CONVERGENCE OF ASYNCHRONOUS VARIATIONAL INTEGRATORS IN LINEAR ELASTODYNAMICS

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ABSTRACT. Within the setting of linear elastodynamics of simple bodies, we prove that the discrete action functional obtained by following the scheme of asynchronous variational integrators converges in time. The convergence in space is assured by standard arguments when the finite element mesh is progressively refined. Our strategy exploits directly the action functional. In particular, we show that, if the asynchronicity of time steps and nodal initial data satisfy a boundedness condition, any sequence of stationary points of the discrete action functional is pre-compact in the weak$-*$ $W^{1,\infty}$ topology and all its cluster points are stationary points for the continuous (in time) action. In this sense our proof is new with respect to existing ones.

1. A convergence theorem

A simple elastic body undergoing infinitesimal deformations occupies a regular region $\mathcal{B}$ of the three-dimensional ambient space. The differentiable map $(x,t) \longmapsto u := u(x,t) \in \mathbb{R}^n$, with $x \in \mathcal{B}$, $t \in [t_0, t_f]$, $n = 1,2,3$, represents the *displacement field*. The map $(x,t) \longmapsto \varepsilon := sym \nabla u(x,t)$ associates the standard *measure* ε *of infinitesimal deformations* to each point at a given instant. In linear elastic constitutive setting and infinitesimal deformation regime, the dynamics of a simple body is governed by the action functional

$$\mathcal{A}(\mathcal{B}, [t_0, t_f]; u) := \int_{t_0}^{t_f} \left(\int_{\mathcal{B}} \frac{1}{2} \rho \left| \dot{u} \right|^2 \, dx - V(\mathcal{B}, u, t) \right) dt \tag{1.1}$$

where ρ is the density of mass and

$$V(\mathcal{B}, u, t) := \int_{\mathcal{B}} \frac{1}{2} (\mathbb{C}\varepsilon) \cdot \varepsilon \, dx - \int_{\mathcal{B}} b \cdot u \, dx - \int_{\partial \mathcal{B}_{\mathtt{t}}} \mathtt{t} \cdot u \, d\mathcal{H}^{n-1} \tag{1.2}$$

the potential, with $\mathbb{C}$ the standard elastic constitutive tensor, b and $\mathtt{t}$ bulk and surface conservative forces respectively, the latter applied over a part $\partial \mathcal{B}_t$ of the boundary $\partial \mathcal{B}$. The fields $x \longmapsto b(x)$ and $x \longmapsto \mathtt{t}(x)$ belong to $L^2(\mathcal{B}, \mathbb{R}^3)$ and $L^2(\partial \mathcal{B}, \mathbb{R}^3)$, respectively. Initial conditions are given by regular fields $(x, t_0) \longmapsto u_0(x, t_0)$ and $(x, t_0) \longmapsto \dot{u}_0(x, t_0)$.

To find approximate solutions for some boundary value problems for $\mathcal{A}$, one needs to select first a suitable tessellation $\mathcal{T}$ of $\mathcal{B}$ of finite elements and, for each element K of it, a discrete time set

$$\Theta_K = \left\{ t_0 = t_K^1 < ... < t_K^{N_K - 1} < t_K^{N_K} = t_f \right\}.$$

We indicate by Θ the entire time set for $[t_0, t_f]$ defined by $\Theta := \cup_{K \in \mathcal{T}} \Theta_K$, saying that it has time size h when the difference between two arbitrary subsequent instants

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in Θ is lesser or equal to h. When appropriate, we write then Θ_h to underline the time size. As a measure of asynchronicity of Θ, we consider the ratio M_Θ between the time size and the minimum of the differences between subsequent instants in Θ.

A reasonable way to construct computational schemes is based on the direct discretization of the action both in space and time. The sole space discretization leads to an action $\mathcal{A}_\mathcal{T}$ defined by

$$\mathcal{A}_\mathcal{T}\left(\left[t_0,t_f\right];u\right) := \sum_{K\in\mathcal{T}} \int_{t_0}^{t_f} \mathcal{A}\left(K;u_K\left(t\right)\right)\ dt, \tag{1.3}$$

where, for each $K\in\mathcal{T}$,

$$\mathcal{A}\left(K;u_K\left(t\right)\right) := \sum_{a\in K} \frac{m_{K,a}}{2}\left|\dot{u}_a\left(t\right)\right|^2 - V_K\left(u_K\left(t\right)\right),$$

with $m_{K,a}$ the *nodal mass* associated with the node a in K endowed with velocity $\dot{u}_a\left(t\right)$ and

$$\begin{aligned} V_K\left(u_K\left(t\right)\right) :=& \int_K \frac{1}{2}\left(\mathbb{C}\nabla u_K\left(x,t\right)\right)\cdot\nabla u_K\left(x,t\right)\ dx \\ & -\int_K b(x)\cdot u_K\left(x,t\right)\ dx - \int_{\partial K\cap\partial\mathcal{B}_t} \mathsf{t}(x)\cdot u_K\left(x,t\right)\ d\mathcal{H}^{n-1}, \end{aligned}$$

with $u_K\left(x,t\right)$ the restriction of $u\left(x,t\right)$ to K. When we select also a time discretization, a discrete action sum

$$\mathcal{A}_{\mathcal{T},\Theta}\left(u,I\right) := \sum_{K\in\mathcal{T}} \sum_{\left\{j\,\middle|\,\left[t_K^j,t_K^{j+1}\right]\subseteq I\right\}} \mathcal{A}^j\left(K;u_K\right) \tag{1.4}$$

arises. Here u_K indicates the nodal displacements in the element K, I any open interval in (t_0,t_f), and $\mathcal{A}^j\left(K;u_K\right)$ is defined by

$$\begin{aligned} \mathcal{A}^j\left(K;u_K\right)\ :=& \sum_{a\in K} \sum_{\left\{i\,\middle|\,t_a^i\in\left[t_K^j,t_K^{j+1}\right)\right\}} \frac{1}{2}m_{K,a}\left(t_a^{i+1}-t_a^i\right)\left|\dot{u}_a\left(t_a^i\right)\right|^2 \\ & -(t_K^{j+1}-t_K^j)V_K(u_K(t_K^{j+1})), \end{aligned}$$

with $V_K(u_K(t_K^{j+1}))$ the value at t_K^{j+1} of $V_K\left(u_K\left(t\right)\right)$ presented above.

