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UNIVERSAL POTENTIAL ESTIMATES

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Abstract. We prove a class of endpoint pointwise estimates for solutions to quasilinear, possibly degenerate elliptic equations in terms of linear and nonlinear potentials of Wolff type of the source term. Such estimates allow to bound size and oscillations of solutions and their gradients pointwise, and entail in a unified approach virtually all kinds of regularity properties in terms of the given datum and regularity of coefficients. In particular, local estimates in Hölder, Lipschitz, Morrey and fractional spaces, as well as Calderón-Zygmund estimates, follow as a corollary in a unified way. Moreover, estimates for fractional derivatives of solutions by mean of suitable linear and nonlinear potentials are also implied. The classical Wolff potential estimate by Kilpeläinen & Malý and Trudinger & Wang as well as recent Wolff gradient bounds for solutions to quasilinear equations embed in such a class as endpoint cases.

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1. Introduction and main results

The aim of this paper is to prove pointwise estimates for solutions to possibly degenerate, quasilinear elliptic equations of the type

$$-\text{div } a(x, Du) = \mu \,, \tag{1.1}$$

considered in a bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, where μ is a Borel measure defined on Ω with finite total mass. The estimates presented here allow to give pointwise *size and oscillation bounds* for solutions and their derivatives in terms of linear and nonlinear potentials of Wolff type of the datum μ. In turn they imply a *completely unified approach to regularity theory* since they essentially capture all the regularity properties of solutions with respect to the regularity properties of the given datum μ and of the coefficients $x \mapsto a(x, \cdot)$. Indeed, as a corollary we will obtain nonlinear Calderón-Zygmund estimates in Sobolev spaces of integer and fractional order as well as (nonlinear) Schauder estimates. In turn, these reduce to the known results when considering linear equations.

Our estimates also recover and extend both the classical pointwise nonlinear estimate obtained by Kilpeläinen & Malý [16] and Trudinger & Wang [36, 37], and the more recent ones for the gradient obtained in [8, 30], and entail endpoint pointwise bounds for fractional derivatives of solutions. Moreover, new finer and optimal regularity estimates in intermediate and non-interpolation spaces are demonstrated. Due to such a *unifying character*, we took the liberty to call the ones found here *universal estimates* to emphasize their principal role.

In the rest of the paper, when considering a measure μ as in (1.1), up to letting $\mu\lfloor_{\mathbb{R}^n\setminus\Omega}= 0$, we shall assume that μ is defined on the whole $\mathbb{R}^n$, having finite total mass. The vector field $a\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be at least measurable in the coefficients x, C^1-regular in the gradient variable $z \in \mathbb{R}^n$ (far from the origin when $p < 2$) and satisfying the following *growth, ellipticity and continuity assumptions*:

$$\begin{cases} |a(x,z)| + |\partial a(x,z)|(|z|^2 + s^2)^{1/2} \leq L(|z|^2 + s^2)^{(p-1)/2} \\ \nu(|z|^2 + s^2)^{(p-2)/2}|\lambda|^2 \leq \langle \partial a(x,z)\lambda, \lambda\rangle \end{cases} \tag{1.2}$$

whenever $x \in \Omega$ and $z, \lambda \in \mathbb{R}^n$; the symbol ∂a in this paper will always denote the gradient of $a(\cdot)$ with respect to the gradient variable z. We shall moreover assume that $\partial a(\cdot)$ is continuous with respect to the gradient variable z when $p \geq 2$ and continuous outside the origin when $p \leq 2$; finally, the partial map $x \mapsto \partial a(x, \cdot)$ is assumed to be measurable. Here and in the rest of the paper we are assuming that ν, L, s are fixed parameters such that $0 < \nu \leq L$ and $s \geq 0$. The prototype of (1.1) is - choosing $s = 0$ - clearly given by the p-Laplacean equation with coefficients

$$-\text{div } (\gamma(x)|Du|^{p-2}Du) = \mu \,, \qquad \nu \leq \gamma(x) \leq L \,, \tag{1.3}$$

while on the other hand the full significance of the results presented in this paper is in the nonlinear situation already when $p = 2$.

We recall that by a weak solution to the equation (1.1) we mean a function $u \in W^{1,p}_{\text{loc}}(\Omega)$ such that the distributional relation

$$\int_\Omega \langle a(x, Du), D\varphi\rangle \, dx = \int_\Omega \varphi \, d\mu$$

holds whenever $\varphi \in C^\infty_0(\Omega)$ has a compact support in Ω. In fact, our results continue to hold for a class of a priori less regular solutions called very weak solutions, via approximation, see discussion in Section 2.2. For the same reason, without loss of generality, we shall assume that solutions will be of class C^1 or C^0, according to the type of estimates treated. In other words, we shall confine ourselves to state the results under the form of *a priori estimates for more regular solutions.*

For the basic notation adopted in this paper we refer to Section 2.1 below; in particular, by B_R we shall indicate a general ball in $\mathbb{R}^n$ with the radius $R > 0$.

1.1. **The case $p \geq 2$, the role of coefficients and general strategy.** Here we present the results for the case $p \geq 2$. By now classical theorems from nonlinear potential theory allow for pointwise estimates of solutions to (1.1) in terms of the (truncated) Wolff potential $\mathbf{W}^{\mu}_{\beta,p}(x,R)$ defined by

$$\mathbf{W}^{\mu}_{\beta,p}(x,R) := \int_0^R \left(\frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta p}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho}\,, \qquad \beta > 0\,. \tag{1.4}$$

These reduce to the standard (truncated) Riesz potentials when $p = 2$

$$\mathbf{W}^{\mu}_{\beta/2,2}(x,R) = \mathbf{I}^{\mu}_{\beta}(x,R) = \int_0^R \frac{\mu(B(x,\varrho))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho}\,, \qquad \beta > 0\,, \tag{1.5}$$

with the first equality being true for non-negative measures.

A fundamental fact due to Kilpeläinen & Malý [16] - later deduced and extended via different approaches by Trudinger & Wang in [36, 37] - is the estimate

$$|u(x)| \leq c\mathbf{W}^{\mu}_{1,p}(x,R) + c \fint_{B(x,R)} (|u| + Rs)\, d\xi\,, \tag{1.6}$$

valid whenever $B(x,R) \subset \Omega$, with x being a Lebesgue point of u. This result has been upgraded to the gradient first in [30] for the case $p = 2$ and then in [8] for the case $p > 2$, where the estimate

$$|Du(x)| \leq c\mathbf{W}^{\mu}_{1/p,p}(x,R) + c \fint_{B(x,R)} (|Du| + s)\, d\xi \tag{1.7}$$

has been proved. See also [22] and Remark 1.2 below for another gradient estimate avoiding the use of nonlinear potentials.

Estimates (1.6) and (1.7) are the nonlinear counterparts of the well-known estimates valid for solutions to the Poisson equation

$$-\triangle u = \mu \tag{1.8}$$

in $\mathbb{R}^n$ - here we take $n \geq 3$, μ being a locally integrable function and u being the only solutions to (1.8) decaying to zero at infinity. Such estimates, an immediate consequence of the representation formula

$$u(x) = \frac{1}{n(n-2)|B_1|} \int_{\mathbb{R}^n} \frac{d\mu(\xi)}{|x-\xi|^{n-2}}\,, \tag{1.9}$$

take on the whole space the form

$$|u(x)| \leq c\mathbf{I}^{|\mu|}_{2}(x,\infty)\,, \qquad \text{and} \qquad |Du(x)| \leq c\mathbf{I}^{|\mu|}_{1}(x,\infty)\,. \tag{1.10}$$

It is important to note here that while (1.6) holds true when the dependence on $x \mapsto a(x,\cdot)$ is just measurable, estimate (1.7) necessitates more regularity from the mapping $x \mapsto a(x,\cdot)$. Indeed, (1.7) implies the gradient boundedness for regular enough measures, for which plain continuity of coefficients is known to be insufficient, while for instance Dini continuity suffices. As we shall see in a few moments, intermediate - and essentially sharp - moduli of continuity of $x \mapsto a(x,\cdot)$ will appear in the next statements according to the estimates considered. Let us notice that Wolff potential estimates are of basic importance to derive further existence theorem for quasilinear equations, as shown for instance by Phuc & Verbitsky [33, 34].

The main aim of this paper is to show that the *estimates* (1.6) *and* (1.7) *are particular instances of more general endpoint estimates.* While (1.6) and (1.7) are size estimates, the new ones derived here will be *oscillation estimates*, allowing to express properties like continuity and to get size bounds for fractional derivatives of solutions to (1.1), ultimately *catching up regularity properties at every function space scale*. There are actually several ways to express the concept of fractional differentiability. It might appear at the beginning vague to extend pointwise estimates (1.6)-(1.7) to fractional derivatives, as these are obviously non-local objects. We shall here use a *notion of fractional differentiability* introduced by DeVore & Sharpley [5] that allows to describe fractional derivatives reducing the non-locality of the definition to a minimal status, i.e. using two points only.

Definition 1. Let $\alpha \in (0,1]$, $q \geq 1$, and let $\Omega \subset \mathbb{R}^n$ be a bounded open subset. A measurable function v, finite a.e. in Ω, belongs to the Calderón space $C^\alpha_q(\Omega)$ if and only if there exists a nonnegative function $m \in L^q(\Omega)$ such that

$$|v(x) - v(y)| \leq [m(x) + m(y)]|x-y|^\alpha \tag{1.11}$$

holds for almost every couple $(x,y) \in \Omega \times \Omega$.

Such spaces are closely related to the usual fractional Sobolev spaces $W^{\alpha,q}$ (see [5]), and actually they coincide with Triebel-Lizorkin spaces for $q > 1$ in the sense that $C^\alpha_q \equiv F^\alpha_{q,\infty}$ when $\alpha \in (0,1)$ and $C^1_q \equiv F^1_{q,2}$. Of course there could be more than one function $m(\cdot)$ working in (1.11). For this reason in their original paper DeVore & Sharpley fix $m(\cdot)$ to be the sharp fractional maximal operator of order α of v, i.e $m = M^\#_\alpha(v)$, see Definition 3 below. Indeed, notice that it follows from the definitions that the validity of (1.11) for some $m \in L^q$ is equivalent to have $M^\#_\alpha(v) \in L^q$ whenever $q > 1$. Here we shall not be interested in the functional theoretic properties of the spaces $C^\alpha_q(\Omega)$, for which we refer to [5], but only in the fact that (1.11) *allows to identify* $m(\cdot)$ *as "a fractional derivative of order* α*"* for v. For this reason, in the following by pointwise estimates on fractional derivatives of a function $v(\cdot)$ we shall mean estimates on a function as $m(\cdot)$ in (1.11). With such a notation, and referring to the discussion at the beginning of Section 1.3 below, we deduce that for the Poisson equation (1.8), and with abuse of notation, it holds that "$|\partial^\alpha u(x)| \leq \mathbf{I}^{|\mu|}_{2-\alpha}(x,\infty)$" with $\alpha \in [0,1]$. In a few lines we shall see that, notwithstanding the absence of representation formulae as (1.9), *this kind of relation holds in the nonlinear case too*, in a way that can be made perfectly precise.

The first result we present upgrades estimate (1.6) to low order fractional derivatives, and actually holds in the case $p < 2$ as well. In fact, our aim here is also to demonstrate a sharp connection between classical De Giorgi's theory and nonlinear potential estimates. Indeed, when considering solutions to homogeneous equations as $\operatorname{div} a(x, Dw) = 0$, with measurable dependence on x, De Giorgi's theory provides the existence of a *universal Hölder continuity exponent* $\alpha_m \in (0,1)$, depending only on n, p, ν, L, such that

$$w \in C^{0,\alpha_m}_{\mathrm{loc}}(\Omega)\,, \qquad |w(x) - w(y)| \leq c \fint_{B_R} (|w| + Rs)\, dx \cdot \left(\frac{|x-y|}{R}\right)^{\alpha_m}, \tag{1.12}$$

where the last inequality holds whenever $x, y \in B_{R/2}$ and $B_R \subset \Omega$. The exponent α_m can be thought as the *maximal Hölder regularity exponent* associated to the vector field $a(\cdot)$, and is actually universal in that it is even independent of $a(\cdot)$ and depends only on n, p, ν, L. It then holds

Theorem 1.1 (De Giorgi's theory via potentials)**.** *Let $u \in C^0(\Omega) \cap W^{1,p}(\Omega)$ be a weak solution to the equation with measurable coefficients* (1.1)*, and let* (1.2) *hold with $p > 2 - 1/n$. Let $B_R \subset \Omega$ be such that $x, y \in B_{R/8}$, then*

$$|u(x) - u(y)| \quad \leq \quad c\left[\mathbf{W}^\mu_{1-\alpha(p-1)/p,p}(x,R) + \mathbf{W}^\mu_{1-\alpha(p-1)/p,p}(y,R)\right]|x-y|^\alpha$$

$$+c \fint_{B_R} (|u| + Rs)\, d\xi \cdot \left(\frac{|x-y|}{R}\right)^{\alpha} \tag{1.13}$$

holds uniformly in $\alpha \in [0, \tilde{\alpha}]$, for every $\tilde{\alpha} < \alpha_m$, where the constant c depends only n, p, ν, L and $\tilde{\alpha}$.

In general, counterexamples show that $\alpha_m \to 0$ when $L/\nu \to \infty$, and this prevents estimate (1.13) to hold in general for the full range $\alpha \in [0,1)$ when in presence of measurable coefficients. Let us remark that the restriction to the case $2-1/n < p$ is motivated by the fact that this is the range in solutions to measure data problems belong to the Sobolev space $W^{1,1}$, and we can talk about the usual gradient. In this respect the lower bound $p > 2-1/n$ is optimal as showed by the (so called nonlinear fundamental) solution

$$G_p(x) := c(n,p) \begin{cases} \left(|x|^{\frac{p-n}{p-1}} - 1\right) & \text{if} \quad 1 < p \neq n \\ \log\, |x| & \text{if} \quad\quad p = n \end{cases}$$

to the equation $-\triangle_p u = \delta$, where δ is the Dirac measure charging the origin.

To proceed with the results, in order to prove estimates for higher order fractional derivatives we shall need more regularity on coefficients. Indeed, certain types of potential estimates will be allowed only in presence of suitably strong regularity of the partial map $x \mapsto a(x, \cdot)$, otherwise counterexamples would not allow for the claimed statements. In this respect, we record in the last years a large interest in weaker forms of continuity of coefficients allowing for Calderón-Zygmund type estimates and here we incorporate and extend also such kind of results. As already in [3], we define the averaged operator

$$(a)_{x,r}(z) := \fint_{B(x,r)} a(\xi, z)\, d\xi\,, \qquad \text{for} \quad z \in \mathbb{R}^n\,, \tag{1.14}$$

whenever $B(x,r) \subseteq \Omega$ and then the averaged (and renormalized) modulus of continuity of $x \mapsto a(x, \cdot)$ as follows:

$$\omega(r) := \left[\sup_{z \in \mathbb{R}^n, B(x,r) \subseteq \Omega} \fint_{B(x,r)} \left(\frac{|a(\xi,z) - (a)_{x,r}(z)|}{(|z| + s)^{p-1}} \right)^2 d\xi \right]^{1/2}. \tag{1.15}$$

Accordingly, we shall consider various decay properties of $\omega(\cdot)$; first, a definition.

Definition 2. A function $h : [0, \infty) \to [0, \infty)$ will be called VMO-regular if

$$\lim_{r \to 0} h(r) = 0\,, \tag{1.16}$$

while it will be called Dini-VMO regular if

$$\int_0^r h(\varrho)\, \frac{d\varrho}{\varrho} < \infty \qquad\qquad \forall\, r > 0\,. \tag{1.17}$$

Finally, $h(\cdot)$ will be called Dini-Hölder regular of order $\alpha \in [0,1]$ if

$$\int_0^r \frac{h(\varrho)}{\varrho^{\alpha}}\, \frac{d\varrho}{\varrho} < \infty \qquad\qquad \forall\, r > 0\,. \tag{1.18}$$

The next result that again holds also when $p < 2$, is

Theorem 1.2 (Fractional nonlinear potential bound)**.** *Let $u \in C^1(\Omega)$ be a weak solution to* (1.1)*, under the assumptions* (1.2) *with $p > 2 - 1/n$. For every $\tilde{\alpha} < 1$ there exists a positive number $\delta \equiv \delta(n, p, \nu, L, \tilde{\alpha})$ such that if*

$$\lim_{r \to 0} \omega(r) \leq \delta, \tag{1.19}$$

then the pointwise estimate (1.13) *holds uniformly in* $\alpha \in [0, \tilde{\alpha}]$*, for a constant* $c \equiv c(n, p, \nu, L, \omega(\cdot), \tilde{\alpha}, \operatorname{diam}(\Omega))$*, as soon as* $x, y \in B_{R/8}$*. In particular, if* $\omega(\cdot)$ *is* VMO *in the sense of Definition 2, then* (1.13) *holds whenever* $\alpha < 1$.

Theorem 1.2 in particular covers the case coefficients $x \mapsto a(x, \cdot)$ are continuous, while in the model case (1.3) we are actually assuming that $\gamma(\cdot)$ is VMO regular - or with small BMO-norm when considering (1.19) - which is known to be an essentially optimal condition in order to get such type of results. Estimate (1.13) fails for the case $\alpha = 1$, already when considering continuous coefficients. Instead, a form of Dini continuity must be assumed as follows:

Theorem 1.3 (Full interpolation estimate)**.** *Let* $u \in C^1(\Omega)$ *be a weak solution to* (1.1) *under the assumptions* (1.2) *with* $p \geq 2$*, and assume also that* $[\omega(\cdot)]^{2/p}$ *is* Dini-VMO *regular, that is*

$$\int_0^r [\omega(\varrho)]^{2/p} \frac{d\varrho}{\varrho} < \infty \qquad \forall\ r < \infty\,. \tag{1.20}$$

Then (1.13) *holds uniformly* $\alpha \in [0, 1]$*, whenever* $B_R \subset \Omega$ *is a ball such that* $x, y \in B_{R/8}$*, where* $c \equiv c(n, p, \nu, L, \omega(\cdot), \operatorname{diam}(\Omega))$.

Theorem 1.3 also improves the classical results concerning Lipschitz continuity in that it relaxes the standard Dini continuity, sufficient to prove pointwise gradient bounds already when $\mu = 0$, to an integrated form of it. Let us remark that assuming (1.20) still implies that $x \to a(x, \cdot)$ is continuous, but not necessarily Dini continuous.

Remark 1.1 (Endpoint/Interpolation nature of the estimates)**.** A main feature of this work is the *endpoint nature* of estimates as (1.13) - as well as of other similar estimates as (1.23), (1.26) and (1.28) below - in that they hold uniformly up to including the borderline cases (1.6)-(1.7) (modulo constants) when this is allowed by the regularity of coefficients. It requires effort to make for instance estimate (1.13) *uniform in* $\alpha \in [0, 1]$, that is to prove that it is a real *interpolation endpoint* estimate between (1.6) and (1.7). A primary goal of the paper is indeed in its unificatory role, also from the point of view of the proofs given.

Remark 1.2. When dealing with pointwise gradient estimates it has been shown in [21, 22] that the Wolff potential estimate (1.7) can be still improved. More precisely *Riesz potentials come back when dealing with gradient estimates.* Since we are here interested in finding a universal estimate which covers both the case of pointwise estimates for solutions and the one of gradient estimates, that is (1.13) with the range $\alpha \in [0, 1]$, we decide, when $p > 2$, to deal only with Wolff potentials avoiding Riesz potentials in one end-point ($C^{0,1}$-estimates). Anyway, Wolff potentials definitely disappear in the subquadratic case $1 - 2/n < p < 2$ as we shall see in the following; in fact, in a dual way, we shall there deal only with Riesz potentials, avoiding Wolff potentials in one end-point (L^∞-estimates). More cases where Wolff potentials are not necessary and weaker (maximal) operators can be considered, are the non-endpoint estimates proposed in Section 1.4 below.

Finally we move towards the maximal regularity of the operator in (1.1). When considering the homogeneous equation

$$\operatorname{div} a(Dv) = 0$$

a version of De Giorgi's theory is available - see [6, 26, 27] for a very neat presentation - ultimately leading to the existence of a *universal maximal regularity exponent* $\alpha_M \in (0, 1)$, depending only on n, p, ν and L such that whenever $x, y \in B_{R/4}$,

$$Dv \in C^{0,\alpha_M}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)\,, \quad |Dv(x) - Dv(y)| \leq c \fint_{B_R} (|Dv| + s)\, d\xi \cdot \left(\frac{|x-y|}{R}\right)^{\alpha_M} \tag{1.21}$$

holds *for any local solution* v. Similarly to (1.12), α_M can be defined as the largest exponent for which (2.5) below - a rigid, self-scaling version of (1.21) that in fact implies (1.21) - holds for every local solution v. We have now:

Theorem 1.4 (Gradient fractional bound)**.** *Let* $u \in C^1(\Omega)$ *be a weak solution to* (1.1)*, under the assumptions* (1.2) *with* $p \geq 2$*, and assume that* $[\omega(\cdot)]^{2/p}$ *is* Dini-Hölder *of order* $\tilde{\alpha} < \alpha_M$*, i.e.*

$$S := \sup_r \int_0^r \frac{[\omega(\varrho)]^{2/p}}{\varrho^{\tilde{\alpha}}} \frac{d\varrho}{\varrho} < \infty \,. \tag{1.22}$$

Then the pointwise estimate

$$\begin{aligned} &|Du(x) - Du(y)| \\ &\quad \leq c \left[\mathbf{W}^{\mu}_{1-(1+\alpha)(p-1)/p,p}(x,R) + \mathbf{W}^{\mu}_{1-(1+\alpha)(p-1)/p,p}(y,R)\right] |x-y|^{\alpha} \\ &\quad\quad + c \fint_{B_R} (|Du| + s)\, d\xi \cdot \left(\frac{|x-y|}{R}\right)^{\alpha} \end{aligned} \tag{1.23}$$

holds uniformly in $\alpha \in [0, \tilde{\alpha}]$*, whenever* $x, y \in \Omega$ *and* $B_R \subset \Omega$ *is a ball such that* $x, y \in B_{R/4}$*, for a constant* c *depending only on* $n, p, \nu, L, \omega(\cdot), \tilde{\alpha}, S$ *and* $\operatorname{diam}(\Omega)$*.*

1.2. **The case** $p < 2$ **and linear potentials.** We shall here restrict to the case $2 - 1/n < p \leq 2$ for the reasons already explained after Theorem 1.1. In [9] the following estimate has been proved:

$$|Du(x)| \leq c \left[\mathbf{I}^{|\mu|}_1(x,R)\right]^{1/(p-1)} + c \fint_{B(x,R)} (|Du| + s)\, d\xi \,, \tag{1.24}$$

which is moreover conjectured to be sharp, and connects with the analogous one in [22] valid for the case $p \geq 2$. Therefore, finding an estimate "interpolating" (1.6) and (1.24) appears to be problematic: while the first one features a nonlinear Wolff potential, the second one includes linear potentials. We therefore opt for an alternative: when looking for an estimate of the type (1.13) for $\alpha \leq \tilde{\alpha} < 1$, i.e. we are not approaching a gradient estimate, we have that Theorems 1.1-1.2 still hold as seen in the previous section. Instead, when looking for an estimate that covers the case (1.24) with a stable constant c remaining bounded as $\alpha \to 1$, we prove an estimate which features only linear potentials. In this case we replace Wolff potentials as $\mathbf{W}^{\mu}_{1,p}$ by slightly larger ones as $[\mathbf{I}^{|\mu|}_p]^{1/(p-1)}$. Nevertheless, the new potentials share the scaling and homogeneity properties of Wolff potentials.

