The Willmore and other L^p-curvature functionals in Riemannian manifolds

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An important problem in Geometric Analysis concerning the *intrinsic* geometry of manifolds sounds roughly as follows: given an n-dimensional smooth manifold find the **"best metrics"** on it, where with "best metric" we mean a metric whose curvature tensors satisfy special conditions (for example some traces of the Riemann curvature tensor are null or constant or prescribed, or minimize some functional; think of the Yamabe Problem, the Uniformization Theorem, etc.).

The analogous problem concerning the *extrinsic* geometry of surfaces sounds roughly as follows: given an abstract 2-dimensional surface Σ (we will always consider Σ closed: compact and without boundary) and a Riemannian n-dimensional manifold (M^n, h), find the **"best immersions"** $f : \Sigma \hookrightarrow M^n$ of Σ into M^n. Here with "best immersion" we mean an immersion whose curvature, i.e. second fundamental form, satisfies special conditions: for example if the second fundamental form is null the immersion is totally geodesic, if the mean curvature is null the immersion is minimal, if the trace-free second fundamental form is null the immersion is totally umbilical, etc.

Before proceeding let us introduce some notation. Given an immersion $f : \Sigma \hookrightarrow (M^n, h)$ let us denote by $g = f^*h$ the pull back metric on Σ (i.e. the metric on Σ induced by the immersion f); the area form $\sqrt{\det g}$ is denoted with $d\mu_g$; the second fundamental form is denoted with A and half of its trace $H := \frac{1}{2} g^{ij} A_{ij}$ is called mean curvature (notice that we use the convention that, in the codimension one case, the mean curvature is the arithmetic mean of the principal curvatures), finally $A^\circ := A - Hg$ is called trace-free second fundamental form.

As explained above, classically the "best immersions" are the ones for which the quantities A, H, A° are null or constant (i.e. parallel) but in many cases such immersions **do not exist**: for example if Σ is a closed surface and $(M, h) = (\mathbb{R}^3, \mathrm{eucl})$ is the Euclidean three dimensional space, by maximum principle there exist no minimal, and in particular totally geodesic, immersion of Σ into $(\mathbb{R}^3, \mathrm{eucl})$; moreover if the ambient manifold is the Heisenberg group or a non constant curvature Berger sphere then there exists no totally umbilic-and a fortiori totally geodesic-immersion.

If such classical special submanifolds do not exist it is interesting to study the minimization of natural integral functionals associated to A, H, A° of the type

$$\int_\Sigma |A|^p d\mu_g, \quad \int_\Sigma |H|^p d\mu_g, \quad \int_\Sigma |A^\circ|^p d\mu_g, \quad \text{for some } p \geq 1.$$

A global minimizer, if it exists, can be seen respectively as a generalized totally geodesic, minimal, or totally umbilic immersion in a natural integral sense. For some recent results regarding the existence of minimizers of such functionals in

Riemannian manifolds (in the class of integral m-varifolds and in case $p > m$) see [4] and the references therein.

An important particular case of such functionals is given by the Willmore functional

$$W(f) := \int_\Sigma H^2 d\mu_g.$$

The topic is classical and goes back to the 1920-'30 when Blaschke and Thomsen, looking for a conformally invariant theory which included the minimal surfaces, discovered the functional and proved its invariance under Moebius transformations of $\mathbb{R}^n$. The functional relative to immersions in $\mathbb{R}^n$, $\mathbb{S}^n$ and more generally in space forms has been deeply studied with remarkable results (for a matter of space we avoid the long citation of articles, let us just quote some authors: Bauer, Bérnard, Chen, Kuwert, Y. Li, P. Li, Marques, Montiel, Neves, Rivière, Ros, Schätlze, Schygulla, Simon, Weiner, Yau).

Let us recall that the Willmore functional has lots of applications: biology (Hellfrich energy), general relativity (Hawking mass), string theory (Polyakov extrinsic action), elasticity theory and optics.