Roughly, this is the way in which one constructs *asynchronous variational integrators* (AVIs).

For ordinary and partial differential equations, a recursive rule that allows one to calculate discrete trajectories starting from initial data is called *variational integrator* if it is the discrete Euler-Lagrange equation of some discrete Lagrangian. For a detailed treatment of variational integrators see [11], [12], the former work, in particular, introduces them in the multisymplectic description of partial differential equations.

When the total potential energy can be additively subdivided into the energy of subsystems, such as finite elements, as usual in mechanics of solids, or groups of particles in molecular dynamics, asynchronous time discretization can be then assigned to each subsystem.

The resulting AVIs have been introduced in [7] and further developed in [8]. Their structure has non trivial features. Some of them are summarized below;

sometimes their are not only peculiar of AVIs but are also common to variational integrators in general. Details and pertinent comments can be found in [7] and [8].

- The determination of individual trajectories displays problems analogous to those of traditional algorithms. On the contrary, the evaluation of statistical quantities (in the sense of time averages for example) is substantially better with respect to traditional algorithms.
- A version of Noether theorem assures that the natural link between symmetries and conserved quantities in the continuous formulation is still present between the discrete action and the corresponding integrator.
- If one chooses in each subsystem (different) constant time steps, the energy is nearly conserved because the algorithm 'preserves' the symplectic nature of the original (continuos) variational formulation. Artificial numerical damping is not introduced. Advantages in tracking the error decay follow.
- The Courant condition (which estimates the stability limit) provides the way to evaluate the time steps. However, in practice, solving for the time steps is not necessary. For non-costant time steps, the energy evaluated numerically oscillates around the average value. The absolute value of the residual energy (the fluctuation in a sense) is small with respect to the overall energy of each finite element (at least for the tests for a neo-Hookian material presented in [8]). In this sense in the item above we write that the energy is nearly preserved.
- Examples developed in [8] show that the computational cost of AVIs is low with respect to the cost in other methods (such as an explicit Newmark condition) for a desired error value.

AVIs share many features in common with subcyclic methods introduced in [3], [4]. Analogies with other methods used in molecular dynamics can be found (see comparisons in [8]).

The utility of AVIs for engineering purposes is evident from the list of their features. They can be used profitably for both solids and fluids. The dynamics of structures and, above all, the fluid-structure interaction can be appropriate field of applications.

In the dynamics of continuous simple bodies, convergence in time of AVIs for a fixed space discretization has been proven in [8] for potentials having uniformly bounded second derivative, substantially potentials with quadratic growth. Such a condition is sufficient for assuring that the integrator is a Lipschitz map. The proof in [8] exploits the discrete version of Euler-Lagrange equations and a recursive use of Gronwall's inequality.

In the case of zero-dimensional oscillators, the convergence of the discrete dynamics to the continuous counterpart has been proven in [13] *via* Γ-convergence arguments. Assumptions of technical nature have been removed in [9].

The aim of this paper is to discuss further the convergence of AVIs. Stability of AVIs is not treated here.

We start from the work in [13] and, for the linear elastodynamics of simple bodies, we develop a new proof for the convergence of AVIs in time, once a spatial discretization is fixed.

- We analyze directly the discrete action rather than the discrete Euler-Lagrange equations. In this sense our proof is eminently variational and exploits the intrinsic structure of the action functional. Of course, the convergence we find is weaker than the one in [8] but it includes cluster points that satisfy the Euler-Lagrange equations only in the weak form.

- Although we restrict the treatment to linear elastodynamics of simple continuous bodies, the proof can be extended to the non-linear case involving potentials with quadratic growth, differences resting only in additional technical details related to estimates. The basic difficulty would be related to the control of the growth of the determinant of the gradient of deformation.
- We consider the trajectory of each node to be piecewise affine in time. Our choice is justified by the sake of simplicity. More general time approximations can be used in principle, paying their presence with additional non-trivial technical difficulties in the path that we follow here.
- The removal of the homogeneous boundary condition on the displacement field is possible, paying it with the introduction of additional technicalities.

The main result of our work is the theorem below.

Theorem 1. (Convergence in time.) *Let Θ_h be entire time sets for $[t_0, t_f]$ indexed by the time size h. Let also $u_h(t_0)$ and $\dot{u}_h(t_0)$ be initial conditions satisfying*

$$\sup_h \left(M_{\Theta_h} + |u_h(t_0)| + |\dot{u}_h(t_0)| \right) < +\infty.$$

Then any sequence (u_h), with u_h a stationary point of the discrete action $\mathcal{A}_{\mathcal{T},\Theta_h}$, is pre-compact in the weak-$*$ $W^{1,\infty}\left((t_0,t_f),\mathbb{R}^N\right)$ topology and all its cluster points are stationary points for the action $\mathcal{A}_{\mathcal{T}}$.

In the statement above N is the total number of nodal degrees of freedom in the tessellation $\mathcal{T}$, and M_{Θ_h} is a quantity controlling the asynchronicity of Θ_h (see (3.5) for the definition).

The theorem above can be extended to the linear elastodynamics of complex bodies, provided that the manifold of substructural shapes is embedded isometrically in a linear space (the paradigmatic case of quasicrystals is treated in [6]; for quasicrystals the standard Cauchy balance is augmented by a balance of substructural actions of parabolic type, both equations arising from a d'Alembert-Lagrange-type variational principle).

2. A summary of linear elastodynamics of simple bodies

Although our results hold generally in $\mathbb{R}^n$, we restrict (only formally) the developments below to the three-dimensional ambient space $\mathbb{R}^3$ in which the region occupied by a body is always denoted by $\mathcal{B}$. By assumption $\mathcal{B}$ is a bounded domain with boundary $\partial\mathcal{B}$ of finite two-dimensional measure, a boundary where the outward unit normal is defined to within a finite number of corners and\or edges. On $\mathcal{B}$ the *displacement field* is defined by $x \longmapsto u := u(x) \in \mathbb{R}^3$, $x \in \mathcal{B}$, and is assumed to be differentiable. The field $x \longmapsto u(x) + x$ is also one-to-one and orientation preserving in the sense that $\det(\nabla u + \mathbf{I}) > 0$ at each x, with $\mathbf{I}$ the second-rank unit tensor. When $|\nabla u| << 1$, the natural measure of infinitesimal deformations is given at each point by the value of the field $x \longmapsto \varepsilon(x) := sym\nabla u(x)$ assigning at each point the strain ε.