Theorem 1.5 (Linear potentials endpoint bound)**.** *Let* $u \in C^1(\Omega)$ *be a weak solution to* (1.1) *under the assumptions* (1.2) *with* $2 - 1/n < p \leq 2$ *and assume also that* $[\omega(\cdot)]^{\sigma}$ *is* Dini-VMO*-regular for some* $\sigma < 1$*, i.e.*

$$\int_0^r [\omega(\varrho)]^{\sigma} \frac{d\varrho}{\varrho} < \infty \qquad \forall\, r < \infty \,. \tag{1.25}$$

Then there exists a constant c *depending only on* $n, p, \nu, L, \omega(\cdot), \sigma, \operatorname{diam}(\Omega)$*, such that*

$$\begin{aligned} |u(x) - u(y)| \quad &\leq \quad c \left[\mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(x,R) + \mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(y,R)\right]^{1/(p-1)} |x-y|^{\alpha} \\ &\quad\quad + c \fint_{B_R} (|u| + Rs)\, d\xi \cdot \left(\frac{|x-y|}{R}\right)^{\alpha} \end{aligned} \tag{1.26}$$

holds uniformly in $\alpha \in [0,1]$*, whenever* $B_R \subset \Omega$ *is a ball such that* $x, y \in B_{R/8}$*.*

Finally, when switching to the gradient estimates we come to a situation which is completely similar to that of the Poisson equation $-\triangle u = \mu$, as equations as for instance (1.3) are *linear* in the *nonlinear field* $|Du|^{p-2}Du$.

Theorem 1.6 (Linear potentials gradient bound)**.** *Let $u \in C^1(\Omega)$ be a weak solution to* (1.1) *under the assumptions* (1.2) *with $2-1/n < p \leq 2$; assume that $[\omega(\cdot)]^\sigma$ is* Dini-Hölder *of order $\tilde{\alpha}$ for some $\sigma < 1$, i.e.*

$$S := \sup_r \int_0^r \frac{[\omega(\varrho)]^\sigma}{\varrho^{\tilde{\alpha}}} \frac{d\varrho}{\varrho} < \infty\,, \qquad \tilde{\alpha} \in (0, \alpha_M)\,. \tag{1.27}$$

Then the pointwise estimate

$$\begin{aligned} |Du(x) - Du(y)| &\leq c \left[\mathbf{I}^{|\mu|}_{1-\alpha}(x,R) + \mathbf{I}^{|\mu|}_{1-\alpha}(y,R)\right]^{1/(p-1)} |x-y|^\alpha \\ &\quad + c \fint_{B_R} (|Du| + s)\, d\xi \cdot \left(\frac{|x-y|}{R}\right)^\alpha \end{aligned} \tag{1.28}$$

holds uniformly in $\alpha \in [0, \tilde{\alpha}]$, whenever $B_R \subset \Omega$ is a ball such that $x, y \in B_{R/4}$, for a constant c depending on $n, p, \nu, L, \omega(\cdot), \tilde{\alpha}, \sigma, S, \operatorname{diam}(\Omega)$.

Situations in which the value $\sigma = 1$ is allowed in Theorems 1.5.-1.6 are presented in Section 8 below. This happens for instance in (1.3) when $\gamma(\cdot)$ is Dini continuous in the classical sense.

1.3. **Connections with the linear theory.** For $p = 2$ estimate (1.13) is

$$|u(x) - u(y)| \leq c \left[\mathbf{I}^{|\mu|}_{2-\alpha}(x,R) + \mathbf{I}^{|\mu|}_{2-\alpha}(y,R) + R^{-\alpha} \fint_{B_R} |u|\, d\xi\right] |x-y|^\alpha\,, \tag{1.29}$$

with actually $c \equiv c(n,p,\nu,L)$. Consider now the Poisson equation (1.8) here we again take $n \geq 3$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying $|u(x)| \leq c|x|^{2-n}$ asymptotically as $|x| \to \infty$ (this for instance happens when μ is compactly supported). The representation formula (1.9) gives

$$|u(x) - u(y)| \leq c\,[I_{2-\alpha}(|\mu|)(x) + I_{2-\alpha}(|\mu|)(y)]\,|x-y|^\alpha \tag{1.30}$$

whenever $x, y \in \mathbb{R}^n$ and $\alpha \in [0,1]$; here

$$I_\beta(|\mu|)(x) := \int_{\mathbb{R}^n} \frac{d|\mu|(\xi)}{|x-\xi|^{\beta-n}}$$

denotes the standard Riesz potential with $\beta \in (0,n]$ (we omit the usual renormalization constant here). We have of course used the elementary inequality

$$\left||x-\xi|^{2-n} - |y-\xi|^{2-n}\right| \leq c(n) \left||x-\xi|^{2-n-\alpha} + |y-\xi|^{2-n-\alpha}\right| |x-y|^\alpha\,.$$

Now, when $\Omega \equiv \mathbb{R}^n$, letting $R \to \infty$ in (1.29) and using the decay of u, inequality (1.30) follows from (1.29). A similar argument works for the gradient when using (1.23) in the case $p = 2$, i.e.

$$|Du(x) - Du(y)| \leq c \left[\mathbf{I}^{|\mu|}_{1-\alpha}(x,R) + \mathbf{I}^{|\mu|}_{1-\alpha}(y,R) + R^{-\alpha} \fint_{B_R} |Du|\, d\xi\right] |x-y|^\alpha \tag{1.31}$$

whenever $x, y \in B_{R/8}$ and $\alpha < \alpha_M$. Assuming again appropriate decay for $|Du|$ and letting $R \to \infty$,

$$|Du(x) - Du(y)| \leq c \left[\mathbf{I}^{|\mu|}_{1-\alpha}(x) + \mathbf{I}^{|\mu|}_{1-\alpha}(y)\right] |x-y|^\alpha\,.$$

follows for $x, y \in \mathbb{R}^n$. In case of (1.8), the same is attainable via estimating the differentiated Riesz kernel as above. It is worth remarking here that, due to the nature of the proofs, in the basic linear case (1.8), we have that (1.31) holds for every $\alpha < 1$, with a constant c depending on α and being uniformly bounded as long as α is bounded away from 1. To see this we remark that for the Laplacean operator in (1.21) we may take $\alpha_M = 1$. This is exactly the same estimate directly obtainable by the standard representation formula via fundamental solutions. Looking for a more general result in this direction we are led to a connection between our approach and the classical Cordes type perturbation theory, and we shall demonstrate an example here. Let us consider equations as

$$-\operatorname{div} a(Du) = \mu \tag{1.32}$$

and a "near-linearity" condition of the type

$$\sup_{z\in\mathbb{R}^n} |\partial a(z) - A| \leq \delta\,, \tag{1.33}$$

where $A \in \mathbb{R}^{n\times n}$ is a fixed, elliptic matrix in the sense that

$$\nu|\lambda|^2 \leq \langle A\lambda, \lambda\rangle \leq L|\lambda|^2 \tag{1.34}$$

holds whenever $\lambda \in \mathbb{R}^n$. We then have

Theorem 1.7 (Cordes type theory via potentials)**.** *Let $u \in C^1(\Omega)$ be a weak solution to the equation* (1.32) *under the assumptions* (1.2) *with $p = 2$. For every $\tilde{\alpha} < 1$ there exists a number $\delta \equiv \delta(n, p, \nu, L, \tilde{\alpha})$ such that if* (1.33) *holds for a certain matrix $A \in \mathbb{R}^{n\times n}$ as in* (1.34)*, then estimate* (1.31) *holds uniformly in $\alpha \in [0, \tilde{\alpha}]$, with $c \equiv c(n, \nu, L, \tilde{\alpha})$.*

It is at this point obvious to remark that in the case of Poisson equation (1.8) assumption (1.33) is satisfied with $\delta = 0$ and $A = I$ (the identity matrix).

1.4. **Maximal estimates.** Preliminary to the proof of the potential estimates there are additional results concerned with the pointwise estimate of certain maximal operators of solutions. We here make a clear connection to classical results in Harmonic Analysis allowing for pointwise estimates of maximal operator of fractional and singular integrals. Further connections are given to the recent developments in the nonlinear case [4, 3, 32] where L^q-estimates are obtained for maximal operators: here we present L^∞ estimates. See Section 2.3 below for the relevant definitions of maximal operators.

Theorem 1.8 (Superquadratic maximal estimates)**.** *Let $u \in C^1(\Omega)$ be a weak solution to* (1.1) *under the assumptions* (1.2) *with $p \geq 2$; let $B_R \subset \Omega$ be a ball centered at x. Then*

- *For every $\tilde{\alpha} < 1$ there exists a positive number $\delta \equiv \delta(n, p, \nu, L, \tilde{\alpha})$ such that if* (1.19) *is satisfied, then the pointwise estimate*

$$\begin{aligned} &M^{\#}_{\alpha,R}(u)(x) + M_{1-\alpha,R}(Du)(x) \\ &\quad\leq c\left[M_{p-\alpha(p-1),R}(\mu)(x)\right]^{1/(p-1)} + cR^{1-\alpha} \fint_{B_R} (|Du| + s)\, d\xi \end{aligned} \tag{1.35}$$

  holds uniformly in $\alpha \in [0, \tilde{\alpha}]$, for a constant $c \equiv c(n, p, \nu, L, \omega(\cdot), \tilde{\alpha}, \operatorname{diam}(\Omega))$
- *In addition, if* (1.20) *is in force, the estimate*

$$\begin{aligned} &M^{\#}_{\alpha,R}(u)(x) + M_{1-\alpha,R}(Du)(x) \\ &\quad\leq c\mathbf{W}^{\mu}_{1-\alpha(p-1)/p,p}(x, R) + cR^{1-\alpha} \fint_{B_R} (|Du| + s)\, d\xi \end{aligned} \tag{1.36}$$

  is satisfied uniformly in $\alpha \in [0, 1]$, with $c \equiv c(n, p, \nu, L, \omega(\cdot), \sigma, \operatorname{diam}(\Omega))$
- *Finally, assume that* (1.20) *is in force together with*

$$\sup_r \frac{[\omega(r)]^{2/p}}{r^{\tilde{\alpha}}} \leq S \tag{1.37}$$

  for some $\tilde{\alpha} \in [0, \alpha_M)$. Then

$$\begin{aligned} M^{\#}_{\alpha,R}(Du)(x) \quad &\leq \quad c\left[M_{1-\alpha(p-1),R}(\mu)(x)\right]^{1/(p-1)} \\ &\qquad + c\mathbf{W}^{\mu}_{1/p,p}(x, R) + cR^{-\alpha} \fint_{B_R} (|Du| + s)\, d\xi \end{aligned} \tag{1.38}$$

holds uniformly in $\alpha \in [0, \tilde{\alpha}]$, *for a constant* c *depending only on the parameters* $n, p, \nu, L, \omega(\cdot), \tilde{\alpha}, \operatorname{diam}(\Omega), S$

Notice that assumption (1.37) weakens (1.20) and refers to the standard Hölder continuity.

Theorem 1.9 (Subquadratic maximal estimates)**.** *Let* $u \in C^1(\Omega)$ *be a weak solution to* (1.1) *under the assumptions* (1.2) *with* $2 - 1/n < p \leq 2$; *let* $B_R \subset \Omega$ *be a ball centered at* x. *Then*

- *For every* $\tilde{\alpha} < 1$ *there exists a positive number* $\delta \equiv \delta(n, p, \nu, L, \tilde{\alpha})$ *such that if* (1.19) *is satisfied, then estimate* (1.35) *holds uniformly in* $\alpha \in [0, \tilde{\alpha}]$, *for a constant* $c \equiv c(n, p, \nu, L, \tilde{\alpha}, \operatorname{diam}(\Omega))$.
- *In addition, if* (1.25) *is in force, the estimate*

$$
\begin{aligned}
& M^{\#}_{\alpha,R}(u)(x) + M_{1-\alpha,R}(Du)(x) \\
& \leq c \left[\mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(x, R)\right]^{1/(p-1)} + cR^{1-\alpha} \fint_{B_R} (|Du| + s)\, d\xi
\end{aligned} \tag{1.39}
$$

holds uniformly in $\alpha \in [0, 1]$, *with* $c \equiv c(n, p, \nu, L, \omega(\cdot), \sigma, \operatorname{diam}(\Omega))$

- *Finally, assume that* (1.25) *is in force together with*

$$
\sup_r \frac{[\omega(r)]^\sigma}{r^{\tilde{\alpha}}} \leq S \tag{1.40}
$$

for some $\sigma < 1$ *and* $\tilde{\alpha} \in [0, \alpha_M)$. *Then*

$$
\begin{aligned}
M^{\#}_{\alpha,R}(Du)(x) \quad & \leq \quad c\,[M_{1-\alpha,R}(\mu)(x)]^{1/(p-1)} \\
& \quad +c \left[\mathbf{I}^{|\mu|}_{1}(x, R)\right]^{1/(p-1)} + cR^{-\alpha} \fint_{B_R} (|Du| + s)\, d\xi
\end{aligned} \tag{1.41}
$$

holds uniformly in $\alpha \in [0, \tilde{\alpha}]$, *for a constant* c *depending only on the parameters* $n, p, \nu, L, \omega(\cdot), \sigma, \tilde{\alpha}, \operatorname{diam}(\Omega), S$.

Suitable versions of estimates (1.36) and (1.39) also follow in the case of measurable coefficients; see Proposition 3.1 below. We also remark that Theorems 1.8-1.9 imply slightly stronger - *but not endpoint* - versions of the results presented in Sections 1.1-1.2. See Theorem 5.1 below.

Finally, we close the section by revisiting a well-known result of Kilpeläinen and Malý [16]; here the classical pointwise estimate is upgraded to a pointwise estimate for the (restricted) Hardy-Littlewood maximal operator. In case that both the solution u and the measure μ are nonnegative, the result is a consequence of (1.6) and the weak Harnack inequality.

Theorem 1.10 (Kilpeläinen & Malý '94 revisited)**.** *Let* $u \in C^0(\Omega) \cap W^{1,p}(\Omega)$ *be a weak solution to* (1.1), *under the assumptions* (1.2) *with* $p > 2 - 1/n$. *Then the inequality*

$$
M_R(u)(x) \leq c\mathbf{W}^{\mu}_{1,p}(x, R) + c \fint_{B_R} (|u| + Rs)\, d\xi
$$

holds for a constant c *depending only* n, p, ν, L, *whenever* $B_R \subset \Omega$.

1.5. **Plan of the paper.** Let us briefly outline the strategy by describing the organization of the paper. In Section 2, after recalling a few preliminary definitions and results, especially concerned with the regularity of homogeneous equations, we derive a few comparison lemmas allowing to treat with low regularity coefficients, as described in Definition 2. Such lemmas require a rather delicate use of certain up-to-the-boundary Calderón-Zygmund type estimates for nonlinear equations recently derived in [18].

In Section 3 we proceed with the proof of Theorems 1.8 and 1.9, the most delicate of which being the proof of the endpoint estimates (1.36) and (1.39). In order to do this we shall use certain precise iteration methods and reference estimates from standard De Giorgi's theory for nonlinear equations. Let us observe that the approach given here gives a pointwise estimate on fractional operators, and therefore allow to get L^∞-bounds. This connects to classical, fundamental work of Tadeusz Iwaniec [14], who was the first to observe the main role of maximal operators in nonlinear problems, and that has been a major source of inspiration for several works in the field (see for instance [7, 17]).

Section 4 contains the main material of the paper, together with the proofs of Theorems 1.1, 1.4, 1.6 and 1.10. Here we shall use pointwise iteration schemes in order to make fractional potentials appear. We shall finally come up with a certain hybrid estimate involving both the desired fractional potential term and an additional error of excess type, i.e. the integral deviation of the solution (or of its gradient) from its average; this last term will be then estimated by means of the sharp maximal function estimates of Section 1.4.

The remaining pointwise estimates, that are those appearing in Theorems 1.2, 1.3 and 1.5, are derived in Section 5, essentially as a corollary of the results previously obtained; moreover a non-endpoint version of the pointwise estimates is presented in Theorem 5.1. In Section 6 we give the proof of Theorem 1.7 using higher order perturbations. In Section 7 we prove a Lipschitz regularity result already used in the proof of the various pointwise estimates. This result might have its own interest in that it relaxes some well-known Dini continuity conditions usually assumed in several papers and holds in the full range $p > 1$ for $W^{1,p}$-solutions. Finally, in Section 8 we describe possible refinements and demonstrate applications by stating a few selected corollaries of our results.

Some of the results of this paper have been reported in the research announcement [20].

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2. Auxiliary results

2.1. **General notation.** In what follows we denote by c a general constant larger (or equal) than one, possibly varying from line to line; special occurrences will be denoted by c_1 etc; relevant dependencies on parameters will be emphasized using parentheses. We also denote by $B(x_0, R) := \{x \in \mathbb{R}^n \, : \, |x - x_0| < R\}$ the open ball with center x_0 and radius $R > 0$; when not important, or clear from the context, we shall omit denoting the center as follows: $B_R \equiv B(x_0, R)$. Unless otherwise stated, different balls in the same context will have the same center. We shall also denote $B \equiv B_1 = B(0, 1)$. With A being a measurable subset with positive measure, and with $g \colon A \to \mathbb{R}^k$ being a measurable map, we shall denote by

$$\fint_A g(x)\, dx := \frac{1}{|A|} \int_A g(x)\, dx$$

its integral average. When considering an L^1-function μ we shall denote $|\mu|(A) := \|\mu\|_{L^1(A)}$, i.e. *thinking of L^1-functions as measures.* Next we recall a few standard consequences of the strict ellipticity of the vector field $a(\cdot)$ assumed in $(1.2)_2$. Indeed - see also [28] - for $c \equiv c(n, p, \nu) > 0$, and whenever $z_1, z_2 \in \mathbb{R}^n$ it holds that

$$c^{-1}(|z_2|^2 + |z_1|^2 + s^2)^{(p-2)/2}|z_2 - z_1|^2 \leq \langle a(x, z_2) - a(x, z_1), z_2 - z_1 \rangle \,. \tag{2.1}$$

Notice that when $z_1 = 0 = z_2$ we shall interpret the left hand side as zero. Obviously, in the case $p \geq 2$, the previous inequality implies

$$c^{-1}|z_2 - z_1|^p \leq \langle a(x,z_2) - a(x,z_1), z_2 - z_1\rangle \, . \tag{2.2}$$

2.2. **On the notion of solution.** A function $u \in W^{1,\min\{p-1,1\}}_{\rm loc}(\Omega)$ is called a very weak (distributional) solution to the equation (1.1) if it satisfies the distributional relation

$$\int_\Omega \langle a(x,Du), D\varphi\rangle \, dx = \int_\Omega \varphi \, d\mu$$

whenever $\varphi \in C^\infty_0(\Omega)$ has a compact support in Ω. Very weak solutions are usually obtained by approximation via problems involving regular data $\mu_\varepsilon \in C^\infty(\Omega)$ converging weakly to μ, and regularized smooth operators a_ε converging to a in a suitably strong sense. Solutions obtained in this way are often called SOLA (Solutions Obtained by Limiting Approximation). The relevant existence theory and compactness properties are developed in the paper of Boccardo & Gallouët [2] to which we refer, together with [8], for the approximation procedures. When μ is nonnegative, an alternative, essentially equivalent, existence theory for equations is developed in [13, 16] based on the concept of p-superharmonic functions. Furthermore, by standard regularity theory, when starting from a vector field satisfying assumptions (1.2), approximating solutions belong to $C^1(\Omega)$ and, in particular, they satisfy regularity assumptions of Theorems 1.1-1.10. By compactness results, statements of corresponding theorems continue to hold also for SOLA almost everywhere. For such reasons, as already remarked in the Introduction, we confine ourselves to state the results under additional regularity assumptions on the solutions and on the data, in the form of uniform a priori estimates.

2.3. **Maximal operators.** Here we recall the definitions of a few maximal operators; a point we want to immediately emphasize here is that for our purposes it will be necessary to consider only *centered maximal operators* as it will clear from the definitions given below. In the following, by f we shall always denote a possibly vector valued map such that $f \in L^1(\Omega;\mathbb{R}^k)$ and $\Omega \subset \mathbb{R}^n$ is a bounded subdomain.

Definition 3. Let $\beta \in [0,n]$, $x \in \Omega$ and $R < \operatorname{dist}(x,\partial\Omega)$, and let f be an $L^1(\Omega)$-function or a measure with finite mass; the function defined by

$$M_{\beta,R}(f)(x) := \sup_{0<r\leq R} r^\beta \frac{|f|(B(x,r))}{|B(x,r)|}$$

is called the restricted (centered) fractional β maximal function of f.

Obviously, when $\beta = 0$ the one defined above is the classical (restricted) Hardy-Littlewood maximal operator, and we shall denote $M_{0,R}(f) \equiv M_R(f)$

Definition 4. Let $\beta \in [0,1]$, $x \in \Omega$ and $R < \operatorname{dist}(x,\partial\Omega)$, and let $f \in L^1(\Omega)$; the function defined by

$$M^\#_{\beta,R}(f)(x) := \sup_{0<r\leq R} r^{-\beta} \fint_{B(x,r)} |f - (f)_{B(x,r)}| \, d\xi$$

is called the restricted (centered) sharp fractional maximal function of f.

Taking $\beta = 0$ in Definition 4 we find the usual Fefferman-Stein sharp maximal operator. Let us observe that, by using the standard Poincaré inequality, when $f \in W^{1,1}(\Omega,\mathbb{R}^k)$ we obtain

$$M^\#_{\alpha,R}(f)(x) \leq cM_{1-\alpha,R}(Df)(x) \qquad \forall \; \alpha \in [0,1] \, . \tag{2.3}$$

2.4. **Regularity properties of $\tilde{a}$-harmonic functions.** Here we are concerned with the regularity of $\tilde{a}$-harmonic functions, that is solutions $v \in W^{1,p}_{\rm loc}(\Omega)$ to homogeneous equations as

$$\operatorname{div} \tilde{a}(Dv) = 0 \tag{2.4}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, with the vector field $\tilde{a}\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfying (1.2). For such equations the maximal regularity is the one outlined in (1.21); for this we refer for instance to [6, 26, 27] and to the related bibliography. The next result that in the present version can be retrieved from [8] - in turn building on [24] - encodes the regularity properties of v in decay estimates for a suitable excess functionals of the gradient.

Theorem 2.1. *Let $v \in W^{1,p}_{\rm loc}(\Omega)$ be a weak solution to* (2.4) *under the assumptions* (1.2) *with $p > 1$. Then there exist constants $\alpha_M \in (0,1]$ and $c \geq 1$, both depending only on n, p, ν, L, but otherwise independent of the solution v and on the vector field $\tilde{a}(\cdot)$, such that the estimate*

$$\fint_{B_\varrho} |Dv - (Dv)_{B_\varrho}|\, dx \leq c \left(\frac{\varrho}{R}\right)^{\alpha_M} \fint_{B_R} |Dv - (Dv)_{B_R}|\, dx \tag{2.5}$$

holds whenever $B_\varrho \subseteq B_R \subseteq \Omega$ are concentric balls. Moreover, it also holds that

$$\fint_{B_\varrho} (|Dv| + s)\, dx \leq c \fint_{B_R} (|Dv| + s)\, dx, \tag{2.6}$$

again for a constant c depending only on n, p, ν, L.

We next turn our attention to the case of solutions to homogeneous equations with measurable coefficients of the type

$$\operatorname{div} a(x, Dw) = 0\,. \tag{2.7}$$

For such equations De Giorgi's theory is available and provides the basic regularity result in (1.12). This last result is encoded in the following Morrey type growth lemma, implying (1.12).

Theorem 2.2. *Let $w \in W^{1,p}(\Omega)$ be a weak solution to equation* (2.7) *under the assumptions* (1.2) *with $p > 1$. Then there exist constants $\alpha_m \in (0,1]$ and $c \geq 1$, both depending only on n, p, ν, L, such that the estimate*

$$\fint_{B_\varrho} (|Dw| + s)\, dx \leq c \left(\frac{\varrho}{R}\right)^{-1+\alpha_m} \fint_{B_R} (|Dw| + s)\, dx \tag{2.8}$$

holds whenever $B_\varrho \subseteq B_R \subseteq \Omega$ are concentric balls.

