WAVE AND MAXWELL'S EQUATIONS IN CARNOT GROUPS

BRUNO FRANCHI
MARIA CARLA TESI

ABSTRACT. In this paper we define Maxwell's equations in the setting of the intrinsic complex of differential forms in Carnot groups introduced by M. Rumin. It turns out that these equations are higher order equations in the horizontal derivatives. In addition, when looking for a vector potential, we have to deal with a new class of higher order evolution equations that replace usual wave equations of the Euclidean setting and that are no more hyperbolic. We prove equivalence of these equations with the "geometric equations" defined in the intrinsic complex, as well as existence and properties of solutions.

CONTENTS

1. INTRODUCTION

Consider the space-time $\mathbb{R} \times \mathbb{R}^3$ of special relativity, where we denote by $s \in \mathbb{R}$ the time variable and by $x \in \mathbb{R}^3$ the space variable. If (Ω^*, d) is the de Rham complex of differential forms in $\mathbb{R} \times \mathbb{R}^3$, classical Maxwell's equations can be formulated in their simplest form as follows: we fix the standard volume form dV in $\mathbb{R}^3$, and we consider a 2-form $F \in \Omega^2$ (Faraday's form), that can be always written as $F = ds \wedge E + B$, where E is the electric field 1-form and B is the magnetic induction 2-form. Then, if we assume for

1991 *Mathematics Subject Classification.* 35R03, 58A10, 35Q61, 35L05, 58D25 .

Key words and phrases. Carnot groups, differential forms, Maxwell's equations, wave equations.

The authors are supported by MURST, Italy, and by University of Bologna, Italy, funds for selected research topics and by EC project GALA.

It is a pleasure to thank warmly Annalisa Baldi for her friendly help and for many stimulating suggestions and discussions.

sake of simplicity all "physical" constants (i.e. magnetic permeability and electric permittivity) to be 1, classical Maxwell's equations become

$$dF = 0 \quad \text{and} \quad d(*_M F) = \mathcal{J}. \tag{1}$$

Here $*_M$ is the Hodge-star operator associated with the space-time Minkowskian metric and the volume form $ds \wedge dV$ in $\mathbb{R} \times \mathbb{R}^3$, and $\mathcal{J} = ds \wedge *J - \rho$ is a closed 3-form in $\mathbb{R} \times \mathbb{R}^3$, where $*J$ and $\rho = \rho_0 dV$ are respectively the current density 2-form and the charge density 3-form (here $*$ is the standard Hodge-star operator in $\mathbb{R}^3$ associated with the Euclidean metric and the volume form dV). Since $dF = 0$, we can always assume that $F = dA$, where A (the electromagnetic potential 1-form) can be written as $A = A_\Sigma + \varphi ds$. If in addition A_Σ and φ satisfy suitable gauge conditions, then they satisfy the wave equations

$$\frac{\partial^2 A_\Sigma}{\partial s^2} = -\Delta A_\Sigma - J \tag{2}$$

$$\frac{\partial^2 \varphi}{\partial s^2} = -\Delta \varphi + \rho_0, \tag{3}$$

where ΔA_Σ is the positive Hodge Laplacian on 1-forms $\Delta A_\Sigma = (d^*d + dd^*)A_\Sigma$.

It is well known that this theory has a natural extension in general relativity to Riemannian manifolds. The aim of the present paper is to extend this theory, as much as possible, to space-time structures based on *non Riemannian* spaces, looking for analogies with classical theory, but, first of all, trying to detect new phenomena.

In this paper we carry on our program in the setting of Carnot groups, that are, as we shall see, on several respect the first natural generalization of Euclidean spaces. In the special case of the first Heisenberg group $\mathbb{H}^1$, a Maxwell theory has been presented in [21].

A Carnot group $\mathbb{G}$ is a connected, simply connected, nilpotent Lie group with stratified Lie algebra $\mathfrak{g}$. More precisely, this means that the Lie algebra $\mathfrak{g}$ has dimension n, and admits a *step κ stratification*, i.e. there exist linear subspaces (so-called layers) $V_1, ..., V_\kappa$ such that

$$\mathfrak{g} = V_1 \oplus ... \oplus V_\kappa, \quad [V_1, V_i] = V_{i+1}, \quad V_\kappa \neq \{0\}, \quad V_i = \{0\} \text{ if } i > \kappa, \tag{4}$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. Clearly, the Lie algebra $\mathfrak{g}$ can be endowed with a scalar product that makes the decomposition (4) orthogonal. We refer to the first layer V_1 as to the *horizontal layer*. It plays a key role in our theory, since it generates the all of $\mathfrak{g}$ by commutations.

Through exponential coordinates, the group $\mathbb{G}$ can be identified with $(\mathbb{R}^n, \cdot)$, the Euclidean space $\mathbb{R}^n$ endowed with a (generally non-commutative) group law.

One of the main properties of Carnot groups is that they are endowed with two family of important transformations: the *(left) translation* $\tau_x : \mathbb{G} \to \mathbb{G}$ defined as $z \mapsto \tau_x z := x \cdot z$, and the non-isotropic *group dilations* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, that are associated with the stratification of $\mathfrak{g}$ and are automorphisms of the group.

It is well known that the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ can be identified with the tangent space at the origin e of $\mathbb{G}$, and hence the horizontal layer of $\mathfrak{g}$ can be identified with a subspace $H\mathbb{G}_e$ of $T\mathbb{G}_e$. By left translation, $H\mathbb{G}_e$ generates a subbundle $H\mathbb{G}$ of the tangent bundle $T\mathbb{G}$, called the horizontal bundle. A section of $H\mathbb{G}$ is called a horizontal vector field.

Obviously, Euclidean spaces are commutative Carnot groups, and, more precisely, the only commutative Carnot groups.

It is well known that Carnot groups are endowed with an intrinsic geometry, the so-called Carnot-Carathéodory geometry (see for instance, choosing in a wide literature, [8], [26] [17]). From now on, the adjective "intrinsic" is meant to emphasize a privileged role played by the horizontal layer and by group translations and dilations. It is worth stressing that Carnot-Carathéodory geometry is not Riemannian at any scale (see [41]).

In fact, Carnot groups can be seen as a particular case of more general structures, the so-called sub-Riemannian spaces. It is worth describing shortly these structures, since this makes more perspicuous the role of Carnot groups for our purposes.

Roughly speaking, a sub-Riemannian structure on a manifold M is defined by a subbundle H of the tangent bundle TM, that defines the "admissible" directions at any point of M (typically, think of a mechanical system with non-holonomic constraints). Usually, H is called the *horizontal* bundle. If we endow each fiber H_x of H with a scalar product, there is a naturally associated Carnot-Carathéodory distance d on M, defined as the Riemannian length of the horizontal curves on M, i.e. of the curves γ such that $\gamma'(t) \in H_{\gamma(t)}$.

In the last few years, sub-Riemannian structures have been largely studied in several respects, such as differential geometry, geometric measure theory, subelliptic differential equations, complex variables, optimal control theory, mathematical models in neurosciences, non-holonomic mechanics, robotics.

Clearly, Carnot groups fit in this more general picture, playing a privileged role, akin to that of Euclidean spaces versus Riemannian manifolds, providing not only some of the most relevant examples, but also acting in some sense as rigid "tangent" spaces to general sub-Riemannian spaces (rigid because they are invariant under left translations and group dilations). Thus, they provide a natural setting for Maxwell's equations, similar to that of special relativity that is, roughly speaking, a "tangent theory" for general relativity.

We want to stress preliminarily that, in spite of the number of various applications of Carnot groups to describe different phenomena in applications, here we are not looking for any application modeling physical situations. Our purpose is to carry on - through the study of Maxwell's equations - the investigation of the peculiar features of the geometry of Carnot groups. Since we are interested in detecting non-Euclidean phenomena more than in the analogies with the classical setting, it has been intriguing to discover - as we shall see - that intrinsic Maxwell's equations yield a new class of "wave equations", with new unexpected properties

In order to develop a theory of Maxwell's equations in Carnot groups, we need a complex of "intrinsic" differential forms. This setting is provided by

Rumin's complex (E_0^*, d_c) of differential forms in a Carnot group $\mathbb{G}$. Rumin's theory needs a quite technical introduction that is sketched in Section 3 to make the paper self-consistent. For a more exhaustive presentation, we refer to original Rumin's papers [40] and [39], as well as to the presentation in [2]. The main properties of (E_0^*, d_c) that we shall use through this paper can be summarized in the following points:

- Intrinsic 1-forms are horizontal 1–forms, i.e. forms that are dual of horizontal vector fields, where by duality we mean that, if v is a vector field in $\mathbb{R}^n$, then its dual form $v^\natural$ acts as $v^\natural(w) = \langle v, w\rangle$, for all $w \in \mathbb{R}^n$.
- The "intrinsic" exterior differential d_c on a smooth function is its horizontal differential (that is the dual operator of the gradient along a basis of the horizontal bundle).
- The complex (E_0^*, d_c) is exact and self-dual under Hodge $*$-duality.

The first two properties above clearly fit our request for an "intrinsic" theory. However an even stronger evidence is provided by Theorem 3.16 that proves what we can call the "weak naturality" of the complex under homogeneous homomorphisms of the group $\mathbb{G}$. Indeed, let T be a homogeneous homomorphism of $\mathbb{G}$ (where homogeneous means that $T(\delta_\lambda x) = \delta_\lambda(Tx)$). In exponential coordinates, T can be identified with linear map $T : \mathbb{R}^n \to \mathbb{R}^n$. Suppose now that also ${}^tT$ is a homogeneous homomorphism. Then the pull-back $T^\#$ maps E_0^* into E_0^* and the following diagram is commutative:

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d_c} & E_0^h & \xrightarrow{d_c} & E_0^{h+1} & \xrightarrow{d_c} & \cdots \\
 & & \Big\downarrow{\scriptstyle T^\#} & & \Big\downarrow{\scriptstyle T^\#} & & \\
\cdots & \xrightarrow{d_c} & E_0^h & \xrightarrow{d_c} & E_0^{h+1} & \xrightarrow{d_c} & \cdots
\end{array}$$

Since the class of homogeneous homomorphisms (denoted by $HL(\mathbb{G})$) well reflects both the group structure and the stratification, the naturality of d_c under homogeneous homomorphisms shows the intimate connection between the complex and the Carnot group. Indeed, homogeneous homomorphisms between Carnot groups appear naturally as Pansu differentials of maps between Carnot groups ([37]). On the other hand, the "artificial assumption" on ${}^tT$ is extensively discussed in Remarks 3.17 and 3.13 below, and is basically motivated by the fact that we are working with classes of "true differential forms" and not with quotient classes. Nevertheless, for the purposes of the present paper, this "weak naturality" suffices, since it yields the invariance of our Maxwell's equations in $\mathbb{G}$ under the action of intrinsic Lorentz transformations.

We stress also that in [1] it is proved that, despite its technical definition, the complex appears naturally through a variational approach. Indeed, on intrinsic 1-forms, the energy associated with d_c, i.e. the functional

$$F(\omega) = \int_{\mathbb{G}} \|d_c\omega\|^2 \, dV,$$

is a suitable Γ-limit (see [11]) of "Riemannian" energies associated with the usual de Rham's exterior differential (this result is akin to some Γ-limit results in elasticity).

If we need to stress that the complex is built on a specific group $\mathbb{G}$, we shall denote it by $(E_{0,\mathbb{G}}^*, d_{c,\mathbb{G}})$, to avoid misunderstandings.

Below we shall go back to the structure of (E_0^*, d_c), but we want to emphasize here a key property of d_c that will play later a crucial role in Maxwell's theory, yielding new unexpected phenomena: in general, d_c is an operator *of higher order (i.e. greater than 1) in the horizontal derivatives* (unlike classical de Rham's exterior differential), and also, in general, not even homogenous.

At this point, we can mimic classical formulation of Maxwell's equations as follows: starting from a Carnot group $\mathbb{G}$, we define a space-time Carnot group $\mathbb{R}\times\mathbb{G}$, just by adding to the first layer of the Lie algebra the vector field $S = \partial/\partial s$, where s is the time variable, and by assuming that S commutes with all the element of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$. To avoid misunderstandings, we denote by a hat $\hat{}$ all objects related to the space-time $\mathbb{R} \times \mathbb{G}$. Thus, for instance, $\hat{\mathfrak{g}}$ will denote its Lie algebra, and $(\hat{E}_0^*, \hat{d}_c)$ its intrinsic complex. In Section 4, we write explicitly $\hat{E}_0^*$ (Lemma 4.4) and $\hat{d}_c$ (Proposition 4.7), and we define a group $HO(\mathbb{G})$ of special homogeneous automorphisms of $\mathbb{R} \times \mathbb{G}$, that plays the role of Lorentz transformations. Again as in the Euclidean setting, $\bigwedge^h \hat{\mathfrak{g}}$, the spaces of h-covectors in $\mathbb{R}\times\mathbb{G}$, can be endowed with a Minkowskian scalar product $\langle\cdot,\cdot\rangle_M$ (see Definition 4.1). In turn, if $dV = \theta_1 \wedge \cdots \wedge \theta_n$ is the canonical volume form in $\mathbb{G}$, the Minkowskian scalar product together with the natural volume form $ds\wedge dV$ define a Hodge duality operator $*_M$.

Once all this machinery is assembled, it is straightforward to write Maxwell's equations in $\mathbb{R} \times \mathbb{G}$ as

$$\hat{d}_c F = 0 \quad \text{and} \quad \hat{d}_c(*_M F) = \mathcal{J}. \tag{5}$$

Here $F \in \hat{E}_0^2$ is the unknown Faraday's form, and $\mathcal{J}$ is a fixed closed intrinsic n-form in $\mathbb{R} \times \mathbb{G}$ (a source form), that can be written as $\mathcal{J} = ds \wedge *J - \rho$, where $J = J(s,\cdot)$ is an intrinsic 1-form on $\mathbb{G}$ and $\rho(s,\cdot) = \rho_0(s,\cdot)\,dV$ is a volume form on $\mathbb{G}$ for any fixed $s \in \mathbb{R}$.

However, we have to point out that the analogies with classical theory basically stop here, and we are facing a series of new unexpected phenomena.

First of all, we notice that equations (5) are invariant under the action of the Lorentz group $HO(\mathbb{G})$, as we should expect, but the rigidity of the structure of $HL(\mathbb{R}\times\mathbb{G})$ substantially reduces the number of intrinsic Lorentz transformations that are allowed. This is due to the fact that homogeneous homomorphisms of a Carnot group $\mathbb{G}$ enjoy the *contact property* (see Theorem 2.10). If we identify $T \in HL(\mathbb{G})$ through exponential coordinates with a linear map $T : \mathfrak{g} \to \mathfrak{g}$, this means basically that T must preserve the layers of $\mathfrak{g}$ (it is a block-matrix). As pointed out in Remark 4.15, this implies that, with the exception of some special groups with a product structure, intrinsic Lorentz transformations "do not mix space and time".

However, the most interesting phenomena come from the fact that the exterior differential $\hat{d}_c$ is a non-homogeneous higher order differential operator. In this perspective, let us give a gist of how non-homogeneous higher order horizontal derivatives appear in $\hat{d}_c$ (but we stress that they may already appear when we restrict ourselves to the "stationary" d_c acting on forms on $\mathbb{G}$). To avoid cumbersome notations, let us consider a Carnot group $\mathbb{M}$, that can be both $\mathbb{G}$ or $\mathbb{R}\times\mathbb{G}$, with Lie algebra $\mathfrak{m}$. We need now the notion of *weight* of vectors in $\mathfrak{m}$ and, by duality, of covectors (see [40]). Elements of the j-th layer of $\mathfrak{m}$ are said to have (pure) weight $w=j$; by duality, a 1-covector that is dual of a vector of (pure) weight $w=j$ will be said to have (pure) weight $w=j$.

This procedure can be extended to h-forms. Clearly, there are forms that have no pure weight, but we can decompose $E^h_{0,\mathbb{M}}$ in the direct sum of orthogonal spaces of forms of pure weight, and therefore we can find a basis of $E^h_{0,\mathbb{M}}$ given by orthonormal forms of increasing pure weights. We refer to such a basis as to a basis adapted to the filtration of $E^h_{0,\mathbb{M}}$ induced by the weight.

Then, once suitable adapted bases of h-forms and $(h+1)$-forms are chosen, $d_{c,\mathbb{M}}$ can be seen as a matrix-valued operator such that, if α has weight p, then the component of weight q of $d_{c,\mathbb{M}}\alpha$ is given by an homogeneous differential operator in the horizontal derivatives of order $q-p\geq 1$, acting on the components of α.

In order to provide a concrete example of these phenomena, let us consider as in [21] the specific case $\mathbb{G}=\mathbb{H}^1\equiv\mathbb{R}^3$, the first Heisenberg group, with variables x,y,t. For sake of simplicity, we set $X:=\partial_x-\frac12 y\partial_t$, $Y:=\partial_y+\frac12 x\partial_t$, $T:=\partial_t$. The stratification of the algebra $\mathfrak{g}$ is given by $\mathfrak{g}=V_1\oplus V_2$, where $V_1=\text{span}\,\{X,Y\}$ and $V_2=\text{span}\,\{T\}$. We have $X^\natural=dx$, $Y^\natural=dy$, $T^\natural=\theta$ (the contact form of $\mathbb{H}^1$). In this case

$$
\begin{aligned}
E^1_{0,\mathbb{H}^1} &= \text{span}\,\{dx,dy\};\\
E^2_{0,\mathbb{H}^1} &= \text{span}\,\{dx\wedge\theta, dy\wedge\theta\};\\
E^3_{0,\mathbb{H}^1} &= \text{span}\,\{dx\wedge dy\wedge\theta\}.
\end{aligned}
$$

The action of d_c on $E^1_{0,\mathbb{H}^1}$ is the following ([38], [20], [4]): let $\alpha=\alpha_1dx+\alpha_2dy\in E^1_{0,\mathbb{H}^1}$ be given. Then

$$
\begin{aligned}
d_{c,\mathbb{H}^1}\alpha &= (X^2\alpha_2-2XY\alpha_1+YX\alpha_1)dx\wedge\theta\\
&\quad+(2YX\alpha_2-Y^2\alpha_1-XY\alpha_2)dy\wedge\theta\\
&:= P_1(\alpha_1,\alpha_2)dx\wedge\theta+P_2(\alpha_1,\alpha_2)dy\wedge\theta.
\end{aligned}
$$

We see that $d_{c,\mathbb{H}^1}$ is a homogeneous operator of order 2 in the horizontal derivatives, since 2-forms have weight 3 and 1-forms have weight 1.

On the other hand, if

$$
\alpha=\alpha_{13}dx\wedge ds+\alpha_{23}dy\wedge ds+\alpha_{14}dx\wedge\theta+\alpha_{24}dy\wedge\theta\in E^2_{0,\mathbb{R}\times\mathbb{H}^1},
$$

then, by Proposition 4.7 below,

$$
\begin{aligned}
d_{c,\mathbb{R}\times\mathbb{H}^1}\alpha &= (X\alpha_{24}-Y\alpha_{14})\,dx\wedge dy\wedge\theta\\
&\quad+(S\alpha_{14}-P_1(\alpha_{13},\alpha_{23}))\,ds\wedge dx\wedge\theta\\
&\quad+(S\alpha_{24}-P_2(\alpha_{13},\alpha_{23}))\,ds\wedge dy\wedge\theta.
\end{aligned}
$$

Here the operator $d_{c,\mathbb{R}\times\mathbb{H}^1}$ is no more homogeneous: indeed, though all 3-forms have weight 4, $X\alpha_{24}-Y\alpha_{14}$, $S\alpha_{14}$, and $S\alpha_{24}$ are operators of order 1, since both $\alpha_{14}dx\wedge\theta$ and $\alpha_{24}dy\wedge\theta$ have weight 3, whereas P_1 and P_2 have order 2, coherently with the fact that they act on the coefficients of forms of weight 2. Incidentally, we notice that this lack of homogeneity always appears in space-time Carnot groups (except in the commutative case).

Let us see now how this peculiarity of the intrinsic differential affects Maxwell's equations (beside to the obvious fact that they are no more first order equations).

Since the complex $(\hat{E}_0^*, \hat{d}_c)$ is exact, if $F \in \hat{E}_0^2$ is a solution of (5), we can write

$$F = \hat{d}_c(A_\Sigma + \varphi\, ds),$$

where A_Σ is an intrinsic 1-form on $\mathbb{G}$ (with coefficients depending also on s), and φ is a scalar function of s and x. As in Euclidean setting, we can proceed now to write explicitly the equations satisfied by A_Σ and φ, under suitable gauge conditions. For sake of simplicity, we assume here $J = 0$ and $\rho_0 = 0$. If we denote by δ_c the formal L^2-adjoint of d_c, using the explicit form of $\hat{d}_c$ provided by Proposition 4.7, an elementary computation shows that

$$\frac{\partial^2 A_\Sigma}{\partial s^2} = -\delta_c d_c A_\Sigma + d_c \frac{\partial \varphi}{\partial s}.$$

In the classical setting, if Lorenz gauge condition holds, we can replace the right hand side by $-(\delta_c d_c + d_c \delta_c) A_\Sigma = -\Delta A_\Sigma$. Keeping also in mind that the usual Hodge Laplacian on 1-forms is diagonal in Cartesian coordinates, it follows that the Cartesian components of A_Σ solve the classical wave equation.

On the other hand, in our case, if we want to repeat a similar argument, we face several difficulties. First of all, the "naif Hodge Laplacian" associated with d_c, i.e.

$$\delta_c d_c + d_c \delta_c, \tag{6}$$

generally is not homogeneous (and therefore, as long as we know, we lack Rockland type hypoellipticity results (see, e.g. [27]) and sharp a priori estimates in a "natural" scale of Sobolev spaces). This because d_c itself may not be homogeneous, but mainly because the two terms in (6) may have different orders. When dealing with intrinsic 1-forms, as in our case, we can recover the homogeneity of d_c with an additional assumption on $\mathbb{G}$: we assume that $\mathbb{G}$ is *free* (see Definition 5.8); then Theorem 5.9 below yields that d_c is an homogeneous differential operator of order κ (the step of the group) in the horizontal derivatives when acting on 1-forms. However, even if d_c is homogeneous, such a "Hodge Laplacian" fails to be homogeneous. For instance, on 1–forms, $\delta_c d_c$ is an operator of order 2κ, while $d_c \delta_c$ is a 2nd order one. This is due to the fact that the order of d_c depends on the order of the forms on which it acts on: d_c on intrinsic 1–forms is an operator of order κ, as well as its adjoint δ_c (which acts on 2–form), while δ_c on intrinsic 1–forms is a first order operator, since it is the adjoint of d_c on 0–forms, which is a first order operator. To overcome this difficulty, we remind that in $\mathbb{H}^1$ (where $\kappa = 2$), M. Rumin in [38] introduces a new homogeneous 4th order operator $\delta_c d_c + (d_c \delta_c)^2$ that satisfies sharp a priori estimates in intrinsic Sobolev spaces of order 4. We apply the same idea in free groups of arbitrary step κ and we obtain an homogeneous operator of order 2κ in the horizontal derivatives acting on intrinsic 1-forms

$$\Delta_{\mathbb{G},1} = \delta_c d_c + (d_c\delta_c)^\kappa.$$

We prove in Theorem 5.10 that $\Delta_{\mathbb{G},1}$ satisfies sharp a priori estimates of order 2κ and is self-adjoint (see Proposition 6.18). Assume now (as we can always do) the higher order gauge condition

$$(-\Delta_{\mathbb{G}})^{\kappa-1}\delta_c A_\Sigma + \frac{\partial\varphi}{\partial s} = 0. \tag{7}$$

where $\Delta_{\mathbb{G}} := \sum_{j=1}^m X_j^2$ is the usual subelliptic Laplacian in $\mathbb{G}$. Then we have:

$$\begin{cases} \dfrac{\partial^2 A_\Sigma}{\partial s^2} & = -\Delta_{\mathbb{G},1} A_\Sigma \\ \dfrac{\partial^2 \varphi}{\partial s^2} & = -(-\Delta_{\mathbb{G}})^\kappa \varphi \ , \end{cases} \tag{8}$$

provided (7) holds (see Theorem 5.12).

Some remarks are now in order: first of all, the equation for A_Σ cannot be diagonalized, and has to be treated as a whole. But the main new phenomenon is that the "wave equations" we obtain utterly differ even in the scalar case from what one could imagine as "wave equations in the group", i.e.

$$\frac{\partial^2\varphi}{\partial s^2} - \Delta_{\mathbb{G}}\varphi = 0. \tag{9}$$

Indeed, the equations we obtain are by no means hyperbolic equations, by [29], Theorem 5.5.2, since they contain second order derivatives in s and 2κ-th order derivatives in x, so that their principal parts are (degenerate) elliptic. Thus, we should not expect any hyperbolic behavior, as, for instance, finite speed of propagation like in (9) (see, e.g., [35], [25]). To retrieve a suggestion of the possible behavior of our solutions, let us notice that the scalar equation for φ can be written through the product of two Schrödinger operators in $\mathbb{G}$, since

$$\frac{\partial^2}{\partial s^2} + \Delta_{\mathbb{G}}^2 = \big(\frac{\partial}{\partial s} + i\Delta_{\mathbb{G}}\big)\big(\frac{\partial}{\partial s} - i\Delta_{\mathbb{G}}\big),$$

and it is natural to expect our equation to inherit intrinsic features of Schrödinger operator that essentially differ from those of the classical wave operator.

Another interesting feature of "wave equations" (8) has been already pointed out in [21]. In case of cylindrical symmetry in $\mathbb{H}^1$ (i.e. when dealing with functions depending only on the horizontal variables), the components of A_Σ as well as φ all solve the equation

$$\frac{\partial^2 u}{\partial s^2} = -\Delta^2 u \quad \text{in } \mathbb{R}^2,$$

with suitable Cauchy data at $s = 0$. If we consider this equation in a cylinder $\Omega \times \mathbb{R}$, where Ω is a (say) bounded open subset of $\mathbb{R}^m$, we can impose boundary conditions on $\partial\Omega$. In this way, we recover a classical equation of elasticity, the so-called Germain-Lagrange equation for the vibration of plates (see e.g. [43], Section 9).