In a time interval $[t_0, t_f]$, a standard motion is described by

$$(x,t) \longmapsto u := u(x,t) \in \mathbb{R}^n,\ x \in \mathcal{B},\ t \in [t_0, t_f],$$

a field twice differentiable in time.

The (contact) interaction, power conjugated with the velocity $\dot{u} := \frac{d}{dt} u(x,t)$ on any virtual smooth surface in the body (Cauchy cut), oriented pointwise by the normal n, is the tension $\mathtt{t}$ which depends linearly on n through Cauchy stress tensor σ, namely $\mathtt{t} = \sigma n$. A tensor field $(x,t) \longmapsto \sigma = \sigma(x,t) \in Hom\left(\mathbb{R}^3, \mathbb{R}^3\right)$ is then defined over $\mathcal{B}$ and is assumed to be differentiable. $Hom\left(\mathbb{R}^3, \mathbb{R}^3\right)$ is the space of linear operators from $\mathbb{R}^3$ to itself. Really, the stress tensor maps $\mathbb{R}^3$ onto its dual because the tension $\mathtt{t}$ is intrinsically a co-vector (that is a force), but here we make tacit use of the isomorphism between $\mathbb{R}^3$ and its dual.

The invariance with respect to isometric changes in observers of the external power of bulk and surface actions on any subset $\mathfrak{b}$ of $\mathcal{B}$ with non-vanishing volume measure and the same regularity of $\mathcal{B}$ itself allows one to get pointwise balances between bulk and contact actions. Moreover, the inertial parts of the body forces are identified by making use of the balance between the rate of the kinetic energy and the power of inertial forces. As a result one gets the standard balance equations

$$b + div\sigma = \rho \ddot{u}, \tag{2.1}$$

$$skw\sigma = 0, \tag{2.2}$$

where ρ is the mass density conserved in time. Natural boundary conditions are given by the prescription of the traction $\mathtt{t}$ on a part $\partial\mathcal{B}_t$ of the boundary $\partial\mathcal{B}$ and of the displacement u on another part $\partial\mathcal{B}_u$ provided that $\partial\mathcal{B}_t \cap \partial\mathcal{B}_u = \varnothing$ and $\overline{\partial\mathcal{B}_t \cup \partial\mathcal{B}_u} = \partial\mathcal{B}$. Precisely, here it is assumed that $u = 0$ along $\partial\mathcal{B}_u$.

When the material is homogeneous and displays a linear hyperelastic behavior one gets

$$\sigma = \mathbb{C}\varepsilon, \tag{2.3}$$

with $\mathbb{C}$ a constant fourth-rank tensor with minor and major symmetries. In particular it is assumed that

$$\mathbb{C}\left(\xi \otimes \eta\right) \cdot \left(\xi \otimes \eta\right) > 0$$

for any pair of vectors ξ and η. Such a condition implies that the balance equations above generate a quasi-contractive semigroup in $H^1 \times L^2$ (see, e.g., [1], [10]).

Under the mixed boundary conditions mentioned above and the constitutive structure (2.3), the balance equation (2.1) can be also obtained by imposing that the first variation of the action functional (1.1) vanishes. From the geometrical aside, the action (1.1) is a fuctional defined on a first jet bundle. In fact, consider a fiber bundle $\mathcal{Y}$ with basis the space-time tube $\mathcal{B} \times [t_0, t_f]$ and canonical projection $\pi : \mathcal{Y} \rightarrow \mathcal{B} \times [t_0, t_f]$, such that the prototype fiber is given by $\pi(x,t) = \mathbb{R}^3$. Sections $\eta : \mathcal{B} \times [t_0, t_f] \rightarrow \mathcal{Y}$ determined by means of the map $x \longmapsto u(x)$, take then values $\eta(x,t) := (x, t, u(x))$. They admit first prolongations $j^1\eta(x,t) := (x, t, u(x,t), \dot{u}(x,t), \nabla u(x,t))$, each one being an element of the first jet bundle $J^1\mathcal{Y}$ of $\mathcal{Y}$. The action (1.1) is then a functional defined over $J^1\mathcal{Y}$. The symplectic structure[1] associated with the action (1.1) is assured by Betti's reciprocity theorem

[1]We remind that a symplectic manifold is a pair $(\mathcal{M}, \omega)$ where $\mathcal{M}$ is a manifold and ω a closed weakly non-degenerate two-form on $\mathcal{M}$. At each point ν of $\mathcal{M}$, the form ω defines an isomorphism between the tangent space $T_\nu\mathcal{M}$ and its dual. Hamiltonian vector fields over $\mathcal{M}$ consist of canonical transformations. Precisely, if $(\mathcal{M}_1, \omega_1)$ and $(\mathcal{M}_2, \omega_2)$ are symplectic manifolds, a smooth mapping $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is called canonical (or symplectic) if ω_1 is the pull-back of ω_2

(see [10]) which implies the existence of closed weakly non-degenerate two-form. Symplecticity is the source of the property that the energy is nearly preserved by the AVI discretization of the action functional.

3. Space-time discretization of the action functional

As anticipated above, a way to obtain algorithms preserving the symplectic structure of the linear elastodynamics of simple bodies consists in constructing a direct discretization of the action functional (1.1) in space and time and attributing different discrete time sequences to each spatial finite element. This is the scheme of asynchronous variational integrators introduced in [7] and further developed in [8].

Discretization in space is obtained by means of standard finite elements given by a tessellation $\mathcal{T}$ of $\mathcal{B}$, a simple triangulation, for example, chosen to be consistent with the partition of the boundary $\partial\mathcal{B}$ into $\partial\mathcal{B}_t$ and $\partial\mathcal{B}_u$ (see, e.g., [3]). By adopting a triangulation, for the sake of simplicity, we assume that $\mathcal{B}$ is endowed with polyhedral shape, a shape compatible with a regular 'triangulation'. For each $K \in \mathcal{T}$ a finite number of points are selected as integration nodes, the generic one being indicated by a. Here we choose as nodes the vertexes of the elements of the triangulation. We consider then the space $PA(\mathcal{T})$ of linear polynomials on each $K \in \mathcal{T}$ and are interested in a vector subspace $\mathcal{V}_{\mathcal{T}}$ of $PA(\mathcal{T}) \otimes H^1\left((t_0, t_f), \mathbb{R}^3\right)$ containing all displacement mappings satisfying $u\,|_{x\in\partial\mathcal{B}_u} = 0$. Any $u \in \mathcal{V}_{\mathcal{T}}$ is then of the form

$$u(x,t) = \sum_{a\in\mathcal{T}} \mathcal{N}_a(x) u_a(t). \tag{3.1}$$

For each finite element K, $\mathcal{N}_a(x)$ is the nodal shape function corresponding to the node $a \in K$ and $u_a(t)$ is the value of the displacement at the generic node a at time t. Of course $u_a(t) \equiv 0$ if $a \in \mathcal{T} \cap \partial\mathcal{B}_u$. As usual, shape functions are selected in a way that they form an orthonormal family in $L^2(\mathcal{B})$. When restricted to a generic finite element K, the map $u(x,t)$ in (3.1) is indicated by $u_K(x,t)$.