While, as we remarked, there is an extensive literature for immersions into $\mathbb{R}^n$ or $\mathbb{S}^n$, up to five years ago very little was known for general ambient manifolds (apart from the case of minimal surfaces).

The first result regarding the existence of Willmore surfaces in non constantly curved spaces is in [2] where we studied the Willmore functional in a perturbative setting: endowed $\mathbb{R}^3$ with the perturbed metric $\delta_{\mu\nu} + \epsilon h_{\mu\nu}$ (where $\delta_{\mu\nu}$ is the Euclidean metric), under generic conditions on the scalar curvature of $(\mathbb{R}^3, \delta_{\mu\nu} + \epsilon h_{\mu\nu})$ and a fast decreasing assumption at infinity on the perturbation $h_{\mu\nu}$ we proved existence and multiplicity of embeddings of $\mathbb{S}^2$ which are critical points for the functional $\int H^2 d\mu_g$. The method was perturbative and the proof relied on a Lyapunov-Schmidt reduction. Using a similar technique, in [3] we studied the conformal Willmore functional $\frac{1}{2}\int |A^\circ|^2 d\mu_g$, which is conformally invariant in Riemannian manifolds, in the same perturbative setting $(\mathbb{R}^3, \delta_{\mu\nu} + \epsilon h_{\mu\nu})$ under generic conditions on the trace-free Ricci tensor $S_{\mu\nu} := Ric_{\mu\nu} - \frac{1}{3} R g_{\mu\nu}$.

The case of Willmore spheres under area constraint in a perturbative setting has been analyzed by Lamm-Metzger and Schulze.

Then using more global techniques coming on one hand from geometric measure theory (the so called "ambient approach" of Simon, involving mainly varifolds as weak objects), and functional analysis on the other hand (the so called "parametric approach" of Rivière involving Sobolev immersions) we investigated the existence of Willmore spheres in Riemannian manifolds.

Adopting the first point of view, together with E. Kuwert and J. Schygulla (see [1]) we proved the existence of a smooth immersion of $\mathbb{S}^2$ into a compact Riemannian 3-manifold, with positive sectional curvature, minimizing the L^2 norm of the second fundamental form.

Using the same approach, in [7] together with Schygulla we extended the above existence theorem to similar L^2 curvature functionals (as $\int |H|^2+1$ and $\int |A|^2+1$) on immersions of $\mathbb{S}^2$ in non compact Riemannian 3-manifolds satisfying asymptotic conditions which are natural in general relativity (as asymptotically Euclidean or Hyperbolic).

Since in higher codimension it is natural to expect existence of branched immersions minimizing the Willmore functional (this follows a fortiori from the existence of branched area minimizing surfaces), together with Rivière in [5] we introduced the notion of weak, possibly branched, immersion: a Lipschitz quasi-conformal map having at most finitely many branched points and finite total curvature. In the same paper we proved compactness results in this framework: in order to have $W^{2,2}$-weak compactness away the branched points it is enough to have uniform bounds on the areas and on the total curvatures and a uniform positive lower bound on the diameter of the images (related compactness results have been obtained independently by Chen-Li).

In the second paper [6] in collaboration with Rivière, we proved the differentiability of the Willmore functional on this space of weak immersions and we showed the regularity of the *critical points*. We then applied the theory to get existence of Willmore surfaces in homotopy groups (this result in particular shows how the Willmore functional can be useful to complete the theory of minimal surfaces, indeed our result complete the classical result of Sacks-Uhlembeck regarding area minimizing spheres in homotopy groups) and under other various conditions and constraints.

Let us stress that the ambient approach of Simon uses in a crucial way that we are dealing with a minimization problem; on the contrary, the regularity theory developed by Rivière for immersions in Euclidean space, and in [5]-[6] for immersions in manifolds uses just that the Willmore PDE is satisfied. It is therefore more appropriate for attacking min-max problems in the future.

References

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