A SURPRISING LINEAR TYPE ESTIMATE FOR NONLINEAR ELLIPTIC EQUATIONS

UNE ESTIMATION DE TYPE LINÉAIRE SURPRENANTE POUR DES ÉQUATIONS ELLIPTIQUES NON LINÉAIRES

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Abstract. Pointwise gradient bounds via Riesz potentials like those available for the Poisson equation actually hold for p-Laplacean type equations.

Resumé: Des bornes ponctuelles par potentiels de Riesz semblables à celles disponibles pour l' équation de Poisson sont valables pour des quations du type du p-laplacien.

1. Results

The aim of this note is to report on a surprising estimate, together with its corollaries and related results, which has been recently obtained in [10]; this is in

Theorem 1.1. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to the equation*

$$-\mathrm{div}\,(|Du|^{p-2}Du) = \mu \tag{1.1}$$

with μ being a Borel measure with finite total mass and $p \geq 2$, where $\Omega \subset \mathbb{R}^n$ is an open set and $n \geq 2$. Then there exists a constant c, depending only on n, p, such that the pointwise estimate

$$|Du(x)|^{p-1} \leq c\mathbf{I}_1^{|\mu|}(x,R) + c\left(\frac{1}{|B(x,R)|}\int_{B(x,R)}|Du|\,dy\right)^{p-1} \tag{1.2}$$

holds whenever $B(x,R) \subseteq \Omega$ is a ball centered at x and with radius R, and x is a Lebesgue point of Du.

In (1.2), $\mathbf{I}_1$ denotes the classical, linear (truncated) Riesz potential of $|\mu|$, which is suited to problems defined in bounded domains, and it is defined by

$$\mathbf{I}_1^{|\mu|}(x,R) := \int_0^R \frac{|\mu|(B(x,\varrho))}{\varrho^{n-1}}\,\frac{d\varrho}{\varrho} \quad \left(\lesssim \int_{\mathbb{R}^n}\frac{|\mu|(y)\,dy}{|x-y|^{n-1}}\right).$$

Observe that we may always assume that μ is defined on $\mathbb{R}^n$.

Remark 1.1. Theorem 1.1, proposed above in the form of an a priori estimate for energy solutions, actually extends to the case when u is a so-called very weak solution not necessarily belonging to $W^{1,p}(\Omega)$. This type of low integrability is typical when dealing with measure data problems; the extension goes via a standard approximation argument; for these issues we refer for instance to [1, 4, 11]. Theorem 1.1 also extends to solutions to more general quasilinear, possibly degenerate equations of the type $-\mathrm{div}\ a(Du) = \mu$, under the usual growth and ellipticity assumptions of Ladyzhenskaya & Ural'tseva type

$$\begin{cases} |a(z)| + |a_z(z)|(|z|^2+s^2)^{1/2} \leq L(|z|^2+s^2)^{(p-1)/2} \\ \nu(|z|^2+s^2)^{(p-2)/2}|\lambda|^2 \leq \langle a_z(x,z)\lambda,\lambda\rangle \end{cases}$$

to hold whenever $z, \lambda \in \mathbb{R}^n$. Here $0 < \nu \le L$ and $s \ge 0$ are fixed parameters. Moreover, the vector field $a\colon \mathbb{R}^n \to \mathbb{R}^n$ is supposed to be C^1. Further extensions are possible for equations with coefficients of the type $-\mathrm{div}\, a(x, Du) = \mu$, where the dependence on x is supposed to be Dini-continuous.

The primary effect of Theorem 1.1 is that in some sense it makes the gradient integrability theory of quasilinear equations completely analogous to that of the basic Poisson equation $-\triangle u = \mu$. For example, we have

Corollary 1.1. *Let $u \in W^{1,p}(\mathbb{R}^n)$ be a weak solution to the equation (1.1) in $\mathbb{R}^n$, with μ being a Borel measure with locally finite mass and $p \ge 2$. Then there exists a constant c, depending only on n, p, such that the following estimate holds for every Lebesgue point $x \in \mathbb{R}^n$ of Du:*

$$|Du(x)|^{p-1} \le c \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}} \,.$$