With a slight abuse of notation, we denote by $u(t)$ a vector in $\mathbb{R}^N$, with N the number of degrees of freedom of all nodal placements at time t, relative to the map $u(\cdot,t)$. Specifically, $N = 3M$, with M the total number of nodes in $\mathcal{T}$. Analogously $u_K(t)$ indicates a vector in $\mathbb{R}^{12}$ with components the displacements of all nodes of the generic element K.

It is possible to check that there exists a constant c, depending only on the tessellation $\mathcal{T}$, such that

$$\sup_{\mathcal{B}} |\nabla u(\cdot,t)| \leq c\,|u(t)|\,. \tag{3.2}$$

The spatial discretization $\mathcal{A}_{\mathcal{T}}$ of $\mathcal{A}$ is obtained simply by restricting to $\mathcal{V}_{\mathcal{T}}$ the proper domain of $\mathcal{A}$. More precisely, if $\mathcal{E}$ denotes the class of open intervals $I \subseteq$

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in $\mathcal{M}_1$ along f. In particular, if $\dim\mathcal{M}_1 = \dim\mathcal{M}_2$, f is volume preserving and is a local diffeomorfism. For the linear elastodynamics of a simple body (say for simplicity a body that occupies the whole $\mathbb{R}^3$), the two-form ω is defined over $H^1\left(\mathbb{R}^3,\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3,\mathbb{R}^3\right)$ by

$$\omega\left((u_1,\dot{u}_1),(u_2,\dot{u}_2)\right) := \int_{\mathbb{R}^3} \rho\left(\dot{u}_2\cdot u_1 - \dot{u}_1\cdot u_2\right)\,dx$$

with suitable decay properties at infinity. Linear elastodynamics is Hamiltonian in the sense it generates a Hamiltonian flow. The fact that ω is closed is evidently associated with Betti's reciprocity.

(t_0, t_f), $\mathcal{A}_{\mathcal{T}} : PA(\mathcal{T}) \otimes H^1\left((t_0, t_f), \mathbb{R}^3\right) \times \mathcal{E} \to [0, +\infty]$ is then defined by

$$\mathcal{A}_{\mathcal{T}}(u, I) := \begin{cases} \mathcal{A}(u, I) & u \in \mathcal{V}_{\mathcal{T}} \\ +\infty & \text{otherwise} \end{cases}.$$

Notice that the vector of nodal placements $u(\cdot)$ belongs to $H^1\left((t_0, t_f), \mathbb{R}^N\right)$ for every map $u \in \mathcal{V}_{\mathcal{T}}$. Hence, the expansion in (3.1) allows to regard $\mathcal{A}_{\mathcal{T}}$ as a functional defined over $H^1\left((t_0, t_f), \mathbb{R}^N\right) \times \mathcal{E}$. We agree to do that in the sequel.

By means of a straightforward computation one gets (1.3) where, for each $K \in \mathcal{T}$, we rewrite

$$\mathcal{A}(K; u_K(t)) := \sum_{a \in K} \frac{m_{K,a}}{2} |\dot{u}_a(t)|^2 - V_K(u_K(t)), \tag{3.3}$$

and

$$\begin{aligned} V_K(u_K(t)) := & \int_K \frac{1}{2} (\mathbb{C}\nabla u_K(x,t)) \cdot \nabla u_K(x,t) \ dx \\ & - \int_K b(x) \cdot u_K(x,t) \ dx - \int_{\partial K \cap \partial \mathcal{B}_t} \mathsf{t}(x) \cdot u_K(x,t) \ d\mathcal{H}^2. \end{aligned} \tag{3.4}$$

We call *stationary point* for $\mathcal{A}_{\mathcal{T}}$ any map $u \in H^1\left((t_0, t_f), \mathbb{R}^N\right)$ satisfying for any time interval $I \in \mathcal{E}$, any node a in $\mathcal{T}$, and any $w_a \in u_a + H_0^1\left(I, \mathbb{R}^3\right)$ the weak balance

$$\begin{aligned} \int_I m_a \dot{u}_a(t) \cdot \dot{w}_a(t) \ dt = & \int_I \sum_{\{K|a \in K\}} \left(\int_K (\mathbb{C}\nabla u_K(x,t)) \nabla \mathcal{N}_a(x) \ dx \right) \cdot w_a(t) \, dt \\ & - \int_I \sum_{\{K|a \in K\}} \left(\int_K \mathcal{N}_a(x) b(x) \ dx + \int_{\partial K \cap \partial \mathcal{B}_t} \mathcal{N}_a(x) \mathsf{t}(x) \, d\mathcal{H}^2 \right) \cdot w_a(t) \, dt, \end{aligned}$$

where $m_a := \sum_{\{K|a \in K\}} m_{K,a}$.

Standard results about finite elements imply the theorem below (see [5]).