We point out also another discrepancy with the Euclidean case, that arises when we want to reverse our result and to derive solutions of Maxwell's equations from "wave equations" (8). This derivation is in fact possible, but, unlike in the classical case, it is not just matter of straightforward computations. Indeed (see Theorem 5.15), to achieve the proof we have to rely on quite delicate Liouville type theorem for subelliptic Laplacians in Carnot groups that are proved in [8].

We notice that one could legitimately ask at what extent equations (8) are "natural". Indeed, the operator $\Delta_{\mathbb{G},1}$ could be considered "artificial", since it is obtained by adding an "artificial" term $(d_c\delta_c)^\kappa$ to the "natural" term $\delta_c d_c$, just to obtain a Rockland type operator. However, we point out that this term comes from our choice of the gauge condition, that is, by its nature, arbitrary, precisely as in the usual choice of Lorenz gauge in $\mathbb{R}^4$ that yields the usual Hodge Laplace operator and the associated d'Alembert equation.

Finally, as for the existence and regularity of solutions of our "wave equations", in Section 6 we prove the existence of solutions in the natural Sobolev spaces by means of the theory of the so-called abstract cosine functions for second order evolution equations in Banach spaces (see Theorems 6.19 and 6.20). If we restrict ourselves to step 2 groups, we also prove in Theorem 6.1 the existence of plane wave solutions of our "wave equations". Again, plane wave has to be understood in the intrinsic sense of the group, since their wave fronts are group hyperplanes (in the sense, e.g., of [18], [17], i.e. maximal subgroups that contain the center of $\mathbb{G}$). Again, the non-Riemannian character of the group geometry yields new different phenomena for plane waves that are described in Proposition 6.14 and Remark 6.16.

2. Multilinear algebra in Carnot groups

Let $(\mathbb{G},\cdot)$ be a *Carnot group of step* κ identified to $\mathbb{R}^n$ through exponential coordinates (see [8] for details). By definition, the Lie algebra $\mathfrak{g}$ has dimension n, and admits a *step* κ *stratification*, i.e. there exist linear subspaces $V_1,...,V_\kappa$ such that

$$\mathfrak{g} = V_1 \oplus ... \oplus V_\kappa, \quad [V_1,V_i] = V_{i+1}, \quad V_\kappa \neq \{0\}, \quad V_i = \{0\} \text{ if } i>\kappa, \tag{10}$$

where $[V_1,V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X,Y]$ with $X\in V_1$ and $Y\in V_i$. Set $m_i=\dim(V_i)$, for $i=1,\dots,\kappa$ and $h_i=m_1+\dots+m_i$ with $h_0=0$. Clearly, $h_\kappa=n$.

We say that $\mathbb{G}$ is a free Carnot group if its algebra $\mathfrak{g}$ is isomorphic the free Lie algebra $\mathfrak{f}_{m_1,\kappa}$ (see, for instance [8], Section 14.1).

Choose now a basis $e_1,\dots,e_n$ of $\mathfrak{g}$ adapted to the stratification, i.e. such that

$$e_{h_{j-1}+1},\dots,e_{h_j} \text{ is a basis of } V_j \text{ for each } j=1,\dots,\kappa.$$

Let $X=\{X_1,\dots,X_n\}$ be the family of left invariant vector fields such that $X_i(0)=e_i$, $i=1,\dots,n$. The Lie algebra $\mathfrak{g}$ can be endowed with a scalar product $\langle\cdot,\cdot\rangle$, making $\{X_1,\dots,X_n\}$ an orthonormal basis. If we are dealing with free groups, choosing a Grayson-Grossman-Hall basis of $\mathfrak{g}$ (see [24] and [8], Theorem 14.1.10) makes several computations simpler.

Since $\mathbb{G}$ is written in exponential coordinates, a point $p \in \mathbb{G}$ is identified with the n-tuple $(p_1, \dots, p_n) \in \mathbb{R}^n$ and we we can identify $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$, where the explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula.

For any $x \in \mathbb{G}$, the *(left) translation* $\tau_x : \mathbb{G} \to \mathbb{G}$ is defined as

$$z \mapsto \tau_x z := x \cdot z.$$

For any $\lambda > 0$, the *dilation* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, is defined as

$$\delta_\lambda(x_1, ..., x_n) = (\lambda^{d_1} x_1, ..., \lambda^{d_n} x_n), \tag{11}$$

where $d_i \in \mathbb{N}$ is called *homogeneity of the variable* x_i in $\mathbb{G}$ (see [16] Chapter 1) and is defined as

$$d_j = i \quad \text{whenever } h_{i-1} + 1 \le j \le h_i. \tag{12}$$

The dilations δ_λ are group automorphisms, since

$$\delta_\lambda x \cdot \delta_\lambda y = \delta_\lambda(x \cdot y).$$

We remind that the generating vector fields $X_1, \dots, X_m$ are homogeneous of degree 1 with respect to group dilations.

As customary, we fix a smooth homogeneous norm $|\cdot|$ in $\mathbb{G}$ such that the gauge distance $d(x, y) := |y^{-1}x|$ is a left-invariant true distance, equivalent to the Carnot-Carathéodory distance in $\mathbb{G}$ (see [42], p.638). We set $B(p, r) = \{q \in \mathbb{G};\ d(p, q) < r\}$.

The Haar measure of $\mathbb{G} = (\mathbb{R}^n, \cdot)$ is the Lebesgue measure $\mathcal{L}^n$ in $\mathbb{R}^n$. If $A \subset \mathbb{G}$ is $\mathcal{L}$-measurable, we write also $|A| := \mathcal{L}^n(A)$.

We denote by Q the *homogeneous dimension* of $\mathbb{G}$, i.e. we set

$$Q := \sum_{i=1}^{\kappa} i \dim(V_i).$$

Since for any $x \in \mathbb{G}$ $|B(x, r)| = |B(e, r)| = r^Q |B(e, 1)|$, Q is the Hausdorff dimension of the metric space $(\mathbb{G}, d)$.

By (10), the subset $X_1, \dots, X_{m_1}$ generates by commutations all the other vector fields. Therefore, the subbundle of the tangent bundle $T\mathbb{G}$ that is spanned by $X_1, \dots, X_{m_1}$ plays a particularly important role in the theory, and it is called the *horizontal bundle* $H\mathbb{G}$; the fibers of $H\mathbb{G}$ are

$$H\mathbb{G}_x = \text{span } \{X_1(x), \dots, X_{m_1}(x)\}, \qquad x \in \mathbb{G}.$$

From now on, for sake of simplicity, we set $m := m_1$.

A subriemannian structure is defined on $\mathbb{G}$, endowing each fiber of $H\mathbb{G}$ with a scalar product $\langle \cdot, \cdot \rangle_x$ making the basis $X_1(x), \dots, X_m(x)$ an orthonormal basis. The sections of $H\mathbb{G}$ are called *horizontal sections*, and a vector of $H\mathbb{G}_x$ is an *horizontal vector*.

The Euclidean space $\mathbb{R}^n$ endowed with the usual (commutative) sum of vectors provides the simplest example of Carnot group. It is a trivial example, since in this case the stratification of the algebra consists of only one layer, i.e. the Lie algebra reduces to the horizontal layer.

Following [16], we also adopt the following multi-index notation for higher-order derivatives. If $I = (i_1, \dots, i_n)$ is a multi–index, we set $X^I = X_1^{i_1} \cdots X_n^{i_n}$. By the Poincaré–Birkhoff–Witt theorem (see, e.g. [9], I.2.7), the differential operators X^I form a basis for the algebra of left invariant differential operators in $\mathbb{G}$. Furthermore, we set $|I| := i_1+\cdots+i_n$ the order of the differential operator X^I, and $d(I) := d_1 i_1 + \cdots + d_n i_n$ its degree of homogeneity with respect to group dilations. From the Poincaré–Birkhoff–Witt theorem, it follows, in particular, that any homogeneous linear differential operator in the horizontal derivatives can be expressed as a linear combination of the operators X^I of the special form above.

Let k be a positive integer, $1 \le p < \infty$, and let Ω be an open set in $\mathbb{G}$. The Folland-Stein Sobolev space $W^{k,p}_{\mathbb{G}}(\Omega)$ associated with the vector fields $X_1, \dots, X_m$ is defined to consist of all functions $f \in L^p(\Omega)$ with distributional derivatives $X^I f \in L^p(\Omega)$ for any X^I as above with $d(I) \le k$, endowed with the natural norm. We keep the subscript $\mathbb{G}$ to avoid misunderstanding with the usual Sobolev spaces $W^{k,p}(\Omega)$.

Again following e.g. [16], we can define a group convolution in $\mathbb{G}$: if, for instance, $f \in \mathcal{D}(\mathbb{G})$ and $g \in L^1_{\mathrm{loc}}(\mathbb{G})$, we set

$$f * g(p) := \int f(q) g(q^{-1}p)\, dq \quad \text{for } p \in \mathbb{G}. \tag{13}$$

We remind that, if (say) g is a smooth function and L is a left invariant differential operator, then $L(f*g) = f * Lg$. In addition

$$\langle f * g | \varphi \rangle = \langle g | {}^{\mathrm{v}} f * \varphi \rangle \quad \text{and} \quad \langle f * g | \varphi \rangle = \langle f | \varphi * {}^{\mathrm{v}} g \rangle \tag{14}$$

for any test function φ. Suppose now $f \in \mathcal{E}'(\mathbb{G})$ and $g \in \mathcal{D}'(\mathbb{G})$. Then, if $\psi \in \mathcal{D}(\mathbb{G})$, we have (all convultions being well defined)

$$\begin{aligned} \langle (X^I f) * g | \psi \rangle &= \langle X^I f | \psi * {}^{\mathrm{v}} g \rangle = (-1)^{|I|} \langle f | \psi * ((X^I)^* \, {}^{\mathrm{v}} g) \rangle \\ &= (-1)^{|I|} \langle f * {}^{\mathrm{v}} (X^I)^* \, {}^{\mathrm{v}} g | \psi \rangle. \end{aligned} \tag{15}$$

We remind now the notion of *kernel of order* α. Following [15], a kernel of order α is a homogeneous distribution of degree $\alpha - Q$ (with respect to group dilations), that is smooth outside of the origin.

Proposition 2.1. *Let $K \in \mathcal{D}'(\Omega)$ be a kernel of order α.*

- i) *${}^{\mathrm{v}} K$ is again a kernel of order α;*
- ii) *$X_\ell K$ is a a kernel of order $\alpha - 1$ for any horizontal derivative $X_\ell K$, $\ell = 1, \dots, m$;*
- iii) *If $\alpha > 0$, then $K \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$;*
- iv) *if $\alpha = 0$, then the map $f \to f * K$ is L^p.continuous for $1 < p < \infty$.*

Proof. Assertions ii) and iii) are contained in [15]. Assertion i) follows since $\delta_t(p^{-1}) = (\delta_t p)^{-1}$ for $t > 0$ and $p \in \mathbb{G}$. As for iv), we refer to [15] Proposition 1.9, or to [32] $\square$

The dual space of $\mathfrak{g}$ is denoted by $\bigwedge^1 \mathfrak{g}$. The basis of $\bigwedge^1 \mathfrak{g}$, dual of the basis $X_1, \cdots, X_n$, is the family of covectors $\{\theta_1, \cdots, \theta_n\}$. We indicate by $\langle \cdot, \cdot \rangle$ also the inner product in $\bigwedge^1 \mathfrak{g}$ that makes $\theta_1, \cdots, \theta_n$ an orthonormal basis. We point out that, except for the trivial case of the commutative group $\mathbb{R}^n$, the forms $\theta_1, \cdots, \theta_n$ may have polynomial (hence variable) coefficients. In addition, if $\mathbb{G}$ is a free group, because of our choice of $X_1, \dots, X_n$ as in Grayson-Grossman [24], we have $\theta_i = dx_i$, $i = 1, \dots, m$.

Following Federer (see [14] 1.3), the exterior algebras of $\mathfrak{g}$ and of $\bigwedge^1 \mathfrak{g}$ are the graded algebras indicated as $\bigwedge_* \mathfrak{g} = \bigoplus_{h=0}^{n} \bigwedge_h \mathfrak{g}$ and $\bigwedge^* \mathfrak{g} = \bigoplus_{h=0}^{n} \bigwedge^h \mathfrak{g}$ where $\bigwedge_0 \mathfrak{g} = \bigwedge^0 \mathfrak{g} = \mathbb{R}$ and, for $1 \leq h \leq n$,

$$\bigwedge_h \mathfrak{g} := \operatorname{span}\{X_{i_1} \wedge \cdots \wedge X_{i_h} : 1 \leq i_1 < \cdots < i_h \leq n\},$$
$$\bigwedge^h \mathfrak{g} := \operatorname{span}\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_h} : 1 \leq i_1 < \cdots < i_h \leq n\}.$$

The elements of $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$ are called *h-vectors* and *h-covectors*.

We denote by Θ^h the basis $\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_h} : 1 \leq i_1 < \cdots < i_h \leq n\}$ of $\bigwedge^h \mathfrak{g}$.

The dual space $\bigwedge^1(\bigwedge_h \mathfrak{g})$ of $\bigwedge_h \mathfrak{g}$ can be naturally identified with $\bigwedge^h \mathfrak{g}$. The action of a h-covector φ on a h-vector v is denoted as $\langle \varphi | v \rangle$.

The inner product $\langle \cdot, \cdot \rangle$ extends canonically to $\bigwedge_h \mathfrak{g}$ and to $\bigwedge^h \mathfrak{g}$ making the bases $X_{i_1} \wedge \cdots \wedge X_{i_h}$ and $\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ orthonormal.

Definition 2.2. We define linear isomorphisms (Hodge duality: see [14] 1.7.8)

$$* : \bigwedge_h \mathfrak{g} \longleftrightarrow \bigwedge_{n-h} \mathfrak{g} \quad \text{and} \quad * : \bigwedge^h \mathfrak{g} \longleftrightarrow \bigwedge^{n-h} \mathfrak{g},$$

for $1 \leq h \leq n$, putting, for $v, w \in \bigwedge_h \mathfrak{g}$ and $\varphi, \psi \in \bigwedge^h \mathfrak{g}$

$$v \wedge *w = \langle v, w \rangle X_1 \wedge \cdots \wedge X_n, \qquad \varphi \wedge *\psi = \langle \varphi, \psi \rangle \theta_1 \wedge \cdots \wedge \theta_n.$$

It is easy to see that

$$\begin{aligned} &**v = (-1)^{h(n-h)} v, \qquad **\varphi = (-1)^{h(n-h)} \varphi, \\ &\langle *\varphi | *v \rangle = \langle \varphi | v \rangle. \end{aligned} \tag{16}$$

From now on, we refer to the n-form

$$dV := \theta_1 \wedge \cdots \wedge \theta_n$$

as to the canonical volume form in $\mathbb{G}$.

Notice that, if $v = v_1 \wedge \cdots \wedge v_h$ is a simple h-vector, then $*v$ is a simple $(n-h)$-vector.

If $v \in \bigwedge_h \mathfrak{g}$ we define $v^\natural \in \bigwedge^h \mathfrak{g}$ by the identity $\langle v^\natural | w \rangle := \langle v, w \rangle$, and analogously we define $\varphi^\natural \in \bigwedge_h \mathfrak{g}$ for $\varphi \in \bigwedge^h \mathfrak{g}$.

To fix our notations, we remind the following definition (see e.g. [23], Section 2.1).

Definition 2.3. If V, W are finite dimensional linear vector spaces and $L : V \to W$ is a linear map, we define

$$\Lambda_h L : \bigwedge_h V \to \bigwedge_h W$$

as the linear map defined by

$$(\Lambda_h L)(v_1 \wedge \cdots \wedge v_h) = L(v_1) \wedge \cdots \wedge L(v_h)$$

for any simple h-vector $v_1 \wedge \cdots \wedge v_h \in \bigwedge_h V$, and

$$\Lambda^h L : \bigwedge^h W \to \bigwedge^h V$$

as the linear map defined by

$$\langle (\Lambda^h L)(\alpha) | v_1 \wedge \cdots \wedge v_h \rangle = \langle \alpha | (\Lambda_h L)(v_1 \wedge \cdots \wedge v_h) \rangle$$

for any $\alpha \in \bigwedge^h W$ and any simple h-vector $v_1 \wedge \cdots \wedge v_h \in \bigwedge_h V$.

Proposition 2.4. *If V, W are finite dimensional linear vector spaces endowed with a scalar product that is naturally extended to the associated graded algebras. Let $L : V \to W$ be a linear map, then*

- i) *if $v \in \bigwedge_1 V$ and $\alpha \in \bigwedge^1 W$, then $(\Lambda_1 L)v = Lv$ and $((\Lambda^1 L)\alpha)^\natural = L^*(\alpha^\natural)$;*
- ii) *if $\alpha \in \bigwedge^k W$ and $\beta \in \bigwedge^h W$, then $(\Lambda^{k+h} L)(\alpha \wedge \beta) = (\Lambda^k L)\alpha \wedge (\Lambda^k L)\beta$;*
- iii) *if $v \in \bigwedge_k V$ and $w \in \bigwedge_h V$, then $(\Lambda_{k+h} L)(v \wedge w) = (\Lambda_k L)v \wedge (\Lambda_h L)w$;*
- iv) *${}^t(\Lambda_h L) = \Lambda_h({}^t L)$ and ${}^t(\Lambda^h L) = \Lambda^h({}^t L)$;*
- v) *if H is another finite dimensional linear vector space and $G : H \to V$ is a linear map, then $\Lambda_h(L \circ G) = (\Lambda_h L) \circ (\Lambda_h G)$ and $\Lambda^h(L \circ G) = (\Lambda^h G) \circ (\Lambda^h L)$;*
- vi) *if $L : V \to V$ is a unitary linear operator, then $\Lambda_h L$ and $\Lambda^h L$ are linear isometries. Moreover*

$$*((\Lambda^h L)\alpha) = (\det L) \cdot (\Lambda^h L) * \alpha.$$

We can define now two families of vector bundles (still denoted by $\bigwedge_* \mathfrak{g}$ and $\bigwedge^* \mathfrak{g}$ over $\mathbb{G}$), by putting

$$\bigwedge\nolimits_{h,p} \mathfrak{g} := (\Lambda_h d\tau_p)(\bigwedge\nolimits_{h,e} \mathfrak{g}) \tag{17}$$

and, respectively,

$$\bigwedge\nolimits_p^h \mathfrak{g} := (\Lambda^h d\tau_{p^{-1}})(\bigwedge\nolimits_e^h \mathfrak{g}) \tag{18}$$

for any $p \in \mathbb{G}$ and $h = 1, \ldots, n$, where we have chosen

$$\bigwedge\nolimits_{h,e} \mathfrak{g} \equiv \bigwedge\nolimits_h \mathfrak{g} \quad \text{and} \quad \bigwedge\nolimits_e^h \mathfrak{g} \equiv \bigwedge\nolimits^h \mathfrak{g}.$$

If, for instance, Θ^h is a basis of $\bigwedge^h \mathfrak{g}$, then $\Theta_p^h := (\Lambda^h d\tau_{p^{-1}})(\Theta^h)$ is a basis of the fiber $\bigwedge_p^h \mathfrak{g}$ of $\bigwedge^h \mathfrak{g}$ over $p \in \mathbb{G}$. We refer to the section $p \to \Theta_p^h$ of $\bigwedge^h \mathfrak{g}$ as to the *left invariant moving frame* associated with Θ^h.

The inner products $\langle \cdot, \cdot \rangle$ on $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$ induce inner products on each fiber $\bigwedge_{h,p} \mathfrak{g}$ and $\bigwedge_p^h \mathfrak{g}$ by the identity

$$\langle \Lambda_h d\tau_p(v), \Lambda_h d\tau_p(w) \rangle_p := \langle v, w \rangle$$

and

$$\langle \Lambda^h d\tau_{p^{-1}}(\alpha), \Lambda^h d\tau_{p^{-1}}(\beta) \rangle_p := \langle \alpha, \beta \rangle.$$

Lemma 2.5. *If $p, q \in \mathbb{G}$, then*

$$\Lambda_h d\tau_q : \bigwedge\nolimits_{h,p} \mathfrak{g} \to \bigwedge\nolimits_{h,qp} \mathfrak{g}$$

and

$$\Lambda^h d\tau_{q^{-1}} : \bigwedge\nolimits_p^h \mathfrak{g} \to \bigwedge\nolimits_{qp}^h \mathfrak{g}$$

are isometries onto.

In general, a subbundle $\mathcal{N}$ of $\bigwedge_h \mathfrak{g}$ is said to be left-invariant if

$$\mathcal{N}_p = (\Lambda_h d\tau_p)(\mathcal{N}_e)$$

for all $p \in \mathbb{G}$. Analogously, a subbundle $\mathcal{N}$ of $\bigwedge^h \mathfrak{g}$ is said to be left-invariant if

$$\mathcal{N}_p := (\Lambda^h d\tau_{p^{-1}})(\mathcal{N}_e)$$

for all $p \in \mathbb{G}$.

From now on, if $\mathcal{U} \subset \mathbb{G}$ is an open set and $h = 0, 1, \dots, n$ we denote by $\Omega_h(\mathcal{U})$ and $\Omega^h(\mathcal{U})$ the sets of all sections of $\bigwedge_h \mathfrak{g}$ and $\bigwedge^h \mathfrak{g}$, respectively. If $\mathcal{U} = \mathbb{G}$ we write only Ω_h and Ω^h. We refer to elements of Ω_h as to fields of h-vectors and to elements of Ω^h as to h-forms.

If X is a vector field and α is a h-form, we denote by $i_X\alpha$ the contraction of α with X given by $(i_X\alpha)(v_1 \wedge \cdots \wedge v_{h-1}) := \alpha(X \wedge v_1 \wedge \cdots \wedge v_{h-1})$.

If d is the usual De Rham's exterior differential, we denote by $\delta = d^*$ its formal adjoint in $L^2(\mathbb{G}, \Omega^*)$. We remind that, when acting on h-forms

$$\delta = (-1)^{n(h+1)+1} * d * . \tag{19}$$

As customary, if $f : \mathbb{G} \to \mathbb{G}$ is a continuously differentiable map, then the pull-back $f^\# \omega$ of a form $\omega \in \Omega^h(\mathbb{G})$ is defined by

$$f^\# \omega(x) := \big(\Lambda^h(df_x)\big)\omega(f(x)).$$

If $v \in \Omega_h(\mathbb{G})$, we set also

$$f_\# v(y) := \big(\Lambda_h(df^{-1}_{f(y)})\big)v(f(y)).$$

We have

$$\langle f^\# \omega | f_\# v\rangle = \langle \omega | v\rangle \circ f. \tag{20}$$

A h-form α on $\mathbb{G}$ is said left-invariant if $\tau_p^\# \alpha = \alpha$ for any $p \in \mathbb{G}$. If in particular $x \in \mathbb{G}$ is arbitrary and we take $p = x^{-1}$, we get

$$\alpha(x) = (\Lambda^h d\tau_{x^{-1}})\alpha(e). \tag{21}$$

Lemma 2.6. *Let $\xi \in \bigwedge^h \mathfrak{g} \equiv \bigwedge^h_e \mathfrak{g}$ be given. If $x \in \mathbb{G}$, we set $I_\xi(x) := (\Lambda^h d\tau_{x^{-1}})\xi$. Then*

i) *the map $x \to I_\xi(x)$ belongs to Ω^h and is left-invariant;*
ii) *any left-invariant form $\alpha \in \Omega^h$ has the form $\alpha = I_{\alpha(e)}$;*

Proof. By (18), $I_\xi(x) \in \bigwedge^h_x \mathfrak{g}$ and therefore the map $x \to I_\xi(x)$ is a section of $\bigwedge^h \mathfrak{g}$. Take now $p \in \mathbb{G}$. Keeping in mind Proposition 2.4, vi), we have

$$\begin{aligned}(\tau_p^\# I_\xi)(x) &= (\Lambda^h d\tau_p) I_\xi(p \cdot x) = (\Lambda^h d\tau_p) \circ (\Lambda^h d\tau_{x^{-1} \cdot p^{-1}})\xi \\ &= (\Lambda^h (d\tau_{x^{-1} \cdot p^{-1}} \circ d\tau_p))\xi = \Lambda^h(d\tau_{x^{-1}})\xi \\ &= I_\xi(x).\end{aligned}$$

This proves that I_ξ is left-invariant. The second assertion follows from (21). $\square$

Remark 2.7. In the sequel, we use also another notation, that has to be clarified to avoid misunderstandigs: let (for instance) α be a differential form in $\mathbb{G}$, and let $\varphi : \mathbb{G} \to \mathbb{G}$ be a (say) continuous map. Once a basis Θ^* of $\bigwedge^* \mathfrak{g}$ (and hence a "moving frame" of the fiber bundle $\bigwedge^* \mathfrak{g}$) is fixed, we denote by $\alpha \circ \varphi$ the section of $\bigwedge^* \mathfrak{g}$ with coefficients with respct to Θ^*_x coinciding with those of $\alpha(\varphi(x))$ with respect to $\Theta^*_{\varphi(x)}$.

Let $\mathbb{G}$ and $\mathbb{M}$ be two Carnot groups, and let $\mathfrak{g} = \oplus_{i=1}^{\kappa_1} V_i$ and $\mathfrak{m} = \oplus_{i=1}^{\kappa_2} W_i$ be their Lie algebras (respectively n-dimensional and N-dimensional).

We denote by $\hat{e}_1, \dots, \hat{e}_N$ an adapted basis of $\mathfrak{m}$, and by $\hat{X}_1, \dots, \hat{X}_N$ the corresponding family of vector fields.