The previous result is classical, and in this low integrability version has been established in [28, Lemma 3.3] for the case $p < n$. The general case $p > 1$ can be obtained with a small variant as described in [29, Remark 11] (in this last reference the case $p = n$ is treated, but the one $p > n$ follows exactly in the same fashion).

We finally state a result concerning boundary regularity and nonlinear Calderón-Zygmund theory (see for instance [31] for more on this subject).

Theorem 2.3. *Let $v \in W^{1,p}(\Omega)$ be a weak solution to the Dirichlet problem*

$$\begin{cases} \operatorname{div}\ \tilde{a}(Dv) = 0 & \text{in } B_R \\ v = w & \text{on } \partial B_R\,, \end{cases} \tag{2.9}$$

where the vector field $\tilde{a}(\cdot)$ satisfies (1.2)*, $B_R \subset \mathbb{R}^n$ is a ball with radius R, and $w \in W^{1,q}(B_R)$ is an assigned boundary datum with $p \leq q < \infty$. Then $v \in W^{1,q}(B_R)$ and moreover the estimate*

$$\|Dv\|_{L^q(B_R)} \leq c(\|Dw\|_{L^q(B_R)} + s) \tag{2.10}$$

holds for a constant c depending only on n, p, ν, L and q.

Proof. This follows from minor modifications from the proof of [18, Theorem 7.7]. Indeed, in [18] estimate (2.10) is proved in the case of a vector valued solution, i.e. when an elliptic system is considered instead of a single equation, provided $q < np/(n-2)$ when $n > 2$. In turn, such a limitation comes from the fact that reverse gradient inequalities, holding for solutions to homogeneous systems $\operatorname{div} a(Dv) = 0$ with homogeneous type lateral boundary datum (see [18, Lemma 7.5] for the specific situation relevant here),

$$\left(\fint_{\Omega(y,\varrho/2)} |Dv|^{\chi}\,dx\right)^{1/\chi} \leq c\left(\fint_{\Omega(y,\varrho)} (|Dv|+s)^p\,dx\right)^{1/p},$$

hold in general only when $\chi \leq np/(n-2)$ when $n \geq 2$. Here $\Omega(y,\varrho/2) = B(y,\varrho)\cap\Omega$, and $y \in \partial B_R$ when $B(y,\varrho) \not\subset B_R$. In the scalar case such a limitation does not take place - compare with the approach of [18] - and the previous inequality follows even for $\chi = \infty$, see also [23]. As a consequence, adapting the arguments of [18] using this new fact now available, the proof of the Theorem follows. □

2.5. **Comparison results.** We start recalling a few known comparison results between solutions of homogeneous and non-homogeneous elliptic equations. In the rest of the section we fix $u \in W^{1,p}(\Omega)$ as a specific solution to (1.1) and we fix, again for the rest of this section, a ball $B_{2R} \equiv B(x_0, 2R) \subseteq \Omega$ with the radius $2R$. Define $w \in u + W^{1,p}_0(B_{2R})$ as the unique solution to the homogeneous Dirichlet problem

$$\begin{cases} \operatorname{div}\, a(x, Dw) = 0 & \text{in } B_{2R} \\ w = u & \text{on } \partial B_{2R}\,. \end{cases} \tag{2.11}$$

Moreover, in the rest of the paper, following a standard notation we denote

$$\chi_{\{p<2\}} = \begin{cases} 0 & \text{if} \quad p \geq 2 \\ 1 & \text{if} \quad p < 2\,. \end{cases}$$

Lemma 2.1 ([9, 22, 28])**.** *Under the assumption* (1.2) *with $p > 2 - 1/n$, let $u \in W^{1,p}(\Omega)$ be a local solution to* (1.1)*, and $w \in u + W^{1,p}_0(B_{2R})$ as in* (2.11)*. Then the following inequality holds for a constant $c \equiv c(n,p,\nu)$:*

$$\begin{aligned} \fint_{B_{2R}} |Du - Dw|\,dx \quad &\leq \quad c\left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]^{1/(p-1)} \\ &\qquad + c\chi_{\{p<2\}}\left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]\left(\fint_{B_{2R}} (|Du|+s)\,dx\right)^{2-p}. \end{aligned} \tag{2.12}$$

With $w \in W^{1,p}(B_{2R})$ defined in (2.11), we then define $v \in w + W^{1,p}_0(B_R)$, on the concentric smaller ball $B_R \equiv B(x_0,R)$, as the unique solution to the homogeneous Dirichlet problem

$$\begin{cases} \operatorname{div}\, (a)_{x_0,R}(Dv) = 0 & \text{in } B_R \\ v = w & \text{on } \partial B_R\,, \end{cases} \tag{2.13}$$

where the averaged vector field $(a)_{x_0,R}(\cdot)$ has been defined in (1.14).

Lemma 2.2. *Let $p > 1$; with $w \in W^{1,p}(B_{2R})$ solving* (2.11)*, and v solving* (2.13) *there exists a constant $c \equiv c(n,p,\nu,L)$ such that the inequality*

$$\begin{aligned} \fint_{B_R} |Dv - Dw|^p\,dx \quad &\leq \quad c\left(\fint_{B_R} [A(Dw,B_R)]^2(|Dv|^2+|Dw|^2+s^2)^{p/2}\,dx\right)^{p/2} \cdot \\ &\qquad \cdot\left(\fint_{B_R} (|Dw|+s)^p\,dx\right)^{(2-p)/2} \end{aligned}$$

holds in the case $1 < p < 2$*, where*

$$A(Dw, B_R) \equiv A(Dw, B_R)(x) := \frac{|a(x, Dw(x)) - (a)_{x_0,R}(Dw(x))|}{(|Dw(x)|^2 + s^2)^{(p-1)/2}} .$$

In the case $p \geq 2$ *it instead holds that*

$$\fint_{B_R} |Dv - Dw|^p \, dx \leq c \fint_{B_R} [A(Dw, B_R)]^2 (|Dw|^2 + s^2)^{p/2} \, dx \,, \tag{2.14}$$

with a similar dependence of the constant c*.*

Proof. By (1.2), using standard monotonicity argument (see (2.2)) or by using the fact that v is a quasi-minimizer of the functional

$$z \mapsto \int_{B_R} |Du|^p \, dx \,,$$

see [12, Theorem 6.1] also for the definition, we have

$$\int_{B_R} |Dv|^p \, dx \leq c(n, p, \nu, L) \int_{B_R} (|Dw|^2 + s^2)^{p/2} \, dx \,. \tag{2.15}$$

Notice that by its very definition the averaged vector field $(a)_{x_0,R}(\cdot)$ still satisfies (1.2). Therefore, using (2.1), the fact that both v and w are solutions, $(1.2)_1$ and again Young's inequality, we have

$$\begin{aligned}
&\int_{B_R} (|Dv|^2 + |Dw|^2 + s^2)^{(p-2)/2} |Dw - Dv|^2 \, dx \\
&\leq c \int_{B_R} \langle (a)_{x_0,R}(Dw) - (a)_{x_0,R}(Dv), Dw - Dv \rangle \, dx \\
&= c \int_{B_R} \langle (a)_{x_0,R}(Dw) - a(x, Dw), Dw - Dv \rangle \, dx \\
&\leq c \int_{B_R} A(Dw, B_R)(|Dw|^2 + s^2)^{(p-1)/2} |Dw - Dv| \, dx \\
&\leq c \int_{B_R} A(Dw, B_R)(|Dw|^2 + |Dv|^2 + s^2)^{(p-1)/2} |Dw - Dv| \, dx \\
&\leq \frac{1}{2} \int_{B_R} (|Dv|^2 + |Dw|^2 + s^2)^{(p-2)/2} |Dw - Dv|^2 \, dx \\
&\qquad + c \int_{B_R} [A(Dw, B_R)]^2 (|Dv|^2 + |Dw|^2 + s^2)^{p/2} \, dx \,.
\end{aligned} \tag{2.16}$$

Ultimately,

$$\begin{aligned}
&\int_{B_R} (|Dv|^2 + |Dw|^2 + s^2)^{(p-2)/2} |Dw - Dv|^2 \, dx \\
&\qquad \leq c \int_{B_R} [A(Dw, B_R)]^2 (|Dv|^2 + |Dw|^2 + s^2)^{p/2} \, dx
\end{aligned} \tag{2.17}$$

follows. We now start analyzing the case $p < 2$. Let us write

$$\begin{aligned}
|Dv - Dw|^p \quad = \quad & \left[(|Du|^2 + |Dw|^2 + s^2)^{(p-2)/2} |Dv - Dw|^2 \right]^{p/2} \\
& \cdot (|Dv|^2 + |Dw|^2 + s^2)^{p(2-p)/4} \,,
\end{aligned}$$

and therefore using the last estimate, together with (2.15) and Hölder's inequality, yields

$$\fint_{B_R} |Dv - Dw|^p \, dx$$

$$\leq c\left(\fint_{B_R}(|Dv|^2+|Dw|^2+s^2)^{(p-2)/2}|Dv-Dw|^2\,dx\,dx\right)^{p/2}$$
$$\cdot\left(\fint_{B_R}(|Dw|^2+s^2)^{p/2}\,dx\right)^{(2-p)/2}.$$

The statement for the case $1<p<2$ now follows matching the last inequality with (2.17). In the case $p\geq 2$ we go back to (2.16) and directly estimate

$$\begin{aligned}
&\frac{1}{2}\int_{B_R}|Dw-Dv|^p\,dx+\frac{1}{2}\int_{B_R}(|Dw|^2+s^2)^{(p-2)/2}|Dw-Dv|^2\,dx\\
&\leq c\int_{B_R}(|Dv|^2+|Dw|^2+s^2)^{(p-2)/2}|Dw-Dv|^2\,dx\\
&\leq c\int_{B_R}A(Dw,B_R)(|Dw|^2+s^2)^{(p-1)/2}|Dw-Dv|\,dx\\
&\leq \frac{1}{4}\int_{B_R}(|Dw|^2+s^2)^{(p-2)/2}|Dw-Dv|^2\,dx\\
&\qquad\qquad+c\int_{B_R}[A(Dw,B_R)]^2(|Dw|^2+s^2)^{p/2}\,dx\,,
\end{aligned}$$

implying the statement of the lemma for the case $p\geq 2$. □

The next Lemma is a corollary of the previous one used together with a suitable version of Gehring's lemma.

Lemma 2.3. *Let $p>1$; with $w\in W^{1,p}(B_{2R})$ solving* (2.11)*, and v solving* (2.13) *there exists a constant $c\equiv c(n,p,\nu,L)$ such that the inequality*

$$\fint_{B_R}|Dv-Dw|\,dx\leq c[\omega(R)]^{\sigma}\fint_{B_{2R}}(|Dw|+s)\,dx\,, \tag{2.18}$$

holds, where $\omega(\cdot)$ has been defined in (1.15) *and σ is a positive ("small") exponent depending only on n,p,ν,L.*

Proof. We start recalling a few basic results from elliptic regularity theory. The first is a classical version of Gehring's lemma, asserting that there exists an exponent $q>p$ and a constant c, both depending only on n,p,ν,L, such that

$$\left(\fint_{B_R}(|Dw|^2+s^2)^{q/2}\,dx\right)^{t/q}\leq c\fint_{B_{2R}}(|Dw|^2+s^2)^{t/2}\,dx \tag{2.19}$$

holds whenever $t>0$ for a constant c depending on n,p,ν,L and also on $t>0$. Actually Gehring's lemma gives the previous inequality for $t=p$; the statement for the general case $t>0$ follows from a standard self-improving property of reverse Hölder inequalities, as explained for instance in [28, Lemma 3.3]; moreover, we remark that although the statement is usually reported for the case $p\leq n$, it continues to hold whenever $p>1$; see also [12, Chapter 6] and [29, Remark 11]. Combining (2.19) - for the choice $t=1$ - with the up-to-the-boundary higher integrability in (2.10) and using also (2.15) yields

$$\left(\fint_{B_R}(|Dw|^2+|Dv|^2+s^2)^{q/2}\,dx\right)^{1/q}\leq c\fint_{B_{2R}}(|Dw|+s)\,dx \tag{2.20}$$

for a constant c depending only on n,p,ν,L. On the other hand, by Hölder's inequality we have

$$\int_{B_R}[A(Dw,B_R)]^2(|Dv|^2+|Dw|^2+s^2)^{p/2}\,dx$$

$$\leq \left(\fint_{B_R} [A(Dw,B_R)]^{2q/(q-p)}\, dx \right)^{(q-p)/q} \left(\fint_{B_R} (|Dv|^2+|Dw|^2+s^2)^{q/2}\, dx \right)^{p/q} .$$

In turn we estimate, by means of $(1.2)_1$ and (1.15), as follows:

$$\fint_{B_R} [A(Dw,B_R)]^{2q/(q-p)}\, dx \leq (2L)^{2p/(q-p)} \fint_{B_R} [A(Dw,B_R)]^2\, dx \leq c[\omega(R)]^2 .$$

Combining the last two estimates with (2.20) gives

$$\begin{aligned} &\int_{B_R} [A(Dw,B_R)]^2 (|Dv|^2+|Dw|^2+s^2)^{p/2}\, dx \\ &\qquad\qquad \leq c\,[\omega(R)]^{2(q-p)/q} \left(\fint_{B_{2R}} (|Dw|+s)\, dx \right)^p . \end{aligned}$$

Using the last estimate together with Lemma 2.2 and (2.19) leads to

$$\left(\fint_{B_R} |Dv-Dw|^p\, dx \right)^{1/p} \leq c[\omega(R)]^{\sigma} \fint_{B_{2R}} (|Dw|+s)\, dx \tag{2.21}$$

with σ defined by

$$\sigma := \begin{cases} \frac{2(q-p)}{pq} & \text{if} \qquad p \geq 2 \\ \frac{(q-p)}{q} & \text{if} \quad 2-1/n < p \leq 2 . \end{cases} \tag{2.22}$$

Finally, (2.18) follows by using (2.21) together with Hölder's inequality. □

In the rest of the paper we shall use the following quantity:

$$\sigma_d := \begin{cases} 2/p & \text{if} \qquad p \geq 2 \\ \sigma < 1 & \text{if} \quad 2-1/n < p < 2 . \end{cases} \tag{2.23}$$

In other words, σ_d is a number that can be chosen *arbitrarily close to* 1 when $p < 2$.

When additional Lipschitz regularity is available on w we can quantify the exponent σ in Lemma 2.3. This leads to the following improvement:

Lemma 2.4. *Let $p > 1$; with $w \in W^{1,p}(B_{2R})$ solving* (2.11)*, and v solving* (2.13)*, assume also that $w \in W^{1,\infty}(B_R)$. Then the following inequality holds:*

$$\fint_{B_R} |Dv-Dw|\, dx \leq c[\omega(R)]^{\sigma_d} (\|Dw\|_{L^\infty(B_R)}+s) , \tag{2.24}$$

where σ_d has been defined in (2.23)*. The constant c depends only on n,p,ν,L,q when $p \geq 2$ and additionally on the number σ chosen in* (2.23) *when $p \in (1,2)$.*

Proof. First the case $1 < p < 2$; we go back to the proof of Lemma 2.3 and, thanks to (2.10), we may now estimate, for every $q < \infty$

$$\begin{aligned} \fint_{B_R} (|Dw|^2+|Dv|^2+s^2)^{q/2}\, dx &\leq c \fint_{B_R} (|Dw|^2+s^2)^{q/2}\, dx \\ &\leq c(\|Dw\|_{L^\infty(B_R)}+s)^q \end{aligned}$$

for a constant $c \equiv c(n,p,\nu,L,q)$. With this last estimate replacing (2.20) we can proceed as in the proof of Lemma 2.3, with the difference that we can now take q to be any positive number; ultimately, this results in the fact that the number σ in (2.22) can be taken arbitrarily close to 1. This ends the proof of the Lemma in the case $p < 2$ in view of the definition in (2.24). In the case $p \geq 2$ the path is straightforward: we take fully advantage of (2.14); recalling again the definition in (1.15) we simply estimate

$$\begin{aligned}\left(\fint_{B_R} |Dv-Dw|^p\,dx\right)^{1/p} &\leq c\left(\fint_{B_R} [A(Dw,B_R)]^2(|Dw|^2+s^2)^{p/2}\,dx\right)^{1/p}\\ &\leq c(\|Dw\|_{L^\infty(B_R)}+s)\left(\fint_{B_R}[A(Dw,B_R)]^2\,dx\right)^{1/p}\\ &\leq c(\|Dw\|_{L^\infty(B_R)}+s)[\omega(R)]^{2/p}\,.\end{aligned}$$

At this point (2.24) follows by using Hölder's inequality and again recalling that $\sigma_d = 2/p$ when $p \geq 2$. □

Finally, when Dini-VMO continuity of coefficients is available, the function w is indeed Lipschitz - a fact we will prove later, see Theorem 7.1 below. Therefore, combining (2.24) with (7.3) we obtain the following:

Lemma 2.5. *Let $p>1$; with $w \in W^{1,p}(B_{2R})$ solving* (2.11)*, and v solving* (2.13)*, let us assume that the function $[\omega(\cdot)]^{\sigma_d}$ is* Dini-VMO*, i.e. that the condition*

$$\int_0^r [\omega(\varrho)]^{\sigma_d}\,\frac{d\varrho}{\varrho} < \infty \qquad \forall\ r<\infty\,, \tag{2.25}$$

is in force, where σ_d has been defined in (2.23)*. Then the following inequality holds:*

$$\fint_{B_R} |Dv-Dw|\,dx \leq c[\omega(R)]^{\sigma_d}\fint_{B_{2R}}(|Dw|+s)\,dx\,. \tag{2.26}$$

The constant c depends only on n,p,ν,L when $p\geq 2$ and additionally on σ when $p\in(1,2)$.

3. Maximal estimates and Theorems 1.8-1.9

In this section we give the proof of the maximal estimates presented in Section 1.4. After a preliminary list of lemmas, we shall present the results in the subquadratic case $2-1/n<p\leq 2$, and then we shall proceed with the case $p\geq 2$. We recall that $\alpha_M \in (0,1]$ indicates the maximal Hölder gradient regularity exponent of solutions to homogeneous equations of the type (2.4), described in (2.5) and (1.21). Accordingly, by $\alpha_m \in (0,1]$ we denote the maximal Hölder regularity exponent of solutions to homogeneous equations with measurable coefficients (2.7) as described in Theorem 2.2 and in (1.12).

Lemma 3.1. *Let u be as in Theorem 1.1, then, with $p>2-1/n$, there exist constants $c_1, c\geq 1$, depending only on n,p,ν,L, such that the following estimate holds whenever $B_\varrho \subseteq B_R \subseteq \Omega$ are concentric balls:*

$$\begin{aligned}\fint_{B_\varrho}(|Du|+s)\,d\xi &\leq c_1\left(\frac{\varrho}{R}\right)^{-1+\alpha_m}\fint_{B_R}(|Du|+s)\,d\xi\\ &\quad +c\left(\frac{R}{\varrho}\right)^n\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]^{1/(p-1)}\\ &\quad +c\chi_{\{p<2\}}\left(\frac{R}{\varrho}\right)^n\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]\left(\fint_{B_R}(|Du|+s)\,d\xi\right)^{2-p}.\end{aligned}$$

Proof. It is based on a comparison argument using strict monotonicity; using Theorem 2.2 - we obviously define the function w in (2.11) as being the solution of the same Dirichlet problem in the ball B_R (instead of B_{2R}) considered here - we have

$$\fint_{B_\varrho}(|Du|+s)\,d\xi \leq \fint_{B_\varrho}(|Dw|+s)\,d\xi + \left(\frac{R}{\varrho}\right)^n\fint_{B_R}|Du-Dw|\,d\xi$$

$$
\begin{aligned}
&\leq c_1\left(\frac{\varrho}{R}\right)^{-1+\alpha_m} \fint_{B_R} (|Dw|+s)\,d\xi + \left(\frac{R}{\varrho}\right)^n \fint_{B_R} |Du-Dw|\,d\xi \\
&\leq c_1\left(\frac{\varrho}{R}\right)^{-1+\alpha_m} \fint_{B_R} (|Du|+s)\,d\xi \\
&\qquad\qquad +c\left[\left(\frac{R}{\varrho}\right)^{1-\alpha_m} + \left(\frac{R}{\varrho}\right)^n\right] \fint_{B_R} |Du-Dw|\,d\xi\,,
\end{aligned}
$$

and the statement follows using (2.12) in the previous inequality. □

In a completely similar way goes the proof of the next lemma. We use (2.6) instead of (2.8) as "reference estimate". Then we make a double comparison: first we use Lemma 2.3 (on B_R) and then Lemma 2.1 (on B_{2R}) twice, and therefore we first compare u with v and then w with v; we also use the fact $\omega(R) \leq c(L)$.

Lemma 3.2. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to* (1.1) *under the assumptions* (1.2) *with $p > 2-1/n$. Then there exist positive constants $c, c_1 > 1$ and $\sigma \in (0,1)$, all depending only on n, p, ν, L such that the following estimate holds whenever $B_\varrho \subseteq B_R \subseteq B_{2R} \subseteq \Omega$ are concentric balls:*

$$
\begin{aligned}
\fint_{B_\varrho} (|Du|+s)\,d\xi \quad &\leq \quad c_1 \fint_{B_{2R}} (|Du|+s)\,d\xi + c\left(\frac{R}{\varrho}\right)^n \left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]^{1/(p-1)} \\
&\quad +c\left(\frac{R}{\varrho}\right)^n [\omega(R)]^\sigma \fint_{B_{2R}} (|Du|+s)\,d\xi \\
&\quad +c\chi_{\{p<2\}}\left(\frac{R}{\varrho}\right)^n \left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right] \left(\fint_{B_{2R}} (|Du|+s)\,d\xi\right)^{2-p}.
\end{aligned}
\tag{3.1}
$$

Finally, using the same comparison scheme of the previous lemma, but taking this time (2.5) as "reference estimate" and using Lemma 2.5, we have:

Lemma 3.3. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to* (1.1) *under the assumptions* (1.2) *with $p > 2-1/n$, and assume that the function $[\omega(\cdot)]^{\sigma_d}$ is* Dini-VMO *regular, i.e.* (2.25) *holds with σ_d defined in* (2.23). *Then there exist constants $c_1, c \geq 1$ depending only on n, p, ν, L, such that the following estimate holds whenever $B_\varrho \subseteq B_R \subseteq B_{2R} \subseteq \Omega$ are concentric balls:*

$$
\begin{aligned}
&\fint_{B_\varrho} |Du-(Du)_{B_\varrho}|\,d\xi \leq c_1\left(\frac{\varrho}{R}\right)^{\alpha_M} \fint_{B_{2R}} |Du-(Du)_{B_{2R}}|\,d\xi \\
&\qquad +c\left(\frac{R}{\varrho}\right)^n \left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]^{1/(p-1)} + c\left(\frac{R}{\varrho}\right)^n [\omega(R)]^{\sigma_d} \fint_{B_{2R}} (|Du|+s)\,d\xi \\
&\qquad +c\chi_{\{p<2\}}\left(\frac{R}{\varrho}\right)^n \left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right] \left(\fint_{B_{2R}} (|Du|+s)\,d\xi\right)^{2-p}.
\end{aligned}
\tag{3.2}
$$

In the case $p < 2$ the constant c depends also on the number $\sigma < 1$ chosen to define σ_d in (2.23).