Theorem 2. *Consider a family $(\mathcal{T}_m)$ of regular triangulations of $\mathcal{B}$, with $m > 0$ the mesh size of $\mathcal{T}_m$. Let also $u_m \in \mathcal{V}_{\mathcal{T}_m}$ be a stationary point for $\mathcal{A}_{\mathcal{T}_m}$. The sequence (u_m) converges in $H^1\left(\mathcal{B} \times (t_0, t_f), \mathbb{R}^3\right)$ to a stationary point of $\mathcal{A}$.*

The discretization of the time interval follows the guidelines indicated in the first section. A partition $\Theta := \{t_i\}_{i=0,\ldots,N_\Theta}$ of $[t_0, t_f]$ with $t_{N_\Theta} = t_f$ is selected. Its *size* is the value $\max_i (t_{i+1} - t_i)$. Each $K \in \mathcal{T}$ is endowed with an *elemental time set* which is an ordered subset Θ_K of Θ. By relabeling the elements we write

$$\Theta_K = \left\{ t_0 = t_K^1 < \ldots < t_K^{N_K - 1} < t_K^{N_K} = t_f \right\}.$$

We assume $\Theta = \cup_{K \in \mathcal{T}} \Theta_K$ and, for the sake of simplicity, we presume also that $\Theta_K \cap \Theta_{K'} \neq \varnothing$ for any K, $K' \in \mathcal{T}$ with $K \neq K'$. We denote also by T_Θ the maximum of the elemental time sizes, namely

$$T_\Theta := \max_K \max_{\Theta_K} (t_K^{j+1} - t_K^j).$$

The circumstance that each finite element can be endowed with a different time set is the basic characteristic of *asynchronous variational integrators* as mentioned in Section 1: if one imposes appropriate choices of elemental time sets, one may prove conservation of energy in discrete time (in the sense specified above). Take note that in principle such choices could not exist (see relevant comments in [7]).

For a node a in $\mathcal{T}$, elemental time sets define also the *nodal time set* Θ_a by

$$\Theta_a := \bigcup_{\{K \mid a \in K\}} \Theta_K = \left\{ t_0 = t_a^1 < ... < t_a^{N_a - 1} < t_a^{N_a} = t_f \right\},$$

by relabeling the elements.

To measure the *asynchronicity of* Θ, we consider the ratio

$$M_\Theta := \frac{\max_{K \in \mathcal{T}} \left(\max_{\Theta_K} \left(t_K^{j+1} - t_K^j \right) \right)}{\min_{K \in \mathcal{T}} \left(\min_{\Theta_K} \left(t_K^{j+1} - t_K^j \right) \right)}. \tag{3.5}$$

Then notice that [2]

$$T_\Theta \leq Card(\mathcal{T}) M_\Theta h, \tag{3.6}$$

where h denotes the size of Θ.

Amid possible choices, we assume that each node $a \in \mathcal{T}$ follows a *linear* trajectory within each time interval with end points that are consecutive instants in Θ_a. Such a choice characterizes the class of AVIs we analyze here. Then we denote by Y_Θ the subspace of functions in $L^2\left((t_0, t_f), \mathbb{R}^N\right)$ which are continuous and with piecewise constant time rates in the intervals in Θ_a. Thus, for each $u \in Y_\Theta$, $a \in \mathcal{T}$, $t_a^i \in \Theta_a$ and $t \in \left[t_a^i, t_a^{i+1}\right)$ we have

$$\dot{u}_a(t) = \frac{u_a\left(t_a^{i+1}\right) - u_a\left(t_a^i\right)}{t_a^{i+1} - t_a^i}.$$

Then, by following [7], the discrete action sums in time is defined for $u \in Y_\Theta$ by (1.4), that is

$$\mathcal{A}_{\mathcal{T},\Theta}(u, I) := \sum_{K \in \mathcal{T}} \sum_{\left\{ j \mid [t_K^j, t_K^{j+1}] \subseteq I \right\}} \mathcal{A}^j(K; u_K)$$

where, for the displacements u_K of the generic finite element K, $\mathcal{A}^j(K; u_K)$ is defined also by

$$\begin{aligned} \mathcal{A}^j(K; u_K) := & \sum_{a \in K} \sum_{\left\{ i \mid t_a^i \in \left[t_K^j, t_K^{j+1}\right) \right\}} \frac{1}{2} m_{K,a} \left(t_a^{i+1} - t_a^i\right) \left|\dot{u}_a\left(t_a^i\right)\right|^2 \\ & - (t_K^{j+1} - t_K^j) V_K(u_K(t_K^{j+1})), \end{aligned}$$

with V_K given by (3.4) and $I \in \mathcal{E}$. Such a choice gives rise to explicit integrators of central-difference type and is only one of the possible schemes that can be used.

It is convenient to define all action sums on the same function space to avoid to link the function space itself to the choice of Θ. For this reason we extend $\mathcal{A}_{\mathcal{T},\Theta}$ to $+\infty$ on $Y \backslash Y_\Theta$, where $Y = L^2\left((t_0, t_f), \mathbb{R}^N\right)$ is endowed with the usual metric.

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[2]Indeed, by definition of time size we have $t_f - t_0 \leq Card(\Theta)h$, and since

$$Card(\Theta) \min_{K \in \mathcal{T}} (\min_{\Theta_K} (t_K^{j+1} - t_K^j)) \leq Card(\mathcal{T})(t_f - t_0)$$

we infer that

$$\min_{K \in \mathcal{T}} (\min_{\Theta_K} (t_K^{j+1} - t_K^j)) \leq Card(\mathcal{T})\, h.$$

Inequality (3.6) then follows by definition of M_Θ (see (3.5)).

Namely, by recalling (1.4) (i.e. the definition of $\mathcal{A}_{\mathcal{T},\Theta}(u,I)$ rewritten above), with a little abuse of notation, we put

$$\mathcal{A}_{\mathcal{T},\Theta}(u,I)=\begin{cases}\mathcal{A}_{\mathcal{T},\Theta}(u,I) & \text{for } u\in Y_\Theta\\ +\infty & \text{for } u\in Y\backslash Y_\Theta\end{cases}\tag{3.7}$$

The discrete variational principle

$$\delta\mathcal{A}_{\mathcal{T},\Theta}=0,$$

with δ indicating the first variation, implies that for all $a\in\mathcal{T}\backslash\partial\mathcal{B}_u$ and $t_a^i\in(t_0,t_f]$ discrete Euler-Lagrange equations have to be satisfied. They read

$$\begin{aligned} m_a\left(\dot u_a\left(t_a^{i-1}\right)-\dot u_a\left(t_a^i\right)\right) &= (t_K^{j+1}-t_K^j)\int_K\left(\mathbb{C}\nabla u_K\left(x,t_K^{j+1}\right)\right)\nabla\mathcal{N}_a(x)\ dx\\ &-(t_K^{j+1}-t_K^j)\left(\int_K\mathcal{N}_a(x)\,b(x)\ dx+\int_{\partial K\cap\partial\mathcal{B}_t}\mathcal{N}_a(x)\,\mathsf{t}(x)\ d\mathcal{H}^2\right),\end{aligned}\tag{3.8}$$

where K is the sole element in $\mathcal{T}$ for which $t_a^i\in\Theta_K$ and $t_a^i=t_K^{j+1}$.

