Malý [7, 8], with a different approach offered by Trudinger & Wang [14], it is a standard fact that solutions to non-homogeneous quasilinear equations with measure data as (1.1) can be pointwise estimated in a natural way by mean of Wolff potentials

\[
W_{\beta,p}^{n}(x,R) := \int_{0}^{R} \left( \frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta p}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho} \quad \beta \in (0, n/p].
\]

In particular, the main result of [7, 8] (see also [14, 9, 4]) claims that the following pointwise estimate hold a.e.:

\[
|u(x)|^{p-1} \leq c \left[ W_{1,p}^{n}(x,R) \right]^{p-1} + c \left( \int_{B(x,R)} |u| \, dy \right)^{p-1}.
\]

The previous result is sharp in the sense that \( W_{1,p}^{n} \) also bounds \( u \) from below when the measure \( \mu \) and the solution \( u \) are nonnegative:

\[
\frac{1}{c} W_{1,p}^{n}(x,R) \leq u(x) \leq c W_{1,p}^{n}(x,R) + c \int_{B(x,R)} |u| \, dy.
\]

We refer to [13] for a global version of the previous estimate. Potential estimates as the one in (1.4) play a fundamental role in the analysis of solutions to degenerate equations.

Estimate (1.4) has been eventually upgraded to the gradient level in [4, 12], where it has been proved that

\[
|Du(x)|^{p-1} \leq c \left[ W_{1,p}^{n}(x,R) \right]^{p-1} + c \left( \int_{B(x,R)} |Du| \, dy \right)^{p-1}.
\]

Estimate (1.2) obviously improves (1.5) as

\[
I_{1,p}^{(\cdot, R)} = \int_{0}^{R} \frac{|\mu|(B(x_{0},\varrho))}{\varrho^{n-1}} \frac{d\varrho}{\varrho}.
\]
where \( c \) when there exists an increasing subsequence for every integer \( i \).

Therefore Theorem 1.1 tells that a change in the type of potential used, i.e. from nonlinear to linear occurs when passing from estimates for \( u \) to those for \( Du \).

Remark 1.2. We also observe that the case \( p < 2 \) of estimate (1.2) has been proved in [5], but in this case it does not improve (1.5) in that (1.6) does not hold in general for \( p < 2 \) and the real difficult case is passing from nonlinear potentials to linear ones when the latter provide better bounds.

Theorem 1.1 opens the way to sharp continuity criteria for the gradient:

Theorem 1.2. Let \( u \in W^{1,p}(\Omega) \) be as in Theorem 1.1. Assume that \( I^\mu_{1}[u](x,R) \to 0 \) as \( R \to 0 \), locally uniformly in \( \Omega \) w.r.t. \( x \); then \( Du \) is continuous in \( \Omega \).

The previous result admits the following relevant corollary, providing a borderline condition for gradient continuity in terms of Lorentz spaces \( L(n,1) \):

Corollary 1.3. Let \( u \in W^{1,p}(\Omega) \) be as in Theorem 1.1. If \( \mu \in L(n,1) \) locally in \( \Omega \), then \( Du \) is continuous in \( \Omega \).

We recall that \( \mu \in L(n,1) \) locally in \( \Omega \) means that

\[
\int_0^{\infty} t^n \{ x \in \Omega' : |\mu(x)| > t \} \frac{dt}{t} < \infty \quad \text{for every open subset } \Omega' \subset \Omega.
\]

Moreover \( L^\gamma \subset L(n,1) \subset L^n \) for every \( \gamma > n \), inclusions being strict. We refer to [11] for relevant definitions and results on Lorentz spaces in this context.

2. A sketch of the proof of Theorem 1.1

The proof of Theorem 1.1 (and that of Theorem 1.2) is very delicate, and virtually involves all the known aspects of the regularity of solutions to \( p \)-Laplacean type equations. The main part is based on a careful local comparison argument with \( p \)-harmonic functions, i.e. solutions to homogeneous equations of the type \( \text{div} (|Du|^p-2Du) = 0 \).

Moreover, with the ball \( B(x,R) \subset \Omega \) fixed in Theorem 1.1, we determine a small shrinking number \( \delta_1 \in (0,1/10^n) \), depending only on \( n,p \), but not on \( \mu \) and of the particular solution \( u \) considered, and then define the sequence of shrinking balls \( B_i := B(x,\delta_i R), \) for every integer \( j \geq 0 \).

Accordingly, we consider \( p \)-harmonic liftings (of \( u \)) \( v_i \in u + W^{1,p}_0(B_i) \), that is \( \text{div} (|Dv_i|^p-2Dv_i) = 0 \) in \( B_i \) and such that \( v_i = u \) on \( \partial B_i \) in Sobolev sense. At this stage it is important to develop a suitable form of the \( C^{1,\alpha} \) regularity estimates for \( v_i \). Next, we define a number

\[
\lambda^{p-1} := c_1 I^\mu_{1}[u](x,R) + c_1 \left( \frac{1}{|B(x,R)|} \int_{B(x,R)} |Du| \, dy \right)^{p-1},
\]

where \( c_1 \geq \delta_1^{-1}n \) is a constant depending only on \( n,p \); moreover, we define

\[
C_i \approx \frac{1}{|B_i|} \int_{B_i} |Du| \, dy
\]

for every integer \( i \geq 2 \). To proceed, we argue on an alternative. The first case is when there exists an increasing subsequence \( \{j_i\} \) such that \( C_{j_i} \leq \lambda/100 \), for every \( i \in \mathbb{N} \), and then, as \( x \) is a Lebesgue point of \( Du \), we have \( |Du(x)| \leq \lim_{i \to \infty} C_{j_i} \leq \lambda/100 \) and the proof is finished. Otherwise we determine an “exit time” index...
\[ i_e \geq 3 \text{ such that } C_{i_e} \leq \lambda/100 \text{ and } C_i \geq \lambda/100 \text{ holds for every } i > i_e. \]

Thanks to this additional information we are now able to get the local comparison estimate
\[ (2.1) \quad \int_{B_{i+1}} |Du - Dv_i| \, dy \leq c(B_i, \lambda, \mu(B_i)). \]

In turn, (2.1) and the good decay/regularity estimates available for the comparison functions \( v_i \) allow to derive the following decay estimate with a remainder:
\[ (2.2) \quad E(Du, B_{i+1}) \leq \frac{1}{4} E(Du, B_i) + \text{controllable error term related to } I_{\mu_i} \]

whenever \( i \geq i_e \). Here
\[ E(Du, B_i) := \int_{B_i} |Du - (Du)_i| \, dy \quad \text{and} \quad (Du)_i := \frac{1}{|B_i|} \int_{B_i} Du \, dy. \]

Next, by summing up (2.2), we prove by induction that actually
\[ |(Du)_i| \leq \lambda \]
holds whenever \( i \geq i_e \). At this stage the proof of (1.2) follows as, \( x \) being a Lebesgue point of \( Du \), we have
\[ |Du(x)| = \lim_{i \to \infty} |(Du)_i| \leq \lambda. \]

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References