Definition 2.8. A map $L : \mathbb{G} \to \mathbb{M}$ is said to be H-linear (and we write $L \in HL(\mathbb{G}, \mathbb{M})$) if

i) it is a group homomorphism;
ii) it is homogeneous, i.e. $\delta_r(Lx) = L(\delta_r x)$ for all $r > 0$.

A H-linear map induces an algebra homomorphism (that we still denote by L) between $\mathfrak{g}$ and $\mathfrak{m}$ by taking $\ln \circ L \circ \exp$. In particular the induced map L is linear.

Since we are using exponential coordinates in $\mathbb{G}$ and $\mathbb{M}$, the map L itself from $\mathbb{G}$ to $\mathbb{M}$ can be written as $N \times n$ real matrix, and we still denote by $HL(\mathbb{G}, \mathbb{M})$ the set of associated matrices. Finally, if $\mathbb{M} = \mathbb{G}$, we write $HL(\mathbb{G}) := HL(\mathbb{G}, \mathbb{G})$.

In addition, we denote by $HU_{\mathbb{R}}(\mathbb{G}) \subset HL(\mathbb{G})$ the subgroup of the unitary real $n \times n$ matrices satisfying i) and ii).

Definition 2.9. Let $\mathbb{G}$ and $\mathbb{M}$ be Carnot groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{m}$. A linear map $L : \mathfrak{g} \to \mathfrak{m}$ is said to have the *contact property* if

$$L(V_i) \subset W_i \quad i = 1, \dots, \kappa_1. \tag{22}$$

If the groups $\mathbb{G}$ and $\mathbb{M}$ are written in exponential coordinates in $\mathbb{R}^n$ and $\mathbb{R}^N$ respectively, and the map $L : \mathfrak{g} \to \mathfrak{m}$ has the contact property, then L can be seen as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^N$ (that in general fails to be an homomorphism).

Theorem 2.10 ([34], Corollary 3.15 and [36])**.** *Let $L : \mathbb{G} \to \mathbb{M}$ be a H-linear map. Then L enjoys the contact property* (22).

Remark 2.11. If a linear map L is a contact map, we have also

$$^t L(W_i) \subset V_i \quad i = 1, \dots, \kappa_2.$$

In other words, ^tL is again a contact map.

Indeed, take $j \neq i$ and $\alpha \in V_j$. If $\theta \in W_i$, then $\langle {}^tL(\theta), \alpha\rangle = \langle \theta, L(\alpha)\rangle = 0$, since $L(\alpha) \in W_j$, by Theorem 2.10.

Example 2.12. In $\mathbb{H}^1$, H-linear maps are associated with 3×3 real matrices of the form (see [36], [31])

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{44}, \end{pmatrix}, \quad \text{with} \quad a_{44} = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

More generally, if we denote by J the symplectic $(2N \times 2N)$-matrix

$$J := \begin{pmatrix} 0_{N\times N} & I_{N\times N} \\ -I_{N\times N} & 0_{N\times N} \end{pmatrix},$$

then the real $(2N+1)\times(2N+1)$ real matrix

$$A := \begin{pmatrix} A_{2N\times 2N} & 0_{2N\times 1} \\ 0_{1\times 2N} & a \end{pmatrix}$$

belongs to $HL(\mathbb{H}^N)$ if and only if

$${}^tAJA = aJ.$$

If $a > 0$, then the above condition reads as

$$\frac{1}{\sqrt{a}}A \text{ belongs to the symplectic group } Sp_N(\mathbb{R}).$$

For characterizations and properties of $Sp_N(\mathbb{R})$ we refer to [7], Section 1.2.

Example 2.13. Later on, we have to deal with a space-time group like $\mathbb{R}\times\mathbb{H}^1$. In this case, a H-linear map $L : \mathbb{R}\times\mathbb{H}^1 \to \mathbb{R}\times\mathbb{H}^1$ has the two following possible structures:

i) either the associated matrix L has the form

$$L = \left(\begin{array}{ccc|c} & & & 0 \\ & L_0 & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right),$$

where L_0 is a 3×3 real matrix with the last two row linearly dependent,

ii) or the associated matrix L has the form

$$L = \begin{pmatrix} a_{00} & a_{01} & a_{02} & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ 0 & 0 & 0 & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix},$$

with $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

More generally, let $\mathbb{G}$ be a step 2 group. First of all, a matrix $A \in HL(\mathbb{G})$ has a block structure of the form

$$A := \begin{pmatrix} A^{(1)}_{m_1\times m_1} & 0_{m_1\times m_2} \\ 0_{m_2\times m_1} & A^{(2)}_{m_2\times m_2} \end{pmatrix}.$$

Denote a point $x \in \mathbb{G}$ as $x = (x', x'')$, with $x' \in \mathbb{R}^{m_1}$ and $x'' \in \mathbb{R}^{m_2}$; then $(x\cdot y)' = x' + y'$ and

$$(x\cdot y)_j = x_j + y_j + \langle Q_j x', y'\rangle_{\mathbb{R}^{m_1}}, \quad j = m_1+1, \dots m_2,$$

where the Q_j's are $m_1 \times m_1$ real matrices (see e.g. [17], Proposition 2.1). If we denote now by $a^{(2)}_{ij}$ the entries of $A^{(2)}_{m_2\times m_2}$, a direct computation shows that $A \in HL(\mathbb{G})$ if and only if

$${}^tA^{(1)}_{m_1\times m_1} Q_i A^{(1)}_{m_1\times m_1} = \sum_j a^{(2)}_{ij} Q_j.$$

3. Weights of forms and Rumin's complex

Definition 3.1. If $\alpha \in \bigwedge^1 \mathfrak{g}$, $\alpha \neq 0$, we say that α has *pure weight* k, and we write $w(\alpha) = k$, if $\alpha^\natural \in V_k$. More generally, if $\alpha \in \bigwedge^h \mathfrak{g}$, we say that α has pure weight k if α is a linear combination of covectors $\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ with $w(\theta_{i_1}) + \cdots + w(\theta_{i_h}) = k$.

Remark 3.2. If $\alpha, \beta \in \bigwedge^h \mathfrak{g}$ and $w(\alpha) \neq w(\beta)$, then $\langle \alpha, \beta \rangle = 0$. Indeed, it is enough to notice that, if $w(\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}) \neq w(\theta_{j_1} \wedge \cdots \wedge \theta_{j_h})$, with $i_1 < i_2 < \cdots < i_h$ and $j_1 < j_2 < \cdots < j_h$, then for at least one of the indices $\ell = 1, \dots, h$, $i_\ell \neq j_\ell$, and hence $\langle \theta_{i_1} \wedge \cdots \wedge \theta_{i_h}, \theta_{j_1} \wedge \cdots \wedge \theta_{j_h} \rangle = 0$.

We have ([2], formula (16))

$$\bigwedge^h \mathfrak{g} = \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \bigwedge^{h,p} \mathfrak{g}, \tag{23}$$

where $\bigwedge^{h,p} \mathfrak{g}$ is the linear span of the h–covectors of weight p and $M_h^{\min}$, $M_h^{\max}$ are respectively the smallest and the largest weight of left-invariant h-covectors.

Keeping in mind the decomposition (23), we can define in the same way several left invariant fiber bundles over $\mathbb{G}$, that we still denote with the same symbol $\bigwedge^{h,p} \mathfrak{g}$.

We notice also that the fiber $\bigwedge_x^h \mathfrak{g}$ (and hence the fiber $\bigwedge_x^{h,p} \mathfrak{g}$) can be endowed with a natural scalar product $\langle \cdot, \cdot \rangle_x$.

We denote by $\Omega^{h,p}$ the vector space of all smooth h–forms in $\mathbb{G}$ of pure weight p, i.e. the space of all smooth sections of $\bigwedge^{h,p} \mathfrak{g}$. We have

$$\Omega^h = \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \Omega^{h,p}. \tag{24}$$

The following crucial property of the weight follows from Cartan identintity: see [40], Section 2.1:

Lemma 3.3. *We have $d(\bigwedge^{h,p} \mathfrak{g}) \subset \bigwedge^{h+1,p} \mathfrak{g}$, i.e., if $\alpha \in \bigwedge^{h,p} \mathfrak{g}$ is a left invariant h-form of weight p with $d\alpha \neq 0$, then $w(d\alpha) = w(\alpha)$.*

Proposition 3.4. *If L enjoys the contact property* (22)*, then*

$$\Lambda^h L : \bigwedge^{h,p} \mathfrak{g} \to \bigwedge^{h,p} \mathfrak{g}$$

for $h = 1, \dots, n$ and $M_h^{\min} \le p \le M_h^{\max}$.

Proof. Let $\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ be an element of the basis $\Theta^{h,p}$ of $\bigwedge^{h,p} \mathfrak{g}$. Since

$$\Lambda^h L(\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}) = (\Lambda^1 L)\theta_{i_1} \wedge \cdots \wedge (\Lambda^1 L)\theta_{i_h},$$

we have only to show that $w((\Lambda^1 L)\theta_\ell) = w(\theta_\ell)$ for $\ell = 1, \dots, n$. But this follows straightforwardly from Theorem 2.10, since, by Proposition 2.4, i),

$$((\Lambda^1 L)\theta_\ell)^\natural = {}^t L(X_\ell) \in V_{w(\theta_\ell)},$$

by Remark 2.11. $\square$

Definition 3.5. Let now $\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i \, \theta_i^h \in \Omega^{h,p}$ be a (say) smooth form of pure weight p. Then we can write

$$d\alpha = d_0\alpha + d_1\alpha + \cdots + d_\kappa\alpha,$$

where

$$d_0\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i d\theta_i^h$$

does not increase the weight,

$$d_1\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \sum_{j=1}^{m_1} (X_j\alpha_i)\theta_j \wedge \theta_i^h$$

increases the weight by 1, and, more generally,

$$d_i\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \sum_{X_j \in V_i} (X_j\alpha_i)\theta_j \wedge \theta_i^h,$$

when $i = 0, 1, \ldots, \kappa$. In particular, d_0 is an algebraic operator.

Lemma 3.6. *$d_0^2 = 0$, i.e. (Ω^*, d_0) is a complex.*

Proof. Take $\alpha \in \Omega^{h,p}$, and write the identity $d^2\alpha = 0$, gathering all terms according their weights. Since terms with different weights are orthogonal, this yields that all groups of given weight vanish. But the group of weight p is precisely $d_0^2\alpha$, and we are done.

□

Lemma 3.7. *Let $\alpha \in \Omega^h$ be left-invariant. We have:*

i) *$d\alpha = d_0\alpha$;*
ii) *$d_0\alpha$ is left-invariant.*

Proof. The first assertion is straightforward by Lemma 3.3. As for ii), if $p \in \mathbb{G}$ we have

$$\tau_p^\#(d_0\alpha) = \tau_p^\#(d\alpha) = d(\tau_p^\#\alpha) = d\alpha = d_0\alpha.$$

□

We denote by δ_i the formal L^2-adjoint of d_i for $i = 0, \ldots, \kappa$. We stress that also δ_0 is an algebric operator.

Proposition 3.8. *If $h = 0, 1, \ldots, n$, $M_h^{\min} \le p \le M_{h-1}^{\max}$, and $i = 0, 1, \ldots, \kappa$, $i \le p - M_{h-1}^{\min}$, we have*

$$\delta_i(\Omega^{h,p}) \subset \Omega^{h-1,p-i} \quad \textit{and} \quad \delta_i = (-1)^{n(h+1)+1} * d_i * .$$

Proof. If $\alpha \in \Omega^{h,p}$ and $\beta \in \Omega^{h-1}$, then

$$\int \langle \delta_1\alpha, \beta\rangle dV = \int \langle \alpha, d_1\beta\rangle dV \neq 0 \quad \text{only if } \beta \in \Omega^{h-1,p-i}.$$

This proves the first assertion. As for the second assertion, by (19),

$$\begin{aligned}\delta_0 + \delta_1 + \cdots = \delta &= (-1)^{n(h+1)+1} * d* \\ &= (-1)^{n(h+1)+1} * d_0 * + (-1)^{n(h+1)+1} * d_1 * + \cdots ,\end{aligned}$$

and the assertion follows since both decompositions are orthogonal. □

The following definition of intrinsic covectors (and therefore of intrinsic forms) is due to M. Rumin ([40], [39]).

Definition 3.9. If $0 \le h \le n$ we set

$$E_0^h := \ker d_0 \cap \ker \delta_0 = \ker d_0 \cap (\mathrm{Im}\, d_0)^\perp \subset \Omega^h$$

In the sequel, we refer to the elements of E_0^h as to *intrinsic h-forms on $\mathbb{G}$*. Since the construction of E_0^h is left invariant, this space of forms can be seen as the space of sections of a fiber subbundle of $\bigwedge^h \mathfrak{g}$, generated by left translation and still denoted by E_0^h. In particular E_0^h inherits from $\bigwedge^h \mathfrak{g}$ the scalar product on the fibers.

Moreover, there exists a left invariant orthonormal basis $\Xi_0^h = \{\xi_j\}$ of E_0^h that is adapted to the filtration (23).

Since it is easy to see that $E_0^1 = \mathrm{span}\,\{\theta_1, \dots, \theta_m\}$, without loss of generality, we can take $\xi_j = \theta_j$ for $j = 1, \dots, m$.

Finally, we denote by $N_h^{\min}$ and $N_h^{\max}$ respectively the lowest and highest weight of forms in E_0^h.

We define now a (pseudo) inverse of d_0 as follows (see [2], Lemma 2.11):

Lemma 3.10. *If $\beta \in \bigwedge^{h+1} \mathfrak{g}$, then there exists a unique $\alpha \in \bigwedge^h \mathfrak{g} \cap (\ker d_0)^\perp$ such that*

$$\delta_0 d_0 \alpha = \delta_0 \beta. \quad \textit{We set} \quad \alpha := d_0^{-1}\beta.$$

In particular

$$\alpha = d_0^{-1}\beta \quad \textit{if and only if} \quad d_0\alpha - \beta \in \ker \delta_0 = \mathcal{R}(d_0)^\perp.$$

In addition, d_0^{-1} preserves the weights.

Proof. The first statement follows by an easy linear algebra argument. As for the second statement, suppose $\beta \in \bigwedge^{h+1,p} \mathfrak{g}$, and, by (23), write $\alpha = \sum_i \alpha_i$, with $\alpha_i \in \bigwedge^{h,i} \mathfrak{g}$. We have $\sum_i d_0 \alpha_i = \beta + \xi$, with $\xi \in \mathcal{R}(d_0)^\perp$. Keeping in mind that covectors of different weights are orthogonal, and that d_0 preserves the weights, if $j \neq p$, we get

$$\|d_0\alpha_j\|^2 = \sum_i \langle \alpha_i, d_0\alpha_j \rangle = \langle \beta, d_0\alpha_j \rangle + \langle \xi, d_0\alpha_j \rangle = 0.$$

Therefore $\alpha_j \in \ker d_0 \perp \alpha$ for $j \neq p$, and hence

$$\|\alpha_j\|^2 = \langle \alpha, \alpha_j \rangle = 0$$

if $j \neq p$. This proves the assertion. $\square$

The following theorem summarizes the construction of the intrinsic differential d_c (for details, see [40] and [2], Section 2) .

Theorem 3.11. *The de Rham complex (Ω^*, d) splits in the direct sum of two sub-complexes (E^*, d) and (F^*, d), with*

$$E := \ker d_0^{-1} \cap \ker(d_0^{-1} d) \quad \textit{and} \quad F := \mathcal{R}(d_0^{-1}) + \mathcal{R}(d d_0^{-1}).$$

We have

i) *Let* Π_E *be the projection on* E *along* F *(that is not an orthogonal projection). Then for any* $\alpha \in E_0^{h,p}$*, if we denote by* $(\Pi_E\alpha)_j$ *the component of* $\Pi_E\alpha$ *of weight* j*, then*

$$
(25) \qquad \begin{aligned} &(\Pi_E\alpha)_p = \alpha \\ &(\Pi_E\alpha)_{p+k+1} = -d_0^{-1}\big(\sum_{1\le\ell\le k+1} d_\ell(\Pi_E\alpha)_{p+k+1-\ell}\big). \end{aligned}
$$

ii) Π_E *is a chain map, i.e.*

$$
d\Pi_E = \Pi_E d.
$$

iii) *Let* Π_{E_0} *be the orthogonal projection from* Ω^* *on* E_0^**, then*

$$
(26) \qquad \Pi_{E_0} = Id - d_0^{-1}d_0 - d_0d_0^{-1}, \quad \Pi_{E_0^\perp} = d_0^{-1}d_0 + d_0d_0^{-1}.
$$

Notice that, since d_0 *and* d_0^{-1} *are algebraic, then formulas* (26) *hold also for covectors.*

iv) $\Pi_{E_0}\Pi_E\Pi_{E_0} = \Pi_{E_0}$ *and* $\Pi_E\Pi_{E_0}\Pi_E = \Pi_E$.

Set now

$$
d_c = \Pi_{E_0}\, d\, \Pi_E : E_0^h \to E_0^{h+1}, \quad h = 0, \dots, n-1.
$$

We have:

v) $d_c^2 = 0$*;*

vi) *the complex* $E_0 := (E_0^*, d_c)$ *is exact;*

vii) *with respect to the bases* Ξ^**, the intrinsic differential* d_c *can be seen as a matrix-valued operator such that, if* α *has weight* p*, then the component of weight* q *of* $d_c\alpha$ *is given by an homogeneous differential operator in the horizontal derivatives of order* $q-p \ge 1$*, acting on the components of* α*.*

Remark 3.12. Let us give a gist of the construction of E. The map $d_0^{-1}d$ induces an isomorphism from $\mathcal{R}(d_0^{-1})$ to itself. Thus, since $d_0^{-1}d_0 = Id$ on $\mathcal{R}(d_0^{-1})$, we can write $d_0^{-1}d = Id + D$, where D is a differential operator that increases the weight. Clearly, $D : \mathcal{R}(d_0^{-1}) \to \mathcal{R}(d_0^{-1})$. As a consequence of the nilpotency of $\mathbb{G}$, $D^k = 0$ for k large enough, and therefore the Neumann series of $d_0^{-1}d$ reduces to a finite sum on $\mathcal{R}(d_0^{-1})$. Hence there exist a differential operator

$$
P = \sum_{k=1}^{N} (-1)^k D^k, \quad N \in \mathbb{N} \text{ suitable},
$$

such that

$$
Pd_0^{-1}d = d_0^{-1}dP = \mathrm{Id}_{\mathcal{R}(d_0^{-1})}.
$$

We set $Q := Pd_0^{-1}$. Then Π_E is given by

$$
\Pi_E = Id - Qd - dQ.
$$

If more Carnot groups are involved, to avoid misunderstandings we write also $(E_{0,\mathbb{G}}^*, d_{c,\mathbb{G}})$, whereas the usual exterior differential is denoted by $d_{\mathbb{G}}$.

Remark 3.13. Our definition of the complex (E_0^*, d_c) is not fully intrinsic, since it depends not only on the group structure of $\mathbb{G}$ and on the stratification of its Lie algebra, but also on the scalar product we have fixed rather arbitrarily at the very beginning. In fact, the elements of E_0^* should better be defined intrinsically as quotient classes, by putting

$$\tilde{E}_0^* = \ker d_0 / \mathcal{R}(d_0),$$

and then by defining coherently the intrinsic differential as an operator between classes.

As pointed out in [40], Section 2.2.2, the choice of the orthogonal complement of $\mathcal{R}(d_0)$ as a representative of the quotient space makes possible to work with "true forms" instead of equivalence classes. Obviously, this advantage must have some negative counterpart. In particular, when we are interested in the invariance of (E_0^*, d_c) under the pull-back, we are forced to add supplementary and "non natural" assumptions related to the scalar product (see Remark 3.17).

The following "integration by parts" formula (that is not a straightforward consequence of Stokes theorem as in $\mathbb{R}^n$) is proved in [2], Remark 3.18. Indeed, the identity $d_c(\alpha \wedge \beta) = d_c\alpha \wedge \beta - \alpha \wedge d_c\beta$ fails to hold for intrinsic forms, as pointed out in [4], Proposition A.7, since $\alpha \wedge \beta$ cannot be defined in a coherent way.

Proposition 3.14. *If $\alpha \in \mathcal{D}(\mathbb{G}, E_0^h)$ and $\beta \in \mathcal{D}(\mathbb{G}, E_0^k)$ with $k + h + 1 = n$, we have*

$$\int_{\mathbb{G}} d_c\alpha \wedge \beta = (-1)^{h+1} \int_{\mathbb{G}} \alpha \wedge d_c\beta.$$

We denote by $\delta_c = \delta_{c,\mathbb{G}} = d_c^* = d_{c,\mathbb{G}}^*$ the formal adjoint of d_c in $L^2(\mathbb{G}, E_0^*)$. Thanks to Proposition 3.14, the following assertion holds.

Proposition 3.15. *We have*

$$\delta_c = (-1)^{n(h+1)+1} * d_c * .$$

Proof of Proposition 3.15. Let $\alpha \in E_0^h$ and $\beta \in E_0^{h-1}$ be smooth compactly supported forms. Then

$$\begin{aligned}
\int_{\mathbb{G}} \langle \delta_c\alpha, \beta\rangle \, dV &= \int_G \langle \alpha, d_c\beta\rangle \, dV \\
&= \int_{\mathbb{G}} d_c\beta \wedge *\alpha = (-1)^h \int_{\mathbb{G}} \beta \wedge * * d_c\,(*\alpha) \\
&= (-1)^{n(h+1)+1} \int_{\mathbb{G}} \beta \wedge * * d_c\,(*\alpha) \\
&= (-1)^{n(h+1)+1} \int_{\mathbb{G}} \langle \beta, * \, d_c * \alpha\rangle \, dV.
\end{aligned}$$

□

The following theorem states the so-called *naturality* of the exterior differential d_c. Since homogeneous homomorphisms of $\mathbb{G}$ appear naturally as intrinsic differentials (Pansu differentials: see [37]) of maps between Carnot groups, we can expect the invariance of (E_0^*, d_c) under pull-back associated with maps in $HL(\mathbb{G})$. However, our statement is weaker than one could wish, and the reason is illustrated below in Remark 3.16.

Theorem 3.16. *If both L and ${}^tL$ belong to $HL(\mathbb{G})$, then*

i) $L^\# : E_0^* \to E_0^*$;

ii) *for any intrinsic h-form $\alpha \in E_0^h$*

$$d_c(L^\# \alpha) = L^\#(d_c\alpha);$$

Proof. The proof is divided in several steps.

Step 1. We prove that, if $\alpha \in \Omega^h$ is left-invariant, then both $L^\#\alpha$ and ${}^tL^\#\alpha$ are left invariant. We prove the assertion for $L^\#\alpha$. Take $p \in \mathbb{G}$. We notice preliminarily that $L \circ \tau_p = \tau_{Lp} \circ L$. Thus, by Proposition 2.4, vi), we have

$$\tau_p^\#(L^\#\alpha) = (L \circ \tau_p)^\#\alpha = (\tau_{Lp} \circ L)^\#\alpha = L^\#\tau_{Lp}^\#\alpha = L^\#\alpha.$$

Step 2. For any h-form $\omega \in \Omega^h$ we have $d_0(L^\#\omega) = L^\#(d_0\omega)$ (again the same assertion still holds for ${}^tL^\#$). Indeed, suppose ω has pure weight p, i.e. suppose $\omega \in \Omega^{h,p}$. We can write now

$$d(L^\#\omega) = d_0(L^\#\omega) + d_1(L^\#\omega) + \cdots ,$$

and, at the same time, by linearity

$$d(L^\#\omega) = L^\#(d\omega) = L^\#(d_0\omega) + L^\#(d_1\omega) + \cdots .$$

But both $d_i(L^\#\omega)$ and $L^\#(d_i\omega)$ belong to $\Omega^{h+1,p+i}$ for $i = 0, 1, \dots$, since the pull back preserves the weights, by Proposition 3.4. Keeping in mind that $\Omega^{h+1,r}$ is orthogonal to $\Omega^{h+1,s}$ for $r \neq s$, it follows that $d_i(L^\#\omega) = L^\#(d_i\omega)$ for $i = 0, 1, \dots$.

Step 3. Suppose $\omega \in \Omega^h$ is left-invariant. Combining Steps 1 and 2, and keeping in mind that d_0 preserves the left-invariance, we obtain that $d_0(L^\#\omega)$ and $L^\#(d_0\omega)$ are left-invariant. If we write the identity $d_0(L^\#\omega) = L^\#(d_0\omega)$ at 0, keeping in mind that $Le = e$, we obtain

$$d_0\big((\Lambda^h L)\omega(0)\big) = (\Lambda^h L)(d_0\omega(0)),$$

since

Step 4. Suppose $\omega \in \Omega^h$ is left-invariant. Then $d_0^{-1}L^\#\omega = L^\#(d_0^{-1}\omega)$. Since both terms are left-invariant, by Step 1, keeping in mind that d_0^{-1} is algebraic, we need only to prove the assertion at 0, i.e to prove that

$$d_0^{-1}((\Lambda^h L)\omega(0)) = (\Lambda^h L)(d_0^{-1}\omega(0)),$$

since $Le = e$. Set now $\omega_0 := \omega(0)$; by the very definition of d_0^{-1}, this is equivalent to show that

(a) $(\Lambda^h L)(d_0^{-1}\omega_0) \perp \ker d_0$;

(b) $d_0\big((\Lambda^h L)(d_0^{-1}\omega_0)\big) - (\Lambda^h L)\omega_0 \in \mathcal{R}(d_0)^\perp$.

To prove (a), take $\xi \in \ker d_0$. We notice that $d_0 I_\xi = I_{d_0\xi}$, since both $d_0 I_\xi$ and $I_{d_0\xi}$ are left-invariant (Lemma 3.7) and coincide at 0 (since d_0 is algebraic).