As a matter of fact, by using the basic results about Riesz potentials, all the classical integrability results available for the solutions to (1.1) are now recovered (for instance, those in [1, 2, 3, 4, 6, 11, 12]). Moreover, delicate and open borderline cases can be fixed. For instance, Theorem 1.1 also provides us with a sharp Lipschitz regularity criterium for solutions to non-homogeneous equations:

Corollary 1.2. *Let $u \in W^{1,p}(\Omega)$ be as in Theorem 1.1. If $\mathbf{I}_1^{|\mu|}(\cdot, R) \in L^\infty(\Omega)$ for some $R > 0$, then $Du \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$.*

The other thing making Theorem 1.1 somehow unexpected is that, starting from the seminal paper of Kilpeläinen & Malý [7, 8], with a different approach offered by Trudinger & Wang [14], it is a standard fact that solutions to non-homogeneous quasilinear equations with measure data as (1.1) can be pointwise estimated in a natural way by mean of Wolff potentials

$$\mathbf{W}^{\mu}_{\beta,p}(x,R) := \int_0^R \left(\frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta p}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho} \qquad \beta \in (0, n/p] \,. \tag{1.3}$$

In particular, the main result of [7, 8] (see also [14, 9, 4]) claims that the following pointwise estimate holds a.e.:

$$|u(x)|^{p-1} \le c \left[\mathbf{W}^{\mu}_{1,p}(x,R) \right]^{p-1} + c \left(\fint_{B(x,R)} |u| \, dy \right)^{p-1} \,. \tag{1.4}$$

The previous result is *sharp* in the sense that $\mathbf{W}^{\mu}_{1,p}$ also bounds u from below when the measure μ and the solution u are nonnegative:

$$\frac{1}{c} \mathbf{W}^{\mu}_{1,p}(x,R) \le u(x) \le c \mathbf{W}^{\mu}_{1,p}(x,R) + c \fint_{B(x,R)} |u| \, dy \,.$$

We refer to [13] for a global version of the previous estimate. Potential estimates as the one in (1.4) play a fundamental role in the analysis of solutions to degenerate equations.

Estimate (1.4) has been eventually upgraded to the gradient level in [4, 12], where it has been proved that

$$|Du(x)|^{p-1} \le c \left[\mathbf{W}^{\mu}_{1/p,p}(x,R) \right]^{p-1} + c \left(\fint_{B(x,R)} |Du| \, dy \right)^{p-1} \,. \tag{1.5}$$

Estimate (1.2) obviously improves (1.5) as

$$\mathbf{I}_1^{|\mu|}(x,R) \quad = \quad \int_0^R \frac{|\mu|(B(x_0,\varrho))}{\varrho^{n-1}} \frac{d\varrho}{\varrho}$$

$$\lesssim \left[\int_0^{2R}\left(\frac{|\mu|(B(x_0,\varrho))}{\varrho^{n-1}}\right)^{1/(p-1)}\frac{d\varrho}{\varrho}\right]^{p-1} = \left[\mathbf{W}^{\mu}_{1/p,p}(x,2R)\right]^{p-1}. \tag{1.6}$$

Therefore Theorem 1.1 tells that *a change in the type of potential used, i.e. from nonlinear to linear* occurs when passing from estimates for u to those for Du. Our interpretation on this fact is that, while an equation as (1.1) looks genuinely nonlinear in terms of the solution u, it rather looks *as a linear one* when considering the nonlinear quantity $|Du|^{p-2}Du$.

Remark 1.2. We also observe that the case $p < 2$ of estimate (1.2) has been proved in [5], but in this case it does not improve (1.5) in that (1.6) does not hold in general for $p < 2$ and the real difficult case is passing from nonlinear potentials to linear ones when the latter provide better bounds.