3.1. **The case $2-1/n < p \leq 2$ and the proof of Theorem 1.9.** Let us first give a general idea of the proof. In order to get the limiting potential estimates (1.6)-(1.7) the idea is to get a bound for the quantities of the type

$$
\fint_{B_i} |u|\,dx \qquad \text{and} \qquad \fint_{B_i} |Du|\,dx\,,
$$

respectively, where B_i are balls geometrically shrinking at the point x. The more general idea here is to get bounds for intermediate, "non-local" quantities as

$$
|B_i|^{(1-\alpha)/n} \fint_{B_i} |Du|\,dx\,, \qquad\qquad 0 \leq \alpha \leq 1\,,
$$

and some higher-order analogs of them related to fractional maximal operators. A main point of interest here, eventually helpful for the proof of the endpoint estimates of the next section, is to show the proper uniform dependence of the estimates with respect to $\alpha \in [0,1]$. The core of the ideas is therefore presented in the proof of estimate (1.39).

Proof of Theorem 1.9. In the rest of the proof all the balls will be concentric and centered at the point $x \in \Omega$ identified by the statement of the Theorem. Most of the times, the considered radii R will be such that $R \leq \tilde{R}$, where the quantity $\tilde{R} > 0$ will be in general chosen along the proof in dependence of the data $n, p, \nu, L, \tilde{\alpha}, \omega(\cdot)$, essentially using conditions as (1.19). More precisely, we shall determine several smallness conditions of the type

$$\omega(\tilde{R}) \leq \delta\,, \tag{3.3}$$

where δ will be a small quantity that will be reduced at several stages, as a *decreasing function* of the quantities n, p, ν, L - and also $\tilde{\alpha}$ according to the statement we will be proving; the quantity δ will be in other words implicitly determined by several choices as (3.3). In this respect, we remark that satisfying an inequality like (3.3) is always possible in the rest of the proof: when dealing with the case $\alpha < \tilde{\alpha}$ this is directly assumed in (1.19), while (3.3) is a consequence of any of (1.20), (1.22) or (1.25) (recall that $\omega(\cdot)$ is non-decreasing).

(**) *Proof of* (1.35). The proof is in two steps and works also in the case $p \geq 2$.

Step 1: Validity of (1.35) *for small radii* $R \leq \tilde{R}$. We shall confine ourselves to prove the estimate

$$M_{1-\alpha,R}(Du)(x) \leq c\left[M_{p-\alpha(p-1),R}(\mu)(x)\right]^{1/(p-1)} + cR^{1-\alpha}\fint_{B_R}(|Du|+s)\,d\xi \tag{3.4}$$

while (1.35) follows from (3.4) by means of (2.3). We take concentric balls $B_\varrho \subset B_{r/2} \subset B_r \subset B_R$ with positive radii, and start observing the following identities, which will be actually used several times throughout the paper:

$$r^{1-\alpha}\left[\frac{|\mu|(B_r)}{r^{n-1}}\right]^{1/(p-1)} = \left[\frac{|\mu|(B_r)}{r^{n-p+\alpha(p-1)}}\right]^{1/(p-1)} \tag{3.5}$$

and

$$\begin{aligned} & r^{1-\alpha}\left[\frac{|\mu|(B_r)}{r^{n-1}}\right]\left(\fint_{B_r}(|Du|+s)\,d\xi\right)^{2-p} \\ & = \frac{|\mu|(B_r)}{r^{n-p+\alpha(p-1)}}\left(r^{1-\alpha}\fint_{B_r}(|Du|+s)\,d\xi\right)^{2-p}\,. \end{aligned} \tag{3.6}$$

We now use Lemma 3.2 (we take $R \equiv r/2$ there) and multiply both sides of (3.1) by $\varrho^{1-\alpha}$; easy manipulations involving (3.5)-(3.6) give

$$\begin{aligned} \varrho^{1-\alpha}\fint_{B_\varrho}(|Du|+s)\,d\xi \leq{}& c_1\left(\frac{\varrho}{r}\right)^{1-\alpha} r^{1-\alpha}\fint_{B_r}(|Du|+s)\,d\xi \\ & +c\left(\frac{r}{\varrho}\right)^{n-1+\alpha}\left[\frac{|\mu|(B_r)}{r^{n-p+\alpha(p-1)}}\right]^{1/(p-1)} \\ & +c\chi_{\{p<2\}}\left(\frac{r}{\varrho}\right)^{n-1+\alpha}\left[\frac{|\mu|(B_r)}{r^{n-p+\alpha(p-1)}}\right]\left(r^{1-\alpha}\fint_{B_r}(|Du|+s)\,d\xi\right)^{2-p} \\ & +c\left(\frac{r}{\varrho}\right)^{n-1+\alpha}[\omega(R)]^{\sigma}\, r^{1-\alpha}\fint_{B_r}(|Du|+s)\,d\xi\,, \end{aligned} \tag{3.7}$$

which is valid whenever $\varrho \le r/2 \le R/2$, for $c, c_1 \equiv c, c_1(n,p,\nu,L)$. We now choose a number $H \equiv H(n,p,\nu,L,\tilde\alpha) > 2$ large enough in order to have

$$c_1\left(\frac{1}{H}\right)^{1-\alpha} \le c_1\left(\frac{1}{H}\right)^{1-\tilde\alpha} = \frac{1}{8} \tag{3.8}$$

so that by taking $\varrho = r/H$ in (3.7) leads to

$$\begin{aligned}\left(\frac{r}{H}\right)^{1-\alpha} & -\!\!\!\!\!\!\int_{B_{r/H}} (|Du|+s)\,dx \le \frac{r^{1-\alpha}}{8} -\!\!\!\!\!\!\int_{B_r} (|Du|+s)\,dx \\ & + cH^n\left[\frac{|\mu|(B_r)}{r^{n-p+\alpha(p-1)}}\right]^{1/(p-1)} \\ & + c\chi_{\{p<2\}}H^n\left[\frac{|\mu|(B_r)}{r^{n-p+\alpha(p-1)}}\right]\left(r^{1-\alpha} -\!\!\!\!\!\!\int_{B_r} (|Du|+s)\,d\xi\right)^{2-p} \\ & + cH^n\left[\omega(R)\right]^\sigma r^{1-\alpha} -\!\!\!\!\!\!\int_{B_r} (|Du|+s)\,d\xi\,.\end{aligned}$$

In turn we now choose $\tilde R \equiv \tilde R(n,p,\nu,L,\tilde\alpha,\omega(\cdot))$ in such a way that

$$cH^n\left[\omega(R)\right]^\sigma \le cH^n\left[\omega(\tilde R)\right]^\sigma \le 1/8 \tag{3.9}$$

and this provides us

$$\begin{aligned}\left(\frac{r}{H}\right)^{1-\alpha} & -\!\!\!\!\!\!\int_{B_{r/H}} (|Du|+s)\,d\xi \le \frac{r^{1-\alpha}}{4} -\!\!\!\!\!\!\int_{B_r} (|Du|+s)\,d\xi \\ & + c\left[\frac{|\mu|(B_r)}{r^{n-p+\alpha(p-1)}}\right]^{1/(p-1)} \\ & + c\chi_{\{p<2\}}\left[\frac{|\mu|(B_r)}{r^{n-p+\alpha(p-1)}}\right]\left(r^{1-\alpha} -\!\!\!\!\!\!\int_{B_r} (|Du|+s)\,d\xi\right)^{2-p}\,,\end{aligned} \tag{3.10}$$

with c depending only on $n,p,\nu,L,\tilde\alpha$; observe that here we have used that H depends also on $\tilde\alpha$ via (3.8). Being r arbitrary and such that $r \le R$, in turn, (3.10) readily implies that

$$\begin{aligned}\sup_{\varrho \le R/H} \varrho^{1-\alpha} & -\!\!\!\!\!\!\int_{B_\varrho} (|Du|+s)\,d\xi \le (1/4)M_{1-\alpha,R}(|Du|+s) \\ & + c\left[M_{p-\alpha(p-1),R}(\mu)(x)\right]^{1/(p-1)} \\ & + c\chi_{\{p<2\}}\left[M_{p-\alpha(p-1),R}(\mu)(x)\right]\left[M_{1-\alpha,R}(|Du|+s)\right]^{2-p}\,,\end{aligned}$$

where c depends only on n,p,ν,L and $\tilde\alpha$. On the other hand, we notice that

$$\sup_{R/H \le \varrho \le R} \varrho^{1-\alpha} -\!\!\!\!\!\!\int_{B_\varrho} (|Du|+s)\,d\xi \le H^n R^{1-\alpha} -\!\!\!\!\!\!\int_{B_R} (|Du|+s)\,d\xi$$

and therefore, recalling that $H \equiv H(n,p,\nu,L,\tilde\alpha)$ as determined in (3.8), matching the last two estimates yields

$$\begin{aligned}M_{1-\alpha,R}(|Du|+s)(x) & \le (1/4)M_{1-\alpha,R}(|Du|+s) + cR^{1-\alpha} -\!\!\!\!\!\!\int_{B_R} (|Du|+s)\,d\xi \\ & + c\left[M_{p-\alpha(p-1),R}(\mu)(x)\right]^{1/(p-1)} \\ & + c\chi_{\{p<2\}}\left[M_{p-\alpha(p-1),R}(\mu)(x)\right]\left[M_{1-\alpha,R}(|Du|+s)(x)\right]^{2-p}\,.\end{aligned}$$

In turn, when $p < 2$ we apply Young's inequality, that is

$$ab \le (p-1)\varepsilon^{(p-2)/(p-1)}a^{1/(p-1)} + \varepsilon b^{1/(2-p)}, \qquad a,b,\varepsilon > 0, \tag{3.11}$$

to get

$$
\begin{aligned}
& c\chi_{\{p<2\}} \left[M_{p-\alpha(p-1),R}(\mu)(x)\right] \left[M_{1-\alpha,R}(|Du|+s)(x)\right]^{2-p} \\
& \qquad \leq (1/4) M_{1-\alpha,R}(|Du|+s)(x) + c\left[M_{p-\alpha(p-1),R}(\mu)(x)\right]^{1/(p-1)} \qquad (3.12)
\end{aligned}
$$

so that combining inequalities above gives

$$
\begin{aligned}
M_{1-\alpha,R}(|Du|+s)(x) \quad \leq \quad & (1/2) M_{1-\alpha,R}(|Du|+s)(x) + cR^{1-\alpha} \fint_{B_R} (|Du|+s)\, d\xi \\
& + c\left[M_{p-\alpha(p-1),R}(\mu)(x)\right]^{1/(p-1)}
\end{aligned}
$$

from which (3.4) finally follows provided we are assuming to deal with small radii $R \leq \tilde{R} \equiv \tilde{R}(n,p,\nu,L,\tilde{\alpha},\omega(\cdot))$ determined in order to meet (3.9). We notice that while the constant in (3.4) blows-up when $p \to 2-1/n$, it instead remains bounded when $p \to 2$ as follows by looking at (3.11). The proof indeed applies to the case $p=2$ when (3.11) is not needed.

Step 2: Removing the condition $R \leq \tilde{R}$. We now, by means of standard arguments, show how to deduce the general form of (3.4), therefore avoiding to consider the restriction $R \leq \tilde{R}$. The main outcome is that the dependence on $\omega(\cdot)$ of $\tilde{R}$ will be transferred to the constant c appearing in the final version of (1.36) together with a dependence on diam (Ω). Assuming (3.4) to hold whenever $R \leq \tilde{R} \equiv \tilde{R}(n,p,\nu,L,\tilde{\alpha},\omega(\cdot))$, we take $R > \tilde{R}$ and observe that

$$
M_{1-\alpha,R}(Du)(x) \leq M_{1-\alpha,\tilde{R}}(Du)(x) + \left(\frac{R}{\tilde{R}}\right)^n R^{1-\alpha} \fint_{B_R} (|Du|+s)\, d\xi\,,
$$

and, trivially, $M_{p-\alpha(p-1),\tilde{R}}(\mu)(x) \leq M_{p-\alpha(p-1),R}(\mu)(x)$. In turn, by using estimate (3.4) with $R \equiv \tilde{R}$ to bound the second quantity appearing in the second-last estimate, and properly enlarging the integrals, that is estimating

$$
\tilde{R}^{1-\alpha} \fint_{B_{\tilde{R}}} (|Du|+s)\, d\xi \leq \left(\frac{R}{\tilde{R}}\right)^n R^{1-\alpha} \fint_{B_R} (|Du|+s)\, d\xi\,,
$$

we have that (3.4) follows with a new constant c, which is obtained from the former one by a magnification factor of $[\operatorname{diam}(\Omega)/\tilde{R}]^n$. Recalling that $\tilde{R}$ depends itself on n,p,ν,L,α and $\omega(\cdot)$ the proof is complete.

(**) *Proof of* (1.39). In the following we shall write the proof in order to report also a few manipulations that will be used later and in particular when proving (1.41) and Theorem 1.6 below. We shall in this way emphasize how a certain set of estimates works in a dual way allowing to get estimates both below and beyond the threshold given by Lipschitz continuity. Moreover, when we shall write that a certain constant c depends on σ_d, keeping the definition (2.22) in mind, we shall mean that it will actually depend on the number $\sigma < 1$ in (2.23), and this will only happen in the case $p < 2$. We divide the proof in three steps.

Step 1: Dyadic sequence. We choose a geometric sequence $\{R_i\}$ whose spread $2H > 1$ will be a certain function of the fixed quantities n,p,ν,L, and will be chosen in due course of the proof. More precisely we set

$$
B_i := B(x, R/(2H)^i) := B(x, R_i)\,, \qquad (3.13)
$$

for $i = 0,1,2,\ldots$, and define

$$
A_i := \fint_{B_i} |Du - (Du)_{B_i}|\, d\xi\,, \qquad k_i := |(Du)_{B_i} - G|\,, \qquad G \in \mathbb{R}^n\,. \qquad (3.14)
$$

Here G is a fixed vector. We now select an integer $H \equiv H(n,p,\nu,L) \geq 1$ large enough to have

$$c_1 \left(\frac{1}{H}\right)^{\alpha_M} \leq \frac{1}{16}\,, \tag{3.15}$$

where α_M is the maximal gradient regularity exponent defined via (2.5). Note that the stated dependence of H on n,p,ν,L also stems from a similar dependence of α_M. Applying (3.2) on arbitrary balls $B_\varrho \equiv B_{R/(2H)^{i+1}} \equiv B_{i+1} \subseteq B_{R_i/2} \subset B_{R_i}$ and using the fact that $\omega(\cdot)$ is non-decreasing we gain

$$\begin{aligned}
&\fint_{B_{i+1}} |Du-(Du)_{B_{i+1}}|\,d\xi \leq \frac{1}{16}\fint_{B_i} |Du-(Du)_{B_i}|\,d\xi \\
&+c(2H)^n\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]^{1/(p-1)} + c\chi_{\{p<2\}}(2H)^n\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]\left(\fint_{B_i}(|Du|+s)\,d\xi\right)^{2-p} \\
&+c(2H)^n[\omega(R_i)]^{\sigma_d}\fint_{B_i}(|Du|+s)\,d\xi\,,
\end{aligned} \tag{3.16}$$

where c depends only on n,p,ν,L,σ_d. We reduce the value of $\tilde{R}$ - in a way depending only on n,p,ν,L,σ_d and $\omega(\cdot)$ - to get

$$c(2H)^n[\omega(R_i)]^{\sigma_d} \leq 1/16 \Longleftarrow 16c(2H)^n[\omega(\tilde{R})]^{\sigma_d} \leq 1\,, \tag{3.17}$$

and using some further elementary estimates - in particular estimating

$$\fint_{B_i}|Du|\,d\xi \leq \fint_{B_i}|Du-(Du)_{B_i}|\,d\xi + k_i + |G|$$

- and taking also (3.14) into account we obtain

$$\begin{aligned}
A_{i+1} &\leq (1/8)A_i + c[\omega(R_i)]^{\sigma_d}(k_i+s+|G|) \\
&+c_2\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]^{1/(p-1)} + c_2\chi_{\{p<2\}}\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]\left(\fint_{B_i}(|Du|+s)\,d\xi\right)^{2-p}
\end{aligned} \tag{3.18}$$

whenever $i \geq 0$. Now, notice that

$$\begin{aligned}
|k_{i+1}-k_i| &\leq |(Du)_{B_{i+1}}-(Du)_{B_i}| \\
&\leq \fint_{B_{i+1}}|Du-(Du)_{B_i}|\,d\xi \\
&\leq (2H)^n\fint_{B_i}|Du-(Du)_{B_i}|\,d\xi = (2H)^n A_i
\end{aligned}$$

holds whenever $i \geq 0$ so that for $m \in \mathbb{N}$ we have

$$k_{m+1} = \sum_{i=0}^{m}(k_{i+1}-k_i) + k_0 \leq (2H)^n\sum_{i=0}^{m}A_i + k_0\,. \tag{3.19}$$

To estimate the right hand side in (3.19) we observe that summing up (3.18) over $i \in \{0,\ldots,m-1\}$ yields

$$\begin{aligned}
\sum_{i=1}^{m}A_i &\leq \frac{1}{2}\sum_{i=0}^{m-1}A_i + c_2\sum_{i=0}^{m-1}\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]^{1/(p-1)} \\
&\quad +c_2\chi_{\{p<2\}}\sum_{i=0}^{m-1}\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]\left(\fint_{B_i}(|Du|+s)\,d\xi\right)^{2-p} \\
&\quad +c_2\sum_{i=0}^{m-1}[\omega(R_i)]^{\sigma_d}(k_i+s+|G|)\,,
\end{aligned}$$

and therefore

$$
\begin{aligned}
\sum_{i=1}^{m} A_i \quad \leq \quad & A_0 + 2c_2 \sum_{i=0}^{m-1} \left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]^{1/(p-1)} \\
& + 2c_2\chi_{\{p<2\}} \sum_{i=0}^{m-1} \left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right] \left(\fint_{B_i} (|Du|+s)\, d\xi\right)^{2-p} \\
& + 2c_2 \sum_{i=0}^{m-1} [\omega(R_i)]^{\sigma_d}(k_i + s + |G|)
\end{aligned} \tag{3.20}
$$

follows. For every integer $m \geq 1$ (3.19) gives

$$
\begin{aligned}
k_{m+1} \quad \leq \quad & cA_0 + ck_0 + c\sum_{i=0}^{m-1} \left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]^{1/(p-1)} \\
& + c\chi_{\{p<2\}} \sum_{i=0}^{m-1} \left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right] \left(\fint_{B_i} (|Du|+s)\, d\xi\right)^{2-p} \\
& + c_3 \sum_{i=0}^{m-1} [\omega(R_i)]^{\sigma_d}(k_i + s + |G|)\,,
\end{aligned} \tag{3.21}
$$

and the constants c, c_3 depend only on n, p, ν, L, σ_d - recall the dependence of H. We also observe that trivially estimating

$$
k_0 + k_1 \leq [1 + (2H)^n] \fint_{B_R} |Du - G|\, d\xi \tag{3.22}
$$

and keeping in mind the definition of A_0 we end up with

$$
\begin{aligned}
k_{m+1} \quad \leq \quad & c\fint_{B_R} (|Du - (Du)_{B_R}| + |Du - G|)\, d\xi + c\sum_{i=0}^{m} \left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]^{1/(p-1)} \\
& + c\chi_{\{p<2\}} \sum_{i=0}^{m} \left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right] \left(\fint_{B_i} (|Du|+s)\, d\xi\right)^{2-p} \\
& + c_3 \sum_{i=0}^{m} [\omega(R_i)]^{\sigma_d}(k_i + s + |G|)\,,
\end{aligned} \tag{3.23}
$$

for every $m \geq 0$. In the previous inequality we choose $G = 0$ and add s to both sides, and finally multiply both sides by $R_m^{1-\alpha}$; taking into account that $R_{m+1} \leq R_i$ we get

$$
\begin{aligned}
R_{m+1}^{1-\alpha}(k_{m+1} + s) \quad \leq \quad & cR^{1-\alpha} \fint_{B_R} (|Du| + s)\, d\xi + c\sum_{i=0}^{m} R_i^{1-\alpha} \left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]^{1/(p-1)} \\
& + c\chi_{\{p<2\}} \sum_{i=0}^{m} R_i^{1-\alpha} \left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right] \left(\fint_{B_i} (|Du|+s)\, d\xi\right)^{2-p} \\
& + c_3 \sum_{i=0}^{m} [\omega(R_i)]^{\sigma_d} R_i^{1-\alpha}(k_i + s)\,.
\end{aligned}
$$

In turn, using again identities (3.5)-(3.6) with $R \equiv R_i$ and the very definition of fractional maximal operator yields

$$
R_{m+1}^{1-\alpha}(k_{m+1} + s) \quad \leq \quad cR^{1-\alpha} \fint_{B_R} (|Du| + s)\, d\xi + c\sum_{i=0}^{m} \left[\frac{|\mu|(B_i)}{R_i^{n-p+\alpha(p-1)}}\right]^{1/(p-1)}
$$

$$
\begin{aligned}
&+c\chi_{\{p<2\}}\left[M_{1-\alpha,R}(|Du|+s)(x)\right]^{2-p}\sum_{i=0}^{m}\frac{|\mu|(B_i)}{R_i^{n-p+\alpha(p-1)}}\\
&+c_3\sum_{i=0}^{m}[\omega(R_i)]^{\sigma_d}R_i^{1-\alpha}(k_i+s)\,. \qquad (3.24)
\end{aligned}
$$

Remark 3.1. Let us observe that if we restart from (3.16) and avoid to estimate as in (3.17) and the subsequent inequality, i.e. we avoid to introduce k_i in (3.16) but we rather keep the integral averages, and eventually proceed as after (3.18), we obtain the following version of (3.23):

$$
\begin{aligned}
k_{m+1} \leq\ & c\fint_{B_R}(|Du-(Du)_{B_R}|+|Du-G|)\,d\xi + c\sum_{i=0}^{m}\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]^{1/(p-1)}\\
&+c\chi_{\{p<2\}}\sum_{i=0}^{m}\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]\left(\fint_{B_i}(|Du|+s)\,d\xi\right)^{2-p}\\
&+c_3\sum_{i=0}^{m}[\omega(R_i)]^{\sigma_d}\fint_{B_i}(|Du|+s)\,d\xi\,. \qquad (3.25)
\end{aligned}
$$

The main difference with (3.23) is that this last inequality does not need any smallness assumption on $\tilde{R}$ as the one required to satisfy (3.17), but it rather works for any ball $B_R \subseteq \Omega$.