By Proposition 2.4, we obtain

$$\langle (\Lambda^h L)(d_0^{-1}\omega_0), \xi\rangle = \langle d_0^{-1}\omega_0, (\Lambda^h\, {}^tL)\xi\rangle.$$

On the other hand, keeping in mind Lemma 2.6,

$$\begin{aligned} d_0(\Lambda^h\ {}^tL)\xi &= d_0\big(\ {}^tL^\# I_\xi(0)\big) = d_0\big(\ {}^tL^\# I_\xi\big)(0) \\ &= \big(\ {}^tL^\#(d_0 I_\xi)\big)(0) \quad \text{(by Step 3)} = \big(\ {}^tL^\#(I_{d_0\xi})\big)(0) = 0. \end{aligned}$$

Therefore $\langle d_0^{-1}\omega_0, (\Lambda^h\ {}^tL)\xi\rangle = 0$, since $d_0^{-1}\omega_0$ is normal to $\ker d_0$, by its very definition. This proves (a).

To prove (b), take $\xi = d_0\eta \in \mathcal{R}(d_0)$. Arguing as above, we obtain

$$\begin{aligned} &\langle d_0\big((\Lambda^h L)(d_0^{-1}\omega_0)\big) - (\Lambda^h L)\omega_0, \xi\rangle \\ &\quad = \langle (\Lambda^h L)(d_0 d_0^{-1}\omega_0) - (\Lambda^h L)\omega_0, \xi\rangle = \langle d_0 d_0^{-1}\omega_0 - \omega_0, (\Lambda^h\ {}^tL)\xi\rangle. \end{aligned}$$

On the other hand, $d_0\eta = (d_0 I_\eta)(0)$, and hence

$$\begin{aligned} (\Lambda^h\ {}^tL)\xi &= \big(\ {}^tL^\#(d_0 I_\eta)\big)(0) = \big(d_0(\ {}^tL^\# I_\eta)\big)(0) \\ &= d_0(\Lambda^{h-1}\ {}^tL\eta) \in \mathcal{R}(d_0). \end{aligned}$$

Thus $\langle d_0 d_0^{-1}\omega_0 - \omega_0, (\Lambda^h\ {}^tL)\xi\rangle = 0$, by the very definition of $d_0^{-1}\omega_0$ and (b) follows.

Step 5. By Steps 3 and 4, $L^\#$ commutes with both d_0 and d_0^{-1} and therefore

$$L^\#\Pi_{E_0} = L^\#(Id - d_0 d_0^{-1} - d_0^{-1}d_0) = \Pi_{E_0}L^\#.$$

In particular, i) follows.

Step 6. In order to prove ii), we have but to show that the pull back commutes with Π_E.

Following now the notations of Lemma **??**, $L^\#$ commutes with $D = d_0^{-1}d_0 - Id$, and hence with $P = \sum_k (-1)^k D^k$, and finally with Q. Thus it commutes with $\Pi_E = Id - Qd - dQ$. This achieves the proof of the theorem.

□

Remark 3.17. The statement of Theorem 3.16 is not as natural as we could wish, though sufficient for our purposes when dealing later with Maxwell's equations. This because of the assumption on ${}^tL$. However, as we already pointed out in Remark 3.13, this is due to the fact that we are not working with the natural complex $(\tilde{E}_0^*, \tilde{d}_c)$, but with an isomorphic complex of "true forms", that depends on the choice of a scalar product in $\mathfrak{g}$. Indeed, we could get rid of the "non natural" assumption by working on quotient spaces as sketched in Remark 3.13, and keeping in mind that, by Step 2 of our previous proof, if $L \in HL(\mathbb{G})$, then $d_0 L^\# = L^\# d_0$. Thus the map

$$\tilde{L}^\# : \ker d_0/\mathcal{R}(d_0) \longrightarrow \ker d_0/\mathcal{R}(d_0)$$

given by

$$\tilde{L}^\#[\alpha] := [L^\#\alpha]$$

is well defined without further assumption on L.

If $\Omega \subset \mathbb{G}$ is an open set, $0 \le h \le n$, $k \in \mathbb{N}$ and $1 \le p \le \infty$, then we denote by $W^{k,p}_{\mathbb{G}}(\Omega, E_0^h)$ the space of all forms in E_0^h with coefficients in $W^{k,p}_{\mathbb{G}}(\Omega)$, endowed with its natural norm. It is easy to see that this definition is independent of the basis of $\bigwedge^h \mathfrak{g}$ we have chosen. The spaces $L^p(\Omega, E_0^h)$ and

$\mathcal{D}(\Omega, E_0^h)$ are defined analogously starting from $L^p(\Omega)$ and from the space of test functions $\mathcal{D}(\Omega)$, respectively.

Definition 3.18. If $\Omega \subset \mathbb{G}$ is an open set and $1 \le h \le n$, we say that T is a h-current on Ω if T is a continuous linear functional on $\mathcal{D}(\Omega, E_0^h)$ endowed with the usual topology. We write $T \in \mathcal{D}'(\Omega, E_0^h)$.

Any (usual) distribution $T \in \mathcal{D}'(\Omega)$ can be identified canonically with an n-current $\tilde{T} \in \mathcal{D}'(\Omega, E_0^n)$ through the formula

$$\langle \tilde{T} | \alpha \rangle := \langle T | *\alpha \rangle \tag{27}$$

for any $\alpha \in \mathcal{D}(\Omega, E_0^n)$. Reciprocally, by (27), any n-current $\tilde{T}$ can be identified with an usual distribution $T \in D'(\Omega)$.

Following [14], 4.1.7, we give the following definition.

Definition 3.19. If $T \in \mathcal{D}'(\Omega, E_0^n)$, and $\varphi \in \mathcal{E}(\Omega, E_0^k)$, with $0 \le k \le n$, we define $T \llcorner \varphi \in D'(\Omega, E_0^{n-k})$ by the identity

$$\langle T \llcorner \varphi | \alpha \rangle := \langle T | \alpha \wedge \varphi \rangle$$

for any $\alpha \in \mathcal{D}(\Omega, E_0^{n-k})$.

We notice that, when $\varphi \in \mathcal{E}(\Omega, E_0^k)$ and $\alpha \in \mathcal{D}(\Omega, E_0^{n-k})$, then the wedge product $\alpha \wedge \varphi$ belongs to $\mathcal{D}(\Omega, E_0^n)$, since $E_0^n = \Omega^n$.

The following result is taken from [3], Propositions 5 and 6, and Definition 10, but we refer also to [12], Sections 17.3 17.4 and 17.5.

Let $\Omega \subset \mathbb{G}$ be an open set. If $1 \le h \le n$, If $\Xi_0^h = \{\xi_1^h, \dots \xi_{\dim E_0^h}^h\}$ is a left invariant basis of E_0^h and $T \in \mathcal{D}'(\Omega, E_0^h)$, then there exist (uniquely determined) $T_1, \dots, T_{\dim E_0^h} \in \mathcal{D}'(\Omega)$ such that

$$T = \sum_j \tilde{T}_j \llcorner (*\xi_j^h).$$

It is well known that currents can be seen as forms with distributional coefficients in the following sense: if $\alpha \in E(\Omega, E_0^h)$, then α can be identified canonically with a h-current T_α through the formula

$$\langle T_\alpha | \varphi \rangle := \int_\Omega *\alpha \wedge \varphi \tag{28}$$

for any $\varphi \in \mathcal{D}(\Omega, E_0^h)$. Moreover, if $\alpha = \sum_j \alpha_j \xi_j^h$ then

$$T_\alpha = \sum_j \tilde{\alpha}_j \llcorner (*\xi_j^h)$$

The notion of convolution can be extended by duality to currents.

4. Space-time Carnot groups

From now on, we denote by x a "space" point in the Carnot group $\mathbb{G}$, and by $s \in \mathbb{R}$ the "time", and we choose in $\mathbb{R} \times \mathbb{G}$ the canonical volume form $ds \wedge dV$, where $dV = \theta_1 \wedge \cdots \wedge \theta_n$ is the canonical volume form in $\mathbb{G}$. Moreover, we denote by $(\Omega^*_{\mathbb{G}}, d_{\mathbb{G}})$ and $(\Omega^*_{\mathbb{R}\times\mathbb{G}}, d_{\mathbb{R}\times\mathbb{G}})$ the de Rham complex of forms on $\mathbb{G}$ and on $\mathbb{R} \times \mathbb{G}$, respectively. For sake of brevity, we write

$$\Omega^* := \Omega^*_{\mathbb{G}} \quad \text{and} \quad \hat{\Omega}^* := \Omega^*_{\mathbb{R}\times\mathbb{G}},$$

$$d := d_{\mathbb{G}} \quad \text{and} \quad \hat{d} := d_{\mathbb{R}\times\mathbb{G}},$$
$$\delta := d^*_{\mathbb{G}} \quad \text{and} \quad \hat{\delta} := d^*_{\mathbb{R}\times\mathbb{G}}.$$

Analogously, we write

$$d_i := d_{\mathbb{G},i} \quad \text{and} \quad \hat{d}_i := d_{\mathbb{R}\times\mathbb{G},i},$$
$$\delta_i := d^*_{\mathbb{G},i} \quad \text{and} \quad \hat{\delta}_i := d^*_{\mathbb{R}\times\mathbb{G},i},$$

$i = 0, \dots, \kappa$ (see Definition 3.5).

When dealing with intrinsic forms, we denote by $(E^*_{0,\mathbb{G}}, d_{c,\mathbb{G}})$ and $(E^*_{0,\mathbb{R}\times\mathbb{G}}, d_{c,\mathbb{R}\times\mathbb{G}})$ the complex of intrinsic forms on $\mathbb{G}$ and on $\mathbb{R}\times\mathbb{G}$, respectively. For sake of brevity, we write

$$E^*_0 := E^*_{0,\mathbb{G}} \quad \text{and} \quad \hat{E}^*_0 := E^*_{0,\mathbb{R}\times\mathbb{G}},$$

as well as

$$d_c := d_{c,\mathbb{G}} \quad \text{and} \quad \hat{d}_c := d_{c,\mathbb{R}\times\mathbb{G}},$$
$$\delta_c := d^*_{c,\mathbb{G}} \quad \text{and} \quad \hat{\delta}_c := d^*_{c,\mathbb{R}\times\mathbb{G}}.$$

Denote by S the vector field $\frac{\partial}{\partial s}$. The Lie group $\mathbb{R}\times\mathbb{G}$ is a Carnot group; its Lie algebra $\hat{\mathfrak{g}}$ admits the stratification

$$\hat{\mathfrak{g}} = \hat{V}_1 \oplus V_2 \oplus \cdots \oplus V_\kappa, \tag{29}$$

where $\hat{V}_1 = \operatorname{span}\{S, V_1\}$. Since the adapted basis $\{X_1, \dots, X_n\}$ has been already fixed once and for all, the associated orthonormal fixed basis for $\hat{\mathfrak{g}}$ will be

$$\{S, X_1, \dots, X_{m_1}, \dots, X_n\} := \{X_0, \dots, X_n\},$$

where we have set $X_0 := S$. Coherently, we write also $\theta_0 := ds$. Consider the Lie derivative $\mathcal{L}_S$ along S. If $f\theta_{i_1}\wedge\cdots\wedge\theta_{i_h}$ is a h-form in $\mathbb{G}$, $1 \le i_1 < \cdots < i_h \le n$, we have $\mathcal{L}_S(f\theta_{i_1}\wedge\cdots\wedge\theta_{i_h}) = (Sf)\theta_{i_1}\wedge\cdots\wedge\theta_{i_h}$. Indeed

- if $h = 0$, by definition $\mathcal{L}_S f = i_S df = \sum_{j=0}^n (X_j f)\theta_j(X_0) = Sf$;
- if $f\theta_{i_1}\wedge\cdots\theta_{i_h}$ is a h-form in $\mathbb{G}$, $h \ge 1$, then $\mathcal{L}_S(f\theta_{i_1}\wedge\cdots\wedge\theta_{i_h}) = (Sf)\theta_{i_1}\wedge\cdots\wedge\theta_{i_h} + f\mathcal{L}_S(\theta_{i_1}\wedge\cdots\wedge\theta_{i_h})$. But $\mathcal{L}_S(\theta_{i_1}\wedge\cdots\wedge\theta_{i_h})$ is a sum of terms of the form $\theta_{i_1}\wedge\cdots\wedge\mathcal{L}_S\theta_{i_\ell}\wedge\cdots\wedge\theta_{i_h} = 0$, since $\mathcal{L}_S\theta_{i_\ell} = 0$.

Thus, when acting on h-forms α in $\mathbb{G}$, without risk of misunderstandings, we write $S\alpha$ for $\mathcal{L}_S\alpha$.

We point out that S commutes with d, the exterior differential in $\mathbb{G}$. Indeed, if $\alpha = \sum_{j=1}^n \alpha_j\theta_j^h$, then

$$Sd\alpha = \sum_{j=1}^n\sum_{\ell=1}^n (SX_\ell\alpha_j)\theta_\ell\wedge\theta_j^h = \sum_{j=1}^n\sum_{\ell=1}^n (X_\ell S\alpha_j)\theta_\ell\wedge\theta_j^h = d(S\alpha).$$

Moreover, if $\alpha \in \Omega^h$ and its coefficients depend on s and x (and is identified with a h-form in $\hat{\Omega}^h$), then

$$\hat{d}\alpha = d\alpha + ds\wedge(S\alpha). \tag{30}$$

As in special relativity, the space-time $\mathbb{R}\times\mathbb{G}$ can be endowed with a Minkowskian scalar product as follows.

Definition 4.1. We denote by $G = (g_{ij})_{i,j=0,\dots,n}$ the $(n+1)\times(n+1)$-matrix in $HL(\mathbb{R}\times\mathbb{G},\mathbb{R}\times\mathbb{G})$ such that $g_{ij}=0$ if $i\neq j$, $g_{ii}=1$ if $i>0$, $g_{00}=-1$, i.e. we set

$$G = \left(\begin{array}{ccccc|ccc|c}
-1 & 0 & & 0 & 0 & & & & \\
0 & 1 & & 0 & 0 & & & & \\
 & & \ddots & & & & 0 & & 0 \\
0 & 0 & & 1 & 0 & & & & \\
0 & 0 & & 0 & 1 & & & & \\
\hline
 & & & & & 1 & & 0 & \\
 & & 0 & & & & \ddots & & 0 \\
 & & & & & 0 & & 1 & \\
\hline
 & & 0 & & & & 0 & & \ddots
\end{array}\right).$$

We can define now "Minkowskian" scalar products $\langle\cdot,\cdot\rangle_M$ in $\bigwedge_*\hat{\mathfrak{g}}$ and $\bigwedge^*\hat{\mathfrak{g}}$ by

$$\langle v,v'\rangle_M = \langle(\Lambda_h G)v,v'\rangle \quad \text{ if } v,v'\in\bigwedge\nolimits_h\hat{\mathfrak{g}}$$

and

$$\langle\alpha,\alpha'\rangle_M = \langle(\Lambda^h G)\alpha,\alpha'\rangle \quad \text{ if } \alpha,\alpha'\in\bigwedge\nolimits^h\hat{\mathfrak{g}}.$$

Notice that the bilinear form $\langle\cdot,\cdot\rangle_M$ is nondegenerate.

Definition 4.2. We denote by $*_M$ the Hodge operator $*_M:\bigwedge^h\hat{\mathfrak{g}}\to\bigwedge^{n-h}\hat{\mathfrak{g}}$ associated with the Minkowskian scalar product in $\bigwedge^*\hat{\mathfrak{g}}$ and with the volume form $ds\wedge dV$ by

$$\alpha\wedge *_M\beta = \langle\alpha,\beta\rangle_M ds\wedge dV.$$

Definition 4.3. If $1\le h\le n$, we denote by $\hat{\delta}_c^M$ the codifferential $\hat{\delta}_c^M:\hat{\Omega}^h\to\hat{\Omega}^{h-1}$ associated with the Minkowskian scalar product by

$$\int_{\mathbb{G}}\langle\hat{\delta}_c^M\alpha,\beta\rangle_M\, ds\wedge dV = \int_{\mathbb{G}}\langle\alpha,\hat{d}_c\beta\rangle_M\, ds\wedge dV$$

for $\alpha\in\mathcal{D}(\mathbb{G},\hat{E}_0^h)$ and $\beta\in\mathcal{D}(\mathbb{G},\hat{E}_0^{h-1})$.

By Proposition 3.14, we have

$$\hat{\delta}_c^M = (-1)^{(n+1)(h+1)+1} *_M \hat{d}_c *_M .$$

Let us state preliminarily a structure lemma for intrinsic forms in $\mathbb{R}\times\mathbb{G}$. The result is proved in [5], but we sketch the proof here for sake of completeness.

Lemma 4.4. *If $1\le h\le n$, then a h-form α belongs to $\hat{E}_0^h$ if and only if it can be written as*

$$\alpha = ds\wedge\beta+\gamma, \tag{31}$$

where $\beta\in E_0^{h-1}$ and $\gamma\in E_0^h$ are respectively intrinsic $(h-1)$-forms and h-forms in $\mathbb{G}$ with coefficients depending on x and s.

Proof. Without loss of generality, we can restrict ourselves to prove an analogous decomposition for covectors (identified with left invariant forms) $\alpha \in \bigwedge^h \hat{\mathfrak{g}}$, $\gamma \in \bigwedge^h \mathfrak{g}$ and $\beta \in \bigwedge^{h-1} \mathfrak{g}$, where, as usual, we identify $\bigwedge^h \mathfrak{g}$ with a linear subspace of $\bigwedge^h \hat{\mathfrak{g}}$.

First of all, notice that, if $\sigma \in \bigwedge^h \mathfrak{g}$ is arbitrary, then $d_0(ds \wedge \sigma) = d(ds \wedge \sigma) = ds \wedge d\sigma = ds \wedge d_0\sigma$. Thus, if $d_0\beta = 0$ and $d_0\gamma = 0$, then $d_0\alpha = 0$.

On the other hand, let $\alpha' = ds \wedge \beta' + \gamma'$ be arbitrarily given in $\bigwedge^{h-1} \hat{\mathfrak{g}}$. Then, keeping in mind that covectors in $\bigwedge^\ell \mathfrak{g}$ are orthogonal in $\bigwedge^h \hat{\mathfrak{g}}$ to covectors of the form $ds \wedge \sigma$ with $\sigma \in \bigwedge^{\ell-1} \mathfrak{g}$, by the very definition of the scalar product, we have

$$\langle \alpha, d_0\alpha' \rangle = \langle \beta, d_0\beta' \rangle + \langle \gamma, d_0\gamma' \rangle = 0,$$

since both β and γ belong to $E^*_{0,\mathbb{G}}$ and hence are orthogonal to the range of d_0. Thus also α is orthogonal to the range of d_0 and eventually $\alpha \in \hat{E}_0^h$.

Suppose now α belongs to $\hat{E}_0^h$. We can always write it as $\alpha = ds \wedge \beta + \gamma$, with $\beta \in \bigwedge^{h-1} \mathfrak{g}$ and $\gamma \in \bigwedge^h \mathfrak{g}$, and $0 = d_0\alpha = ds \wedge d_0\beta + d_0\gamma$. But $ds \wedge d_0\beta$ and $d_0\gamma$ are orthogonal, and hence $d_0\beta = 0$ and $d_0\gamma = 0$. Let now $\gamma' \in \bigwedge^h \mathfrak{g}$ be given. Since $d_0\gamma'$ is orthogonal to any h-covector of the form $ds \wedge \beta'$ with $\beta' \in \bigwedge^{h-1} \mathfrak{g}$, we have $0 = \langle \alpha, d_0\gamma' \rangle = \langle ds \wedge \beta + \gamma, d_0\gamma' \rangle = \langle \gamma, d_0\gamma' \rangle$, i.e. γ is orthogonal to the range of d_0 and then $\gamma \in E^h_{0,\mathbb{G}}$. Analogously, if $\beta' \in \bigwedge^{h-1} \mathfrak{g}$, then $0 = \langle \alpha, d_0(ds \wedge \beta') \rangle = \langle \alpha, ds \wedge d_0\beta' \rangle = \langle ds \wedge \beta, ds \wedge d_0\beta' \rangle + \langle \gamma, ds \wedge d_0\beta' \rangle = \langle \beta, d_0\beta' \rangle$. Thus β is orthogonal to the range of d_0 and then $\beta \in E^{h-1}_{0,\mathbb{G}}$. □

Remark 4.5. If $1 \le \ell \le n$, by Lemma 4.4, keeping in mind that forms in $\hat{E}_0^\ell$ are orthogonal to the forms $ds \wedge \sigma$ with $\sigma \in E_0^{\ell-1}$, we have

$$\langle ds \wedge \beta + \gamma, ds \wedge \beta' + \gamma' \rangle_M := \langle \gamma, \gamma' \rangle - \langle \beta, \beta' \rangle.$$

Remark 4.6. If $\alpha = ds \wedge \beta + \gamma \in \hat{E}_0^h$, then

$$*_M \alpha = (-1)^h ds \wedge *\gamma - *\beta.$$

Indeed, if $\alpha' = ds \wedge \beta' + \gamma' \in \hat{E}_0^h$, by Lemma 4.4 we have

$$\begin{aligned}
&(ds \wedge \beta' + \gamma') \wedge ((-1)^h ds \wedge *\gamma - *\beta) \\
&\quad = -ds \wedge \beta' \wedge *\beta + (-1)^h \gamma' \wedge ds \wedge *\gamma \\
&\qquad \text{(since } \gamma' \wedge *\beta \text{ vanishes, being a } (n+1)\text{-form in } \mathbb{G}) \\
&\quad = -ds \wedge \beta' \wedge *\beta + ds \wedge \gamma' \wedge *\gamma \\
&\quad = \langle \alpha, \alpha' \rangle_M \, ds \wedge dV.
\end{aligned}$$

Proposition 4.7. *If $1 \le h \le n$, and $\alpha = ds \wedge \beta + \gamma \in \hat{E}_0^h$, then*

$$\hat{d}_c \alpha = ds \wedge (S\gamma - d_c\beta) + d_c\gamma. \tag{32}$$

and

$$\hat{\delta}_c^M \alpha = -ds \wedge \delta_c\beta + \delta_c\gamma + S\beta. \tag{33}$$

Proof. The proof will be articulated in several lemmata. □

Lemma 4.8. *If* $\alpha = ds \wedge \beta + \gamma \in \hat{\Omega}^h$, *then*

$$\hat{d}\alpha = ds \wedge (S\gamma - d\beta) + d\gamma. \tag{34}$$

Proof. By (30) we have:

$$\begin{aligned}\hat{d}\alpha &= -ds \wedge \hat{d}\beta + \hat{d}\gamma = -ds \wedge \big(d\beta + ds \wedge (S\beta)\big) + d\gamma + ds \wedge (S\gamma) \\ &= ds \wedge (S\gamma - d\beta) + d\gamma.\end{aligned}$$

□

Lemma 4.9. *If* $\alpha = ds \wedge \beta + \gamma \in \hat{E}_0^h$, *then*

$$\hat{d}_1\alpha = ds \wedge (S\gamma - d_1\beta) + d_1\gamma, \tag{35}$$

and

$$\hat{d}_i\alpha = -ds \wedge d_i\beta + d_i\gamma \quad \textit{for } i \neq 1. \tag{36}$$

Proof. Suppose α has weight p, i.e. suppose β has weight $p-1$ and γ has weight p. By Lemma 4.8, and keeping in mind that $S\gamma$ has weight p too, we can write

$$\begin{aligned}\hat{d}\alpha &= ds \wedge (S\gamma - d\beta) + d\gamma \\ &= ds \wedge (S\gamma - d_0\beta - d_1\beta - \cdots) + d_0\gamma + d_1\gamma + \cdots \\ &= \{-ds \wedge d_0\beta + d_0\gamma\} + \{ds \wedge (S\gamma - d_1\beta) + d_1\gamma\} \\ &\quad + \{-ds \wedge d_2\beta + d_2\gamma\} + \cdots .\end{aligned}$$

This proves the assertion.

□

Lemma 4.10. *If* $\alpha = ds \wedge \beta + \gamma \in \hat{E}_0^h$, *then*

$$\hat{d}_0^{-1}\alpha = -ds \wedge d_0^{-1}\beta + d_0^{-1}\gamma. \tag{37}$$

Proof. To prove the assertion, by the very definition of d_0^{-1}, we have to show that $-ds \wedge d_0^{-1}\beta + d_0^{-1}\gamma \perp \ker \hat{d}_0$ and $\hat{d}_0(-ds \wedge d_0^{-1}\beta + d_0^{-1}\gamma) - \alpha \in \mathcal{R}(\hat{d}_0)^\perp$. Thus, take first $ds \wedge \sigma + \tau \in \ker \hat{d}_0$. Then $d_0\sigma = 0$, $d_0\tau = 0$, and hence, $\langle -ds \wedge d_0^{-1}\beta + d_0^{-1}\gamma, ds \wedge \sigma + \tau\rangle = \langle d_0^{-1}\beta, \sigma\rangle + \langle d_0^{-1}\gamma, \tau\rangle = 0$, since $d_0^{-1}\beta \perp \sigma$, and $d_0^{-1}\gamma \perp \tau$, by definition.

Take now $ds \wedge \sigma + \tau \in \hat{\Omega}^{h-1}$. By (36), $\hat{d}_0(-ds \wedge d_0^{-1}\beta + d_0^{-1}\gamma) - \alpha = ds \wedge (d_0 d_0^{-1}\beta - \beta) + (d_0 d_0^{-1}\gamma - \gamma)$, so that

$$\begin{aligned}&\langle \hat{d}_0(-ds \wedge d_0^{-1}\beta + d_0^{-1}\gamma) - \alpha, d_0(ds \wedge \sigma + \tau)\rangle \\ &\quad = -\langle d_0 d_0^{-1}\beta - \beta, d_0\sigma\rangle + \langle d_0 d_0^{-1}\gamma - \gamma, d_0\tau\rangle = 0,\end{aligned}$$

since, by definition, both $d_0 d_0^{-1}\beta - \beta \perp \mathcal{R}(d_0)$ and $d_0 d_0^{-1}\gamma - \gamma \perp \mathcal{R}(d_0)$.