Theorem 1.1 opens the way to sharp continuity criteria for the gradient:

Theorem 1.2. *Let $u \in W^{1,p}(\Omega)$ be as in Theorem 1.1. Assume that $\mathbf{I}_1^{|\mu|}(x,R) \to 0$ as $R \to 0$, locally uniformly in Ω w.r.t. x; then Du is continuous in Ω.*

The previous result admits the following relevant corollary, providing a borderline condition for gradient continuity in terms of Lorentz spaces $L(n,1)$:

Corollary 1.3. *Let $u \in W^{1,p}(\Omega)$ be as in Theorem 1.1. If $\mu \in L(n,1)$ locally in Ω, then Du is continuous in Ω.*

We recall that $\mu \in L(n,1)$ locally in Ω means that

$$\int_0^{\infty} t^n|\{x \in \Omega' \,:\, |\mu(x)| > t\}|\,\frac{dt}{t} < \infty \quad \text{for every open subset } \Omega' \Subset \Omega\,.$$

Moreover $L^{\gamma} \subset L(n,1) \subset L^n$ for every $\gamma > n$, inclusions being strict. We refer to [11] for relevant definitions and results on Lorentz spaces in this context.

2. A sketch of the proof of Theorem 1.1

The proof of Theorem 1.1 (and that of Theorem 1.2) is very delicate, and virtually involves all the known aspects of the regularity of solutions to p-Laplacean type equations. The main part is based on a careful local comparison argument with p-harmonic functions, i.e. solutions to homogeneous equations of the type $\operatorname{div}(|Dv|^{p-2}Dv) = 0$. More precisely, with the ball $B(x,R) \subset \Omega$ fixed in Theorem 1.1, we determine a small shrinking number $\delta_1 \in (0, 1/10^8)$, depending only on n, p, but not on μ and of the particular solution u considered, and then define the sequence of shrinking balls $B_i := B(x, \delta_1^i R)$, for every integer $j \geq 0$. Accordingly, we consider p-harmonic liftings (of u) $v_i \in u + W_0^{1,p}(B_i)$, that is $\operatorname{div}(|Dv_i|^{p-2}Dv_i) = 0$ in B_i and such that $v_i = u$ on ∂B_i in Sobolev sense. At this stage it is important to develop a suitable form of the $C^{1,\alpha}$ local regularity estimates for v_i. Next, we define a number

$$\lambda^{p-1} := c_1\mathbf{I}_1^{|\mu|}(x,R) + c_1\left(\frac{1}{|B(x,R)|}\int_{B(x,R)}|Du|\,dy\right)^{p-1},$$

where $c_1 \geq \delta_1^{-5n}$ is a constant depending only on n, p; moreover, we define

$$C_i \approx \frac{1}{|B_i|}\int_{B_i}|Du|\,dy$$

for every integer $i \geq 2$. To proceed, we argue on an alternative. The first case is when there exists an increasing subsequence $\{j_i\}$ such that $C_{j_i} \leq \lambda/100$, for every $i \in \mathbb{N}$, and then, as x is a Lebesgue point of Du, we have $|Du(x)| \leq \lim_{i\to\infty} C_{j_i} \leq \lambda/100$ and the proof is finished. Otherwise we determine an "exit time" index

$i_e \geq 3$ such that $C_{i_e} \leq \lambda/100$ and $C_i \geq \lambda/100$ holds for every $i > i_e$. Thanks to this additional information we are now able to get the local comparison estimate

$$\fint_{B_{i+1}} |Du - Dv_i| \, dy \leq c(B_i, \lambda, \mu(B_i)). \tag{2.1}$$

In turn, (2.1) and the good decay/regularity estimates available for the comparison functions v_i allow to derive the following *decay estimate with a remainder*:

$$E(Du, B_{i+1}) \leq \frac{1}{4} E(Du, B_i) + \text{“controllable error term related to } \mathbf{I}_1^{|\mu|}\text{”} \tag{2.2}$$

whenever $i \geq i_e$. Here

$$E(Du, B_i) := \fint_{B_i} |Du - (Du)_i| \, dy \qquad \text{and} \qquad (Du)_i := \frac{1}{|B_i|} \int_{B_i} Du \, dy \, .$$

Next, by summing up (2.2), we prove by induction that actually

$$|(Du)_i| \leq \lambda$$

holds whenever $i \geq i_e$. At this stage the proof of (1.2) follows as, x being a Lebesgue point of Du, we have

$$|Du(x)| = \lim_{i \to \infty} |(Du)_i| \leq \lambda \, .$$

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