Step 2: A uniform upper bound. Here we really focalize on the case $2-1/n < p \leq 2$, therefore the exponent σ_d in (2.23) coincides with σ which is in turn a number we may choose to be strictly smaller than one; the associated constants will depend on the choice of σ and will blow-up as $\sigma \to 1$. Starting from (3.24) we shall by induction prove the following:

Lemma 3.4. *There exists a constant c depending only on n, p, ν, L, σ, and a radius $\tilde{R}$ depending on $n, p, \nu, L, \omega(\cdot), \sigma$, but both independent of α, such that*

$$
R_m^{1-\alpha}(k_{m+1}+s) \leq cM \qquad (3.26)
$$

holds for every integer $m \geq 0$ and $R \leq \tilde{R}$, where

$$
\begin{aligned}
M :=\ & R^{1-\alpha}\fint_{B_R}(|Du|+s)\,d\xi\\
&+\left[M_{1-\alpha,R}(|Du|+s)(x)\right]^{2-p}\mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(x,2R)+\left[\mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(x,2R)\right]^{1/(p-1)}\,. \qquad (3.27)
\end{aligned}
$$

Proof. The choice of the radius $\tilde{R}$ is of course the one determined in Step 1. Note that in the previous statement we are evaluating the Riesz potential on balls not necessarily contained in Ω; this is not restrictive in that we are assuming without loss of generality that the measure μ is defined on the whole $\mathbb{R}^n$. We start with some preliminary estimates. Let us recall the elementary inequality

$$
\sum_{k=0}^{\infty}a_k^q \leq \left(\sum_{k=0}^{\infty}a_k\right)^q\,, \qquad q \geq 1\,, \qquad (3.28)
$$

valid for any nonnegative sequence $\{a_k\}$. We apply it with the choice $q = 1/(p-1)$ - obviously $q \geq 1$ as $p \leq 2$ - to get

$$
\sum_{i=0}^{m}\left[\frac{|\mu|(B_i)}{R_i^{n-p+\alpha(p-1)}}\right]^{1/(p-1)} \leq \left[\sum_{i=0}^{\infty}\frac{|\mu|(B_i)}{R_i^{n-p+\alpha(p-1)}}\right]^{1/(p-1)}\,. \qquad (3.29)
$$

In turn, with $c_* := \max\{1, (2H)^{n-p+\alpha(p-1)}\}$ - here keep in mind that $n-p+\alpha(p-1)$ is also allowed to be negative - we deduce

$$
\begin{aligned}
\sum_{i=0}^{\infty} \frac{|\mu|(B_i)}{R_i^{n-p+\alpha(p-1)}} &\le \frac{c_*}{\log 2}\int_R^{2R} \frac{|\mu|(B(x_0,\varrho))}{\varrho^{n-p+\alpha(p-1)}}\,\frac{d\varrho}{\varrho} + \sum_{i=0}^{\infty} \frac{|\mu|(B_{i+1})}{R_{i+1}^{n-p+\alpha(p-1)}}\\
&\le \frac{c_*}{\log 2}\int_R^{2R} \frac{|\mu|(B(x_0,\varrho))}{\varrho^{n-p+\alpha(p-1)}}\,\frac{d\varrho}{\varrho}\\
&\qquad + \frac{c_*}{\log 2H}\sum_{i=0}^{\infty}\int_{R_{i+1}}^{R_i} \frac{|\mu|(B(x_0,\varrho))}{\varrho^{n-p+\alpha(p-1)}}\,\frac{d\varrho}{\varrho}\\
&\le \left(\frac{c_*}{\log 2}+\frac{c_*}{\log 2H}\right)\mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(x,2R)\,. \qquad (3.30)
\end{aligned}
$$

Therefore, referring to (3.29) we have

$$
\sum_{i=0}^{m}\left[\frac{|\mu|(B_i)}{R_i^{n-p+\alpha(p-1)}}\right]^{1/(p-1)} \le c\left[\mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(x,2R)\right]^{1/(p-1)} \qquad (3.31)
$$

valid for every $m \in \mathbb{N}$, where $c \equiv c(n,p,\nu,L)$. For later use we also record that, similarly to (3.30), we obtain

$$
\sum_{i=0}^{\infty}[\omega(R_i)]^{\sigma} \le c\int_0^{2R}[\omega(\varrho)]^{\sigma}\,\frac{d\varrho}{\varrho} =: d(2R)\,, \qquad (3.32)
$$

where again we have $c \equiv c(n,p,\nu,L)$. With M defined as in (3.27), using (3.31) in (3.24) gives

$$
R_{m+1}^{1-\alpha}(k_{m+1}+s) \le c_4 M + c_3\sum_{i=0}^{m}[\omega(R_i)]^{\sigma}\,R_i^{1-\alpha}(k_i+s)\,, \qquad (3.33)
$$

whenever $m \ge 0$, where $c_3, c_4 \ge 1$ are new constants depending only on n,p,ν,L,σ.

We now prove by induction that

$$
R_{m+1}^{1-\alpha}(k_{m+1}+s) \le [2c_4 + (2H)^n]M \qquad (3.34)
$$

holds for every $m \ge 0$, provided we further reduce the size of $\tilde{R}$; this will prove Lemma 3.4. Choose $\tilde{R}$ small enough in such a way that

$$
d(2R) = \int_0^{2R}[\omega(\varrho)]^{\sigma}\,\frac{d\varrho}{\varrho} \le \int_0^{2\tilde{R}}[\omega(\varrho)]^{\sigma}\,\frac{d\varrho}{\varrho} \le \frac{1}{2c_3} \qquad (3.35)
$$

holds. This choice still makes $\tilde{R}$ depending on n,p,ν,L,σ and $\omega(\cdot)$.

Now, the case $m=0$ of (3.34) follows trivially by (3.22) (recall that here $|G|=0$). On the other hand, let us assume that $R_i^{1-\alpha}(k_i+s) \le [2c_4+(2H)^n]M$ holds whenever $i \le m$, then using (3.33), (3.32) and (3.35) we conclude with

$$
\begin{aligned}
R_{m+1}^{1-\alpha}(k_{m+1}+s) &\le c_4 M + c_3[2c_4+(2H)^n]d(2R)M\\
&\le (2c_4 + 2^{n-1}H^n)M \le [2c_4+(2H)^n]M\,.
\end{aligned}
$$

Therefore (3.34) follows for every integer $m \ge 0$ and Lemma 3.4 is proved.

Step 3: Maximal inequality and conclusion. We let, for every integer $m \ge 0$

$$
\begin{cases}
C_m := R_m^{1-\alpha}A_m = R_m^{1-\alpha}\displaystyle⨍_{B_m}|Du-(Du)_{B_m}|\,d\xi\\
h_m := \displaystyle⨍_{B_m}|Du|\,d\xi
\end{cases} \qquad (3.36)
$$

and now our aim is to prove that, for a constant $c \equiv c(n,p,\nu,L,\sigma)$, it holds that

$$
R_m^{1-\alpha}h_m \le cM\,. \qquad (3.37)
$$

Obviously, (3.26) implies

$$R_m^{1-\alpha}h_m \le R_m^{1-\alpha}k_m + C_m \le cM + C_m\,, \tag{3.38}$$

where M has been defined in (3.27), and therefore we look for a bound on C_m. For this we manipulate (3.18). Let us observe that, using (3.5) and (3.31) and keeping the definition of M in (3.27) in mind, it follows that

$$\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]^{1/(p-1)} \le cR_i^{\alpha-1}\left[\mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(x,2R)\right]^{1/(p-1)} \le cR_i^{\alpha-1}M \tag{3.39}$$

for a constant c depending on n, p, ν, L. By (3.6) we similarly have

$$\begin{aligned}&\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]\left(\fint_{B_i}(|Du|+s)\,d\xi\right)^{2-p}\\ &\le cR_i^{\alpha-1}\left[M_{1-\alpha,R}(|Du|+s)(x)\right]^{2-p}\mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(x,2R) \le cR_i^{\alpha-1}M\,.\end{aligned}$$

Using the last two estimates in (3.18) yields

$$A_{m+1} \le (1/8)A_m + c(k_i+s) + cR_m^{\alpha-1}M\,. \tag{3.40}$$

In turn, using (3.26) to estimate

$$k_m + s \le cR_m^{\alpha-1}M\,, \tag{3.41}$$

inequality (3.40) rewrites as

$$C_{m+1} \le \frac{1}{8}\left(\frac{R_{m+1}}{R_m}\right)^{1-\alpha}C_m + c_5\left(\frac{R_{m+1}}{R_m}\right)^{1-\alpha}M \le (1/8)C_m + c_5M \tag{3.42}$$

with $c_5 \ge 1$ being a constant depending only on n, p, ν, L, σ. Now, by means of the previous relation, we shall prove by induction that

$$C_m \le 2c_5M \tag{3.43}$$

holds whenever $m \ge 0$. When $m = 0$ the previous inequality is a trivial consequence of the definitions:

$$C_0 \le R^{1-\alpha}\fint_{B_R}|Du-(Du)_{B_R}|\,d\xi \le 2M_{1-\alpha,R}(|Du|+s)(x) \le 2M \le 2c_5M\,.$$

On the other hand, assuming (3.43) and then using (3.42) gives

$$C_{m+1} \le (c_5/4)M + c_5M \le 2c_5M\,,$$

and therefore (3.43) follows for every integer $m \ge 0$. In turn, merging (3.43) with (3.38), we conclude with the proof of (3.37) for every integer $m \ge 0$.

Now, let us now observe that, for a new constant c, still depending on n, p, ν, L and σ, it holds that

$$M_{1-\alpha,R}(Du)(x) \le cM\,. \tag{3.44}$$

In fact, let us consider $r \le R$ and determine the integer $i \ge 0$ such that $R_{i+1} < r \le R_i$; we then have

$$r^{1-\alpha}\fint_{B_r}|Du|\,d\xi \le \left(\frac{R_i}{R_{i+1}}\right)^n R_i^{1-\alpha}\fint_{B_i}|Du|\,d\xi \le c(2H)^nR_i^{1-\alpha}h_i \le cM\,, \tag{3.45}$$

and (3.44) follows. Recalling the definition of M in (3.27) we in turn obtain

$$\begin{aligned}M_{1-\alpha,R}(|Du|+s)(x) \quad &\le \quad cR^{1-\alpha}\fint_{B_R}(|Du|+s)\,d\xi + c\left[\mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(x,2R)\right]^{1/(p-1)}\\ &\quad +c\left[M_{1-\alpha,R}(|Du|+s)(x)\right]^{2-p}\mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(x,2R)\,.\end{aligned} \tag{3.46}$$

When $p<2$, by using Young's inequality with conjugate exponents

$$\left(\frac{1}{2-p},\frac{1}{p-1}\right)$$

in (3.11), the last term in the previous inequality can be estimated by

$$(1/2)M_{1-\alpha,R}(|Du|+s)(x)+c\left[\mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(x,2R)\right]^{1/(p-1)}\,,$$

and c depends only on n,p,ν,L and σ, remaining bounded as p approaches 2. Using the last inequality with (3.46), we finally conclude with a preliminary form of (1.39), that is

$$M_{1-\alpha,R}(Du)(x)\leq cR^{1-\alpha}\fint_{B_R}(|Du|+s)\,d\xi+c\left[\mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(x,2R)\right]^{1/(p-1)}\,, \tag{3.47}$$

which is valid whenever $R\leq\tilde{R}$ and $\tilde{R}\equiv\tilde{R}(n,p,\nu,L,\omega(\cdot),\sigma)$ has been determined according to the various restrictions imposed on the size of quantities like for instance $c(n,p,\nu,L,\sigma)[\omega(\tilde{R})]^{\sigma}$. Keep in mind that to estimate the first term appearing in (1.39) it is sufficient to use (2.3). The passage to the general case, i.e. when R is not necessarily smaller than the determined $\tilde{R}$, follows now along the lines of the proof of (1.36), Step 2, modulo the obvious modifications. The final outcome is an estimate where the constant involved depends on $n,p,\nu,L,\sigma,\omega(\cdot)$ and $\operatorname{diam}(\Omega)$.

Remark 3.2. No dependence on $\operatorname{diam}(\Omega)$ appears in the constants of the proofs of (1.36) and (1.39) when the vector field is independent of x, i.e. $a(x,Du)\equiv a(Du)$ as in this case we obviously do not need to operate restrictions on the size of radii. Indeed, we take $\tilde{R}$ small enough to satisfy (3.17) and (3.32); when no dependence on x is allowed these are automatically satisfied for every radius as $\omega(\cdot)=0$.

(**) *Proof of* (1.41). We restart as in Step 2 of the proof of (1.39), by considering the sequence of shrinking balls in (3.13) with $H\geq 4$ to be determined later, and this time defining, in connection to (3.14),

$$\tilde{A}_i:=R_i^{-\alpha}\fint_{B_i}|Du-(Du)_{B_i}|\,d\xi\,. \tag{3.48}$$

Next, we a priori restrict to the case $R\leq 1$, so that $R_i\leq 1$ for every $i\geq 0$; using this fact, and since $p\leq 2$ we may estimate

$$R_i^{-\alpha}\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]^{1/(p-1)}=\left[\frac{|\mu|(B_i)}{R_i^{n-1+\alpha(p-1)}}\right]^{1/(p-1)}\leq\left[\frac{|\mu|(B_i)}{R_i^{n-1+\alpha}}\right]^{1/(p-1)}\,. \tag{3.49}$$

Moreover, from the case $\alpha=1$ of inequality (1.39) we have

$$\begin{aligned}\fint_{B_i}(|Du|+s)\,d\xi &\leq c\left[\mathbf{I}^{|\mu|}_1(x,R)\right]^{1/(p-1)}+c\fint_{B_R}(|Du|+s)\,d\xi\\ &\leq c\left[\mathbf{I}^{|\mu|}_1(x,R)\right]^{1/(p-1)}+cR^{-\alpha}\fint_{B_R}(|Du|+s)\,d\xi=:K \qquad (3.50)\end{aligned}$$

whenever $i\geq 0$, and for a constant $c\equiv c(n,p,\nu,L,\omega(\cdot),\sigma,\operatorname{diam}(\Omega))$. Consequently

$$R_i^{-\alpha}\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]\left(\fint_{B_i}(|Du|+s)\,d\xi\right)^{2-p}\leq\left[\frac{|\mu|(B_i)}{R_i^{n-1+\alpha}}\right]K^{2-p}$$

holds. Using Lemma 3.3 with $\varrho\equiv R_{i+1}$ and $2R\equiv R_i$, multiplying both sides of the resulting inequality (3.2) by $R_{i+1}^{-\alpha}$, and finally using (3.49), (3.50) and the last two inequalities, we have

$$\tilde{A}_{i+1}\leq c_1\left(\frac{R_{i+1}}{R_i}\right)^{\alpha_M-\alpha}\tilde{A}_i+c\left(\frac{R_i}{R_{i+1}}\right)^{n+\alpha}\left[\frac{|\mu|(B_i)}{R_i^{n-1+\alpha}}\right]^{1/(p-1)}$$

$$+c\left(\frac{R_i}{R_{i+1}}\right)^{n+\alpha}\left[\frac{|\mu|(B_i)}{R_i^{n-1+\alpha}}\right]K^{2-p}+c\left(\frac{R_i}{R_{i+1}}\right)^{n+\alpha}\left(\frac{[\omega(R_i)]^{\sigma}}{R_i^{\alpha}}\right)K \tag{3.51}$$

valid for every $i \geq 0$, where $c \equiv c(n,p,\nu,L,\omega(\cdot),\sigma,\operatorname{diam}(\Omega))$ and moreover $c_1 \equiv c_1(n,p,\nu,L)$. Now we choose $H \equiv H(n,p,\nu,L,\tilde{\alpha}) > 1$ large enough in order to obtain

$$c_1\left(\frac{R_{i+1}}{R_i}\right)^{\alpha_M-\alpha}=c_1\left(\frac{1}{2H}\right)^{\alpha_M-\alpha}\leq c_1\left(\frac{1}{2H}\right)^{\alpha_M-\tilde{\alpha}}\leq \frac{1}{4}\,.$$

By using Young's inequality (3.11) when $p<2$ we estimate

$$\left[\frac{|\mu|(B_i)}{R_i^{n-1+\alpha}}\right]K^{2-p}\leq\left[\frac{|\mu|(B_i)}{R_i^{n-1+\alpha}}\right]^{1/(p-1)}+cK\leq [M_{1-\alpha,R}(\mu)(x)]^{1/(p-1)}(x)+cK$$

and using assumption (1.40)

$$\frac{[\omega(R_i)]^{\sigma}}{R_i^{\alpha}}\leq\frac{[\omega(R_i)]^{\sigma}}{R_i^{\tilde{\alpha}}}\leq S\,.$$

The last three estimates used in (3.51) give

$$\tilde{A}_{i+1}\leq(1/2)\tilde{A}_i+c_6\left\{[M_{1-\alpha,R}(\mu)(x)]^{1/(p-1)}+K\right\} \tag{3.52}$$

for every $i\geq 0$, where $c_6\equiv c_6(n,p,\nu,L,\omega(\cdot),\sigma,\operatorname{diam}(\Omega),S)$. Iterating the previous relation, it is easy to prove that

$$\tilde{A}_i\leq 2^{-i}\tilde{A}_0+c_6\sum_{j=0}^{i-1}2^{-j}\left\{[M_{1-\alpha,R}(\mu)(x)]^{1/(p-1)}+K\right\} \tag{3.53}$$

holds for every $i\geq 1$. In turn, recalling (3.48), (3.50) and that $R\leq 1$, we also have

$$\begin{aligned}\sup_{i\geq 0}\tilde{A}_i &\leq cR^{-\alpha}\fint_{B_R}|Du|\,dx+c_6\left\{[M_{1-\alpha,R}(\mu)(x)]^{1/(p-1)}+K\right\}\\ &\leq c\left\{[M_{1-\alpha,R}(\mu)(x)]^{1/(p-1)}+K\right\}\end{aligned}$$

with a constant c depending only on $n,p,\nu,L,\omega(\cdot),\tilde{\alpha},S$ and $\operatorname{diam}(\Omega)$. We now observe that, in view of the previous inequality and of the definition of K in (3.50), and recalling that H depends only on $n,p,\nu,L,\tilde{\alpha}$, in order to complete the proof for the case $R\leq 1$ it is sufficient to prove that

$$M^{\#}_{\alpha,R}(Du)(x)\leq c(n)H^{n+\alpha}\left[\sup_{i\geq 0}\tilde{A}_i\right]\,. \tag{3.54}$$

To this aim, with $\varrho\in(0,R]$, let $i\in\mathbb{N}$ be such that $R_{i+1}<\varrho\leq R_i$; then it holds

$$\begin{aligned}\varrho^{-\alpha}\fint_{B_\varrho}|Du-(Du)_{B_\varrho}|\,d\xi &\leq c\varrho^{-\alpha}\fint_{B_\varrho}|Du-(Du)_{B_i}|\,d\xi\\ &\leq c(2H)^{n+\alpha}R_i^{-\alpha}\fint_{B_i}|Du-(Du)_{B_i}|\,d\xi\\ &\leq c(n)H^{n+\alpha}\left[\sup_{i\geq 0}\tilde{A}_i\right]\,,\end{aligned}$$

proving (3.54), and in turn (1.41) follows in the case $R\leq 1$. We finally remove the constraint $\tilde{R}\leq 1$ as already done for estimate (1.35). The proof is complete. $\square$

3.2. **The case $p \geq 2$ and proof of Theorem 1.8.** The proof follows the one given for Theorem 1.9, and we shall give the suitable modifications, keeping the notation thereby introduced. The proof of (1.35) is exactly the same as for the case $p \leq 2$, as already noticed above. As for the proof of (1.36) we start observing that we can restart from Step 2, as the content of Step 1 also works for the case $p \geq 2$. Lemma 3.4 must be now replaced by the following:

Lemma 3.5. *There exists a constant c, depending only on n, p, ν, L and a positive radius $\tilde{R}$ depending only on n, p, ν, L and $\omega(\cdot)$, but both independent of α, such that*

$$R_m^{1-\alpha} k_{m+1} \leq cR^{1-\alpha} \fint_{B_R} (|Du| + s)\, d\xi + c\mathbf{W}^{\mu}_{1-\alpha(p-1)/p,p}(x, 2R) =: cM \tag{3.55}$$

holds for every integer $m \geq 0$ and whenever $R \leq \tilde{R}$.

The proof is essentially the same as for the case $p < 2$, but we replace (3.30) by

$$\begin{aligned}
\sum_{i=0}^{\infty} \left[\frac{|\mu|(B_i)}{R_i^{n-p+\alpha(p-1)}}\right]^{1/(p-1)} &\leq c\int_R^{2R} \left[\frac{|\mu|(B(x_0,\varrho))}{\varrho^{n-p+\alpha(p-1)}}\right]^{1/(p-1)} \frac{d\varrho}{\varrho} \\
&\quad + c\sum_{i=0}^{\infty} \int_{R_{i+1}}^{R_i} \left[\frac{|\mu|(B(x_0,\varrho))}{\varrho^{n-p+\alpha(p-1)}}\right]^{1/(p-1)} \frac{d\varrho}{\varrho} \\
&\leq c\mathbf{W}^{\mu}_{1-\alpha(p-1)/p,p}(x, 2R)\,,
\end{aligned} \tag{3.56}$$

with $c \equiv c(n, p, \nu, L)$. We therefore arrive at (3.33) with the new definition of M in (3.55). We also note that everywhere $[\omega(\cdot)]^{\sigma}$ is replaced by $[\omega(\cdot)]^{2/p}$. From this point on the rest of the proof of the Lemma is as for the case $p \leq 2$. We then proceed with Step 3; we adopt the definitions in (3.36) and keeping (3.56) in mind we replace (3.39) by the new estimate

$$\left[\frac{|\mu|(B_i)}{R_i^{n-1}}\right]^{1/(p-1)} \leq cR_i^{\alpha-1}\mathbf{W}^{\mu}_{1-\alpha(p-1)/p,p}(x, 2R) \leq cR_i^{\alpha-1}M$$

with M now being defined in (3.55). With this estimate and (3.18) we find once again the validity of (3.42) and from this point on the proof follows exactly the one of Theorem 1.9.

Finally, we provide the modifications for the proof of (1.38). First, we notice that (3.50) has to be replaced by

$$\fint_{B_r} (|Du| + s)\, d\xi \leq c\mathbf{W}^{\mu}_{1/p,p}(x, R) + c\fint_{B_R} (|Du| + s)\, d\xi$$

as we are now using estimate (1.36) with $\alpha = 1$. Taking into account the first equality in (3.49) and using Lemma 3.3 we arrive at the following analog of (3.51):

$$\begin{aligned}
\varrho^{-\alpha} \fint_{B_\varrho} |Du - (Du)_{B_\varrho}|\, dx &\leq c_1 \left(\frac{\varrho}{r}\right)^{\alpha_M - \alpha} r^{-\alpha} \fint_{B_r} |Du - (Du)_{B_r}|\, dx \\
&\quad + c\left(\frac{r}{\varrho}\right)^{n+\alpha} \left[\frac{|\mu|(B_r)}{r^{n-1+\alpha(p-1)}}\right]^{1/(p-1)} \\
&\quad + c\left(\frac{r}{\varrho}\right)^{n+\alpha} \frac{[\omega(r)]^{2/p}}{r^{\alpha}} \left\{\mathbf{W}^{\mu}_{1/p,p}(x, R) + c\fint_{B_R} (|Du| + s)\, d\xi\right\},
\end{aligned}$$

which is valid for every $\varrho \leq r/2 \leq R/2$. Here $c \equiv c(n, p, \nu, L, \omega(\cdot), \operatorname{diam}(\Omega))$ and $c_1 \equiv c_1(n, p, \nu, L)$. Using this last inequality, taking $\varrho = R_{i+1}$ and $r = R_i$, and proceeding as after (3.51) estimate (1.38) follows too and the proof of Theorem 1.8 is complete. $\square$

We finally proceed with further maximal estimates concerning the case when the equation has measurable coefficients, and therefore Hölder continuity of solutions with any exponent is not expected even in the case of zero right hand side.

Proposition 3.1. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to the equation with measurable coefficient* (1.1)*, and let* (1.2) *hold with $p > 2 - 1/n$. Then estimate* (1.35) *holds uniformly in $\alpha \in [0, \tilde{\alpha}]$, whenever $\tilde{\alpha} < \alpha_m$, for a constant c depending only n, p, ν, L and $\tilde{\alpha}$.*

Proof. The proof follows the lines of the proof of (1.35) of Theorem 1.8, at the beginning of this section. The main difference is that we have to use Lemma 3.1 instead of Lemma 3.2. Here we give the suitable modifications. Instead of (3.7) we have that

$$
\begin{aligned}
\varrho^{1-\alpha} \fint_{B_\varrho} (|Du| + s)\, dx &\le c_1 \left(\frac{\varrho}{r}\right)^{\alpha_m - \alpha} r^{1-\alpha} \fint_{B_r} (|Du| + s)\, dx \\
&\quad + c \left(\frac{r}{\varrho}\right)^{n-1+\alpha} \left[\frac{|\mu|(B_r)}{r^{n-p+\alpha(p-1)}}\right]^{1/(p-1)} \\
&\quad + c\chi_{\{p<2\}} \left(\frac{r}{\varrho}\right)^{n-1+\alpha} \left[\frac{|\mu|(B_r)}{r^{n-p+\alpha(p-1)}}\right] \left(r^{1-\alpha} \fint_{B_r} (|Du| + s)\, dx\right)^{2-p}, \qquad (3.57)
\end{aligned}
$$

holds whenever $\varrho \le r \le R$. Next, we choose this time $H \equiv H(n, p, \nu, L, \tilde{\alpha})$ large enough in order to have

$$
c_1 \left(\frac{1}{H}\right)^{\alpha_m - \alpha} \le c_1 \left(\frac{1}{H}\right)^{\alpha_m - \tilde{\alpha}} \le \frac{1}{4}\,.
$$

Note that here we are using the fact that $\tilde{\alpha} < \alpha_m$. By using this relation in (3.57) we arrive at (3.10) and the rest of the proof proceeds as for Theorem 1.8. Note that here we do not need to choose small radii, therefore in the final estimate no dependence on $\operatorname{diam}(\Omega)$ occurs. $\square$

4. Endpoint estimates and Theorems 1.1, 1.4, 1.6 and 1.10

In this section we give the proof of a certain number of theorems characterized by the fact of featuring "endpoint estimates". This means we shall prove estimates with fractional potentials depending on α, and catching the borderline case $\alpha = 0$, with all the constants involved in the estimates *being stable*, i.e. remaining bounded when α approaches zero. This time we start with the gradient estimates.