□

We want to express the lifting operator Π_E and the orthogonal projection Π_{E_0} in $\hat{\Omega}^*$ in terms of its counterpart in $\Omega^*_{\mathbb{G}}$. We denote by $\hat{E}$ and $\hat{F}$ the complexes E and F in $\hat{\Omega}^*$. These notations are coherent with our previous notations when we had to distinguish between $\mathbb{R} \times \mathbb{G}$ and $\mathbb{G}$.

Lemma 4.11. *If* $\alpha = ds \wedge \beta + \gamma \in \hat{E}_0^h$, *then*

$$\Pi_{\hat{E}}\alpha = ds \wedge \Pi_E\beta + \Pi_E\gamma. \tag{38}$$

Proof. The proof consists of two steps: first we shall prove that

$$\hat{d}_0^{-1}(ds \wedge \Pi_E\beta + \Pi_E\gamma) = 0 \tag{39}$$

and

$$\hat{d}_0^{-1}\hat{d}(ds \wedge \Pi_E\beta + \Pi_E\gamma) = 0 \tag{40}$$

This proves that $ds \wedge \Pi_E\beta + \Pi_E\gamma \in \hat{E}$. The second step will consist of showing that

$$\alpha - (ds \wedge \Pi_E\beta + \Pi_E\gamma) \in \mathcal{R}(\hat{d}_0^{-1}) + \mathcal{R}(\hat{d}\hat{d}_0^{-1}) = \hat{F}. \tag{41}$$

Clearly, (39), (40), and (41) yield (38).

Now, by (35),

$$\hat{d}_0^{-1}(ds \wedge \Pi_E\beta + \Pi_E\gamma) = -ds \wedge d_0^{-1}\Pi_E\beta + d_0^{-1}\Pi_E\gamma = 0,$$

i.e. (39). As for (40), by (34) (keeping in mind that S commutes with Π_E, since Π_E is a linear differential operator on $\mathbb{G}$)

$$\hat{d}(ds \wedge \Pi_E\beta + \Pi_E\gamma) = ds \wedge (-d\Pi_E\beta + \Pi_E S\gamma) + d\Pi_E\gamma.$$

Therefore

$$\begin{aligned}&\hat{d}_0^{-1}\hat{d}(ds \wedge \Pi_E\beta + \Pi_E\gamma)\\ &\qquad = ds \wedge (d_0^{-1}d\Pi_E\beta - d_0^{-1}\Pi_E S\gamma) + d_0^{-1}d\Pi_E\gamma = 0,\end{aligned}$$

since $\Pi_E\beta \in E \subset \ker(d_0^{-1}d)$, $\Pi_E S\gamma \in E \subset \ker(d_0^{-1})$, and $\Pi_E\gamma \in E \subset \ker(d_0^{-1}d)$. This proves (40).

In order to prove (41), we write

$$\alpha - (ds \wedge \Pi_E\beta + \Pi_E\gamma) = ds \wedge (\beta - \Pi_E\beta) + (\gamma - \Pi_E\gamma).$$

We know that

$$\beta - \Pi_E\beta = \Pi_F\beta = d_0^{-1}\sigma_1 + dd_0^{-1}\sigma_2$$

for suitable $\sigma_1 \in \Omega^h_{\mathbb{G}}$ and $\sigma_2 \in \Omega^{h-1}_{\mathbb{G}}$, and

$$\gamma - \Pi_E\gamma = \Pi_F\gamma = d_0^{-1}\tau_1 + dd_0^{-1}\tau_2$$

for suitable $\tau_1 \in \Omega^{h+1}_{\mathbb{G}}$ and $\tau_2 \in \Omega^h_{\mathbb{G}}$.

We show that

$$\begin{aligned}&\hat{d}_0^{-1}(ds \wedge (-\sigma_1 + S\tau_2) + \tau_1) + \hat{d}\hat{d}_0^{-1}(ds \wedge \sigma_2 + \tau_2)\\ &\qquad = ds \wedge (\beta - \Pi_E\beta) + (\gamma - \Pi_E\gamma).\end{aligned}$$

This will achieve the proof of (41). By Lemmata 4.10 and 4.8, and keeping into account that S commutes with d_0^{-1} that is a linear algebraic operator, we have

$$\begin{aligned}&\hat{d}_0^{-1}(ds \wedge (-\sigma_1 + S\tau_2) + \tau_1) + \hat{d}\hat{d}_0^{-1}(ds \wedge \sigma_2 + \tau_2)\\ &\quad = ds \wedge (d_0^{-1}\sigma_1 - d_0^{-1}S\tau_2) + d_0^{-1}\tau_1 + \hat{d}(-ds \wedge d_0^{-1}\sigma_2 + d_0^{-1}\tau_2)\\ &\quad = ds \wedge d_0^{-1}\sigma_1 - ds \wedge d_0^{-1}S\tau_2 + d_0^{-1}\tau_1\\ &\qquad + ds \wedge dd_0^{-1}\sigma_2 + ds \wedge Sd_0^{-1}\tau_2 + dd_0^{-1}\tau_2\\ &\quad = ds \wedge (d_0^{-1}\sigma_1 + dd_0^{-1}\sigma_2) + d_0^{-1}\tau_1 + dd_0^{-1}\tau_2\\ &\quad = ds \wedge (\beta - \Pi_E\beta) + (\gamma - \Pi_E\gamma).\end{aligned}$$

This achieves the proof of the lemma.

$\square$

Lemma 4.12. *If $\alpha = ds\wedge\beta+\gamma\in \hat E_0^h$, then*

$$\Pi_{\hat E_0}\alpha = ds\wedge \Pi_{E_0}\beta + \Pi_{E_0}\gamma. \tag{42}$$

Proof. By Lemma 4.4, $ds\wedge\Pi_{E_0}\beta+\Pi_{E_0}\gamma$ belongs to $\hat E_0^h$. On the other hand, if $ds\wedge\sigma+\tau\in\hat E_0^h$, then

$$\begin{aligned}&\langle \alpha - ds\wedge\Pi_{E_0}\beta - \Pi_{E_0}\gamma, ds\wedge\sigma+\tau\rangle\\ &\quad= \langle ds\wedge(\beta-\Pi_{E_0}\beta)+\gamma-\Pi_{E_0}\gamma, ds\wedge\sigma+\tau\rangle\\ &\quad= \langle \beta-\Pi_{E_0}\beta,\sigma\rangle + \langle\gamma-\Pi_{E_0}\gamma,\tau\rangle = 0,\end{aligned}$$

since $\beta-\Pi_{E_0}\beta\perp E_0^{h-1}$, $\gamma-\Pi_{E_0}\gamma\perp E_0^h$, and, again by Lemma 4.4, $\sigma\in E_0^{h-1}$ and $\tau\in E_0^h$.

$\square$

End of the proof of Proposition 4.7. By Lemma 4.4, if $\alpha = ds\wedge\beta+\gamma\in\hat E_0^h$, then $\beta\in E_0^{h-1}$ and $\gamma\in E_0^h$. Moreover, by Lemma 4.11, keeping in mind that S commutes with Π_E, since Π_E is a linear differential operator on $\mathbb{G}$, and that $S(E_0^h)\subset E_0^h$, since S acts only on the coefficients of a form in E_0^h, we have

$$\begin{aligned}\hat d\Pi_{\hat E}\alpha &= \hat d\big(ds\wedge\Pi_E\beta\ +\Pi_E\gamma\big)\\ &= ds\wedge(-d\Pi_E\beta + S\Pi_E\gamma) + d\Pi_E\gamma\\ &= ds\wedge(-d\Pi_E\beta+\Pi_E\Pi_{E_0}S\gamma)+d\Pi_E\gamma.\end{aligned}$$

Thus, by Lemma 4.12 and Theorem 3.11, iv),

$$\begin{aligned}\Pi_{\hat E_0}\hat d\Pi_{\hat E}\alpha &= ds\wedge(-\Pi_{E_0}d\Pi_E\beta + \Pi_{E_0}\Pi_E\Pi_{E_0}S\gamma)\\ &\quad+\Pi_{E_0}d\Pi_E\gamma\\ &= ds\wedge(-d_c\beta+S\gamma)+d_c\gamma.\end{aligned}$$

This proves (32).

Let us now prove (33). Take $\alpha = ds\wedge\beta+\gamma\in\mathcal{D}(\mathbb{R}\times\mathbb{G},\hat E_0^h), \alpha' = ds\wedge\beta'+\gamma'\in\mathcal{D}(\mathbb{R}\times\mathbb{G},\hat E_0^{h-1})$. We have

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{G}}\langle\hat\delta_c^M\alpha,\alpha'\rangle_M\,ds\wedge dV &:= \int_{\mathbb{R}\times\mathbb{G}}\langle\alpha,\hat d_c\alpha'\rangle_M\,ds\wedge dV\\ &= \int_{\mathbb{R}\times\mathbb{G}}\langle\alpha, ds\wedge(S\gamma'-d_c\beta')+d_c\gamma'\rangle_M\,ds\wedge dV\\ &= \int_{\mathbb{R}\times\mathbb{G}}\big[\langle\gamma,d_c\gamma'\rangle - \langle\beta,-d_c\beta'+S\gamma'\rangle\big]\,ds\wedge dV\\ &= \int_{\mathbb{R}\times\mathbb{G}}\big[\langle\delta_c\gamma,\gamma'\rangle - \langle-\delta_c\beta,\beta'\rangle+\langle S\beta,\gamma'\rangle\big]\,ds\wedge dV\\ &= \int_{\mathbb{R}\times\mathbb{G}}\langle -ds\wedge\delta_c\beta+\delta_c\gamma+S\beta,\alpha'\rangle_M\,ds\wedge dV.\end{aligned}$$

$\square$

Definition 4.13. We denote by $HO(\mathbb{G})$ the group of all $(n+1)\times(n+1)$ matrices $L \in HL(\mathbb{R}\times\mathbb{G}, \mathbb{R}\times\mathbb{G})$ such that ${}^tLGL = G$, where G is defined in Definition 4.1.

We refer to $HO(\mathbb{G})$ as to the contact Lorentzian group of $\mathbb{G}$. If $L \in HO(\mathbb{G})$, then $\det L = \pm 1$. In particular, L is an homogeneous automorphism.

Example 4.14. As in Example 2.12, consider the first Heisenberg group. A matrix as in i) does not belong to $HO(\mathbb{G})$, since it has zero determinant. Thus, a matrix L belongs to $HO(\mathbb{G})$ if and only if it has the form

$$L = \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ 0 & 0 & 0 & \det A \end{pmatrix},$$

where

$$A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{is a unitary matrix.}$$

Remark 4.15. The previous example shows that, due to the rigidity of the contact structure in $\mathbb{H}^1$, Lorentz transformations in $\mathbb{H}^1$ "do not mix space and time". In fact, it turns out that this phenomenon is not peculiar to Heisenberg groups, but is common to most of the non-commutative Carnot groups. For instance, in the case of free Carnot groups, this property is not at all unexpected, keeping into account the lack of homogeneity of the associated "wave equation" (see Theorem 5.12 below).

Theorem 4.16. *If $L \in HO(\mathbb{G})$, then*

i) $L^{\#} : \hat{E}_0^* \to \hat{E}_0^*$;
ii) $\hat{d}_c L^{\#} = L^{\#}\hat{d}_c$;
iii) $*_M L^{\#} = (\det L)\cdot L^{\#}(*_M)$.

Proof. Assertions i) and ii) are already contained in Theorem 3.16, since ${}^tL = GL^{-1}G$ is an homogeneous automomorphism. Indeed, both L^{-1} and G are both homogeneous automomorphisms. As for iii), given a h-forma α, we have but to show that

$$\beta \wedge L^{\#}(*_M\alpha) = (\det L)\cdot\langle \beta, L^{\#}\alpha\rangle_M ds\wedge dV \tag{43}$$

for any $(n+1-h)$-form β. Indeed

$$\begin{aligned}\beta \wedge L^{\#}(*_M\alpha) &= L^{\#}\big((L^{-1})^{\#}\beta \wedge *_M\alpha\big) = (\det L)\cdot\big((L^{-1})^{\#}\beta\wedge *_M\alpha\big)\circ L \\ &= (\det L)\cdot\langle(\Lambda^h G)(\Lambda^h L^{-1})\beta, \alpha\circ L\rangle\, ds\wedge dV \\ &= (\det L)\cdot\langle\beta, (\Lambda^h L)\alpha\circ L\rangle\, ds\wedge dV \\ &= (\det L)\cdot\langle\beta, L^{\#}\alpha\rangle_M ds\wedge dV.\end{aligned}$$

□

5. Maxwell's equations

Let $\mathcal{J}$ be a fixed closed intrinsic n-form in $\mathbb{R}\times\mathbb{G}$ (a source form). By Lemma 4.4, $\mathcal{J} = ds\wedge *J - \rho$, where $J = J(s,\cdot)$ is an intrinsic 1-form on $\mathbb{G}$ and $\rho(s,\cdot) = \rho_0(s,\cdot)\,dV$ is a volume form on $\mathbb{G}$ for any fixed $s\in\mathbb{R}$.

If $F \in \hat{E}_0^2$, we call *Maxwell's equations in* $\mathbb{G}$ the system

$$\hat{d}_c F = 0 \quad \text{and} \quad \hat{d}_c(*_M F) = \mathcal{J} \tag{44}$$

(for sake of simplicity, we assume all "physical" constants to be 1). This system corresponds to a particular choice of the so-called constitutive relations.

Remark 5.1. The source form $\mathcal{J}$ is closed; thus, by Proposition 4.7,

$$\begin{aligned} 0 = \hat{d}_c\mathcal{J} &= ds \wedge (-S\rho - d_c(*J)) - d_c\rho \\ &= -ds \wedge (S\rho + d_c(*J)) \quad \text{(since } \rho \text{ is a volume form in } \mathbb{G}) \\ &= -(S\rho_0) ds \wedge dV - ds \wedge * * d_c(*J) \quad \text{(by (16))} \\ &= -(S\rho_0 - \delta_c J) ds \wedge dV \quad \text{(by Proposition 3.15)}, \end{aligned}$$

that is equivalent to the continuity equation

$$\frac{\partial \rho_0}{\partial s} - \delta_c J = 0. \tag{45}$$

Since J is an intrinsic 1-form in $\mathbb{G}$, we can assume $J = (\vec{J})^\natural$, where $\vec{J}$ is a horizontal vector field. Thus, equation (45) takes the more familiar form of the continuity equation

$$\frac{\partial \rho_0}{\partial s} + \operatorname{div}_{\mathbb{G}} \vec{J} = 0.$$

Remark 5.2. For sake of simplicity, Maxwell's equations, as they appear in (44), are formulated for smooth forms, though their natural formulation should be given in the sense of distributions, or, better, in the sense of intrinsic currents. Indeed, differential operators among intrinsic forms naturally extend by duality to differential operators among currents. Nevertheless, in this note we do not really need to deal with these equations in such a generality; therefore, to avoid cumbersome notations, whenever it is possible we write our equations in terms of differential forms.

Let us clarify the meaning of the last statement: our equations derive their intrinsic character from their formulation (44), but can be written alternatively "in coordinates"; to this end, we remind that we have fixed once for all a basis of $\mathfrak{g}$, i.e. a system of coordinates in $\mathbb{G}$, so that, by Theorem 3.11, vii), the system (44) can be read as a system of differential equations for the coefficients of the differential form F. On the other hand, by duality, this system for a current F can be read as the same system for the distributional coefficients of F.

A crucial property of Maxwell's equations relies in their invariance under the action of Lorentz group. The same property holds in Carnot groups: thanks to Theorem 4.16, equations (44) are invariant under the action of $HO(\mathbb{G})$, i.e.

Theorem 5.3. *If* $L \in HO(\mathbb{G})$, *F satisfies* (44), *and we set*

$$\tilde{F} := L^\# F \quad \textit{and} \quad \tilde{\mathcal{J}} := (\det L) \cdot L^\# \mathcal{J}$$

then

$$\hat{d}_c(\tilde{F}) = 0 \quad \textit{and} \quad \hat{d}_c(*_M \tilde{F}) = \tilde{\mathcal{J}}. \tag{46}$$

Remark 5.4. We stress that, by Theorem 4.16, ii), $\tilde{\mathcal{J}}$ is a closed form.

The following equivalence is well known in classical Maxwell's theory, and has been proved in Heisenberg groups in [5].

Theorem 5.5. *An intrinsic 2-form $F \in \hat{E}_0^2$, $F = ds \wedge E + B$ (with $E \in E_0^1$ and $B \in E_0^2$) satisfies* (44) *if and only if*

$$\frac{\partial(*B)}{\partial s} = *d_c E \quad , \quad \delta_c(*B) = 0 \tag{47}$$

and

$$\frac{\partial E}{\partial s} = (-1)^n * d_c(*B) - J \quad , \quad \delta_c E = -\rho_0. \tag{48}$$

Remark 5.6 (see [5]). In $\mathbb{H}^1$ (as well as in $\mathbb{R}^3$) let $\vec{E}, \vec{B}$ and $\vec{J}$ be horizontal vector fields. Set $E = (\vec{E})^\natural$, $-B = *(\vec{B})^\natural$ and $J = (\vec{J})^\natural$. Moreover, as in [20], [5], if $\vec{V}$ is a horizontal vector field, define $\mathrm{curl}_{\mathbb{H}}\, \vec{V} := (*d_c(\vec{V})^\natural)^\natural$. Then equations (47), (48) take the more familiar form

$$\frac{\partial \vec{B}}{\partial s} = -\mathrm{curl}_{\mathbb{H}}\, \vec{E} \quad , \quad \mathrm{div}_{\mathbb{H}}\, \vec{B} = 0$$

and

$$\frac{\partial \vec{E}}{\partial s} = \mathrm{curl}_{\mathbb{H}}\, \vec{B} - \vec{J} \quad , \quad \mathrm{div}_{\mathbb{H}}\, E = \rho_0.$$

Remark 5.7. By Proposition 3.15, (48) can be written also as

$$\frac{\partial E}{\partial s} = -\delta_c B - J \quad , \quad \delta_c E = -\rho_0. \tag{49}$$

Proof of Theorem 5.5. Suppose F satisfies (44). Keeping in mind Proposition 4.7, the first equation in (44) can be written as

$$ds \wedge (SB - d_c E) + d_c B = 0, \tag{50}$$

that is equivalent to

$$\frac{\partial(*B)}{\partial s} = *d_c E \quad \text{and} \quad \delta_c(*B) = 0.$$

Analogously, by Remark 4.6, the second equation in (44) can be written as

$$\begin{aligned} ds \wedge *J - \rho_0 dV &= \hat{d}_c(ds \wedge (*B) - *E) \\ &= ds \wedge \big(- *SE - d_c(*B)\big) - d_c(*E), \end{aligned}$$

that is equivalent (by (16) and Proposition 3.15) to

$$\frac{\partial E}{\partial s} = (-1)^n * d_c(*B) - J \quad \text{and} \quad \delta_c E = -\rho_0.$$

The reverse implication can be proved in the same way. □

If F is a solution of (44), then it is in particular a closed form. Therefore it admits a vector potential

$$A := A_\Sigma + \varphi\, ds \in \hat{E}_0^1 \quad \text{such that} \quad \hat{d}_c A = F.$$

We want to show that, under suitable gauge conditions, A_Σ and φ satisfy intrinsic "wave equations". To this end, we must restrict ourselves to a particular class of Carnot groups, i.e. to the class of the so-called *free groups*. Let us remind the following definition.

Definition 5.8. Let $m \geq 2$ and $\kappa \geq 1$ be fixed integers. We say that $\mathfrak{f}_{m,\kappa}$ is the free Lie algebra with m generators $x_1, \ldots, x_m$ and nilpotent of step κ if:

i) $\mathfrak{f}_{m,\kappa}$ is a Lie algebra generated by its elements $x_1, \ldots, x_m$, i.e. $\mathfrak{f}_{m,\kappa} = \mathrm{Lie}(x_1, \ldots, x_m)$;
ii) $\mathfrak{f}_{m,\kappa}$ is nilpotent of step κ;
iii) for every Lie algebra $\mathfrak{n}$ nilpotent of step κ and for every map φ from the set $\{x_1, \ldots, x_m\}$ to $\mathfrak{n}$, there exists a (unique) homomorphism of Lie algebras Φ from $\mathfrak{f}_{m,\kappa}$ to $\mathfrak{n}$ which extends φ.

The Carnot group $\mathbb{G}$ is said free if its Lie algebra $\mathfrak{g}$ is isomorphic to a free Lie algebra.

The technical reason for restricting ourselves to the class of free groups relies in the following property.

Theorem 5.9. *Let $\mathbb{G}$ be a free group of step κ with m generators ($m > 1$). Then all forms in E_0^1 have weight 1 and all forms in E_0^2 have weight $\kappa + 1$.*

In particular, the differential $d_c : E_0^1 \to E_0^2$ can be identified, with respect to the adapted bases Ξ_0^1 and Ξ_0^2, with a homogeneous matrix-valued differential operator of degree κ in the horizontal derivatives.

Proof. The first assertion is well known, since $E_0^1 = \mathrm{span}\,\{\theta_1, \ldots, \theta_m\}$. On the other hand, the last assertion follows by [2], Theorem 2.15, iii). Let now $\alpha \wedge \beta \neq 0$ be a left-invariant 2-forms with $w(\alpha) = p_\alpha$, $w(\beta) = p_\beta$, $p_\alpha \leq \kappa$, $p_\beta \leq \kappa$. First of all, we can assume without loss of generality that

$$\beta \perp \alpha. \tag{51}$$

Indeed, we can write $\beta = \lambda\alpha + \beta'$, with $\beta' \perp \alpha$, and, obviously, $\beta' \neq 0$, since, if $\beta' = 0$, $\alpha \wedge \beta = \lambda\alpha \wedge \alpha = 0$, being α of degree 1. Thus, for the same reason, $\alpha \wedge \beta = \alpha \wedge \beta'$, proving (51). Suppose now $p_\alpha + p_\beta \leq \kappa$; we can show that $\alpha \wedge \beta \notin (\mathrm{Im}\ d_0)^\perp$. Indeed, remember first d_0 when acting on left-invariant forms coincides with d (see [2], Lemma 2.8 or [40], Section 2.1). Put $X := [\alpha^\natural, \beta^\natural]$ and $\xi := X^\natural$. The left-invariant vector field X belongs to the layer $V_{p_\alpha + p_\beta}$ and does not vanishes since $p_\alpha + p_\beta \leq \kappa$ and the group $\mathbb{G}$ is free. By Cartan's Lemma ([28], identity (1) p.136, with a different normalization), we have

$$\langle d_0\xi, \alpha \wedge \beta\rangle = \langle d\xi, \alpha \wedge \beta\rangle = \langle d\xi | \alpha^\natural \wedge \beta^\natural\rangle = -\langle \xi | X\rangle = -\langle X, X\rangle \neq 0.$$

This shows that $\alpha \wedge \beta \notin (\mathrm{Im}\ d_0)^\perp$ and therefore that $\alpha \wedge \beta \notin E_0^2$.

Assume now that $p_\alpha + p_\beta > 1 + \kappa$. We want to show that

$$d_0(\alpha \wedge \beta) = d(\alpha \wedge \beta) \neq 0 \tag{52}$$

This will imply that $\alpha \wedge \beta \notin E_0^2$, achieving the proof of the theorem.

To prove (52), suppose $\kappa \geq p_\alpha \geq p_\beta$. Since $p_\alpha + p_\beta > \kappa + 1$, we know that $p_\beta > 1$, so that we can write $\beta^\natural = [W_3, W_1]$, with $W_3 \in V_1$ and $W_1 \in V_{p_\beta - 1}$. We set also $W_2 := \alpha^\natural$.

Again by Cartan's identity

$$
\begin{aligned}
\langle d_0(\alpha\wedge\beta)|W_1\wedge W_2\wedge W_3\rangle &= \langle d(\alpha\wedge\beta)|W_1\wedge W_2\wedge W_3\rangle \\
&= -\Big\{\langle\alpha\wedge\beta|[W_1,W_2]\wedge W_3\rangle + \langle\alpha\wedge\beta|[W_2,W_3]\wedge W_1\rangle \\
&\qquad + \langle\alpha\wedge\beta|[W_3,W_1]\wedge W_2\rangle\Big\} \\
&:= -\Big\{I_1+I_2+I_3\Big\}.
\end{aligned}
$$

Notice now that $[W_1,W_2]\in V_{p_\alpha+p_\beta-1}=\{0\}$, since $p_\alpha+p_\beta-1>\kappa$. Thus $I_1=0$. On the other hand

$$
I_2=\det\begin{pmatrix}\langle\alpha|[W_2,W_3]\rangle & \langle\alpha|W_1\rangle \\ \langle\beta|[W_2,W_3]\rangle & \langle\beta|W_1\rangle\end{pmatrix}.
$$

But $\langle\beta|W_1\rangle = \langle\beta^\natural,W_1\rangle = 0$, since $W_1\in V_{p_\beta-1}$ and $\beta^\natural\in V_{p_\beta}$. Moreover, $[W_2,W_3]\in V_{p_\alpha+1}$ and $\beta^\natural\in V_{p_\beta}$. But $p_\alpha+1>p_\alpha\ge p_\beta$, so that $\langle\beta|[W_2,W_3]\rangle=\langle\beta^\natural,[W_2,W_3]\rangle=0$ and then $I_2=0$.