Given initial conditions $u(t_0)$ and $\dot u(t_0)$, the discrete Euler-Lagrange equations (3.8) define inductively a trajectory u piecewise affine in time, a trajectory which is a (discrete) stationary point for $\mathcal{A}_{\mathcal{T},\Theta}$.

Extensions of AVIs are possible: their possibility generates open problems. As regards this aspect, in [8] the possible formulation of implicit AVIs and the parallel implementation of AVI methods are mentioned as possible extensions of the actual proposals (some details are also given). Briefly, here we may add the indication of some other possible paths. AVIs seem to be a useful tool for analyzing problems with multiple time scales, such as chemical reactions. Also, they could be profitably used in circumstances in which defects or phase interfaces have their own dynamics relative to the rest of the body. From the numerical analysis aside, a non-trivial point could be the construction of AVIs for the dynamics of maps taking values in the unit sphere $\mathbb{S}^2$ (a related mechanical example is the dynamics of bodies with magnetic spins). As regards the application of AVIs to the dynamics of complex bodies[3], application mentioned amid potentialities even in [7] (at the beinning of Section 5 there), we mention that in general the peculiar kinetic energy of material microstructure could not have in principle the standard simple quadratic form in (1.1) (in general it can be considered as a Finsler metric over the manifold of substructural shapes). This last circumstance may generate additional computational difficulties in special cases.

4. Proof of Theorem 1

4.1. **Strategy.** To prove the convergence of asynchronous variational integrators in linear elastodynamics, we adapt here the strategy used in [13] and [9] for analyzing the convergence of variational integrators for the zero-dimensional oscillator (a mass point connected to an elastic massless spring). The essential structure is listed below.

[3]We remind that bodies are called *complex* when their material texture at various scales (substructure) prominently influences the gross mechanical behavior by means of interactions power-conjugated with substuctural changes. Liquid crystals, quasicrystals, ferroelectrics, polymeric bodies are common examples used in engineering applications.

- ***(i)*:** First we need to establish a priori L^∞ estimates for the velocity of stationary points of the discrete actions. Of course here stationarity does not mean independence of time.
- ***(ii)*:** Then we show that stationary points of the discrete action sums are minimizers in short-time intervals.
- ***(iii)*:** Finally we analyze the convergence of such stationary points in the limit $h \to 0^+$, that is as the time size of the discretization goes to zero.

4.2. **Preliminary results.** In the developments below we find inequalities involving constants that depend on the data of the problem, on the tessellation $\mathcal{T}$ of finite elements selected and are independent of the time set Θ. Since it is not essential to distinguish from one specific constant to another, we indicate all of them by the same letter c, leaving understood that c changes from one inequality to another.

As suggested in [9], L^∞ estimates on the velocity of stationary points can be derived by exploiting directly the discrete Euler-Lagrange equations and the growth conditions of the potential energy density.

Proposition 1. *Given initial conditions $u(t_0)$ and $\dot{u}(t_0)$, there exists a constant $k > 0$ depending on the initial conditions themselves and on the data of the problem such that, for every entire time set Θ and $u \in Y_\Theta$ solution to the discrete Euler-Lagrange equations, it satisfies the inequality*

$$\|\dot{u}\|_{L^\infty((t_0,t_f),\mathbb{R}^N)} \leq k \exp(kM_\Theta). \tag{4.1}$$

Proof. Fix a node a in the tessellation $\mathcal{T}$ and a nodal time $t_a^i \in \Theta_a$. By assumption there exists a unique $K \in \mathcal{T}$ such that $t_a^i \in \Theta_K$, with $t_a^i = t_K^{j+1}$. Moreover, the definition of nodal time set yields $t_a^i = t_\alpha^{i(\alpha)} \in \Theta_\alpha$ for all the other nodes $\alpha \in K$. Thus the discrete Euler-Lagrange equations (3.8) entail

$$m_a |\dot{u}_a(t_a^i) - \dot{u}_a(t_a^{i-1})| \leq c(t_K^{j+1} - t_K^j)(\sum_{\alpha\in K} |u_\alpha(t_\alpha^{i(\alpha)})| + 1).$$

By taking into account that the maps u_α are piecewise affine in time, it is easy to get

$$\begin{aligned} &\left|\dot{u}_a(t_a^i) - \dot{u}_a(t_a^{i-1})\right| \\ &\quad\leq c(t_K^{j+1} - t_K^j) \sum_{\alpha\in K} (|u_\alpha(t_0)| + \sum_{l=0}^{i(\alpha)-1} (t_\alpha^{l+1} - t_\alpha^l) \left|\dot{u}_\alpha(t_\alpha^l)\right|) + c(t_K^{j+1} - t_K^j) \\ &\quad\leq c(t_K^{j+1} - t_K^j) \sum_{a\in K} (|u_\alpha(t_0)| + (t_f - t_0) \|\dot{u}_\alpha\|_{L^\infty\left([t_0,t_\alpha^{i(\alpha)-1}],\mathbb{R}^N\right)}) + c(t_K^{j+1} - t_K^j). \end{aligned}$$

By recalling that $\Theta := \{t_l\}$, $l = 1, ..., N_\Theta$, and assuming $t_a^i = t_{s+1} \in \Theta$, the latter inequality yields

$$\|\dot{u}_\alpha\|_{L^\infty((t_0,t_{s+1}),\mathbb{R}^3)} \leq c(t_K^{j+1} - t_K^j) + (1 + c(t_K^{j+1} - t_K^j)) \|\dot{u}\|_{L^\infty((t_0,t_s),\mathbb{R}^N)}. \tag{4.2}$$

In particular, by setting $\beta_l := \|\dot{u}\|_{L^\infty((t_0,t_l),\mathbb{R}^N)}$, we infer from (4.2)

$$\beta_{s+1} \leq cT_\Theta + (1 + cT_\Theta)\beta_s.$$

By iterating such an inequality we obtain

$$\begin{aligned}\beta_{s+1} &\leq cT_\Theta \sum_{i=0}^{s} (1+cT_\Theta)^i + \beta_0 (1+cT_\Theta)^{s+1} \\ &\leq (1+\beta_0)(1+cT_\Theta)^{s+1} \leq c(1+cT_\Theta)^{Card(\Theta)}, \end{aligned} \tag{4.3}$$

where $Card(\cdot)$ denotes the cardinality of the relevant set.