4.1. Proof of Theorems 1.4 and 1.6.

Before the proof let us state a lemma whose proof we include for the sake of completeness.

Lemma 4.1. *Let μ be a Borel measure with finite total mass on Ω; let $\gamma \in (0,1)$, $\beta \in [0, n]$, $p > 1$, and $B_R \subset \Omega$. Then*

$$
[M_{\beta,\gamma R}(\mu)(x)]^{1/(p-1)} \le \frac{\max\{\gamma^{(\beta-n)/(p-1)}, 1\}}{(-\log \gamma)|B_1|^{1/(p-1)}} \mathbf{W}^{\mu}_{\beta/p,p}(x, R)
$$

and

$$
M_{\beta,\gamma R}(\mu)(x) \le \frac{\max\{\gamma^{\beta-n}, 1\}}{(-\log \gamma)|B_1|} \mathbf{I}^{|\mu|}_{\beta}(x, R)
$$

hold.

Proof. For all $\varepsilon > 0$ there is $0 < r \le R$ such that

$$
M_{\beta,\gamma R}(\mu)(x) \le |B_1|^{-1} \frac{|\mu|(B_{\gamma r})}{(\gamma r)^{n-\beta}} + \varepsilon.
$$

We have

$$\begin{aligned}\frac{|\mu|(B_{\gamma r})}{(\gamma r)^{n-\beta}} &= \left[\left(\frac{|\mu|(B_{\gamma r})}{(\gamma r)^{n-\beta}}\right)^{1/(p-1)}\frac{1}{-\log\gamma}\int_{\gamma r}^{r}\frac{d\varrho}{\varrho}\right]^{p-1}\\ &\leq \frac{\max\{\gamma^{\beta-n},1\}}{(-\log\gamma)^{p-1}}\left[\int_{\gamma r}^{r}\left(\frac{|\mu|(B_\varrho)}{\varrho^{n-\beta}}\right)^{1/(p-1)}\frac{d\varrho}{\varrho}\right]^{p-1}\\ &\leq \frac{\max\{\gamma^{\beta-n},1\}}{(-\log\gamma)^{p-1}}\left[\mathbf{W}^{\mu}_{\beta/p,p}(x,R)\right]^{p-1}\end{aligned}$$

and thus the first inequality stated in the lemma follows, while the second also follows via a completely similar argument. □

Proof of Theorem 1.6. In the rest of the proof the points x, y and the radius R will be those fixed in the statement of the theorem, therefore $x, y \in B_{R/4}$. We go back to the proof of Theorem 1.9, proof of (1.39), Step 1, and we adopt the notation introduced there. We restart from estimate (3.25) that we apply to a ball with a general radius r (i.e. we change notation and denote r instead of R in (3.25)) to be determined in a few lines; in particular $r_i := r/(2H)^i$ and $B_i = B(x, r_i)$. Moreover, we shall use the restriction $r \leq R/2$, where R is now the radius appearing in the statement of the Theorem, so that in any case $B(x,r) \subset B_R$. From the case $\alpha = 1$ of inequality (1.39) we have that

$$\fint_{B_i}(|Du|+s)\,d\xi \leq c\left[\mathbf{I}^{|\mu|}_1(x,r)\right]^{1/(p-1)} + c\fint_{B(x,r)}(|Du|+s)\,d\xi =: K(x,r)$$

holds whenever $i \in \mathbb{N}$. By using the last inequality, easy manipulations to (3.25) then lead to

$$\begin{aligned}k_{m+1} &\leq c\fint_{B_r}|Du-(Du)_{B(x,r)}|+|Du-G|)\,d\xi\\ &\quad +cr^\alpha\sum_{i=0}^{m}\left[\frac{|\mu|(B_i)}{r_i^{n-1+\alpha(p-1)}}\right]^{1/(p-1)} + c\chi_{\{p<2\}}r^\alpha\sum_{i=0}^{m}\left[\frac{|\mu|(B_i)}{r_i^{n-1+\alpha}}\right][K(x,r)]^{2-p}\\ &\quad +cr^\alpha\sum_{i=0}^{m}\frac{[\omega(r_i)]^\sigma}{r_i^\alpha}\left[K(x,r)+s+|G|\right].\end{aligned}\tag{4.1}$$

Observe now that, as $p \leq 2$ in the case under examination, estimating as in (3.31) we obtain

$$\begin{aligned}\sum_{i=0}^{m}\left[\frac{|\mu|(B_i)}{r_i^{n-1+\alpha(p-1)}}\right]^{1/(p-1)} &\leq r^{\frac{\alpha(2-p)}{p-1}}\sum_{i=0}^{m}\left[\frac{|\mu|(B_i)}{r_i^{n-1+\alpha}}\right]^{1/(p-1)}\\ &\leq c\left[\mathbf{I}^{|\mu|}_{1-\alpha}(x,2r)\right]^{1/(p-1)}\end{aligned}$$

for a constant c depending only on $n, p, \nu, L, \mathrm{diam}(\Omega), \sigma, H$ and therefore, as in the proof of Theorem 1.9, ultimately depending only $n, p, \nu, L, \mathrm{diam}(\Omega)$. Similarly

$$\sum_{i=0}^{m}\frac{[\omega(r_i)]^\sigma}{r_i^\alpha} \leq c\int_0^{2r}\frac{[\omega(\varrho)]^\sigma}{\varrho^\alpha}\frac{d\varrho}{\varrho} \leq cR^{\tilde{\alpha}-\alpha}\int_0^{2r}\frac{[\omega(\varrho)]^\sigma}{\varrho^{\tilde{\alpha}}}\frac{d\varrho}{\varrho} \leq cS$$

again holds for a constant depending only on $n, p, \nu, L, \mathrm{diam}(\Omega), \sigma$. Merging the last two inequalities with (4.1) we have

$$\begin{aligned}k_{m+1} &\leq c\fint_{B(x,r)}(|Du-(Du)_{B(x,r)}|+|Du-G|)\,d\xi + cr^\alpha\left[\mathbf{I}^{|\mu|}_{1-\alpha}(x,R)\right]^{1/(p-1)}\\ &\quad +cr^\alpha\left[\mathbf{I}^{|\mu|}_{1-\alpha}(x,2r)\right][K(x,r)]^{2-p} + cr^\alpha\left[K(x,r)+s+|G|\right],\end{aligned}$$

where $c \equiv c(n, p, \nu, L, S, \operatorname{diam}(\Omega), \sigma)$. Letting $m \to \infty$ in the previous estimate, and applying Young's inequality with conjugate exponents

$$\left(\frac{1}{2-p}, \frac{1}{p-1}\right)$$

in (3.11) when $p < 2$, we deduce

$$\begin{aligned} |Du(x) - G| &= \lim_{m\to\infty} k_{m+1} \le c \fint_{B(x,r)} (|Du - (Du)_{B(x,r)}| + |Du - G|)\, d\xi \\ &\quad + cr^{\alpha} \left[\mathbf{I}^{|\mu|}_{1-\alpha}(x, R)\right]^{1/(p-1)} + cr^{\alpha} \left[K(x, r) + s + |G|\right] . \end{aligned}$$

Arguing in the same way for the point y we also obtain

$$\begin{aligned} |Du(y) - G| &\le c \fint_{B(y,r)} (|Du - (Du)_{B(y,r)}| + |Du - G|)\, d\xi \\ &\quad + cr^{\alpha} \left[\mathbf{I}^{|\mu|}_{1-\alpha}(y, R)\right]^{1/(p-1)} + cr^{\alpha} \left[K(y, r) + s + |G|\right] , \end{aligned}$$

and therefore, summing up the last two relations gives

$$\begin{aligned} |Du(x) - Du(y)| &\le c \fint_{B(x,r)} (|Du - (Du)_{B(x,r)}| + |Du - G|)\, d\xi \\ &\quad + c \fint_{B(y,r)} (|Du - (Du)_{B(y,r)}| + |Du - G|)\, d\xi \\ &\quad + cr^{\alpha} \left[\mathbf{I}^{|\mu|}_{1-\alpha}(x, R) + \mathbf{I}^{|\mu|}_{1-\alpha}(y, R)\right]^{1/(p-1)} \\ &\quad + cr^{\alpha} \left[K(x, r) + K(y, r) + s + |G|\right] . \end{aligned} \tag{4.2}$$

We now fix G and r by taking

$$G := (Du)_{B(x,3r)} , \qquad r := |x - y|/2 , \tag{4.3}$$

so that $B(y, r) \subset B(x, 3r)$ and therefore

$$\begin{aligned} &\fint_{B(x,r)} (|Du - (Du)_{B(x,r)}| + |Du - G|)\, d\xi \\ &+ c \fint_{B(y,r)} (|Du - (Du)_{B(y,r)}| + |Du - G|)\, d\xi \\ &\le c(n) \fint_{B(x,3r)} |Du - (Du)_{B(x,3r)}|\, d\xi . \end{aligned} \tag{4.4}$$

Now, notice that as we are assuming that the initial ball B_R is such that $x, y \in B_{R/4}$, then we have necessarily $|x - y| \le R/2$ so that now $r \le R/4$ and $B(x, 3r) \subset B(x, 3R/4) \subset B_R$. Therefore we use (1.41) to estimate

$$\begin{aligned} &\fint_{B(x,3r)} |Du - (Du)_{B(x,3r)}|\, d\xi \le cr^{\alpha} M^{\#}_{\alpha,3R/4}(Du)(x) \\ &\quad \le cr^{\alpha} \left[M_{1-\alpha,3R/4}(\mu)(x)\right]^{1/(p-1)} + cr^{\alpha} \left[\mathbf{I}^{|\mu|}_{1}(x, 3R/4)\right]^{1/(p-1)} \\ &\qquad + c\left(\frac{r}{R}\right)^{\alpha} \fint_{B(x,3R/4)} (|Du| + s)\, d\xi \\ &\quad \le cr^{\alpha} \left[M_{1-\alpha,3R/4}(\mu)(x)\right]^{1/(p-1)} + cR^{\alpha/(p-1)} r^{\alpha} \left[\mathbf{I}^{|\mu|}_{1-\alpha}(x, R)\right]^{1/(p-1)} \\ &\qquad + c\left(\frac{r}{R}\right)^{\alpha} \fint_{B_R} (|Du| + s)\, d\xi . \end{aligned}$$

We remark that the use of (1.41) is justified here as (1.40) is a consequence of (1.27) for a new constant depending on the number S used in (1.27), on $\omega(\cdot)$ and on $\operatorname{diam}(\Omega)$, so that the final dependence on the constants remains unvaried; see Remark 4.1 below. The last two estimates and (4.2) give

$$\begin{aligned} |Du(x)-Du(y)| \quad &\leq \quad cr^{\alpha}\left[M_{1-\alpha,3R/4}(\mu)(x)\right]^{1/(p-1)} + c\left(\frac{r}{R}\right)^{\alpha} \fint_{B_R} (|Du|+s)\,d\xi \\ &\quad + cr^{\alpha}\left[\mathbf{I}^{|\mu|}_{1-\alpha}(x,R)+\mathbf{I}^{|\mu|}_{1-\alpha}(y,R)\right]^{1/(p-1)} \\ &\quad + cr^{\alpha}\left[K(x,r)+K(y,r)+s+|G|\right] \end{aligned} \tag{4.5}$$

for a constant c depending on $n, p, \nu, L, \tilde{\alpha}, \omega(\cdot), S, \operatorname{diam}(\Omega)$ and σ; the dependence on $\tilde{\alpha} \geq \alpha$ comes from the use of inequality (1.41). We devote ourselves to estimate the various terms involved in the right hand side of (4.5). We again use inequality (1.39) with $\alpha = 1$ and recall that $B(x, 3R/4) \subset B_R$ to estimate as follows:

$$\begin{aligned} K(x,r) \quad &\leq \quad c\left[\mathbf{I}^{|\mu|}_{1}(x,3R/4)\right]^{1/(p-1)} + c\fint_{B(x,3R/4)} (|Du|+s)\,d\xi \\ &\leq \quad cR^{\alpha/(p-1)}\left[\mathbf{I}^{|\mu|}_{1-\alpha}(x,R)\right]^{1/(p-1)} + cR^{-\alpha}\fint_{B_R} (|Du|+s)\,d\xi \end{aligned}$$

with a similar estimate being obviously true for $K(y, R)$, i.e.

$$K(y,r) \leq cR^{\alpha/(p-1)}\left[\mathbf{I}^{|\mu|}_{1-\alpha}(y,R)\right]^{1/(p-1)} + cR^{-\alpha}\fint_{B_R} (|Du|+s)\,d\xi\,.$$

Moreover, again by (1.39), we have

$$\begin{aligned} |G| \quad &\leq \quad c\fint_{B(x,3r)} |Du|\,d\xi \\ &\leq \quad cR^{\alpha/(p-1)}\left[\mathbf{I}^{|\mu|}_{1-\alpha}(x,R)\right]^{1/(p-1)} + cR^{-\alpha}\fint_{B_R} (|Du|+s)\,d\xi\,. \end{aligned}$$

Now, by Lemma 4.1 with the choice $\gamma = 3/4$, we obtain

$$M_{1-\alpha,3R/4}(\mu)(x) \leq c(n,\alpha)\mathbf{I}^{|\mu|}_{1-\alpha}(x,R)$$

Using the last four inequalities in (4.5) we conclude with

$$|Du(x)-Du(y)| \leq cr^{\alpha}\left[\mathbf{I}^{|\mu|}_{1-\alpha}(x,R)+\mathbf{I}^{|\mu|}_{1-\alpha}(y,R)\right]^{1/(p-1)} + c\left(\frac{r}{R}\right)^{\alpha}\fint_{B_R} (|Du|+s)\,d\xi$$

from which (1.28) follows taking into account (4.3). □

Remark 4.1 (Trivial)**.** Assumption (1.27) implies (1.40) for a new constant depending on S considered in (1.27), and $\operatorname{diam}(\Omega)$. In fact, observe that if $0 < r \leq \operatorname{diam}(\Omega)/2$ as $\omega(\cdot)$ is nondecreasing then

$$\frac{[\omega(r)]^{\sigma}}{r^{\tilde{\alpha}}} \leq \frac{2^{\tilde{\alpha}}}{\log 2}\int_r^{2r} \frac{[\omega(\varrho)]^{\sigma}}{\varrho^{\tilde{\alpha}}}\frac{d\varrho}{\varrho} \leq cS$$

and therefore

$$\sup_r \frac{[\omega(r)]^{\sigma}}{r^{\tilde{\alpha}}} \leq \frac{2^{\tilde{\alpha}}[\omega(\operatorname{diam}(\Omega))]^{\sigma}}{[\operatorname{diam}(\Omega)]^{\tilde{\alpha}}} + cS\,.$$

Proof of Theorem 1.4. The proof follows the one for Theorem 1.6, and we report the necessary modifications. First, instead of estimate (1.39) we obviously have to use (1.36), with $\alpha = 1$; consequently, the definition of $K(\cdot, r)$ changes in

$$K(x,r) := \mathbf{W}^{\mu}_{1/p,p}(x,r) + c\fint_{B(x,r)} (|Du|+s)\,d\xi\,,$$

while in (4.1), and everywhere later on, instead of $[\omega(\cdot)]^{\sigma}$ it appears $[\omega(\cdot)]^{2/p}$. Yet, as already in (3.30) and (3.56), we estimate

$$\sum_{i=0}^{m}\left[\frac{|\mu|(B_i)}{r_i^{n-1+\alpha(p-1)}}\right]^{1/(p-1)} \leq c\mathbf{W}^{\mu}_{1-(1+\alpha)(p-1)/p,p}(x,R)\,,$$

thereby obtaining

$$\begin{aligned} k_{m+1} \quad \leq \quad & c \fint_{B(x,r)} (|Du-(Du)_{B(x,r)}|+|Du-G|)\,d\xi \\ & +cr^{\alpha}\mathbf{W}^{\mu}_{1-(1+\alpha)(p-1)/p,p}(x,R)+cr^{\alpha}\left[K(x,r)+s+|G|\right] \end{aligned}$$

as a consequence of the previous estimates and (4.1). Writing the similar relation for y and proceeding as in the proof of Theorem 1.6 we arrive eventually at

$$\begin{aligned} |Du(x)-Du(y)| \quad \leq \quad & c\fint_{B(x,r)} (|Du-(Du)_{B(x,r)}|+|Du-G|)\,d\xi \\ & +c\fint_{B(y,r)} (|Du-(Du)_{B(y,r)}|+|Du-G|)\,d\xi \\ & +cr^{\alpha}\mathbf{W}^{\mu}_{1-(1+\alpha)(p-1)/p,p}(x,R)+cr^{\alpha}\mathbf{W}^{\mu}_{1-(1+\alpha)(p-1)/p,p}(y,R) \\ & +cr^{\alpha}\left[K(x,r)+K(y,r)+s+|G|\right]\,. \end{aligned} \tag{4.6}$$

We make the same choice as in (4.3) and estimate as in (4.4); eventually, by this time using (1.38) - with $\alpha=1$ - we have

$$\begin{aligned} & \fint_{B(x,3r)} |Du-(Du)_{B(x,3r)}|\,d\xi \leq cr^{\alpha}M^{\#}_{\alpha,3R/4}(Du)(x) \\ & \leq cr^{\alpha}\left[M_{1-\alpha(p-1),3R/4}(\mu)(x)\right]^{1/(p-1)}+cr^{\alpha}\mathbf{W}^{\mu}_{1/p,p}(x,3R/4) \\ & \qquad +c\left(\frac{r}{R}\right)^{\alpha}\fint_{B_R}(|Du|+s)\,d\xi \\ & \leq cr^{\alpha}\left[M_{1-\alpha(p-1),3R/4}(\mu)(x)\right]^{1/(p-1)}+cR^{\alpha}r^{\alpha}\mathbf{W}^{\mu}_{1-(1+\alpha)(p-1)/p,p}(x,R) \\ & \qquad +c\left(\frac{r}{R}\right)^{\alpha}\fint_{B_R}(|Du|+s)\,d\xi\,. \end{aligned}$$

Finally, we estimate

$$\begin{aligned} K(x,r)+|G| \quad \leq \quad & c\mathbf{W}^{\mu}_{1/p,p}(x,3R/4)+c\fint_{B(x,3R/4)}(|Du|+s)\,d\xi \\ \leq \quad & cR^{\alpha}\mathbf{W}^{\mu}_{1-(1+\alpha)(p-1)/p,p}(x,R)+cR^{-\alpha}\fint_{B_R}(|Du|+s)\,d\xi\,, \end{aligned}$$

while Lemma 4.1, with the choice $\gamma=3/4$, implies

$$\left[M_{1-\alpha,3R/4}(\mu)(x)\right]^{1/(p-1)} \leq c(n,p,\alpha)\mathbf{W}^{\mu}_{1-(1+\alpha)(p-1)/p,p}(x,R)\,.$$

The last three inequalities - and the analog for $K(y,R)$ - used in (4.2) give the assertion. $\square$

4.2. **Proof of Theorems 1.1 and 1.10.**

Proof of Theorem 1.1. Let r be such that $r\leq R/2$; we take a geometric sequence $\{r_i\}$ of radii, whose spread $4H>1$ will be chosen later as a function of the parameters n,p,ν,L. The points $x,y\in\Omega$ are those in the statement of the theorem. More precisely, with $H\geq 1$, we set

$$B_i := B(x,r/(4H)^i) := B(x,r_i)\,, \qquad \tilde{B}_i := B(x,r_i/2)\,, \quad i\geq 0\,, \tag{4.7}$$

so that $B_{i+1} \subset \tilde{B}_i \subset B_i$ for every $i \geq 0$, and moreover

$$A_i := \fint_{B_i} |u - (u)_{B_i}|\, d\xi\,, \qquad\qquad k_i := |(u)_{B_i}|\,.$$

Then we start observing that, by the Poincaré inequality and Lemma 3.1 - applied with $\varrho \equiv r_{i+1}$ and $R \equiv r_i/2$ - we have, after some easy manipulations

$$\begin{aligned} A_{i+1} &\leq cr_{i+1} \fint_{B_{i+1}} |Du|\, d\xi \\ &\leq c_1 c \left(\frac{r_{i+1}}{r_i}\right)^{\alpha_m} r_i \fint_{\tilde{B}_i} (|Du| + s)\, d\xi + c\left(\frac{r_i}{r_{i+1}}\right)^n \left[\frac{|\mu|(B_i)}{r_i^{n-p}}\right]^{1/(p-1)} \\ &\quad + c\chi_{\{p<2\}} \left(\frac{r_i}{r_{i+1}}\right)^n \left[\frac{|\mu|(B_i)}{r_i^{n-p}}\right] \left(r_i \fint_{\tilde{B}_i} (|Du| + s)\, d\xi\right)^{2-p}. \end{aligned}$$

To estimate to the integrals appearing in the previous inequality we use Caccioppoli's inequality (4.13) below in the form

$$r_i \fint_{\tilde{B}_i} (|Du| + s)\, d\xi \leq c(A_i + r_i s) + c\left[\frac{|\mu|(B_i)}{r_i^{n-p}}\right]^{1/(p-1)}$$

so that the last two inequalities together give

$$\begin{aligned} A_{i+1} &\leq c_2 \left(\frac{r_{i+1}}{r_i}\right)^{\alpha_m} A_i + cH^n \left[\frac{|\mu|(B_i)}{r_i^{n-p}}\right]^{1/(p-1)} \\ &\quad + c\chi_{\{p<2\}} H^n \left[\frac{|\mu|(B_i)}{r_i^{n-p}}\right] (A_i + r_i s)^{2-p} + cr_i s\,. \end{aligned}$$

Applying Young's inequality (3.11) when $p < 2$ gives, for $\varepsilon \in (0, 1)$

$$A_{i+1} \leq c_2 \left[\left(\frac{1}{H}\right)^{\alpha_m} + \varepsilon\right] A_i + c_3 \left(H^n + H^{n/(p-1)}\right) \left[\frac{|\mu|(B_i)}{r_i^{n-p}}\right]^{1/(p-1)} + cr_i s\,.$$