Finally

$$
I_3=\det\begin{pmatrix}\langle\alpha|\beta^\natural\rangle & \langle\alpha|W_2\rangle \\ \langle\beta|\beta^\natural\rangle & \langle\beta|W_2\rangle\end{pmatrix}.
$$

But $\langle\alpha|\beta^\natural\rangle=\langle\alpha,\beta\rangle=0$, $\langle\beta|\beta^\natural\rangle=\langle\beta,\beta\rangle=|\beta^2|\neq0$, and $\langle\alpha|W_2\rangle=\langle\alpha|\alpha^\natural\rangle=\langle\alpha,\alpha\rangle=|\alpha^2|\neq0$, so that $I_3=|\alpha^2||\beta^2|\neq0$.

This shows that (52) eventually holds. □

We need now a Hodge-Laplace operator on intrinsic forms. We already discussed extensively in the Introduction the problem of the lack of homogeneity of the "naif Laplacian" $d_c\delta_c+\delta_c d_c$, showing that the homogeneity of the exterior differential d_c in a free group $\mathbb{G}$ stated in Theorem 5.9 enables us to build a good "homogeneous Hodge Laplacian" on intrinsic 1-forms on $\mathbb{G}$.

Theorem 5.10. *Let $\mathbb{G}$ be a Carnot group. Suppose $N_2^{\max}=N_2^{\min}:=N_2$, i.e. suppose all intrinsic 2-forms have the same weight N_2 (by Theorem 5.9 this holds true for any free group $\mathbb{G}$, but also for all Heisenberg groups $\mathbb{H}^n$ with $n\ge1$). Set $N_2-1:=r$. Denote by $\Delta_{\mathbb{G},1}:=\delta_c d_c+(d_c\delta_c)^r$ the homogeneous Hodge Laplacian on intrinsic 1-forms in $\mathbb{G}$. Then the following result holds: if $\theta_1,\dots\theta_m$ is the fixed left invariant orthonormal basis of E_0^1, then for $j=1,\dots,m$ there exists*

$$
K_j=\sum_i \tilde K_{ij}\llcorner(*\theta_i)\in\mathcal{D}'(\mathbb{G},E_0^1)\cap\mathcal{E}(\mathbb{G}\setminus\{0\},E_0^1), \tag{53}
$$

with $K_{ij}\in\mathcal{D}'(\mathbb{G})$, $i,j=1,\dots,m$ such that

- i) *$\Delta_{\mathbb{G},1}K_j=\tilde\delta\llcorner(*\theta_j)$, $j=1,\dots,m$;*
- ii) *If $2r<Q$, then the K_{ij}'s are kernels of type $2r$ in the sense of [15], for $i,j=1,\dots,N$ (i.e. they are smooth functions outside of the origin, homogeneous of degree $2r-Q$, and hence belonging to $L^1_{\mathrm{loc}}(\mathbb{G})$, by Corollary 1.7 of [15]). If $2r=Q$, then the K_{ij}'s satisfy the logarithmic estimate $|K_{ij}(p)|\le C(1+|\ln\rho(p)|)$ and hence belong to $L^1_{\mathrm{loc}}(\mathbb{G})$. Moreover, their horizontal derivatives (i.e. $X_\ell K_{ij}$ for $\ell=1,\dots,m$) are kernels of type $Q-1$ in the sense of [15].*

iii) *When* $\alpha \in \mathcal{D}(\mathbb{H}^n, E_0^1)$, *if we set*

$$\mathcal{K}\alpha := \sum_{ij} (\alpha_j * \tilde{K}_{ij}) \llcorner (*\xi_i), \tag{54}$$

then $\Delta_{\mathbb{G},1}\mathcal{K}\alpha = \alpha$.

iv) *If* $2r < Q$, *also* $\mathcal{K}\Delta_{\mathbb{G},1}\alpha = \alpha$. *If* $2r = Q$, *then for any* $\alpha \in \mathcal{D}(\mathbb{G}, E_0^1)$ *there exists a "constant coefficient form"* $\beta_\alpha \in E_0^1$, *such that*

$$\mathcal{K}\Delta_{\mathbb{G},1}\alpha - \alpha = \beta_\alpha.$$

v) $\Delta_{\mathbb{G},1}$ *is maximal subelliptic, i.e. there exists* $C > 0$ *such that for any multi–index* I *with* $d(I) = r$

$$\|X^I\alpha\|_{L^2(\mathbb{G},E_0^1)} \le C\left(\langle \Delta_{\mathbb{G},1}\alpha, \alpha\rangle_{L^2(\mathbb{G},E_0^1)} + \|\alpha\|_{L^2(\mathbb{G},E_0^1)}\right) \tag{55}$$

for any $\alpha \in \mathcal{D}(\mathbb{G}, E_0^1)$.

vi) *If* $1 < p < \infty$ *is fixed, then there exists* $C > 0$ *such that for any multi–index* I *with* $d(I) = 2r$ *we have*

$$\|X^I\alpha\|_{L^p(\mathbb{G},E_0^1)} \le C\left(\|\Delta_{\mathbb{G},1}\alpha\|_{L^p(\mathbb{G},E_0^1)} + \|\alpha\|_{L^p(\mathbb{G},E_0^1)}\right) \tag{56}$$

for any $\alpha \in \mathcal{D}(\mathbb{G}, E_0^1)$ *(if* $p = 2$ *this means that* $\Delta_{\mathbb{G},1}$ *is maximal hypoelliptic in the sense of* [27]*),.*

If we replace $\mathbb{G}$ *by a bounded open set* $\Omega \subset \mathbb{G}$, *then, by Poincaré inequality (*[30]*), in* (56) *and in* (55)*), we can replace* $d(I) = 2r$ *by* $d(I) \le 2r$ *and* $d(I) = r$ *by* $d(I) \le r$, *respectively.*

Proof. If we prove that $\Delta_{\mathbb{G},1}$ is hypoelliptic, then the statements follow by [6], Theorems 3.1 (see also [4], Theorem 4.7). We notice that statements v) and vi) are proved in [6] with constants depending on supp α. But we can easily get rid of this dependence taking the assertion for supp $\alpha \subset U(e, 1)$ and by applying a rescaling argument.

On the other hand, the proof of the (maximal) hypoellipticity of $\Delta_{\mathbb{G},1}$ follows verbatim the scheme of that of [40], Theorem 2.5. Let π be a nontrivial irreducible unitary representation of $\mathbb{G}$. Without loss of generality, if S_π is the space of $\mathbf{C}^\infty$ vectors of the representation, we may assume that

$$S_\pi = \mathcal{S}(\mathbb{R}^k),$$

for a suitable $k \in \mathbb{N}$.

First of all, we remind that, for any $m \in \mathbb{N}$, $\Delta_{\mathbb{G}}^m$ is maximal hypoelliptic, and therefore $\pi(\Delta_{\mathbb{G}}^m)$ is injective on S_π. Indeed, since $\Delta_{\mathbb{G}}^m$ is a left invariant $\mathbb{G}$-homogeneous differential operator, it is enough to notice that $\Delta_{\mathbb{G}}^m$ is hypoelliptic (see, e.g., [6]).

Since we have already fixed two bases of E_0^*, all differential operators among intrinsic forms (and $\Delta_{\mathbb{G},1}$ in particular) can be seen as a matrix-valued differential operators. Thus, by [27] (see also [10], p. 63, Remark 5), the hypoellipticity of $\Delta_{\mathbb{G},1}$ is equivalent to the injectivity of $\pi\big(\Delta_{\mathbb{G},1}\big)$ on $S_\pi^{N_1}$.

Let now $u \in \big(\mathcal{S}(\mathbb{R}^k)\big)^{N_1}$ be such that

$$\pi(d_c)^*\pi(d_c)u + \big(\pi(d_c)\pi(d_c)^*\big)^r u = 0.$$

Integrating by parts, we get

$$\pi(d_c)u = 0 \tag{57}$$

and

$$\begin{aligned} &\pi(d_c^* d_c)\cdots\pi(d_c^* d_c)\pi(d_c^*)u \\ &\qquad = \pi((-\Delta_{\mathbb{G}})^h)\pi(d_c^*)u = 0 \quad \text{if } r = 2h+1 \text{ is odd} \end{aligned} \tag{58}$$

or

$$\begin{aligned} &\pi(d_c)\pi(d_c^* d_c)\cdots\pi(d_c^* d_c)\pi(d_c^*)u \\ &\qquad = \pi(d_c)\pi((-\Delta_{\mathbb{G}})^h)\pi(d_c^*)u = 0 \quad \text{if } r = 2h+2 \text{ is even.} \end{aligned} \tag{59}$$

By [40], proof of Theorem 5.2, there exists $X \in \mathfrak{g}$ such that, for any $v \in \big(\mathcal{S}(\mathbb{R}^k)\big)^{N_1}$,

$$v = Q_X\pi(d_c)v + \pi(d_c)Q_Xv, \tag{60}$$

where

$$Q_X := \pi(\Pi_{E_0}\Pi_E)P_Xi_X\pi(\Pi_E\Pi_{E_0}).$$

Here P_X is the inverse of $\pi(\mathcal{L}_X)$, $\mathcal{L}_X$ being the Lie derivative along X.

Replacing (57) in (60), we get

$$u = \pi(d_c)Q_Xu. \tag{61}$$

Thus, if r is odd, we replace (61) in (58) and we get

$$\pi((-\Delta_{\mathbb{G}})^{h+1})Q_Xu = 0,$$

yielding eventually $u = 0$, since $(-\Delta_{\mathbb{G}})^{h+1}$ is maximal hypoelliptic in the sense of [27] and then $\pi((-\Delta_{\mathbb{G}})^{h+1})$ is injective. Thus $Q_Xu = 0$ and therefore $u = 0$, by (61). On the other hand, if r is even, we replace (61) in (59) and we apply $\pi(d_c^*)$ to both sides of the identity we obtain in this way. We get

$$\pi((-\Delta_{\mathbb{G}})^{h+2})Q_Xu = 0,$$

and we conclude in the same way. □

Remark 5.11. The previous result was proved in [38] and [4] when $\mathbb{G} = \mathbb{H}^n$, $n \geq 1$.

Now we can define our intrinsic "wave equations" for Carnot groups satisfying the assumptions of Theorem 5.10.

Theorem 5.12. *Let $\mathbb{G}$ be a Carnot group satisfying the assumption of Theorem 5.10. Suppose $F \in \hat{E}_0^2$ satisfies (44). Then $F = \hat{d}_cA$ with $A = \sum_{j=1}^m A_j\theta_j + \varphi\, ds := A_\Sigma + \varphi\, ds \in \hat{E}_0^1$, where*

$$\frac{\partial^2 A_\Sigma}{\partial s^2} = -\Delta_{\mathbb{G},1}A_\Sigma - J \tag{62}$$

$$\frac{\partial^2 \varphi}{\partial s^2} = -(-\Delta_{\mathbb{G}})^r\varphi + (-\Delta_{\mathbb{G}})^{r-1}\rho_0, \tag{63}$$

where $\Delta_{\mathbb{G}} := \sum_{j=1}^m X_j^2 (= -\Delta_{\mathbb{G},0})$ is the usual subelliptic Laplacian in $\mathbb{G}$, provided the following gauge condition holds:

$$\delta_c(d_c\delta_c)^{r-1}A_\Sigma + \frac{\partial\varphi}{\partial s} = 0. \tag{64}$$

Notice condition (64) *can also be written as*

$$(-\Delta_{\mathbb{G}})^{r-1}\delta_c A_\Sigma + \frac{\partial\varphi}{\partial s} = 0. \tag{65}$$

Remark 5.13. The gauge condition (64) is always satisfied if we replace A by $A + \hat{d}_c f$, with f satisfying

$$\frac{\partial^2 f}{\partial s^2} = -(-\Delta_{\mathbb{G}})^r f - \Big(\delta_c(d_c\delta_c)^{r-1}A_\Sigma + \frac{\partial\varphi}{\partial s}\Big)$$

(see for instance Section 6.3 for abstract existence results for this equation).

Proof of Theorem 5.12. As in Theorem 5.5, we can write $F = ds\wedge E + B$, with $d_c B = 0$ (by (50)), so that $B := d_c A_\Sigma$. Again by (50)

$$d_c\big(\frac{\partial A_\Sigma}{\partial s}\big) - d_c E = \frac{\partial B}{\partial s} - d_c E = 0,$$

so that we can write

$$\frac{\partial A_\Sigma}{\partial s} = E + d_c\varphi \tag{66}$$

for a suitable scalar function φ. Thus, by Proposition 4.7 we have

$$F = \hat{d}_c(A_\Sigma + \varphi\, ds) \tag{67}$$

Combining now (49) and (66) we get

$$\frac{\partial^2 A_\Sigma}{\partial s^2} = -\delta_c d_c A_\Sigma + d_c\frac{\partial\varphi}{\partial s} - J,$$

and, eventually, by (64)

$$\begin{aligned}\frac{\partial^2 A_\Sigma}{\partial s^2} &= -\delta_c d_c A_\Sigma - (d_c\delta_c)^r - J\\ &= -\Delta_{\mathbb{G},1}A_\Sigma - J,\end{aligned}$$

i.e. (62).

On the other hand, by (66) and the second equation in (48),

$$\delta_c\big(\frac{\partial A_\Sigma}{\partial s} - d_c\varphi\big) = -\rho_0,$$

i.e.

$$-\delta_c\frac{\partial A_\Sigma}{\partial s} = -\delta_c d_c\varphi \ + \rho_0. \tag{68}$$

Differentiating now (64) with respect to s we get

$$\delta_c(d_c\delta_c)^{r-1}\frac{\partial A_\Sigma}{\partial s} + \frac{\partial^2\varphi}{\partial s^2} = 0. \tag{69}$$

If $r = 1$ we can replace (68) in (69) and we get

$$\frac{\partial^2\varphi}{\partial s^2} = -\delta_c d_c\varphi \ + \rho_0,$$

i.e.

$$\frac{\partial^2\varphi}{\partial s^2} = \Delta_{\mathbb{G}}\varphi \ + \rho_0.$$

On the other hand, if $r > 1$ we can write (69) as

$$(\delta_c d_c)^{r-1}\delta_c\frac{\partial A_\Sigma}{\partial s} + \frac{\partial^2\varphi}{\partial s^2} = 0. \tag{70}$$

Replacing again (68) in (70), we get

$$\frac{\partial^2 \varphi}{\partial s^2} = -(\delta_c d_c)^r \varphi \ + (\delta_c d_c)^{r-1} \rho_0,$$

i.e

$$\frac{\partial^2 \varphi}{\partial s^2} = -(-\Delta_{\mathbb{G}})^r \varphi \ + (-\Delta_{\mathbb{G}})^{r-1} \rho_0.$$

This proves (63).

□

Proposition 5.14. *Let $\mathbb{G}$ be a Carnot group satisfying the assumption of Theorem 5.10, and let $A_\Sigma \in E_0^1$ and φ satisfy* (62), (63) *and* (64). *Then*

$$V := \delta_c \frac{\partial A_\Sigma}{\partial s} + \Delta_{\mathbb{G}} \varphi + \rho_0$$

is independent of $s \in \mathbb{R}$.

Proof. We take the s-derivative of V and we replace (62) in $\dfrac{\partial V}{\partial s}$. We get

$$\begin{aligned}
\frac{\partial V}{\partial s} &= \delta_c \frac{\partial^2 A_\Sigma}{\partial s^2} + \Delta_{\mathbb{G}} \frac{\partial \varphi}{\partial s} + \frac{\partial \rho_0}{\partial s} \\
&= \delta_c(-\delta_c d_c A_\Sigma - (d_c\delta_c)^r A_\Sigma) - \delta_c J + \frac{\partial \rho_0}{\partial s} + \Delta_{\mathbb{G}} \frac{\partial \varphi}{\partial s} \\
&= \delta_c(-\delta_c d_c A_\Sigma - (d_c\delta_c)^r A_\Sigma) + \Delta_{\mathbb{G}} \frac{\partial \varphi}{\partial s} \quad \text{(by the continuity equation (45))} \\
&= -(-\Delta_{\mathbb{G}})^r \delta_c A_\Sigma + \Delta_{\mathbb{G}} \frac{\partial \varphi}{\partial s} \quad (\text{since } \delta_c^2 = 0) \\
&= \Delta_{\mathbb{G}}\big((-\Delta_{\mathbb{G}})^{r-1} \delta_c A_\Sigma + \frac{\partial \varphi}{\partial s}\big) = 0
\end{aligned}$$

by the gauge condition (65).

□

Theorem 5.15. *Let $\mathbb{G}$ be a Carnot group satisfying the assumption of Theorem 5.10, and let $A_\Sigma \in E^1_{0,\mathbb{H}^1}$ and φ satisfy* (62), (63) *and* (64).

If $r > 1$, suppose in addition

i) *$V \in \mathcal{S}'(\mathbb{G})$;*
ii) *there exists $m_0 \in [0, 2r-1)$ such that*

$$V(x) = O(\|x\|_{\mathbb{G}}^{m_0}) \quad \textit{as } \|x\|_{\mathbb{G}} \to \infty.$$

Then

i) *if $r = 1$, then $V \equiv 0$;*
ii) *if $r > 1$, then V is a polynomial of homogeneous degree at most $[m_0]$, and there exist a 1-form $G_0 \in E^1_{0,\mathbb{G}}$ with polynomial coefficients of homogeneous degree at most $2r-1$ such that, if we set $G := sG_0$, then*

$$F = d_c A := d_c(A_\Sigma + G + \varphi\, ds)$$

satisfies (44).

In addition, if

$$V = o(1) \quad \text{as } \|x\|_{\mathbb{G}} \to \infty, \tag{71}$$

(therefore in particular when $r = 1$, by i))

$$F = d_c A := d_c(A_\Sigma + \varphi\, ds).$$

Proof. By (65) in (63), we get

$$(-\Delta_{\mathbb{G}})^{r-1}\big(\delta_c S A_\Sigma + \Delta_{\mathbb{G}}\varphi + \rho_0\big) = 0.$$

If $r = 1$, this yields $\delta_c S A_\Sigma + \Delta_{\mathbb{G}}\varphi + \rho_0 \equiv 0$. Suppose now $r > 1$. By [22], V is an homogeneous polynomial, that, arguing as in the proof of [8], Theorem 5.8.8, has homogeneous degree at most $[m_0] \le 2r - 2$. Let $G_0 \in E^1_{0,\mathbb{G}}$ be a 1-form such that $\delta_c G_0 = V$. We can choose G_0 with polynomial coefficients of homogeneous degree at most $2r-1$. Set now $G := sG_0$. We have (keeping in mind that $d_c* = *\delta_c$ on 2-forms)

$$\begin{aligned}
&\hat d_c *_M \hat d_c\big(A_\Sigma + \varphi\, ds + G\big)\\
&\quad = \hat d_c *_M \big(ds \wedge (S(A_\Sigma + G) - d_c\varphi) + d_c(A_\Sigma + G)\big)\\
&\quad = \hat d_c\big(ds \wedge (*d_c(A_\Sigma + G)) - *(S(A_\Sigma + G) - d_c\varphi)\big)\\
&\quad = -ds \wedge \big(d_c * d_c(A_\Sigma + G) - S(*(d_c\varphi)) + S^2(*(A_\Sigma + G))\big)\\
&\qquad + \big(d_c * d_c\varphi - d_c S(*(A_\Sigma + G))\big)\\
&\quad = -ds \wedge \big(* \big(\delta_c d_c(A_\Sigma + G) - S(d_c\varphi + S^2(A_\Sigma + G)\big)\big)\\
&\qquad + \big(d_c * d_c\varphi - d_c S(*(A_\Sigma + G))\big)\\
&\quad = -ds \wedge \big(* \big(\delta_c d_c A_\Sigma - d_c S\varphi + S^2 A_\Sigma\big)\big)\\
&\qquad + \big(d_c * d_c\varphi - d_c S(*(A_\Sigma + G))\big),
\end{aligned}$$

since $S^2 G = 0$ and $\delta_c d_c G = 0$. Now, by gauge condition (64)

$$\begin{aligned}
&\delta_c^* d_c A_\Sigma - d_c(S\varphi) + S^2 A_\Sigma\\
&\quad = (\Delta_{\mathbb{H},1} + S^2) A_\Sigma\\
&\quad = (\Delta_{\mathbb{H},1} + S^2) A_\Sigma = -J.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&d_c \; * d_c\varphi - d_c S(*(A_\Sigma + G))\\
&\quad = *\big(* d_c * d_c\varphi - *d_c * S A_\Sigma - *d_c * (g_1 x dx + g_2 y dy)\big)\\
&\quad = *\big(\Delta_{\mathbb{H}}\varphi + \delta_c^* S A_\Sigma + \delta_c G_0\big)\\
&\quad = *\big(V - \rho_0 + \delta_c G_0\big) = -\rho_0\, dV.
\end{aligned}$$

Thus

$$\hat d_c *_M \hat d_c\big(A_\Sigma + \varphi\, ds + G\big) = \mathcal{J}.$$

This achieves the proof of the theorem, since the last statement is trivial: indeed, an homogeneous polynomial vanishing at infinity must be identically zero, and therefore we can choose $G_0 \equiv 0$.

□

Remark 5.16. Assumption (71) can be replaced by alternative assumptions that guarantee that a polynomial vanishes identically. For instance, we could replace (71) by

$$V \in L^p(\mathbb{G}) \quad \text{for some } p \in [1, \infty). \tag{72}$$

Indeed, suppose (72) holds and $V \not\equiv 0$. We denote by V_0 the homogeneous part of V with $\deg V_0 = \deg V$, then there exists $x_0 \in \mathbb{G}$ with $\|x_0\|_{\mathbb{G}} = 1$ such that (say) $V(x) \geq c > 0$ for $x \in U \cap \{\|x\|_{\mathbb{G}} = 1\}$, where U is a small neighborhood of x_0. Thus, if $\delta_{1/\|x\|_{\mathbb{G}}} x \in U$ and $\|x\|_{\mathbb{G}}$ is sufficiently large, then $V(x) \geq \frac{c}{2} \|x\|_{\mathbb{G}}^{\deg V}$, yielding a contradiction, and the assertion follows.

Remark 5.17. Assumptions (71) can be better formulated when we associate with (62) and (63) a Cauchy problem. Suppose $V \in \mathcal{S}(\mathbb{G})$. Keeping in mind Proposition 5.14, for the Cauchy problem

$$\begin{cases} A_{\Sigma|s=0} = A_{\Sigma,0}, & \frac{\partial A_\Sigma}{\partial s}_{|s=0} = 0; \\ \varphi_{|s=0} = \varphi_0, & \frac{\partial \varphi}{\partial s}_{|s=0} = 0, \end{cases} \tag{73}$$

(71) becomes

$$\Delta_{\mathbb{H}} \varphi_0 + \rho_0 = o(1) \quad \text{as } \|x\|_{\mathbb{G}} \to \infty \tag{74}$$

and (72) becomes (for instance)

$$\Delta_{\mathbb{G}} \varphi_0 + \rho_0 \in L^2(\mathbb{G}), \tag{75}$$

whereas, for the Cauchy problem

$$\begin{cases} A_{\Sigma|s=0} = 0, & \frac{\partial A_\Sigma}{\partial s}_{|s=0} = A_{\Sigma,1}; \\ \varphi_{|s=0} = 0, & \frac{\partial \varphi}{\partial s}_{|s=0} = \varphi_1, \end{cases} \tag{76}$$

(71) becomes

$$\delta_c A_{\Sigma,1} + \rho_0 = o(1) \quad \text{as } \|x\|_{\mathbb{G}} \to \infty \tag{77}$$

and (72) becomes (for instance)

$$\delta_c A_{\Sigma,1} + \rho_0 \in L^2(\mathbb{G}). \tag{78}$$

6. Solutions of wave equations

6.1. **Plane waves.** Let $\mathbb{G}$ be a free Carnot group of step 2 (but see also Remark 6.13 below). As in the Euclidean case, we can consider here suitable "plane waves". Assume $J \equiv 0$ and $\rho_0 \equiv 0$. Since intrinsic hyperplanes in $\mathbb{G}$ are "vertical planes" (i.e. laterals of subgroups $\mathbb{M}$ of the form

$$\mathbb{M} = \exp(\{\xi \in \mathfrak{g},\ \langle \xi, K \rangle = 0\}),$$

where $K = (k_1, \dots, k_m, 0, \dots, 0) \in V_1$: see e.g. [18], [19]), we look for solutions of the wave equations of the form

$$A(s,x) = e^{i(\omega s - k\cdot x)} A_0 + e^{i(\omega s - k\cdot x)} \varphi_0 \, ds := A_\Sigma + \varphi \, ds,$$

where

- ω (the angular frequency) is a real number;
- $k = \exp(K) = \sum_{j=1}^m k_j X_j$ and $k \cdot x := \sum_{j=1}^m k_j x_j$
- $A_0 := \sum_{j=1}^m A_{0,j} \, \theta_j$ is a (complex coefficients) intrinsic 1-covector of $\mathbb{G}$;

- φ_0 is a complex number.

Theorem 6.1. *Let $\mathbb{G}$ be a free group of step 2 (but see also Remark 6.13 below). If we choose $\omega = \pm|k|^2$ and $\varphi_0 = \mp\langle A_0|k\rangle$, then A satisfies* (62), (63) *and* (64).

In addition

$$F := \hat{d}_c A$$

satisfies (44).