Eventually, since $Card(\Theta) \leq Card(\mathcal{T})(t_f - t_0) M_\Theta / T_\Theta$, with M_Θ defined in (3.5), the inequality (4.3) implies

$$\|\dot{u}\|_{L^\infty((t_0,t_f),\mathbb{R}^N)} = \beta_{Card\Theta} \leq c(1+cT_\Theta)^{Card(\mathcal{T})(t_f-t_0)M_\Theta/T_\Theta},$$

and (4.1) follows. □

Proposition 2. *There exists a constant $\kappa > 0$ depending only on the tessellation $\mathcal{T}$ such that given initial conditions $u(t_0)$, $\dot{u}(t_0)$, for every entire time set Θ and $u \in Y_\Theta$ solution to the discrete Euler-Lagrange equations, u is a local minimizer of the functional $\mathcal{A}_{\mathcal{T},\Theta}(\cdot, I)$, namely $\mathcal{A}_{\mathcal{T},\Theta}(u, I) \leq \mathcal{A}_{\mathcal{T},\Theta}(v, I)$ for any $v \in u + H_0^1(I, \mathbb{R}^N)$, provided that $I \in \mathcal{E}$ satisfies $\mathcal{L}^1(I) \leq \kappa$.*

Proof. Fix an interval $I = (t_1, t_2) \in \mathcal{E}$ and define for any $K \in \mathcal{T}$ the time interval $I_K^\Theta = \cup_j \left\{[t_K^j, t_K^{j+1}] | [t_K^j, t_K^{j+1}] \subseteq I\right\}$. We consider local perturbations $v = u + w$ with $w \in Y_\Theta \cap H_0^1(I, \mathbb{R}^N)$. For $w \in H_0^1(I, \mathbb{R}^N) \setminus Y_\Theta$, the action $\mathcal{A}_{\mathcal{T},\Theta}(u + w, I)$ would be infinite (see (3.7)).

Since $u \in Y_\Theta$ satisfies the Euler-Lagrange equations (3.8), a direct computation yields

$$\mathcal{A}_{\mathcal{T},\Theta}(u+w, I) - \mathcal{A}_{\mathcal{T},\Theta}(u, I) = \frac{1}{2} \sum_{K\in\mathcal{T}} \int_{I_K^\Theta} \sum_{a\in K} m_{K,a} |\dot{w}_a(t)|^2 \, dt$$

$$- \sum_{K\in\mathcal{T}} \sum_{\left\{j | [t_K^j, t_K^{j+1}] \subseteq I\right\}} (t_K^{j+1} - t_K^j) \int_K (\mathbb{C}\nabla w_K(x, t_K^{j+1})) \cdot \nabla w_K(x, t_K^{j+1}) \, dx.$$

Furthermore, once an element K in $\mathcal{T}$ and a node a in K are selected, by taking into account that w_a is piecewise affine and belongs also to $H_0^1(I, \mathbb{R}^3)$, we have that w_a vanishes outside I_K^Θ. The reason is that $I \setminus int(I_K^\Theta)$, with $int(I_K^\Theta)$ the set of *interior* points of I_K^Θ, does not contain any closed interval with endpoints in Θ_K.

By taking into account that $\sup_{\mathcal{B}} |\nabla w(\cdot, t)| \leq c|w(t)|$, Hölder inequality gives

$$\int_K (\mathbb{C}\nabla w_K(x, t_K^{j+1})) \cdot \nabla w_K(x, t_K^{j+1}) \, dx \leq c(t_2 - t_1) \int_{I_K^\Theta} |\dot{w}_K(t)|^2 \, dt,$$

with $\dot{w}_K$ the vector of nodal velocities for nodes in K.

In turn the last inequality implies for $m := \min_{a\in\mathcal{T}} m_{K,a}$

$$\mathcal{A}_{\mathcal{T},\Theta}(u+w, I) - \mathcal{A}_{\mathcal{T},\Theta}(u, I) \geq \sum_{K\in\mathcal{T}} \int_{I_K^\Theta} \left(\frac{m}{2} - c(t_2 - t_1)^2\right) |\dot{w}_K(t)|^2 \, dt.$$

The statement of Proposition 2 then follows provided that $(t_2 - t_1)^2 \leq \frac{m}{2c}$. □

Before giving the proof of Theorem 1 we need to establish some technical results. The first one is an easy consequence of Lemma 4.3 in [13].

Lemma 1. *Let Θ_h be any entire time sets for $[t_0, t_f]$ with time size h. For any $I \in \mathcal{E}$ and $u \in H^1\left(I, \mathbb{R}^N\right)$ there exists $u_h \in H^1\left(I, \mathbb{R}^N\right) \cap Y_{\Theta_h}$ such that $u_h \to u$ strongly in $H^1\left(I, \mathbb{R}^N\right)$ as h goes to zero.*

Lemma 2. *Given entire time sets Θ_h for $[t_0, t_f]$ indexed by the time size h and characterized by $\sup_h M_{\Theta_h} < +\infty$, for every sequence (u_h) such that $u_h \in Y_{\Theta_h}$, $|\dot{u}_h|^2$ is weakly convergent in $L^1\left(I, \mathbb{R}^N\right)$, and $\sup_h \|u_h\|_{L^\infty(I,\mathbb{R}^N)} < +\infty$, one gets*

$$\lim_{h \to 0^+} \left(\mathcal{A}_{\mathcal{T}}\left(u_h, I\right) - \mathcal{A}_{\mathcal{T},\Theta_h}\left(u_h, I\right)\right) = 0.$$

Proof. Let Θ be a generic time set with maximum elemental time size T_Θ. For every $K \in \mathcal{T}$ consider the interval $I_K^\Theta := \cup_j \left\{[t_K^j, t_K^{j+1}] | [t_K^j, t_K^{j+1}] \subseteq I\right\}$, and notice that $\mathcal{L}^1\left(I \backslash I_K^\Theta\right) \leq 2T_\Theta$.