In the previous inequality constants c, c_2 depend only on n, p, ν, L while c_3 depends on such quantities and on ε, too. In view of this we choose $H \equiv H(n, p, \nu, L)$ large enough and $\varepsilon \equiv \varepsilon(n, p, \nu, L)$ small enough in order to have

$$\left(\frac{1}{H}\right)^{\alpha_m} + \varepsilon \leq \frac{1}{2c_2}$$

and this in turn also determines the value of the constant c_3; notice that here we have used that α_m depends only on n, p, ν, L. All in all we have proved

$$A_{i+1} \leq (1/2)A_i + c\left[\frac{|\mu|(B_i)}{r_i^{n-p}}\right]^{1/(p-1)} + cr_i s$$

for $c \equiv c(n, p, \nu, L)$ and for all integers $i \geq 0$. We may now proceed exactly as after (3.18), thereby getting relations analog to (3.20) and (3.21), that are

$$\sum_{i=1}^{m} A_i \leq A_0 + c\sum_{i=0}^{m-1} \left[\frac{|\mu|(B_i)}{r_i^{n-p}}\right]^{1/(p-1)} + crs \tag{4.8}$$

and

$$k_{m+1} \leq cA_0 + ck_0 + c\sum_{i=0}^{m-1} \left[\frac{|\mu|(B_i)}{r_i^{n-p}}\right]^{1/(p-1)} + crs\,, \tag{4.9}$$

respectively; the previous two inequalities hold whenever $m \geq 0$. In turn, estimating as in (3.56), the previous relation implies

$$\begin{aligned} k_{m+1} &\leq cA_0 + ck_0 + cr^{\alpha} \sum_{i=0}^{m-1} \left[\frac{|\mu|(B_i)}{r_i^{n-p+\alpha(p-1)}} \right]^{1/(p-1)} + crs \\ &\leq c \fint_{B(x,r)} (|u| + rs)\, d\xi + cr^{\alpha} \mathbf{W}^{\mu}_{1-\alpha(p-1)/p,p}(x,R)\,. \end{aligned}$$

Letting $m \to \infty$ now yields

$$|u(x)| = \lim_{m\to\infty} k_m \leq c \fint_{B(x,r)} (|u| + rs)\, d\xi + cr^{\alpha} \mathbf{W}^{\mu}_{1-\alpha(p-1)/p,p}(x,R)\,.$$

We observe that if u solves (1.1) then $u - g$ is still a solution to the same equation whenever $g \in \mathbb{R}$; therefore we gain

$$|u(x) - g| \leq c \fint_{B(x,r)} (|u-g| + rs)\, d\xi + cr^{\alpha} \mathbf{W}^{\mu}_{1-\alpha(p-1)/p,p}(x,R)\,,$$

for a constant c depending only on n, p, ν, L. Writing the previous relation for y i.e.

$$|u(y) - g| \leq c \fint_{B(y,r)} (|u-g| + rs)\, d\xi + cr^{\alpha} \mathbf{W}^{\mu}_{1-\alpha(p-1)/p,p}(y,R)\,,$$

and summing up the last two inequalities, yields

$$\begin{aligned} |u(x) - u(y)| &\leq c \fint_{B(x,r)} |u-g|\, d\xi + c \fint_{B(y,r)} |u-g|\, d\xi \\ &\quad + cr^{\alpha} \left[\mathbf{W}^{\mu}_{1-\alpha(p-1)/p,p}(x,R) + \mathbf{W}^{\mu}_{1-\alpha(p-1)/p,p}(y,R) \right] + crs\,. \end{aligned} \tag{4.10}$$

Then, similarly to the proof of Theorem 1.6, we take $g = (u)_{B(x,3r)}$ and $r = |x-y|/2$ and therefore, as with such a choice it follows that $B(y,r) \subset B(x,3r)$, we can estimate

$$\fint_{B(x,r)} |u-g|\, d\xi + \fint_{B(y,r)} |u-g|\, d\xi \leq 6^n \fint_{B(x,3r)} |u - (u)_{B(x,3r)}|\, d\xi\,.$$

In turn, using estimate (1.35) - in the variant provided by Proposition 3.1 since we are now dealing with equations with measurable coefficients - we have

$$\begin{aligned} &\fint_{B(x,3r)} |u - (u)_{B(x,3r)}|\, d\xi \leq r^{\alpha} M^{\#}_{\alpha,R/2}(u)(x) \\ &\leq cr^{\alpha} \left[M_{p-\alpha(p-1),R/2}(\mu)(x) \right]^{1/(p-1)} + c \left(\frac{r}{R} \right)^{\alpha} R \fint_{B(x,R/2)} (|Du| + s)\, d\xi\,, \end{aligned}$$

for a new constant c depending on n, p, ν, L and now also on $\tilde{\alpha} < \alpha_m$; notice that we have used that $r \leq R/8$. To estimate the last integral we use Caccioppoli's inequality (4.13) below (with a suitable choice of the radii) to have

$$\begin{aligned} &R \fint_{B(x,R/2)} (|Du| + s)\, d\xi \\ &\qquad \leq c \fint_{B(x,2R/3)} (|u| + Rs)\, d\xi + cR^{\alpha} \left[\frac{|\mu|(B(x,2R/3))}{R^{n-p+\alpha(p-1)}} \right]^{1/(p-1)} \\ &\qquad \leq c \fint_{B_R} (|u| + Rs)\, d\xi + cR^{\alpha} \left[M_{p-\alpha(p-1),2R/3}(\mu)(x) \right]^{1/(p-1)}\,. \end{aligned} \tag{4.11}$$

We have used the fact that since $x \in B_{R/8}$ then $B(x, 2R/3) \subset B_R$. Merging the last three inequalities to (4.10), and noting that $rs = (r/R)Rs \leq (r/R)^{\alpha}Rs$, yields

$$\begin{aligned} |u(x) - u(y)| \quad &\leq \quad c\left(\frac{r}{R}\right)^{\alpha} \fint_{B_R} (|u| + Rs)\, d\xi \\ &\quad +cr^{\alpha}\left[M_{p-\alpha(p-1),2R/3}(\mu)(x)\right]^{1/(p-1)} \\ &\quad +cr^{\alpha}\left[\mathbf{W}^{\mu}_{1-\alpha(p-1)/p,p}(x,R) + \mathbf{W}^{\mu}_{1-\alpha(p-1)/p,p}(y,R)\right]\,, \end{aligned}$$

while in turn Lemma 4.1 gives

$$\left[M_{p-\alpha(p-1),2R/3}(\mu)(x)\right]^{1/(p-1)} \leq c(n,p,\tilde{\alpha})\mathbf{W}^{\mu}_{1-\alpha(p-1)/p,p}(x,R)\,. \tag{4.12}$$

The last two inequalities together give (1.13) and the proof is complete. □

Proof of Theorem 1.10. Adapting the notation from the proof of Theorem 1.1, for every $0 < \varrho \leq R/2$ there is $i \geq 0$ such that $r_{i+1} < \varrho \leq r_i$ and we have

$$\fint_{B(x,\varrho)} |u|\, d\xi \leq (2H)^n \fint_{B_i} |u|\, d\xi.$$

Thus it is sufficient to prove that

$$\fint_{B_i} |u|\, d\xi \leq c\mathbf{W}^{\mu}_{1,p}(x,R) + c\fint_{B_R} (|u| + Rs)\, d\xi$$

holds for every $i \geq 0$. In turn, this is a consequence of estimates (4.8) and (4.9) in view of

$$\fint_{B_i} |u|\, d\xi \leq |(u)_{B_i}| + \fint_{B_i} |u - (u)_{B_i}|\, d\xi = k_i + A_i$$

and an estimate similar to (3.56). □

We conclude with the Caccioppoli type estimate used in the proof of Theorem 1.1. Here we present a short proof, based on some self-improving properties of reverse Hölder inequalities and on the comparison estimate (2.12). Although the result has a standard flavor we could not retrieve the next statement in the form needed anywhere in the literature, while the proof presented here is particularly straightforward.

Proposition 4.1. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to* (1.1) *with measurable coefficients, and let* (1.2) *hold with $p > 2 - 1/n$. Then, for every $\gamma \in (0,1)$ there exists a constant $c \equiv c(n,p,\nu,L,\gamma)$ such that*

$$\fint_{B_{\gamma R}} |Du|\, d\xi \leq \frac{c}{R}\fint_{B_R} |u - (u)_{B_R}|\, d\xi + c\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]^{1/(p-1)} + cs \tag{4.13}$$

holds whenever $B_{\gamma R} \subset B_R \subset \Omega$ are concentric balls.

Proof. First, let us observe that a standard scaling argument allows to reduce to the case $R = 1$; see [28, Lemma 4.1]. Then we may assume that $(u)_{B_1} = 0$ as if u solves (1.1) also $u - (u)_{B_1}$ does. Let $\gamma < r \leq 1$ and $w_r \in u + W^{1,p}_0(B_r)$ be defined as the unique solution to

$$\begin{cases} \operatorname{div}\, a(x, Dw_r) = 0 & \text{in } B_r \\ w_r = u & \text{on } \partial B_r\,. \end{cases}$$

The standard energy estimate under assumptions (1.2) gives

$$\fint_{B_r} |D(w_r\phi)|^p\, d\xi \leq c\left(\fint_{B_r} |w_r|^p|D\phi|^p\, d\xi + \fint_{B_r} s^p|\phi|^p\, d\xi\right) \tag{4.14}$$

for all $\phi \in C_0^\infty(B_r)$, where $c \equiv c(n,p,\nu,L)$ and thus Sobolev's embedding yields

$$\left(\fint_{B_r} |w_r|^\kappa |\phi|^\kappa \, d\xi\right)^{p/\kappa} \leq c\left(\fint_{B_r} |w_r|^p |D\phi|^p \, d\xi + \fint_{B_r} s^p |\phi|^p \, d\xi\right)$$

for some $\kappa = \kappa(n,p) > p$. Taking ϱ such that $\gamma \leq \varrho < r$ and a cut-off function $\phi \in C_0^\infty(B_r)$ such that $0 \leq \phi \leq 1$, $\phi = 1$ in B_ϱ, and $|D\phi| \leq 4/(r-\varrho)$, we arrive at

$$\left(\fint_{B_\varrho} |w_r|^\kappa \, d\xi\right)^{1/\kappa} \leq \frac{c}{r-\varrho}\left(\fint_{B_r} |w_r|^p \, d\xi\right)^{1/p} + cs$$

with a constant $c = c(n,p,\nu,L)$. Reverse Hölder inequalities have a self-improving nature - see for example [13, Lemma 3.38] - therefore from the previous inequality we gain

$$\left(\fint_{B_\varrho} |w_r|^\kappa \, d\xi\right)^{1/\kappa} \leq \frac{c}{(r-\varrho)^q}\fint_{B_r} |w_r| \, d\xi + cs,$$

where $q = q(n,p) > 1$ and $c = c(n,p,\nu,L)$. With the previous inequality in our hands, going back to (4.14), choosing a suitable cut-off function ϕ, and applying Hölder's inequality we obtain

$$\fint_{B_\varrho} |Dw_r| \, d\xi \leq \frac{c}{(r-\varrho)^{1+q}}\left(\fint_{B_r} |w_r| \, d\xi + s\right), \tag{4.15}$$

where $\gamma \leq \varrho < r \leq 1$ and $c \equiv c(n,p,\nu,L)$. Applying the triangle inequality repeatedly gives

$$\int_{B_\varrho} (|Du| + s) \, d\xi \leq \frac{c}{(r-\varrho)^{1+q}}\left(\int_{B_r} |u| \, d\xi + \fint_{B_r} |Du - Dw_r| \, d\xi + s\right). \tag{4.16}$$

Notice that in the last estimate we have also used Poincaré type inequality as $u \equiv w_r$ on ∂B_r. To estimate the last integral in (4.16) we appeal to (2.12). When $p < 2$, Young's inequality with conjugate exponents

$$\left(\frac{1}{2-p}, \frac{1}{p-1}\right)$$

in (3.11) implies

$$\frac{c[|\mu|(B_1)]}{(r-\varrho)^{1+q}}\left(\int_{B_r} (|Du| + s) \, d\xi\right)^{2-p} \leq \frac{1}{2}\int_{B_r} (|Du| + s) \, d\xi + \frac{c[|\mu|(B_1)]^{1/(p-1)}}{(r-\varrho)^{(1+q)/(p-1)}}$$

and therefore, by (2.12)

$$\frac{c}{(r-\varrho)^{1+q}}\int_{B_r} |Du - Dw_r| \, d\xi \leq \frac{1}{2}\int_{B_r} (|Du| + s) \, d\xi + \frac{c[|\mu|(B_1)]^{1/(p-1)}}{(r-\varrho)^{(1+q)/(p-1)}}$$

holds. Substituting this last estimate into (4.16) yields, whenever $p > 2 - 1/n$

$$\begin{aligned}\int_{B_\varrho} (|Du| + s) \, d\xi &\leq \frac{1}{2}\fint_{B_r} (|Du| + s) \, d\xi \\ &\quad + \frac{c}{(r-\varrho)^{(1+q)/(p-1)}}\left(\int_{B_1} |u| \, d\xi + [|\mu|(B_1)]^{1/(p-1)} + s\right)\end{aligned}$$

for all $\gamma \leq \varrho < r \leq 1$ and for a constant c depending only on n,p,ν,L. The result, that is (4.13) in the case $R = 1$, now follows applying the iteration Lemma 4.2 below with the obvious choice $\varphi(t) := \||Du| + s\|_{L^1(B_t)}$ and $R = 1$. $\square$

Lemma 4.2. ([12, Chapter 6]) *Let $\varphi : [\gamma R, R] \to [0,\infty)$, with $\gamma \in (0,1)$, be a bounded function such that the inequality*

$$\varphi(\varrho) \le \frac{1}{2}\varphi(r) + \frac{\mathcal{A}}{(r-\varrho)^\kappa}$$

holds whenever $\gamma R < \varrho < r < R$, for fixed constants $\mathcal{A}, \kappa \ge 0$. Then we have

$$\varphi(\gamma R) \le \frac{c\mathcal{A}}{(1-\gamma)^\kappa R^\kappa}$$

for a constant c depending only on κ.

5. Further oscillation estimates and Theorems 1.2, 1.3 and 1.5

We shall need the following standard lemma (see for instance [5]).

Proposition 5.1. *Let $f \in L^1(\Omega)$; for every $\alpha \in (0,1]$ the inequality*

$$|f(x)-f(y)| \le (c/\alpha)\left[M^{\#}_{\alpha,R}(f)(x) + M^{\#}_{\alpha,R}(f)(y)\right]|x-y|^\alpha \tag{5.1}$$

holds whenever x, y such that $x, y \in B_{R/4}$, for a constant c depending only on n.

Proof of Theorem 1.2. With $\tilde\alpha < 1$ being fixed in the statement, we have to proof the *uniform validity* with respect to $\alpha \in [0,\tilde\alpha]$ of inequality (1.13) as long as $p > 2-1/n$ and (1.19) holds for a suitable number δ. Without loss of generality we may assume that $\tilde\alpha \ge \alpha_m/2$, where α_m is the maximal Hölder regularity exponent of the operator determined by the vector field $a(\cdot)$. In fact, when restricting to the interval $[0,\alpha_m/2]$ the result is a consequence of Theorem 1.1; we again recall that $\alpha_m > 0$ depends on n, p, ν, L, and this serves to obtain the desired dependence of the constants. Therefore it remains to prove that (1.13) holds uniformly in $\alpha \in [\alpha_m/2, \tilde\alpha]$. With $x, y \in B_{R/8}$ this is in turn a consequence of estimate (1.35) that yields, after easy manipulations,

$$\begin{aligned}
&|u(x)-u(y)| \\
&\le (c/\alpha_m)\left[M_{p-\alpha(p-1),R/2}(\mu)(x) + M_{p-\alpha(p-1),R/2}(\mu)(y)\right]^{1/(p-1)}|x-y|^\alpha \\
&\quad +(c/\alpha_m)\left[R\fint_{B(x,R/2)}(|Du|+s)\,d\xi + R\fint_{B(y,R/2)}(|Du|+s)\,d\xi\right]\left(\frac{|x-y|}{R}\right)^\alpha .
\end{aligned}$$

At this point (1.13) follows using Lemma 4.1 to estimate the terms involving the maximal operators as in (4.12), and Caccioppoli's inequality (4.13) as after (4.11) to estimate the two integrals in the formula above. □

Proof of Theorems 1.3 and 1.5. The proof goes exactly as the one for Theorem 1.2 but estimates (1.36) and (1.39) must be used instead of (1.35) to cover the whole interval $[\alpha_m/2, 1]$. Notice that in the case $2-1/n < p \le 2$, when using Theorem 1.1 to cover the interval $[0,\alpha_m/2]$, we also need the inequality

$$\mathbf{W}^{\mu}_{1-\alpha(p-1)/p,p}(\cdot,R) \le c(n,p)\left[\mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(\cdot,R)\right]^{1/(p-1)}$$

which in fact holds when $p \le 2$. This is turn is based on (3.28) and the fact that $1/(p-1) \ge 1$ when $p \le 2$. □

Proceeding exactly as in the proof of Theorem 1.2, but without introducing potentials, and in particular without making use of Theorem 1.1, we have the following maximal version of the results in the Introduction, which is of course non-endpoint, and therefore does not admit (1.6)-(1.7) as borderline cases.

Theorem 5.1 (Non-endpoint estimates)**.** *Let $u \in C^1(\Omega)$ be a weak solution to* (1.1)*, under the assumptions* (1.2) *with $p > 2 - 1/n$. Let B_R be a ball such that $x, y \in B_{R/4}$, then*

- *If $\omega(\cdot)$ is* VMO*, then*

$$
\begin{aligned}
|u(x) - u(y)| \quad &\leq \quad c\left[M_{p-\alpha(p-1),R}(\mu)(x) + M_{p-\alpha(p-1),R}(\mu)(y)\right]^{1/(p-1)}|x-y|^{\alpha} \\
&\qquad + c \fint_{B_R} (|u| + Rs)\, d\xi \cdot \left(\frac{|x-y|}{R}\right)^{\alpha} \qquad (5.2)
\end{aligned}
$$

  holds for every $\alpha \in (0,1)$, where $c \equiv c(n,p,\nu,L,\omega(\cdot),\operatorname{diam}(\Omega),\alpha)$

- *If $p \geq 2$ and*

$$
\sup_r \frac{[\omega(r)]^{2/p}}{r^{\alpha}} \leq S \qquad 0 < \alpha < \alpha_M
$$

  holds, then

$$
\begin{aligned}
|Du(x) - Du(y)| \quad &\leq \quad c\left[M_{1-\alpha(p-1),R}(\mu)(x) + M_{1-\alpha(p-1),R}(\mu)(y)\right]^{1/(p-1)}|x-y|^{\alpha} \\
&\qquad + c \fint_{B_R} (|Du| + s)\, d\xi \cdot \left(\frac{|x-y|}{R}\right)^{\alpha} \qquad (5.3)
\end{aligned}
$$

  holds for a constant $c \equiv c(n,p,\nu,L,\omega(\cdot),\operatorname{diam}(\Omega),\alpha,S)$, where $\alpha \in (0,\alpha_M)$

- *If $p \leq 2$ and*

$$
\sup_r \frac{[\omega(r)]^{\sigma}}{r^{\alpha}} \leq S \qquad 0 < \alpha < \alpha_M
$$

  is satisfied for some $\sigma \in (0,1)$, then (5.3) *holds, provided the operator $M_{1-\alpha(p-1),R}(\mu)$ is replaced by $M_{1-\alpha,R}(\mu)$*

6. Cordes type theory via potentials and Theorem 1.7

The proof of Theorem 1.7 is based on higher order perturbation of the reference solution. Indeed, derivatives of solutions are themselves solutions to linear equations with slowly oscillating coefficients and this allows for application of more efficient perturbation arguments.

Proof of Theorem 1.7. The proof is in two steps.

Step 1: The first decay estimate. We start referring to the material presented in Section 2.5, and we keep the notation used there; in particular, w is the function introduced in (2.11). In the following all the balls will be concentric and will be centered at a fixed point x; we are assuming here that $B_{2R} \subset \Omega$. Let us immediately notice that standard regularity theory implies that $w \in W^{2,2}_{\mathrm{loc}}(B_{2R})$ and that moreover $\tilde{w} := D_i w$ solves

$$
\operatorname{div}\left(\partial a(Dw) D\tilde{w}\right) = 0 \qquad (6.1)
$$

in B_{2R}, which is a linear elliptic equation with measurable coefficients. As such, the following Caccioppoli type inequality holds for every $\lambda \in \mathbb{R}^n$:

$$
\int_{B_{5R/4}} |D^2 w|\, d\xi \leq \frac{c}{R}\int_{B_{2R}} |Dw - \lambda|\, d\xi \qquad (6.2)
$$

for a constant c depending on n, ν, L. We refer to [18], where this type of estimate is presented in L^2; for the L^1-version in (6.2) we refer to [28] and in particular to [30, Proposition 2.1]. Now, we define $\tilde{v} \in W^{1,2}(B_R)$ as the unique solution to the Dirichlet problem

$$
\begin{cases} \operatorname{div} AD\tilde{v} = 0 & \text{in } B_R \\ \tilde{v} = \tilde{w} & \text{on } \partial B_R \end{cases} \qquad (6.3)
$$

where the elliptic matrix A is the one from (1.33). We notice that the function $\tilde{v}$ is smooth in the interior of B_R, and in particular it satisfies the decay estimate

$$\fint_{B_\varrho} |\tilde{v} - (\tilde{v})_{B_\varrho}| \, d\xi \le c \left(\frac{\varrho}{R}\right) \fint_{B_R} |\tilde{v} - (\tilde{v})_{B_R}| \, d\xi \,, \tag{6.4}$$

whenever $B_\varrho \subset B_R$ are concentric balls. We refer for instance to [12, Chapter 10], and again to [28] for the L^1-version of the estimates used. On the other hand, by the ellipticity of A and by equations of $\tilde{v}$ and $\tilde{w}$ we have

$$\fint_{B_R} |D\tilde{v} - D\tilde{w}|^2 \, d\xi \le c \fint_{B_R} |\partial a(Dw) - A|^2 |D\tilde{w}|^2 \, dx, \tag{6.5}$$

where c depends only on n, ν. Indeed, (1.34) and (6.1) yield

$$\begin{aligned} \int_{B_R} |D\tilde{v} - D\tilde{w}|^2 \, d\xi &\le \frac{1}{\nu} \int_{B_R} \langle A(D\tilde{v} - D\tilde{w}), D\tilde{v} - D\tilde{w} \rangle \, d\xi \\ &= \frac{1}{\nu} \int_{B_R} \langle (\partial a(Dw) - A) D\tilde{w}, D\tilde{v} - D\tilde{w} \rangle \, d\xi \\ &\le \frac{1}{\nu} \int_{B_R} |\partial a(Dw) - A| |D\tilde{w}| |D\tilde{v} - D\tilde{w}| \, d\xi \qquad (6.6) \end{aligned}$$

and (6.5) follows via Young's inequality. In turn, using (1.33) we have

$$\fint_{B_R} |D\tilde{v} - D\tilde{w}|^2 \, d\xi \le c\delta^2 \fint_{B_R} |D\tilde{w}|^2 \, d\xi \,.$$

At this point, proceeding as for Lemma 2.3 we lower the previous estimates at the L^1-level via reverse Hölder inequalities (as w solves a linear elliptic equation), that is we obtain the following analog of inequality (2.18):

$$\fint_{B_R} |D\tilde{v} - D\tilde{w}| \, d\xi \le c\delta \fint_{B_{5R/4}} |D\tilde{w}| \, d\xi \,.$$

Note that we are are using different enlarging of radii here, something that was already possible in Lemma 2.3. Use of the Poincaré inequality (recalling that $\tilde{w} = D_i w$) eventually yields

$$\fint_{B_R} |\tilde{v} - \tilde{w}| \, d\xi \le cR \fint_{B_R} |D\tilde{v} - D\tilde{w}| \, d\xi \le c\delta R \fint_{B_{5R/4}} |D^2 w| \, d\xi \,.$$

Finally, applying the previous estimate together with (6.2) we get the comparison estimate we were looking for, i.e.

$$\fint_{B_R} |\tilde{v} - \tilde{w}| \, d\xi \le c\delta \fint_{B_{2R}} |Dw - \lambda| \, d\xi \,.$$

This last estimate holds whenever $\lambda \in \mathbb{R}^n$, and for every choice of $i \in \{1, \ldots, n\}$, where $\tilde{w} = D_i w$. Arguing as in the proof of Lemmata 3.1-3.3, that is using (6.4) and comparing $\tilde{v}$ and $\tilde{w}$ via the last inequality, we have that

$$\fint_{B_\varrho} |Dw - (Dw)_{B_\varrho}| \, d\xi \le \left[c_1 \left(\frac{\varrho}{R}\right) + c \left(\frac{R}{\varrho}\right)^n \delta \right] \fint_{B_{2R}} |Dw - \lambda| \, d\xi$$

holds for every $\lambda \in \mathbb{R}^n$. Eventually comparing u and w via Lemma 2.1, and choosing $\lambda = (Du)_{B_{2R}}$ in the previous inequality, yields

$$\begin{aligned} \fint_{B_\varrho} |Du - (Du)_{B_\varrho}| \, d\xi &\le \left[c_1 \left(\frac{\varrho}{R}\right) + c \left(\frac{R}{\varrho}\right)^n \delta \right] \fint_{B_{2R}} |Du - (Du)_{B_{2R}}| \, d\xi \\ &\quad + \left[c_1 \left(\frac{\varrho}{R}\right) + c \left(\frac{R}{\varrho}\right)^n (1 + \delta) \right] \frac{|\mu|(B_{2R})}{R^{n-1}} \qquad (6.7) \end{aligned}$$

whenever $B_\varrho \subset B_R \subset B_{2R} \subseteq \Omega$ are concentric balls, for new constants c, c_1 still depending only on n, ν, L. The previous estimate will play in the following the same role played by estimate (3.2) in the previous proofs.