Proof. The proof requires an explicit computation of $\Delta_{\mathbb{G},1} e^{i(\omega s - k\cdot x)} A_0$ that in turn relies on several technical lemmata. This computation will be postponed to Subsection 6.2, and the explicit form of $\Delta_{\mathbb{G},1} e^{i(\omega s - k\cdot x)} A_0$ appears in Theorem 6.12 below. If we assume for a while Theorem 6.12 holds, we have

$$\begin{aligned}\Delta_{\mathbb{G},1} A_\Sigma &= \sum_{j=1}^m (\Delta^2_{\mathbb{R}^m} e^{i(\omega s - k\cdot x)}) A_{0,j}\, dx_j \\ &= \sum_{j=1}^m (|k|^4 e^{i(\omega s - k\cdot x)}) A_{0,j}\, dx_j = |k|^4 A_\Sigma.\end{aligned}$$

On the other hand

$$\frac{\partial^2 A_\Sigma}{\partial s^2} = -\omega^2 A_\Sigma,$$

and (62) holds. The proof of (63) is straightforward since the group Laplacian on scalar functions depending only on the first m variables is nothing but the usual Euclidean Laplacian in $\mathbb{R}^m$. As for (64), we have

$$\begin{aligned}-\Delta_{\mathbb{G}} \delta_c A_\Sigma &= \Delta_{\mathbb{R}^m} (\sum_{j=1}^m (X_j e^{i(\omega s - k\cdot x)}) A_{0,j}) \\ &= -i\Delta_{\mathbb{R}^m} (\sum_{j=1}^m k_j e^{i(\omega s - k\cdot x)} A_{0,j}) = i|k|^2 e^{i(\omega s - k\cdot x)} \langle A_0|k\rangle,\end{aligned}$$

wheras

$$\frac{\partial \varphi}{\partial s} = i\omega e^{i(\omega s - k\cdot x)} \varphi_0,$$

and the gauge condition is satisfied. Finally (44) is satisfied by F since the function $V := \delta_c S A_\Sigma + \Delta_{\mathbb{G}} \varphi$ of Proposition 5.14 vanishes identically. $\square$

6.2. **Forms depending only on horizontal variables.** In this section, we look for an explicit form of $\Delta_{\mathbb{G},1} e^{i(\omega s - k\cdot x)} A_0$ that in turn relies on several technical lemmata.

A simple property of free groups that will be crucial in the sequel is that, according to our choices of the scalar producy in $\mathfrak{g}$, $\{[X_i, X_j], X_i, X_j \in V_1, i<j\}$ provides an (orthonormal) Hall basis of V_2.

Lemma 6.2. *Let $\mathbb{G}$ be a Carnot group of any step. If $X_i, X_j \in V_1$ with $i<j$, then $d_0([X_i, X_j]^\natural) = -\theta_i \wedge \theta_j$.*

Proof. If $X, Y \in \mathfrak{g}$, we want to show that

$$\langle d_0([X_i, X_j]^\natural)|X \wedge Y\rangle = -\langle \theta_i \wedge \theta_j | X \wedge Y\rangle.$$

Since d_0 preserves the weights, we may assume that $X \wedge Y$ has weight 2. Therefore, without loss of generality, we can take $X = X_k, Y = X_h$, with $X_k, X_h \in V_1$. Therefore

$$\begin{aligned}&\langle d_0([X_i, X_j]^\natural)|X_k \wedge X_h\rangle = \langle d([X_i, X_j]^\natural)|X_k \wedge X_h\rangle \\ &= -\langle [X_i, X_j]^\natural | [X_k, X_h]\rangle = -\langle [X_i, X_j], [X_k, X_h]\rangle.\end{aligned}$$

On the other hand, $\langle [X_i, X_j], [X_k, X_h]\rangle = 0$ if $\{i, j\} \neq \{k, h\}$, $\langle [X_i, X_j], [X_k, X_h]\rangle = 1$ if $(i, j) = (k, h)$, and $\langle [X_i, X_j], [X_k, X_h]\rangle = -1$ if $(i, j) = (h, k)$, whereas

$$\langle \theta_i \wedge \theta_j | X_k \wedge X_h\rangle = \det \begin{pmatrix} \langle \theta_i | X_k\rangle & \langle \theta_i | X_h\rangle \\ \langle \theta_j | X_k\rangle & \langle \theta_j | X_h\rangle \end{pmatrix}.$$

□

Lemma 6.3. *If $\mathfrak{g}$ is a free algebra of step 2, then*

(1) $d_0(\bigwedge^1 \mathfrak{g}) = \bigwedge^{2,2} \mathfrak{g}$;
(2) *if $\theta_i \wedge \theta_j \in \bigwedge^{2,2} \mathfrak{g}$, then $d_0^{-1}(\theta_i \wedge \theta_j) = -[X_i, X_j]^\natural$;*
(3) *if $\theta_i \wedge \theta_j \in \bigwedge^{1,2} \mathfrak{g}$, then $d_0 d_0^{-1}(\theta_i \wedge \theta_j) = \theta_i \wedge \theta_j$;*
(4) *if $\theta_i \wedge \theta_j \in \bigwedge^{2,3} \mathfrak{g}$ or $\theta_i \wedge \theta_j \in \bigwedge^{2,4} \mathfrak{g}$ then $d_0^{-1}(\theta_i \wedge \theta_j) = 0$, so that again $d_0^{-1}(\theta_i \wedge \theta_j) = -[X_i, X_j]^\natural$.*

Proof. Assertions (1) and (2) follow from previous lemma since $d_0(\bigwedge^{1,1} \mathfrak{g}) = \{0\}$. As for (3), we have but to notice that $d_0 d_0^{-1}(\theta_i \wedge \theta_j) - \theta_i \wedge \theta_j \in \mathcal{R}(d_0)^\perp \cap \bigwedge^{2,2} \mathfrak{g} = \{0\}$, by (1). Finally, assertion (4) holds since, by (1), any 2-form of weight greater than 2 is orthogonal to the range of d_0. □

Remark 6.4. We point out that Lemma 6.3 basically relies on the fact that, when $\kappa = 2$, the basis $\Theta^{1,2}$ of $\bigwedge^{1,\kappa} \mathfrak{g}$ is carried by d_0 onto the dual basis $\Theta^{2,2}$ of $\bigwedge^{2,\kappa} \mathfrak{g}$. If we consider a free group with 2 generators of step 3, the same assertion holds, in the sense that

$$d_0(\Theta^{1,i}) = \Theta^{2,i}, \quad i = 2, 3 \quad (3 = \kappa).$$

The same property fails to hold for more complicated free groups because of Jacobi identity.

Lemma 6.5. *Let $\mathbb{G}$ be a Carnot group of any step. If $\beta = \sum_{i,j} \beta_{ij} \theta_i \wedge \theta_j \in \Omega^{2,2}(\mathbb{G})$, then*

$$(d_0^{-1})^* d_0^{-1} \beta = \beta.$$

Proof. Since we are dealing with algebraic operators, we have but to show that

$$\langle (d_0^{-1})^* d_0^{-1}(\theta_i \wedge \theta_j), \theta_k \wedge \theta_h\rangle = \langle \theta_i \wedge \theta_j, \theta_k \wedge \theta_h\rangle$$

for any choice of (i, j) and (k, h) with $i < j$ and $k < h$. Moreover, since $(d_0^{-1})^* d_0^{-1}(\theta_i \wedge \theta_j)$ has still weight 2, then it is enough to assume $w(\theta_h) = w(\theta_k) = 1$. By Lemma 6.3, we have

$$\begin{aligned}&\langle (d_0^{-1})^* d_0^{-1}(\theta_i \wedge \theta_j), \theta_k \wedge \theta_h\rangle = \langle d_0^{-1}(\theta_i \wedge \theta_j), d_0^{-1}(\theta_k \wedge \theta_h)\rangle \\ &= \langle [X_i, X_j]^\natural, [X_k, X_h]^\natural\rangle = \langle [X_i, X_j], [X_k, X_h]\rangle.\end{aligned}$$

Thus, the assertion follows since $\{[X_i, X_j],\ X_i, X_j \in V_1,\ i < j\}$ is an orthonormal Hall basis of V_2. $\square$

We say that a h-form $\alpha = \sum_j \alpha_j \theta_j^h \in \Omega^h(\mathbb{G})$ *depends only on the horizontal variables* and we write $\alpha \in \Omega_H^h(\mathbb{G})$ if its coefficients depend only on $x_1, \dots, x_m$. The classes $\Omega_H^h(\mathbb{G})$ are invariant under Hodge duality. Moreover

$$d\big(\Omega_H^h(\mathbb{G})\big) \subset \Omega_H^{h+1}(\mathbb{G}). \tag{79}$$

Since $\bigwedge^h \mathbb{R}^m$ can be identified with a subspace of $\bigwedge^h \mathfrak{g}$, any h-form $\alpha \in \Omega^h(\mathbb{R}^m)$,

$$\alpha = \sum_{1 \le i_1 < i_2 < \cdots < i_h \le m} \alpha_{i_1, i_2, \cdots, i_h}(x_1, \dots, x_m) dx_{i_1} \wedge \cdots \wedge dx_{i_h}$$

can be identified with a form in $\Omega_H^h(\mathbb{G})$ (remember $\theta_i = dx_i$, $i = 1, \dots, m$ because of our choice of the basis of $\mathfrak{g}$). Notice the reciprocal is false, but, trivially,

$$\Omega^0(\mathbb{R}^m) = \Omega_H^0(\mathbb{G}) \cap E_0^0 \quad \text{and} \quad \Omega^1(\mathbb{R}^m) = \Omega_H^1(\mathbb{G}) \cap E_0^1. \tag{80}$$

If $\alpha = \sum_j \alpha_j \theta_j^h \in \Omega_H^h(\mathbb{G})$, then

$$d_1 \alpha = \sum_j \sum_{i=1}^m (X_i \alpha_j) \theta_i \wedge \theta_j^h,$$

and

$$d_2 \alpha = 0, \quad \dots, \quad d_\kappa \alpha = 0.$$

Moreover, if $\alpha \in \Omega_H^h(\mathbb{G}) \cap E_0^h$, then we have also $d_0 \alpha = 0$, so that, if we set $\delta_1 := (d_1)^*$, keeping in mind Lemma 3.8 and the invariance of both $\Omega_H^*(\mathbb{G})$ and E_0^* under Hodge duality, we have

$$d_1 \alpha = d\alpha \quad \text{and} \quad \delta_1 \alpha = \delta \alpha \quad \text{if } \alpha \in \Omega_H^h(\mathbb{G}) \cap E_0^h. \tag{81}$$

Take now $\alpha \in \Omega_H^1(\mathbb{G}) \cap E_0^1$ identified with $\Omega^1(\mathbb{R}^m)$. Since the horizontal vector fields on $\Omega_H^0(\mathbb{G})$ reduce to usual derivatives (see e.g. [17], Propositions 2.2), by (81)

$$d_1 \alpha = d_{\mathbb{R}^m} \alpha \quad \text{and} \quad \delta_1 \alpha = \delta_{\mathbb{R}^m} \alpha. \tag{82}$$

Moreover, the following assertion follows straightforwardly by direct computation.

Lemma 6.6. *If $f \in \Omega^0(\mathbb{R}^m)$ and $\alpha = \sum_{j=1}^m \alpha_j dx_j \in \Omega^1(\mathbb{R}^m)$ are identified with forms in E_0^0 and E_0^1, respectively, that depend only on the horizontal variables, then*

1. $d_1 f = d_c f$;
2. $\delta_1 \alpha = \delta_c \alpha$.

Lemma 6.7. *Let $\mathbb{G}$ be a Carnot group of any step. If $\beta = \sum_{i,j} \beta_{ij} \theta_i \wedge \theta_j \in \Omega^2(\mathbb{G})$, then*

$$\delta_1 \beta = -\sum_{h=1}^n \sum_{k=1}^m X_k(\beta_{kh} - \beta_{hk}) \theta_h.$$

Proof. Take $\alpha = \alpha_\ell \theta_\ell$ with compact support, $1 \le \ell \le n$. We have

$$\int \langle \sum_{h=1}^{n} \big(\sum_{k=1}^{m} (X_k(\beta_{kh} - \beta_{hk})) \big) \theta_h, \alpha \rangle dV = \sum_{k=1}^{m} \int (X_k(\beta_{k\ell} - \beta_{\ell k})) \alpha_\ell \, dV$$
$$= - \sum_{k=1}^{m} \int (\beta_{k\ell} - \beta_{\ell k})(X_k \alpha_\ell) \, dV.$$

On the other hand

$$\int \langle \beta, d_1 \alpha \rangle dV = \sum_{k=1}^{m} \int \langle \beta, (X_k \alpha_\ell) \theta_k \wedge \theta_\ell \rangle dV$$
$$= \sum_{k=1}^{m} \int (\beta_{k\ell} - \beta_{\ell k})(X_k \alpha_\ell) dV,$$

and the assertion is proved. □

Remark 6.8. Notice in particular

$$\delta_1(b\, \theta_i \wedge \theta_j) = (X_j b)\theta_i - (X_i b)\theta_j.$$

Remark 6.9. Notice in particular that, whenever $\beta \in \Omega^2(\mathbb{R}^m) = \Omega^2_H(\mathbb{G}) \cap \Omega^{2,2}(\mathbb{G})$

$$\delta_1 \beta = \delta_{\mathbb{R}^m} \beta.$$

Finally, we have:

Lemma 6.10. *Let $\mathbb{G}$ be a Carnot group of any step. If $p \in \mathbb{N}$ and $\alpha \in \Omega^1_H(\mathbb{G}) \cap E_0^1$, then*

$$(\delta_1 d_1 + d_1 \delta_1)^p \alpha = \big((\delta_1 d_1)^p + (d_1 \delta_1)^p \big) \alpha.$$

Proof. We notice first that, keeping in mind that X_i, X_j commute on functions depending only on $x_1, \dots, x_m$, then $d_1^2 f = 0$ when $f \in \Omega^0_M$. Analogously, keeping in mind Remark 6.8, then $\delta_1^2 \beta = 0$ when $\beta \in \Omega^2_H$.

We argue now by induction on p. The assertion is trivial if $p = 1$. Suppose it holds for $p \in \mathbb{N}$. We have

$$\begin{aligned}
(\delta_1 d_1 + d_1 \delta_1)^{p+1} \alpha &= (\delta_1 d_1 + d_1 \delta_1)(\delta_1 d_1 + d_1 \delta_1)^p \alpha \\
&= (\delta_1 d_1 + d_1 \delta_1)\big((\delta_1 d_1)^p + (d_1 \delta_1)^p \big) \alpha \\
&= \big((\delta_1 d_1)^{p+1} + \delta_1 d_1 (d_1 \delta_1)^p + d_1 \delta_1 (\delta_1 d_1)^p + (d_1 \delta_1)^{p+1} \big) \alpha \\
&= \big((\delta_1 d_1)^{p+1} + (d_1 \delta_1)^{p+1} \big) \alpha.
\end{aligned}$$

□

Lemma 6.11. *Let $\mathbb{G}$ be a Carnot group of step 2. If $\alpha = \sum_j \alpha_j \theta_j \in \Omega^1_H(\mathbb{G})$, then*

$$\delta_1 d_1 d_0^{-1} d_1 \alpha = d_0^{-1} d_1 \delta_1 d_1 \alpha.$$

Proof. Using repeatedly the fact that the coefficients of α depend only on the horizontal variables, we have:

$$
\begin{aligned}
d_0^{-1}d_1\delta_1 d_1\alpha &= \sum_{i=1}^{m}\sum_{j=1}^{n} d_0^{-1}d_1\delta_1((X_i\alpha_j)\theta_i\wedge\theta_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} d_0^{-1}d_1\delta_1((X_i\alpha_j)\theta_i\wedge\theta_j)\\
&= -\sum_{j=1}^{n}\sum_{i=1}^{n} d_0^{-1}d_1((X_i^2\alpha_j)\theta_j - (X_jX_i\alpha_j)\theta_i) \quad \text{(by Remark 6.8)}\\
&= -d_0^{-1}(\sum_{\ell=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}(X_\ell X_i^2\alpha_j)\theta_\ell\wedge\theta_j) + d_0^{-1}(\sum_{\ell=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}(X_\ell X_jX_i\alpha_j)\theta_\ell\wedge\theta_i)\\
&= -d_0^{-1}(\sum_{\ell=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}(X_\ell X_i^2\alpha_j)\theta_\ell\wedge\theta_j) + d_0^{-1}(\sum_{\ell=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}(X_\ell X_jX_i\alpha_j)\theta_\ell\wedge\theta_i).
\end{aligned}
$$

We notice now that the second summand in the last line above vanishes. Indeed, let $j = 1,\dots,n$ be fixed. Remember that the vector fields X_k, $k = 1,\dots,n$ commute on functions depending only on the horizontal variables, since their commutators $[X_k, X_h]$ belong at least to the second layer of the algebra. Since all the indices run from 1 to n, the term

$$
\sum_{\ell=1}^{n}\sum_{i=1}^{n}(X_\ell X_jX_i\alpha_j)\theta_\ell\wedge\theta_i
$$

can then be written as

$$
\sum_{\ell=1}^{n}\sum_{i=1}^{n}(X_\ell X_i(X_j\alpha_j))\theta_\ell\wedge\theta_i = \sum_{\ell<i}\big\{X_\ell X_i(X_j\alpha_j) - X_iX_\ell(X_j\alpha_j)\big\}\theta_\ell\wedge\theta_i = 0,
$$

again by the commutativity of X_ℓ and X_i. Thus eventually (by Lemma 6.3)

$$
\begin{aligned}
d_0^{-1}d_1\delta_1 d_1\alpha &= -d_0^{-1}(\sum_{\ell=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}(X_\ell X_i^2\alpha_j)\theta_\ell\wedge\theta_j)\\
&= \sum_{i=1}^{n}\sum_{\ell=1}^{n}\sum_{j=1}^{n}(X_\ell X_i^2\alpha_j)[X_\ell, X_j]^\natural.
\end{aligned}
\tag{83}
$$

On the other hand, by Lemma 6.3 we have

$$\delta_1 d_1 d_0^{-1} d_1\alpha = \sum_{i=1}^{m}\sum_{j=1}^{n}\delta_1 d_1 d_0^{-1}((X_i\alpha_j)\theta_i\wedge\theta_j) = \sum_{i=1}^{n}\sum_{j=1}^{n}\delta_1 d_1 d_0^{-1}((X_i\alpha_j)\theta_i\wedge\theta_j)$$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_1 d_1((X_i\alpha_j)[X_i,X_j]^\natural) = -\sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_1((X_kX_i\alpha_j)\theta_k\wedge[X_i,X_j]^\natural)$$

$$= -\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\delta_1((X_kX_i\alpha_j)\theta_k\wedge[X_i,X_j]^\natural)$$

$$= \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\big\{(X_k^2X_i\alpha_j)[X_i,X_j]^\natural - ([X_i,X_j]X_kX_i\alpha_j)\theta_k\big\} \quad \text{(by Remark 6.8)}$$

$$= \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}(X_k^2X_i\alpha_j)[X_i,X_j]^\natural \quad \text{(since the } \alpha_j\text{'s depend only on the horizontal variables)}$$

$$= \sum_{i=1}^{n}\sum_{\ell=1}^{n}\sum_{j=1}^{n}(X_i^2X_\ell\alpha_j)[X_\ell,X_j]^\natural \quad \text{(just renaming the indices)}$$

Combining this identity with (83) and keeping in mind again the commutativity of all our vector fields on functions depending only on the horizontal variables, we achieve the proof of the lemma.

□

Theorem 6.12. *Let $\mathbb{G}$ be a Carnot group of step 2. If $\alpha = \sum_{j=1}^m \alpha_j\, dx_j \in E_0^1\cap\Omega_H^1(\mathbb{G})$ (therefore identified with a form in $\Omega^1(\mathbb{R}^m)$), then*

$$\Big(\delta_c d_c + (d_c\delta_c)^2\Big)\alpha = \Delta^2_{\mathbb{R}^m,1}\alpha = \sum_{j=1}^{m}((-\Delta_{\mathbb{R}^m})^2\alpha_j)\,dx_j.$$

Proof. Take $\alpha = \alpha_j dx_j = \alpha_j\theta_j$, $j=1,\dots,m$, that has weight 1. Then

$$(\Pi_E\alpha)_1 = \alpha,$$
$$(\Pi_E\alpha)_2 = -d_0^{-1}(d_1\alpha),$$

so that we can take

$$(\Pi_E\alpha)_2 = \sum_{i=1}^{m}(X_i\alpha_j)[X_i,X_j]^\natural.$$

Thus

$$\Pi_E\alpha = \alpha - d_0^{-1}(d_1\alpha),$$

and hence (since $d_0\alpha = 0$ and $d_2\Omega_H^* = \{0\}$)

$$\begin{aligned} d(\Pi_E\alpha) &= (d_0+d_1)(\Pi_E\alpha)\\ &= -d_0d_0^{-1}(d_1\alpha) + d_1\alpha - d_1d_0^{-1}(d_1\alpha)\\ &= -d_1\alpha + d_1\alpha - d_1d_0^{-1}(d_1\alpha) = d_1d_0^{-1}(-d_1\alpha),\end{aligned}$$

by Lemma 6.3 (3), since $d_1\alpha$ is a 2-form of weight 2.

We want to show that $d_1d_0^{-1}(d_1\alpha)$ belongs to E_0^2; this would yield

$$d_c\alpha = d_1d_0^{-1}(-d_1\alpha).$$

First of all, $d_1 d_0^{-1}(d_1\alpha)$ has weight 3, and hence, by Lemma 6.3, is orthogonal to the range of d_0. Let us prove now that $d_0 d_1(-d_0^{-1}d_1)\alpha = 0$. Keeping in mind that $X_1, \dots, X_n$ commute on α_j, we have

$$
\begin{aligned}
& d_0 d_1(-d_0^{-1}d_1)\alpha \\
&= d_0\Big(\sum_{k_1,k_2=1}^{m} (X_{k_2}X_{k_1}\alpha_j)\theta_{k_2}\wedge [X_{k_1},X_j]^\natural\Big) \\
&= \sum_{k_1,k_2=1}^{m} (X_{k_2}X_{k_1}\alpha_j) d_0(\theta_{k_2}\wedge [X_{k_1},X_j]^\natural) \quad \text{(since } d_0 \text{ is algebraic)} \\
&= -\sum_{k_1,k_2=1}^{m} (X_{k_2}X_{k_1}\alpha_j)\theta_{k_2}\wedge d_0[X_{k_1},X_j]^\natural \\
&\qquad \text{(since both } \theta_{k_2}\wedge [X_{k_1},X_j]^\natural \text{ and } [X_{k_1},X_j]^\natural \text{ are left invariant)} \\
&= -\Big(\sum_{k_1,k_2=1}^{m} (X_{k_2}X_{k_1}\alpha_j)\theta_{k_2}\wedge \theta_{k_1}\Big)\wedge\theta_j \\
&\qquad \text{(by Lemma 6.3, since both } X_{k_1} \text{ and } X_j \text{ belong to } V_1) \\
&= 0,
\end{aligned}
$$

since, by the commutativity of X_{k_1} and X_{k_2} on a function depending only on the horizontal variables, then $X_{k_2}X_{k_1}\alpha_j$ multiplies both $\theta_{k_1}\wedge\theta_{k_2}$ and $\theta_{k_2}\wedge\theta_{k_1}$.

This proves that $d_1(-d_0^{-1}d_1)\alpha \in E_0^2$. Thus

$$
\delta_c d_c \alpha = \delta_1 (d_0^{-1})^* \delta_1 d_1 d_0^{-1} d_1 \alpha. \tag{84}
$$

Keep in mind that $d_0^{-1}, (d_0^{-1})^*, d_1, \delta_1$ preserve the class of forms that depend only on the horizontal variables. If we apply Lemmata 6.11 and 6.5 to (84) (taking into account that $d_1\delta_1 d_1\alpha$ has weight 2, by Proposition 3.8), we get

$$
\begin{aligned}
\delta_c d_c \alpha &= \delta_1 (d_0^{-1})^* d_0^{-1} d_1\delta_1 d_1\alpha \\
&= \delta_1 d_1 \delta_1 d_1 \alpha \\
&= (\delta_1 d_1)^2\alpha.
\end{aligned} \tag{85}
$$

We notice now that $(d_c\delta_c)^2\alpha = (d_1\delta_1)^2\alpha$. Indeed $d_c\delta_c\alpha \in E_0^1 \cap \Omega_H^1$, so that, by Lemma 6.6, $d_c\delta_c d_c\delta_c\alpha = d_1\delta_1 d_c\delta_c\alpha = d_1\delta_1 d_1\delta_1\alpha$.

Eventually we have:

$$
\begin{aligned}
\Big(\delta_c d_c + (d_c\delta_c)^2\Big)\alpha &= \Big((\delta_1 d_1)^2 + (d_c\delta_c)^2\Big)\alpha \quad \text{(by (85))} \\
&= \Big((\delta_1 d_1)^2 + (d_1\delta_1)^2\Big)\alpha \quad \text{(by Lemma 6.6)} \\
&= (\delta_1 d_1 + d_1\delta_1)^2\alpha \quad \text{(by Lemma 6.10)} \\
&= (\delta d + d\delta)^2\alpha = \Delta^2_{\mathbb{R}^m,1}\alpha
\end{aligned}
$$

by (81), (82), and Remark 6.9. □

Remark 6.13. As in Remark 6.4, a theorem akin to Theorem 6.12 can be proved for the Carnot group $\mathbb{G}$ of step 3 defined therein. More precisely, if

$\alpha = \alpha_1 dx_1 + \alpha_2 dx_2 \in E_0^1(\mathbb{G}) \cap \Omega_H^1(\mathbb{G})$ (therefore identified with a form in $\Omega^1(\mathbb{R}^2)$), then

$$\Big(\delta_c d_c + (d_c\delta_c)^3\Big)\alpha = \Delta^3_{\mathbb{R}^2,1}\alpha = -(\Delta^3_{\mathbb{R}^2}\alpha_1)dx_1 - (\Delta^3_{\mathbb{R}^2}\alpha_2)dx_2.$$

Again, as in Theorem 6.1, we can obtain plane waves of the form

$$A(s,x) = e^{i(\omega s - k\cdot x)}A_0 + e^{i(\omega s - k\cdot x)}\varphi_0\, ds := A_\Sigma + \varphi\, ds,$$

where

- $k = \exp(K) = (k_1, k_2, 0, 0, 0)$, with $K \in V_1$;
- $\omega = \pm |k|^3$;
- $\varphi_0 = \mp |k| \langle A_0 | k\rangle$.