For any $u \in Y_\Theta$ we have

$$\begin{aligned} |\mathcal{A}_{\mathcal{T}}(u, I) - \mathcal{A}_{\mathcal{T},\Theta}(u, I)| \leq & \sum_{K \in \mathcal{T}} \int_{I \backslash I_K^\Theta} (\sum_{a \in K} \frac{m_{K,a}}{2} \left|\dot{u}_a(t)\right|^2 + V_K(u_K(t)))\, dt \\ & + \sum_{K \in \mathcal{T}} \sum_{\left\{j | [t_K^j, t_K^{j+1}] \subseteq I\right\}} \int_{t_K^j}^{t_K^{j+1}} \left| V_K(u_K(t)) - V_K(u_K(t_K^{j+1}))\right| dt =: J_1 + J_2. \end{aligned}$$

A direct computation and (3.2), namely $\sup_{\mathcal{B}} |\nabla u(\cdot, t)| \leq c\, |u(t)|$, yield

$$J_1 \leq c \sum_{K \in \mathcal{T}} \left(\|\dot{u}_K\|^2_{L^2\left(I \backslash I_K^\Theta, \mathbb{R}^{12}\right)} + 2T_\Theta \|u_K\|^2_{L^\infty(I, \mathbb{R}^{12})}\right). \tag{4.4}$$

For what the term J_2 is concerned notice that by using Hölder inequality we have for every $t \in [t_K^j, t_K^{j+1}]$

$$\begin{aligned} \left| V_K(u_K(t)) - V_K(u_K(t_K^{j+1}))\right| & \leq c \|u_K\|_{L^\infty(I, \mathbb{R}^{12})} |u_K(t) - u_K(t_K^{j+1})| \\ & \leq c (t_K^{j+1} - t)^{1/2} \|u_K\|_{L^\infty((t_K^j, t_K^{j+1}), \mathbb{R}^{12})} \|\dot{u}_K\|_{L^2((t_K^j, t_K^{j+1}), \mathbb{R}^{12})}. \end{aligned}$$

Integrating over (t_K^j, t_K^{j+1}) and summing on j give

$$J_2 \leq c \sum_{K \in \mathcal{T}} \left(T_\Theta^{1/2} \mathcal{L}^1(I_K^\Theta) \|u_K\|_{L^\infty(I, \mathbb{R}^{12})} \|\dot{u}_K\|_{L^2(I, \mathbb{R}^{12})}\right). \tag{4.5}$$

The conclusion follows straightforward from (4.4) and (4.5) when we choose sequences (Θ_h) and (u_h) as those in the statement of the lemma by taking into account that (3.6) implies the convergence of (T_{Θ_h}) to zero as the time size h vanishes if $\sup_h M_{\Theta_h} < +\infty$. □

4.3. **Proof of Theorem 1.** Pre-compactness of (u_h) in $W^{1,\infty}\left((t_0, t_f), \mathbb{R}^N\right)$ follows easily from Proposition 1 since the ratios M_{Θ_h} are bounded uniformly with respect to h by assumption. Then denote by $u \in W^{1,\infty}\left((t_0, t_f), \mathbb{R}^N\right)$ a cluster point of (u_h). By Ascoli-Arzelà theorem we may suppose $u_h \to u$ uniformly on $[t_0, t_f]$ up to a subsequence not relabeled for convenience.

The proof that u is a stationary point for $\mathcal{A}_{\mathcal{T}}$ follows by showing that, for every $I = (t_1, t_2) \in \mathcal{E}$ with $\mathcal{L}^1(I) \leq \kappa$, κ the constant in Proposition 2, one gets

$$\mathcal{A}_{\mathcal{T}}(u, I) \leq \mathcal{A}_{\mathcal{T}}(w, I), \tag{4.6}$$

for any $w \in u + H_0^1\left(I, \mathbb{R}^N\right)$.

Take note first that $\mathcal{A}_{\mathcal{T}}(\cdot, I)$ is lower semicontinuous under weak$-*$ convergence in $W^{1,\infty}$. As a consequence, the weak$-*$ convergence in $W^{1,\infty}\left(I, \mathbb{R}^N\right)$ of (u_h) and Lemma 2 imply

$$\mathcal{A}_{\mathcal{T}}(u, I) \leq \liminf_{h \to 0^+} \mathcal{A}_{\mathcal{T}}(u_h, I) = \liminf_{h \to 0^+} \mathcal{A}_{\mathcal{T},\Theta_h}(u_h, I). \tag{4.7}$$

Furthermore, Lemma 1 provides a sequence (w_h), with $w_h \in Y_{\Theta_h}$, converging to w strongly in $H^1\left(I, \mathbb{R}^N\right)$. Consequently the action $\mathcal{A}_{\mathcal{T}}(\cdot, I)$ is continuous along the sequence (w_h). Then the use of Lemma 2 implies

$$\mathcal{A}_{\mathcal{T}}(w, I) = \lim_{h \to 0^+} \mathcal{A}_{\mathcal{T}}(w_h, I) = \lim_{h \to 0^+} \mathcal{A}_{\mathcal{T},\Theta_h}(w_h, I). \tag{4.8}$$

Thus, if $w_h - u_h \in H_0^1\left(I, \mathbb{R}^N\right)$, the result in Proposition 2, the estimate (4.7) and the limit (4.8) imply

$$\mathcal{A}_{\mathcal{T}}(u, I) \leq \liminf_{h \to 0^+} \mathcal{A}_{\mathcal{T},\Theta_h}(u_h, I) \leq \lim_{h \to 0^+} \mathcal{A}_{\mathcal{T},\Theta_h}(w_h, I) = \mathcal{A}_{\mathcal{T}}(w, I).$$

To cover cases in which the boundary values are not matched on ∂I, notice that, in view of Sobolev embedding, $u_h - w_h \to 0$ uniformly on $\bar{I}$, hence there exists p_h vectors, with components that are linear polynomials, such that $u_h = w_h + p_h$ on ∂I and also $p_h \to 0$ strongly in $W^{1,\infty}\left(I, \mathbb{R}^N\right)$. Such a circumstance and (4.8) imply

$$\mathcal{A}_{\mathcal{T}}(w, I) = \lim_{h \to 0^+} \mathcal{A}_{\mathcal{T},\Theta_h}(w_h, I) = \liminf_{h \to 0^+} \mathcal{A}_{\mathcal{T},\Theta_h}(w_h + p_h, I). \tag{4.9}$$

By Proposition 2 and by collecting (4.7) and (4.9), it follows that

$$\begin{aligned} \mathcal{A}_{\mathcal{T}}(u, I) &\leq \liminf_{h \to 0^+} \mathcal{A}_{\mathcal{T},\Theta_h}(u_h, I) \\ &\leq \lim_{h \to 0^+} \mathcal{A}_{\mathcal{T},\Theta_h}(w_h + p_h, I) = \mathcal{A}_{\mathcal{T}}(w, I), \end{aligned}$$

which concludes the proof. □

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