Step 2: Maximal inequality and conclusion. We only have to prove the statements for "large" α, i.e. when α is far from 0, otherwise the assertion is already contained for instance in Theorem 1.4, where the uniform validity of (1.31) is proved on compact subsets of $[0, \alpha_M)$. In turn, recalling the proof of Theorem 1.4, we remark that it is sufficient to prove the uniform validity for $\alpha \in [\alpha_M/2, \tilde{\alpha}]$ of the maximal inequality

$$M^{\#}_{\alpha,R}(Du)(x) \leq cM_{1-\alpha,R}(\mu)(x) + cR^{-\alpha} \fint_{B_R} |Du|\, d\xi \tag{6.8}$$

whenever $\alpha_M/2 \leq \tilde{\alpha} < 1$. This will in turn ensure the uniform validity of (1.31) on compact subsets of $(0, 1)$ as observed on the proof of Theorem 1.2. We also observe that (6.8) is actually a form of (1.41) adapted to the particular case under consideration. In order to prove (6.8) we go back to Theorem 1.9, proof of (1.41). We perform the same choice as in (3.48) and we select the sequence of radii $R_i = R/(2H)^i$ with $H > 1$ to be selected as usual in a few lines. This time we rely on (6.7) that we multiply by $R_{i+1}^{-\alpha}$, after taking $\varrho = R_{i+1}$ and $2R = R_i$. Proper manipulations then yield

$$\tilde{A}_{i+1} \leq \left[c_1 H^{\alpha-1} + c_2\delta H^{n+\alpha}\right] \tilde{A}_i + c\left[H^{-1+\alpha} + \delta H^{n+\alpha}\right]\left[\frac{|\mu|(B_i)}{R_i^{n-1+\alpha}}\right],$$

where c, c_1 and c_2 depend on n, ν, L. By first choosing $H \equiv H(n, \nu, L, \tilde{\alpha})$ large enough in order to have $c_1 H^{\alpha-1} \leq c_1 H^{\tilde{\alpha}-1} \leq 1/4$ and then determining $\delta \equiv \delta(n, \nu, L, H) \equiv \delta(n, \nu, L, H)$ small enough to get $c_2 H^{n+\alpha}\delta \leq c_2 H^{n+\tilde{\alpha}}\delta \leq 1/4$ we conclude with

$$\tilde{A}_{i+1} \leq (1/2)\tilde{A}_i + cM_{1-\alpha,R}(\mu)(x)\,,$$

where c depends now on $n, \nu, L, \tilde{\alpha}$. Notice that at this point we are determining the dependence of the constant δ appearing in (1.33) as a function of the parameters $n, \nu, L, \tilde{\alpha}$, as prescribed by the statement of Theorem 1.7. The last inequality is totally similar to (3.52), and from this point on we may proceed as in the proof of (1.41) to reach (6.8). The proof of Theorem 1.7 is therefore complete. □

7. A priori regularity estimates

In this section we prove the local Lipschitz regularity results for solutions to homogeneous equations that we used to prove the pointwise potential bounds. We found suitable to put this material at the end of the paper both because presenting them earlier would have interrupted the proof of the main results and because Theorem 7.1, being actually a *particular case of the general potential estimates stated in the Introduction* when $p > 2 - 1/n$, admits a shorter proof in view of the methods previously presented elsewhere. Specifically, we consider homogeneous equations of type

$$\operatorname{div} a(x, Dw) = 0 \tag{7.1}$$

with Dini-VMO coefficients and prove Theorems 7.1 below, which extends similar results available in the literature where the usual Dini continuity is considered. We recall that the number σ_d has been defined in (2.23).

Theorem 7.1. *Let $w \in W^{1,p}(\Omega)$ be a weak solution to* (7.1) *under the assumptions* (1.2) *with $p > 1$, and assume that the function $[\omega(\cdot)]^{\sigma_d}$ is* Dini-VMO *regular, i.e.*

$$\int_0^r [\omega(\varrho)]^{\sigma_d}\, \frac{d\varrho}{\varrho} < \infty \qquad \forall\ r < \infty\,. \tag{7.2}$$

Then $Dw \in L^\infty_{\rm loc}(\Omega)$ and moreover, for every $B_R \subset \Omega$ it holds that

$$\|Dw\|_{L^\infty(B_{R/2})} \leq c \fint_{B_R} (|Dw| + s)\, dx \tag{7.3}$$

for a constant c depending only on $n, p, \nu, L, \omega(\cdot), \operatorname{diam}(\Omega)$ and the number $\sigma < 1$ chosen to define σ_d in (2.23) in the case $p < 2$.

Proof. Step 1: Reduction to the case $Dw \in C^0(\Omega)$. We briefly sketch how to reduce to the case that Dw is continuous, by means of a standard approximation argument. Observe that in the rest of the proof we can without loss of generality assume that $B_R \Subset \Omega$. Let us denote $\Omega' \Subset \Omega$ a Lipschitz regular subdomain of Ω such that $B_R \subset \Omega'$; we denote by $\tilde w_\varepsilon = w * \phi_\varepsilon$ a mollification of w via a standard smoothing mollifiers $\{\phi_\varepsilon\}_\varepsilon$ (obtained by scaling from a single one $\phi_\varepsilon(y) := \varepsilon^{-n}\phi(y/\varepsilon)$) with $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega)$. Here $\phi \in C^\infty(\mathbb{R}^n)$ and it is such that $\operatorname{supp}\phi = \overline{B_1}$ and $\|\phi\|_{L^1} = 1$. In the same way we define the smoothed vector fields by

$$a_\varepsilon(x,z) := (a * \phi_\varepsilon)(x,z) = \int_{\mathbb{R}^n} a(x + \varepsilon y, z)\phi(y)\, dy$$

whenever $x \in \Omega'$ and $z \in \mathbb{R}^n$. We then denote by $w_\varepsilon \in \tilde w_\varepsilon + W^{1,p}_0(\Omega')$ the unique solution to the following Dirichlet problem:

$$\begin{cases} \operatorname{div} a_\varepsilon(x, Dw_\varepsilon) = 0 & \text{in } \Omega' \\ w_\varepsilon \equiv \tilde w_\varepsilon & \text{on } \partial\Omega'\,. \end{cases}$$

By using standard monotonicity arguments we get $w_\varepsilon \to w$ strongly in $W^{1,p}(\Omega')$. Moreover, standard regularity theory gives $Dw_\varepsilon \in C^0(\Omega')$. Assuming now that estimate (7.3) works uniformly for Dw_ε - i.e. assuming that the theorem works for a priori locally Lipschitz solutions - we can easily infer the validity of (7.3) for w by using the strong convergence of the $\{w_\varepsilon\}_\varepsilon$.

Step 2: Pointwise estimate. We take $B_R \equiv B(x,R) \subset \Omega$ and define $v \in W^{1,p}(B_R)$ as the unique solution to the Dirichlet problem (2.13) (where w is now the solutions of (7.1) we are considering) and turn our attention to Lemma 2.4. This gives (2.24); in turn, combining this estimate with (2.5) - see for instance the proof of Lemma 3.1 - yields

$$\begin{aligned} \fint_{B_\varrho} |Dw - (Dw)_{B_\varrho}|\, dy \quad &\leq \quad c\left(\frac{\varrho}{R}\right)^{\alpha_M} \fint_{B_R} |Dw - (Dw)_{B_R}|\, dy \\ &\quad + c\left(\frac{R}{\varrho}\right)^{n} [\omega(R)]^{\sigma_d}(\|Dw\|_{L^\infty(B_R)} + s)\,, \end{aligned}$$

whenever $0 < \varrho < R$, with $c \equiv c(n,p,\nu,L,\sigma_d)$. We now want to proceed as in the proof of (1.39), Step 1. We choose a dyadic sequence of balls $B_i := B(x, R/H^i) := B(x, R_i)$ with H to be a (large) integer to be chosen a in a few lines, and we set

$$A_i := \fint_{B_i} |Dw - (Dw)_{B_i}|\, d\xi \qquad \text{and} \qquad k_i := |(Du)_{B_i}|\,.$$

Proceeding as in the proof of (1.39), Step 1, and therefore selecting $H \equiv H(n,p,\nu,L,\sigma_d)$ large enough in order to have $cH^{-\alpha_M} \leq 1$ we have

$$A_{i+1} \leq (1/2)A_i + c[\omega(R_i)]^{\sigma_d}(\|Dw\|_{L^\infty(B_R)} + s)\,,$$

and therefore, summing up the previous relation and again proceeding as in the proof of (1.39), Step 1, we come to

$$k_{m+1} \quad \leq \quad ck_0 + c(\|Dw\|_{L^\infty(B_R)} + s)\sum_{i=0}^{m}[\omega(R_i)]^{\sigma_d}$$

$$\leq \quad ck_0 + c(\|Dw\|_{L^\infty(B_R)} + s)\int_0^{2R} [\omega(\varrho)]^{\sigma_d}\,\frac{d\varrho}{\varrho}\,.$$

We then restrict to consider radii $R \leq \tilde{R}$, with

$$c\int_0^{2\tilde{R}} [\omega(\varrho)]^{\sigma_d}\,\frac{d\varrho}{\varrho} \leq \frac{1}{2}$$

so that, we ultimately get

$$|Dw(x)| \leq \lim_{m\to\infty} k_{m+1} \leq c ⨍_{B_R} (|Dw| + s)\,d\xi + (1/2)\|Dw\|_{L^\infty(B_R)}\,. \tag{7.4}$$

Step 3: Iteration and conclusion. We now take a ball $B_R \subset \Omega$ as in the statement, and consider concentric balls $B_{R/2} \subset B_\varrho \subset B_r \subset B_R$. We apply estimate (7.4) on balls $B(x,(r-\varrho))$ with $x \in B_\varrho$; this yields

$$|Dw(x)| \leq c ⨍_{B(x,(r-\varrho))} (|Dw| + s)\,d\xi + (1/2)\|Dw\|_{L^\infty(B(x,(r-\varrho)))}\,, \tag{7.5}$$

for every $x \in B_\varrho$. Observing that $B(x,(r-\varrho)) \subset B_r$, and therefore trivially estimating as

$$⨍_{B(x,(r-\varrho))} (|Dw| + s)\,d\xi \leq \frac{c}{(r-\varrho)^n}\int_{B_R} (|Dw| + s)\,d\xi$$

and $\|Dw\|_{L^\infty(B(x,(r-\varrho)))} \leq \|Dw\|_{L^\infty(B_r)}$, and finally taking the sup over B_ϱ in (7.5), leads to

$$\|Dw\|_{L^\infty(B_\varrho)} \leq \frac{c}{(r-\varrho)^n}\int_{B_R} (|Dw| + s)\,d\xi + (1/2)\|Dw\|_{L^\infty(B_r)}\,.$$

In turn applying Lemma 4.2 with the choice $\varphi(t) = \|Dw\|_{L^\infty(B_t)}$, we finally get that estimate (7.3) holds for a constant c depending only on n, p, ν, L and provided $R \leq \tilde{R}$, where $\tilde{R}$ depends only on n, p, ν, L and $\omega(\cdot)$. The full statement now follows arguing as in the proof (1.39), Step 2, to get rid of the constraint $R \leq \tilde{R}$, so that finally a dependence on $\omega(\cdot)$ and diam (Ω) of the constant c occurs in the final form of (7.3). □

8. Selected corollaries and refinements

In this final section we want to point out a few possible additional results and corollaries directly related to the theorems presented in the paper. In all the rest of the section, as usual we deal with a priori estimates valid for a priori regular solutions, while general statements can be as usual obtained by approximation [2, 8]. One of the main aims here is to establish *an intermediate Calderón-Zygmund type theory*, where fractional derivatives are bounded by the natural nonlinear fractional potentials.

8.1. **Estimates in fractional spaces.** Let us first outline how to get estimates in Nikolskii and Hölder spaces. The first application we present is about local regularity in fractional Sobolev spaces. We recall that a function $v \in L^q(A)$ belongs to the Nikolskii space $N^{\alpha,q}(A)$ for $\alpha \in [0,1]$ and $q \geq 1$ iff

$$[v]^q_{\alpha,q;A} := \sup_{|\mathrm{e}|=1}\sup_{|h|\neq 0}\int_{A_h} \frac{|v(x+h\mathrm{e}) - v(x)|^q}{|h|^{\alpha q}}\,dx < \infty\,, \tag{8.1}$$

where $A \subset \mathbb{R}^n$ is an open subset and $A_h \subset A$ denotes the subset of A consisting of all point having distance to the boundary larger than $|h|$. Such spaces are a subclass of a larger family of interpolation spaces called Besov spaces (see [1] and related references). We observe that $N^{\alpha,\infty} \equiv C^{0,\alpha}$, so that estimates in this class of spaces imply those in Hölder spaces, and therefore *nonlinear Schauder estimates.*

By using for instance Theorem 1.3 we see that under the assumptions considered there we have, up to a standard covering argument, that the estimate

$$[u]_{\alpha,q;B_{R/2}} \leq c\|\mathbf{W}_{1-\alpha(p-1)/p,p}(\cdot,R)\|_{L^q(B_R)} + \frac{c}{R^\alpha}\int_{B_R}(|u|+Rs)\,dx \tag{8.2}$$

holds with a constant c depending only on n, p, ν, L. The previous estimate tells us that in order to look for fractional differentiability one can confine himself to require the needed integrability properties of the potential. In turn, via (8.4) below, this immediately yields the necessary integrability assumptions on μ. Indeed, let us recall that the Wolff potential is dominated by the so called Havin-Mazya potential, that is the composition of standard Riesz potentials appearing on the right hand side of the next inequality

$$\mathbf{W}_{\beta,p}(x,R) \leq I_\beta\left[I_\beta(|\mu|)^{1/(p-1)}\right](x)\,, \qquad \beta p < n\,, \qquad R>0\,. \tag{8.3}$$

In turn, the last inequality implies for instance bounds in Lebesgue spaces:

$$\|\mathbf{W}^\mu_{\beta,p}\|_{L^{\frac{n\gamma(p-1)}{n-\beta\gamma p}}(\Omega)} \leq c\|\mu\|_{L^\gamma(\Omega)}\,, \qquad \beta\gamma p < n\,, \tag{8.4}$$

in any open subset $\Omega \subset \mathbb{R}^n$; similar bounds are actually available in several other rearrangement invariant functions spaces. We also observe that when instead applying Theorem 1.2 we end up with an estimate similar to (8.2), but for the case $\alpha = 1$, the same cannot be covered when coefficients are simply VMO (an assumption that in this respect appears to be optimal to reach the regularity scale in question here).

We further remark that estimate (8.2) is an endpoint estimate in that, for the cases $\alpha = 0, 1$, it gives back the basic estimates in Lebesgue spaces, and in particular those for the gradient. Obviously, another similar, slightly sharper estimate can be obtained by using Theorem 5.1, and this involve maximal functions of the datum. Such estimates are anyway not of endpoint type.

Needless to say, the theorems stated in the Introduction provide regularity criteria in the Calderón spaces C^α_q described in Definition 1. Such estimates appear to be new even for linear equations. We remark that such spaces are relevant in several contexts, as for instance when considering the boundary regularity in elliptic vectorial problems [19].

8.2. **Nonlinear Calderón-Zygmund and Schauder theories.** Schauder estimates allow to get the Hölder continuity of the gradient in a sharp way when coefficients are Hölder continuous; this is a classical topic (see for instance [11]) and by the years several approaches to them have been developed. Let us first show how the approach found here allows to recover the well-known linear results for equations of the type

$$\operatorname{div}(B(x)Du) = f$$

where $B(\cdot)$ is an elliptic matrix with bounded and measurable entries. Indeed, in this case, then it turns out that $Du \in C^{0,\alpha}_{\mathrm{loc}}$ iff $B \in C^{0,\alpha}_{\mathrm{loc}}$ and $f \in L^{n/(1-\alpha)}$. This result immediately follows from Theorem 5.1, and in particular requires the weaker Lorentz type assumption $f \in L(n/(1-\alpha),\infty) \equiv \mathcal{M}^{n/(1-\alpha)}$. By using instead Theorem 1.4 we need slightly more stringent assumptions on coefficients and a condition of the type $f \in L(n/(1-\alpha),1)$, but we gain an endpoint estimate that catches up the case $\alpha = 0$ yielding gradient boundedness. Similar results can now be obtained in the nonlinear case by imposing suitable conditions on nonlinear potentials or on maximal operators via inequalities as for instance the ones in (5.3).

When considering solutions to general equations as in (1.1) it is useful to consider measures with a density property of Morrey type as for instance

$$|\mu|(B_\varrho) \leq c\varrho^{n-\theta}\,, \qquad \theta \in [0,n] \tag{8.5}$$

which immediately implies the boundedness of restricted maximal operators

$$M_{\theta,R}(\mu) \in L^\infty\,. \tag{8.6}$$

Moreover, we recall that - see [29] for many references and notation about Morrey spaces - $I_\alpha(\mu) \in L^{\theta/(\theta-\alpha),\theta} \subset L^{\theta/(\theta-\alpha)}$ whenever $\alpha < \theta$; as a consequence, again via (8.3) one derives and generalizes the classical one in Morrey spaces available in the literature for linear problems. It is worth noticing here that such results cannot be obtained via interpolation methods as Morrey spaces - i.e. conditions as (8.5) - are not encodable via interpolation methods. Furthermore, when going back to Hölder estimates, Theorem 5.1 implies that under condition (8.5), $1 < \theta < \max\{p, n\}$, the solutions to equations with VMO coefficients are Hölder continuous with the exponent $\alpha = (p-\theta)/(p-1)$, giving, for instance, a quantitative version of [16, Corollary 4.17]. Similarly, if $1-\alpha_M(p-1) < \theta < 1$ and coefficients are regular enough, say Lipschitz, then the gradient is Hölder continuous with the exponent $\alpha = (1-\theta)/(p-1) < \alpha_M$. The results obtainable here under the condition (8.5) extend those previously obtained in [10, 25, 15, 35].

Finally, by using Theorems 1.8-1.9, Proposition 3.1 and Theorem 5.1, and yet recalling (8.6), we immediately obtain the following corollary, which gives regularity properties of u in terms of regularity of coefficients and familiar Marcinkiewicz (weak Lebesgue) spaces M^γ defined as

$$f \in \mathcal{M}^\gamma(A) \Longleftrightarrow \sup_{\lambda \geq 0} \lambda^\gamma |\{x \in A \,:\, |f| > \lambda\}| < \infty\,.$$

where $A \subset \mathbb{R}^n$ is an open subset.

Corollary 8.1. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to the equation with measurable coefficients* (1.1)*, and let* (1.2) *hold with $p > 2 - 1/n$. Then*

- *$u \in BMO_{\mathrm{loc}}$ when $\mu \in \mathcal{M}^{n/p}(\Omega)$ as long as $p < n$ and $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ if $\mu \in \mathcal{M}^{n/(p-\alpha(p-1))}_{\mathrm{loc}}$ as long as $\alpha < \alpha_m$ and $p - \alpha(p-1) < n$*
- *assume that the dependence $x \mapsto a(x,\cdot)$ is* VMO *in the sense of Theorem 1.2. Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ if $\mu \in \mathcal{M}^{n/(p-\alpha(p-1))}_{\mathrm{loc}}$ as long as $\alpha < 1$ and $p - \alpha(p-1) < n$*
- *assume for simplicity that $a(\cdot)$ is independent i.e. $a(x,z) \equiv a(z)$; then $Du \in C^{0,\alpha}_{\mathrm{loc}}$ if $\mu \in \mathcal{M}^{n/(1-q\alpha)}_{\mathrm{loc}}$ as long as $\alpha < \min\{1/q, \alpha_M\}$ with $q := \max\{1, p-1\}$.*

Explicit local estimates in the various function spaces considered also follow from those in Theorems 1.8-1.9. Let us also remark that, since in the case considered in Corollary 8.1 the right hand side actually belongs to the dual of $W^{1,p}$ then a different comparison argument can eventually lead to omit the lower bound $p > 2-1/n$. We refer to [21, 22] for further criteria for general gradient continuity.

8.3. **A refinement.** In (1.20) and (1.27) it is sometimes possible to take $\sigma = 1$ when $2 - 1/n < p \leq 2$. This happens for instance when the partial map

$$x \to \frac{a(x,z)}{(|z|+s)^{p-1}}$$

is truly Dini-continuous uniformly with respect to the gradient variable z in the sense that

$$\sup_{z \in \mathbb{R}^n} \frac{|a(x,z) - a(y,z)|}{(|z|+s)^{p-1}} \leq \omega(|x-y|) \quad \text{and} \quad \lim_{r \to 0} \omega(r) = 0\,. \tag{8.7}$$

and

$$\sup_r \int_0^r \omega(\varrho)\,\frac{d\varrho}{\varrho} < \infty$$

hold. When considering the model case (1.3) this amounts to assume basically that $\gamma(\cdot)$ is Dini-continuous in the usual sense. To check this we just observe that going back to Section 2.5 and in particular to (2.14) and estimating

$$\begin{aligned} \fint_{B_R} |Dv - Dw|^p \, dx &\leq c\|A(Dw, B_R)\|_{L^\infty}^p \fint_{B_R} (|Dv|^2 + |Dw|^2 + s^2)^{p/2} \, dx \\ &\leq c[\omega(R)]^p \fint_{B_R} (|Dw|^2 + s^2)^{p/2} \, dx \,, \end{aligned}$$

we eventually obtain, via Hölder's inequality, that

$$\fint_{B_R} |Dv - Dw| \, dx \leq c\omega(R)\| |Dw| + s\|_{L^\infty(B_R)} \,.$$

Using this last inequality (as an a priori estimate) in Theorem 7.1 we first obtain that $w \in W^{1,\infty}(B_R)$ and that estimate (7.3) actually holds (with the new and obvious choice of radii). This in turn allows to finally estimate

$$\fint_{B_R} |Dv - Dw| \, dx \leq c\omega(R) \fint_{B_{2R}} (|Dw| + s) \, dx \,.$$

This last estimate can be now used instead of (2.26) and the rest of the proof follows as for the case we where assuming the Dini type decay of the integral modulus of continuity.

Another possible refinement follows by using additional assumptions on the operator $a(\cdot)$ considered in (1.1), as for instance done in [9]. In this case, combining the methods introduced in this paper with some of the estimates in [9], it is possible to refine some the results presented in the Introduction - in particular those for the gradient - using slightly smaller potentials.

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