Proposition 6.14. *Let $\mathbb{G}$, $A = A_\Sigma + \varphi ds$ be as in Theorem 6.1. For sake of simplicity, assume $\omega = |k|^2$ and $\varphi_0 = -\langle A_0|k\rangle$. If*

$$ds \wedge E + B := \hat{d}_c A,$$

then (with the notations of [14], *1.5.1)*

i) $\langle E|k\rangle \equiv 0$*;*

ii) $B^\natural \llcorner E \equiv 0$.

Moreover, if k and $A_0^\natural$ commute, then

iii) $B^\natural \llcorner k^\natural \equiv 0$.

On the other hand

iv) $B \wedge k^\natural \equiv 0$,

and, if k and $A_0^\natural$ commute, then

v) $B \wedge E \equiv 0$.

Proof. By Proposition 32

$$E = ie^{i(\omega s - k\cdot x)} \sum_{j=1}^m \big(|k|^2 A_{0,j} - k_j \langle A_0|k\rangle\big)\theta_j$$

and

$$B = e^{i(\omega s - k\cdot x)} \sum_{\ell,\lambda,h=1}^m k_h k_\ell A_{0,\lambda}\, \theta_h \wedge [X_\ell, X_\lambda]^\natural.$$

Now

$$\begin{aligned}\langle E|k\rangle &= ie^{i(\omega s - k\cdot x)} \sum_{j=1}^m \big(|k|^2 A_{0,j} - k_j\langle A_0|k\rangle\big)k_j \\ &= ie^{i(\omega s - k\cdot x)}\big(|k|^2\langle A_0|k\rangle - |k|^2\langle A_0|k\rangle\big) = 0,\end{aligned}$$

proving i). On the other hand

$$\begin{aligned}B^\natural \llcorner E &= ie^{2i(\omega s - k\cdot x)} \sum_{\ell,\lambda,h=1}^m k_h k_\ell A_{0,\lambda}(|k|^2 A_{0,h} - k_h\langle A_0|k\rangle)[X_\ell, X_\lambda] \\ &= ie^{2i(\omega s - k\cdot x)} \sum_{\ell,\lambda=1}^m k_\ell A_{0,\lambda}\Big(\sum_{h=1}^m k_h(|k|^2 A_{0,h} - k_h\langle A_0|k\rangle)\Big)[X_\ell, X_\lambda] = 0,\end{aligned}$$

proving ii). Analogously

$$
\begin{aligned}
B^\natural \llcorner k^\natural &= e^{i(\omega s - k\cdot x)} \sum_{\ell,\lambda,h=1}^{m} k_h k_\ell A_{0,\lambda} k_h [X_\ell, X_\lambda] \\
&= e^{i(\omega s - k\cdot x)} |k|^2 \sum_{\ell,\lambda}^{m} k_\ell A_{0,\lambda} [X_\ell, X_\lambda] = e^{i(\omega s - k\cdot x)} |k|^2 [k, A_0^\natural],
\end{aligned}
$$

proving iii). To prove iv) we notice that

$$
\begin{aligned}
B \wedge k^\natural &= e^{i(\omega s - k\cdot x)} \sum_{\ell,\lambda,h,i=1}^{m} k_h k_\ell A_{0,\lambda} k_i \, \theta_i \wedge \theta_h \wedge [X_\ell, X_\lambda]^\natural \\
&= e^{i(\omega s - k\cdot x)} \sum_{\ell,\lambda=1}^{m} k_\ell A_{0,\lambda} \big(\sum_{h,i=1}^{m} k_h k_i \, \theta_i \wedge \theta_h \big) \wedge [X_\ell, X_\lambda]^\natural = 0.
\end{aligned}
$$

On the other hand, arguing as above,

$$
\begin{aligned}
B \wedge E = i e^{2i(\omega s - k\cdot x)} &\sum_{\ell,\lambda,h,j=1}^{m} k_h k_\ell A_{0,\lambda} \big(|k|^2 A_{0,j} - k_j \langle A_0 | k \rangle \big) \, \theta_j \wedge \theta_h \wedge [X_\ell, X_\lambda]^\natural \\
= i e^{2i(\omega s - k\cdot x)} &\sum_{\ell,\lambda,h,j=1}^{m} k_h k_\ell A_{0,\lambda} |k|^2 A_{0,j} \, \theta_j \wedge \theta_h \wedge [X_\ell, X_\lambda]^\natural \\
= i e^{2i(\omega s - k\cdot x)} & A_0 \wedge k^\natural \wedge [X_\ell, X_\lambda]^\natural = 0
\end{aligned}
$$

if (and only if) $[X_\ell, X_\lambda] = 0$.

□

Remark 6.15. Notice that ii) and iii) could be derived from iv) and v) arguing as in [14], 1.5.3, keeping in mind i) and that B has weight 3 (Theorem 5.9).

Remark 6.16. Plane waves in groups may exhibit totally unexpected phenomena if we keep in mind classical Maxwell's theory in the Euclidean space. For instance, if $\mathbb{G} = \mathbb{H}^1$, with the notations of Remark 5.6, $\vec{E} \perp k$, $\vec{B} \perp k$, so that $\vec{E}$ and $\vec{B}$(that are horizontal vector fields as well as k) cannot be orthogonal as in the classical setting, since the first layer of $\mathbb{H}^1$ has dimension 2.

Remark 6.17. The arguments of this subsection enable us to study another class of special solutions, that we may call "cylindrical waves", discovering also some unexpected relationships with the equations of classical elasticity. For sake of simplicity, let us restrict ourselves to the case $\mathbb{G} = \mathbb{H}^1$. Keeping in mind the characterization of the homogeneous group homomorphisms of $\mathbb{H}^1$ given in [31], we see that the only homogeneous homomorphisms of $\mathbb{H}^1$ preserving the sublaplacian $\Delta_{\mathbb{H}}$ are the rotations around the t-axis. Thus it is natural to look for cylindrically symmetric waves, i.e. for cylindrically symmetric solutions of (62) and (63). For sake of simplicity we restrict ourselves to consider the case of zero charges, i.e. we assume $\rho_0 \equiv 0$ and $J \equiv 0$. But cylindrically symmetric solutions do not depend on the central variables, and thus we can start by attacking the case A_Σ and φ independent of central variables.

By Theorem 6.12, in case of cylindrical symmetry, the components of A_Σ as well as φ all solve the equation

$$\frac{\partial^2 u}{\partial s^2} = -\Delta^2 u, \tag{86}$$

with suitable Cauchy data at $s = 0$. If we consider this equation in a cylinder $\Omega \times \mathbb{R}$, where Ω is a (say) bounded open subset of $\mathbb{R}^2$, we can impose suitable Dirichlet conditions on $\partial\Omega$. In this way, we recover a classical elasticity equation, the so-called Germain-Lagrange equation for the vibration of plates (see e.g. [43], Section 9). We refer to [43], Section 9 and to the references of that chapter for explicit solutions adapted to particular choices of Ω, and in particular for cylindrically symmetric solutions when we choose Ω to be an open disk of the plane (see Section 9.4).

6.3. **An abstract theory of wave equations in a Carnot group.** In this section, we prove an abstract existence result for our wave equation by means of the well established theory of second order differential equations in Banach spaces, as presented for instance in [13].

Proposition 6.18. *Let $\mathbb{G}$ be a Carnot group satisfying the assumption of Theorem 5.10. The unbounded operator in $L^2(\mathbb{G}, E_0^1)$*

$$-\Delta_{\mathbb{G},1} \quad \textit{with domain} \quad W^{2r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)$$

is self-adjoint and nonpositive.

Proof. Clearly, $\Delta_{\mathbb{G},1}$ is densely defined, since $\mathcal{D}(\mathbb{G}, E_0^1) \subset W^{2r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)$ is dense in $L^2(\mathbb{G}, E_0^1)$. In addition, it is symmetric. Indeed, if $\alpha, \beta \in W^{2r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)$ and $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{D}(G, E_0^1)$ converging to α in $W^{2r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)$, then

$$\begin{aligned}\langle \Delta_{\mathbb{G},1}\beta, \alpha\rangle_{L^2(\mathbb{G},E_0^1)} &= \lim_{n\to\infty} \langle \Delta_{\mathbb{G},1}\beta, \alpha_n\rangle_{L^2(\mathbb{G},E_0^1)} \\ &= \lim_{n\to\infty} \langle \beta, \Delta_{\mathbb{G},1}\alpha_n\rangle_{L^2(\mathbb{G},E_0^1)} = \langle \beta, \Delta_{\mathbb{G},1}\alpha\rangle_{L^2(\mathbb{G},E_0^1)}.\end{aligned}$$

This shows that $(\Delta_{\mathbb{G},1})^*$ is an extension of $\Delta_{\mathbb{G},1}$. Thus, arguing as in [13], Chapter IV, Lemma 1.1, to achieve the proof it is enough to show that $1 \in \rho(\Delta_{\mathbb{G},1})$. Consider now the quadratic form in $W^{r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)$ defined by

$$\begin{aligned}Q(\alpha,\beta) :=& \langle d_c\alpha, d_c\beta\rangle_{L^2(\mathbb{G},E_0^1)} + \langle (d_c\delta_c)^{r/2}\alpha, (d_c\delta_c)^{r/2}\beta\rangle_{L^2(\mathbb{G},E_0^1)} \\ &+ \langle \alpha,\beta\rangle_{L^2(\mathbb{G},E_0^1)} \quad \text{if } r \text{ is even,}\end{aligned}$$

and

$$\begin{aligned}Q(\alpha,\beta) :=& \langle d_c\alpha, d_c\beta\rangle_{L^2(\mathbb{G},E_0^1)} + \langle \delta_c(d_c\delta_c)^{(r-1)/2}\alpha, \delta_c(d_c\delta_c)^{r-1/2}\beta\rangle_{L^2(\mathbb{G},E_0^1)} \\ &+ \langle \alpha,\beta\rangle_{L^2(\mathbb{G},E_0^1)} \quad \text{if } r \text{ is odd.}\end{aligned}$$

If $\alpha \in \mathcal{D}(\mathbb{G}, E_0^1)$, then

$$Q(\alpha,\alpha) = \langle \Delta_{\mathbb{G},1}\alpha, \alpha\rangle_{L^2(\mathbb{G},E_0^1)} + \|\alpha\|^2_{L^2(\mathbb{G},E_0^1)}.$$

Therefore, thanks to the density of $\mathcal{D}(\mathbb{G}, E_0^1)$ in $W^{r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)$, by Theorem 5.10, Q is coercive on $W^{r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)$. Then, by Lax-Milgram theorem, if $\gamma \in L^2(\mathbb{G}, E_0^1)$ there exists $\alpha_\gamma \in W^{r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)$ satisfying $Q(\alpha_\gamma, \beta) = \langle \gamma, \beta\rangle_{L^2(\mathbb{G},E_0^1)}$ for any $\beta \in W^{r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)$. In particular, $\Delta_{\mathbb{G},1}\alpha_\gamma = \gamma - \alpha_\gamma := \gamma_0 \in L^2(\mathbb{G}, E_0^1)$

in the sense of distributions. We write $\gamma_0 = \sum_j \gamma_{0,j}\theta_j$. To achieve the proof, we have but to show that $\alpha_\gamma \in W^{2r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)$. To this end, let I be a multi-index with $d(I) = 2r$, and take $\varphi = \sum_j \varphi_j\theta_j \in \mathcal{D}(\mathbb{G}, E_0^1)$. We have

$$\langle X^I\alpha_\gamma|\varphi\rangle_{\mathcal{S}',\mathcal{S}} = \langle\alpha_\gamma|(X^I)^*\varphi\rangle_{\mathcal{S}',\mathcal{S}} = \langle\alpha_\gamma|\Delta_{\mathbb{G},1}\mathcal{K}((X^I)^*\varphi)\rangle_{\mathcal{S}',\mathcal{S}}. \tag{87}$$

On the other hand,

$$\mathcal{K}((X^I)^*\varphi) = \sum_{ij}((X^I)^*\varphi_j * \tilde{K}_{ij}) \llcorner (*\xi_i). \tag{88}$$

Notice now that $(X^I)^*\varphi_j * K_{ij} = \varphi_j * \big({}^{\mathrm{v}}(X^I)^{*\mathrm{v}}K_{ij}\big)$ belongs to $\mathcal{S}(\mathbb{G})$. Indeed, by Proposition 2.1, ${}^{\mathrm{v}}(X^I)^{*\mathrm{v}}K_{ij}$ is a Folland kernel of order 0, that is L^2-continuous. Hence, if $\tilde{X}^J_\ell$ is an arbitrary right invariant monomial, then $\tilde{X}^J_\ell\big(\varphi_j * \big({}^{\mathrm{v}}(X^I)^{*\mathrm{v}}K_{ij}\big)\big) \in L^2(\mathbb{G})$. Therefore, $\varphi_j * \big({}^{\mathrm{v}}(X^I)^{*\mathrm{v}}K_{ij}\big) \in W^{m,2}(\mathbb{G})$ for any $m \in \mathbb{N}$, and eventually belongs to $\mathcal{S}(\mathbb{G})$. Thus, by Proposition 28, we can write (88) as

$$\mathcal{K}((X^I)^*\varphi) = \sum_{ij}\big(\varphi_j * \big({}^{\mathrm{v}}(X^I)^{*\mathrm{v}}K_{ij}\big)\big)\theta_j, \tag{89}$$

that, together with (87) yields

$$\begin{aligned}
\langle X^I\alpha_\gamma|\varphi\rangle_{\mathcal{S}',\mathcal{S}} &= \langle\Delta_{\mathbb{G},1}\alpha_\gamma|\sum_{ij}\big(\varphi_j * \big({}^{\mathrm{v}}(X^I)^{*\mathrm{v}}K_{ij}\big)\big)\theta_j\rangle_{\mathcal{S}',\mathcal{S}} \\
&= \sum_{ij}\langle\gamma_{0,j}\, , \varphi_j * \big({}^{\mathrm{v}}(X^I)^{*\mathrm{v}}K_{ij}\big)\rangle_{L^2,L^2} \\
&\le C\|\gamma_0\|_{L^2(\mathbb{G},E_0^1)}\|\varphi\|_{L^2(\mathbb{G},E_0^1)},
\end{aligned} \tag{90}$$

again by Proposition 2.1. This achieves the proof of the proposition. $\square$

By [13], Chapter II and Exercise 5 in particular, the following result follows easily from Proposition 6.18.

Theorem 6.19. *The homogeneous Cauchy problem*

$$\begin{cases} \frac{\partial^2\alpha}{\partial s^2} = -\Delta_{\mathbb{G},1} \quad \textit{for} \quad t > 0, \\ \alpha_{|s=0} = \alpha_0, \quad \frac{\partial\alpha}{\partial s}_{|s=0} = \alpha_1 \end{cases} \tag{91}$$

is uniformly well posed in $L^2(\Omega, E_0^1)$. The propagators are explicitly given by

$$C(s) = \int_{-\infty}^0 \cos(s|\lambda|^{1/2})\, dE(\lambda)$$

and

$$S(s) = \int_{-\infty}^0 |\lambda|^{-1/2}\sin(s|\lambda|^{1/2})\, dE(\lambda),$$

where $dE(\lambda)$ is the spectral measure associated with $-\Delta_{\mathbb{G},1}$.

Suppose now the map $s \to J(\cdot, s)$ is continuously differentiable from $\mathbb{R}$ to $L^2(\Omega, E_0^1)$. By Lemma 5.1 of [13], *Chapter II, if $\alpha_0, \alpha_1 \in W^{2r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)$, then*

$$\alpha(s) := \mathcal{C}(s)\alpha_0 + \mathcal{S}(s)\alpha_1 + \int_0^s \mathcal{S}(s-\sigma)J(\sigma)\, d\sigma$$

is a strong solution of

$$
(92)\qquad \begin{cases} \frac{\partial^2\alpha}{\partial s^2} = -\Delta_{\mathbb{G},1} + J \quad \textit{for} \quad t>0, \\ \alpha_{|s=0} = \alpha_0, \quad \frac{\partial\alpha}{\partial s}_{|s=0} = \alpha_1, \end{cases}
$$

i.e. is a twice continuously differentiable function from $(0,\infty)$ *to* $L^2(\mathbb{G}, E_0^1)$ *such that* $\alpha(s) \in W^{2r,2}(\mathbb{G}, E_0^1)$ *for all* $s>0$.

Arguing as in [13], Chapter IV, we can obtain the following stronger statement.

Theorem 6.20. *Let* $\alpha_0 \in W^{r,2}_{\mathbb{G}}(\mathbb{G}, E_0^1)$ *and* $\alpha_1 \in L^2(\mathbb{G}, E_0^1)$ *be given. Then all strong solutions of the homogeneous Cauchy problem*

$$
(93)\qquad \begin{cases} \frac{\partial^2\alpha}{\partial s^2} = -\Delta_{\mathbb{G},1} \quad \textit{for} \quad t>0, \\ \alpha_{|s=0} = \alpha_0, \quad \frac{\partial\alpha}{\partial s}_{|s=0} = \alpha_1 \end{cases}
$$

have the form

$$
(94)\qquad \begin{bmatrix} u(s) \\ u'(s) \end{bmatrix} = \begin{bmatrix} C(s) & S(s) \\ -\Delta_{\mathbb{G},1}S(s) & C(s) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1. \end{bmatrix}
$$

Proof. In fact, the proof of the theorem can be carried out by repeating verbatim the arguments of [13], Chapter IV, provided we prove that

$$
\mathrm{Dom}\,((I+\Delta_{\mathbb{G},1})^{1/2}) = W^{r,2}(\mathbb{G}, E_0^1).
$$

To this end, if $\alpha = \sum_j \alpha_j\theta_j \in E_0^1$, we denote by $B_{\mathbb{G},r}$ the diagonal selfadjoint operator

$$
B_{\mathbb{G},r}\alpha = \sum_j ((I-\Delta_{\mathbb{G}})^{r/2}\alpha_j)\theta_j.
$$

By [15], Proposition 4.1, Dom $(B_{\mathbb{G},r}) = W^{r,2}(\mathbb{G}, E_0^1)$. In addition, again by [15], Proposition 4.1 and by Proposition 6.18

$$
\mathrm{Dom}\,(B^2_{\mathbb{G},r}) = W^{2r,2}(\mathbb{G}, E_0^1) = \mathrm{Dom}\,(I+\Delta_{\mathbb{G},1}).
$$

Therefore, by a classical interpolation argument ([33])

$$
\begin{aligned}
\mathrm{Dom}\,((I+\Delta_{\mathbb{G},1})^{1/2}) &= [\mathrm{Dom}\,(I+\Delta_{\mathbb{G},1}), L^2(\mathbb{G}, E_0^1)]_{1/2} \\
&= [\mathrm{Dom}\,(B^2_{\mathbb{G},r}), L^2(\mathbb{G}, E_0^1)]_{1/2} = \mathrm{Dom}\,(B_{\mathbb{G},r}) \\
&= W^{r,2}(\mathbb{G}, E_0^1).
\end{aligned}
$$

This achieves the proof of the theorem. □

References

[1] Annalisa Baldi and Bruno Franchi. Differential forms in carnot groups: a Γ-convergence approach. preprint, 2010.

[2] Annalisa Baldi, Bruno Franchi, Nicoletta Tchou, and Maria Carla Tesi. Compensated compactness for differential forms in Carnot groups and applications. *Adv. Math.*, 223(5):1555–1607, 2010.

[3] Annalisa Baldi, Bruno Franchi, and Maria Carla Tesi. Fundamental solution and sharp L^p estimates for Laplace operators in the contact complex of Heisenberg groups. *Ricerche Mat.*, 55(1):119–144, 2006.

[4] Annalisa Baldi, Bruno Franchi, and Maria Carla Tesi. Compensated compactness in the contact complex of Heisenberg groups. *Indiana Univ. Math. J.*, 57:133–186, 2008.

[5] Annalisa Baldi, Bruno Franchi, and Maria Carla Tesi. Differential forms, Maxwell equations and compensated compactness in Carnot groups. *Lecture Notes of Seminario Interdisciplinare di Matematica*, 7:21–40, 2008.

[6] Annalisa Baldi, Bruno Franchi, and Maria Carla Tesi. Hypoellipticity, fundamental solution and Liouville type theorem for matrix–valued differential operators in Carnot groups. *J. Eur. Math. Soc.*, 11(4):777–798, 2009.

[7] Rolf Berndt. *An introduction to symplectic geometry*, volume 26 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Translated from the 1998 German original by Michael Klucznik.

[8] Andrea Bonfiglioli, Ermanno Lanconelli, and Francesco Uguzzoni. *Stratified Lie groups and potential theory for their sub-Laplacians*. Springer Monographs in Mathematics. Springer, Berlin, 2007.

[9] Nicolas Bourbaki. *Éléments de mathématique. XXVI. Groupes et algèbres de Lie. Chapitre 1: Algèbres de Lie*. Actualités Sci. Ind. No. 1285. Hermann, Paris, 1960.

[10] Michael Christ, Daryl Geller, Paweł Głowacki, and Larry Polin. Pseudodifferential operators on groups with dilations. *Duke Math. J.*, 68(1):31–65, 1992.

[11] Gianni Dal Maso. *An introduction to* Γ*-convergence*. Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston Inc., Boston, MA, 1993.

[12] Jean Dieudonné. *Éléments d'analyse. Tome III: Chapitres XVI et XVII*. Cahiers Scientifiques, Fasc. XXXIII. Gauthier-Villars Éditeur, Paris, 1970.

[13] H. O. Fattorini. *Second order linear differential equations in Banach spaces*, volume 108 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 99.

[14] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[15] Gerald B. Folland. Subelliptic estimates and function spaces on nilpotent Lie groups. *Ark. Mat.*, 13(2):161–207, 1975.

[16] Gerald B. Folland and Elias M. Stein. *Hardy spaces on homogeneous groups*, volume 28 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1982.

[17] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. On the structure of finite perimeter sets in step 2 Carnot groups. *J. Geom. Anal.*, 13(3):421–466, 2003.

[18] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. Regular hypersurfaces, intrinsic perimeter and implicit function theorem in Carnot groups. *Comm. Anal. Geom.*, 11(5):909–944, 2003.

[19] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. Regular submanifolds, graphs and area formula in Heisenberg groups. *Adv. Math.*, 211(1):152–203, 2007.

[20] Bruno Franchi, Nicoletta Tchou, and Maria Carla Tesi. Div-curl type theorem, H-convergence and Stokes formula in the Heisenberg group. *Commun. Contemp. Math.*, 8(1):67–99, 2006.

[21] Bruno Franchi and Maria Carla Tesi. Faraday's form and Maxwell's equations in the Heisenberg group. *Milan J. Math.*, 77:245–270, 2009.

[22] Daryl Geller. Local solvability and homogeneous distributions on the Heisenberg group. *Comm. Partial Differential Equations*, 5(5):475–560, 1980.

[23] Mariano Giaquinta, Giuseppe Modica, and JiříSouček. *Cartesian currents in the calculus of variations. I*, volume 37 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1998.

[24] Matthew Grayson and Robert Grossman. Models for free nilpotent Lie algebras. *J. Algebra*, 135(1):177–191, 1990.

[25] Peter C. Greiner, David Holcman, and Yakar Kannai. Wave kernels related to second-order operators. *Duke Math. J.*, 114(2):329–386, 2002.

[26] Mikhael Gromov. Carnot-Carathéodory spaces seen from within. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 79–323. Birkhäuser, Basel, 1996.

[27] Bernard Helffer and Jean Nourrigat. *Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs*, volume 58 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1985.

[28] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.

[29] Lars Hörmander. *Linear partial differential operators*. Springer Verlag, Berlin, 1976.

[30] David Jerison. The Poincaré inequality for vector fields satisfying Hörmander's condition. *Duke Math. J.*, 53(2):503–523, 1986.

[31] Adam Korányi and Hans Martin Reimann. Quasiconformal mappings on the Heisenberg group. *Invent. Math.*, 80(2):309–338, 1985.

[32] Adam Korányi and Stephen Vági. Singular integrals on homogeneous spaces and some problems of classical analysis. *Ann. Scuola Norm. Sup. Pisa (3)*, 25:575–648 (1972), 1971.

[33] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I*. Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[34] Valentino Magnani. Differentiability and area formula on stratified Lie groups. *Houston J. Math.*, 27(2):297–323, 2001.

[35] Richard Melrose. Propagation for the wave group of a positive subelliptic second-order differential operator. In *Hyperbolic equations and related topics (Katata/Kyoto, 1984)*, pages 181–192. Academic Press, Boston, MA, 1986.

[36] Pierre Pansu. *Géometrie du group de Heisenberg*. PhD thesis, Université Paris VII, 1982.

[37] Pierre Pansu. Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math. (2)*, 129(1):1–60, 1989.

[38] Michel Rumin. Formes différentielles sur les variétés de contact. *J. Differential Geom.*, 39(2):281–330, 1994.

[39] Michel Rumin. Differential geometry on C-C spaces and application to the Novikov-Shubin numbers of nilpotent Lie groups. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(11):985–990, 1999.

[40] Michel Rumin. Around heat decay on forms and relations of nilpotent Lie groups. In *Séminaire de Théorie Spectrale et Géométrie, Vol. 19, Année 2000–2001*, volume 19 of *Sémin. Théor. Spectr. Géom.*, pages 123–164. Univ. Grenoble I, Saint, 2001.

[41] Stephen Semmes. On the nonexistence of bi-Lipschitz parameterizations and geometric problems about A_∞-weights. *Rev. Mat. Iberoamericana*, 12(2):337–410, 1996.

[42] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[43] Eduard Ventsel and Theodor Krauthammer. *Thin Plates and Shells Theory: Analysis, and Applications*. Marcel Dekker, Inc., New York, 2001.

Bruno Franchi and Maria Carla Tesi
Dipartimento di Matematica
Piazza di Porta S. Donato 5
40126 Bologna, Italy;
e-mail: franchib@dm.unibo.it, tesi@dm.unibo.it