APPROXIMATION OF PIECEWISE AFFINE HOMEOMORPHISMS BY DIFFEOMORPHISMS

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Abstract. We prove that any countably piecewise affine homeomorphism from an open set of $\mathbb{R}^2$ can be approximated, together with its inverse, by diffeomorphisms in the $W^{1,p}$ and the L^∞ norms.

1. Introduction

The question of approximating invertible maps by diffeomorphisms appears naturally in the theory of nonlinear elasticity. Indeed, an elastic deformation is typically modelled as a Sobolev map $u : \Omega \to \mathbb{R}^n$. Here Ω is a bounded open set of $\mathbb{R}^n$ representing the reference configuration of the body, and $u(x)$ is the position that the particle $x \in \Omega$ occupies in the deformed configuration. Of course, $n \in \{2, 3\}$ are the physically relevant cases, but it is mathematically interesting to consider the general case $n \in \mathbb{N}$. In order to avoid non-interpenetration of matter, the deformation u is required to satisfy a suitable invertibility condition as well as the orientation-preserving constraint $\det Du > 0$, where Du denotes the derivative of u. A natural invertibility condition is to impose u to be a homeomorphism onto its image. Indeed, results by [2, 6] ensure that, under somewhat strong assumptions, minimizers of nonlinear elastic energies are indeed homeomorphisms, while results by [1] show that solutions of elliptic systems of the harmonic type are homeomorphisms as well. Apart from homeomorphisms, other more measure-theoretic notions of invertibility have been studied in the context of nonlinear elasticity (see [10, 27, 14, 22] and their many generalizations).

There are several open problems in the theory of elasticity whose main difficulty ultimately relies on the fact that there are few available results on the density of diffeomorphisms in the Sobolev class of invertible and orientation-preserving maps. To fix ideas, if $p \in [1, \infty)$ and the set of admissible deformations $\mathcal{A}$ is the class of those $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ such that u is a homeomorphism onto its image and $\det Du > 0$ almost everywhere, one would like to prove that the set of diffeomorphisms in $W^{1,p}(\Omega, \mathbb{R}^n)$ is dense in $\mathcal{A}$ in the $W^{1,p}$ norm. A positive answer to this question will shed light on the understanding of some important open problems in nonlinear elasticity, such as the numerical approximation of minimizers, the regularity of minimizers under quasiconvexity, or the satisfaction of the Euler–Lagrange equations. These open problems are nicely reviewed in [3, 4, 25, 5].

Only little is known about the approximation of homeomorphisms by diffeomorphisms or by piecewise affine homeomorphisms. In the approximation by piecewise affine homeomorphisms, the dimension n plays a crucial role. Moise [19] proved that, in dimension 3, a homeomorphism can be approximated by piecewise affine homeomorphims in the supremum norm. The analogue result in dimension 2 was known earlier (see, e.g., [20]). In dimension 4, the result is false, as shown by Donaldson and Sullivan [13]. For dimensions 5 and higher, the result for contractible spaces follows from theorems of Connell [11], Bing [8] and Kirby [17] (for a proof see, e.g., [24, 18]). Until recently, the only results on the approximation of homeomorphisms by piecewise affine homeomorphisms in a norm stronger than the supremum norm were due to Bellido and Mora-Corral [7] and Mora-Corral [21].

The problem of approximation of homeomorphisms by diffeomorphisms turns to be even more subtle than approximation by piecewise homeomorphisms. While a simple linear interpolation shows that the result for diffeomorphisms would imply the corresponding one for homeomorphisms, the converse is not immediately true, since standard arguments such as mollification fail, due ultimately to the non-convexity of the determinant; see [26] for an explicit example of a piecewise affine homeomorphism whose mollification is not a diffeomorphism.

In this paper we show that a planar countably piecewise affine homeomorphism can indeed be approximated by a diffeomorphism in the Sobolev norm; related results were proved in [23], but without explicit estimates. Even though our result is proved in dimension 2, it is reasonable to expect that the techniques can be adapted to any dimension.

The theorem in this paper thus complements the recent one by Daneri and Pratelli [12], written in parallel to the current manuscript, in which they show that any bi-Lipschitz planar homeomorphism can be approximated uniformly and in the $W^{1,p}$ norm, together with its inverse, with smooth diffeomorphisms or countably piecewise affine homeomorphisms, for any $1 \leq p < \infty$. The passage from countably piecewise affine homeomorphisms to smooth diffeomorphisms makes use of the main result of this paper (Theorem A below).

Finally, the recent paper of Iwaniec, Kovalev and Onninen [16], of which we learnt while a preliminary version of this paper was being written, shows that any homeomorphism $u \in W^{1,p}(\Omega, \mathbb{R}^2)$ can be approximated uniformly and in the $W^{1,p}$ norm with smooth diffeomorphisms and with countably piecewise affine homeomorphisms, for all $1 < p < \infty$. We refer to [12] for a comparison between these two recent results, as well as for a broader background.

In order to present the statement of our main result, a couple of definitions are needed. Let I be a set of indices. We say that $\{T_i\}_{i\in I}$ is a *triangulation* of Ω if $\{T_i\}_{i\in I}$ is a locally finite family of closed triangles contained in $\overline{\Omega}$, the interior of those triangles form a pairwise disjoint family, the union of the triangles contains Ω, and each edge of any triangle is either contained in $\partial\Omega$ or is the edge of a different triangle of the triangulation. Except for the fact that this definition allows for infinitely many triangles, it essentially coincides with the usual concept of triangulation in the finite element literature (see, e.g., [9]). Notice that any triangulation of an open set Ω contains at most countably many triangles, and in particular Ω admits a finite triangulation if and only if it is a bounded set whose boundary is a finite union of points and segments.

For any $x \in \Omega$, we denote the *natural neigborhood* $J(x)$ of x as the union of all the triangles intersecting some triangle containing x, that is,

$$J(x) := \Big\{ y \in \Omega : \exists i, j \in I,\, x \in T_i,\, y \in T_j,\, T_i \cap T_j \neq \emptyset \Big\}. \tag{1.1}$$

Let $u : \Omega \to \mathbb{R}^2$ be a homeomorphism. We say that u is *countably piecewise affine* if there exists a triangulation $\{T_i\}_{i\in I}$ of Ω such that u is affine on each T_i; note that such a homeomorphism need not be bounded or Sobolev, but it is locally Lipschitz continuous. A *piecewise affine homeomorphism* is a particular case, corresponding to a finite triangulation; any such homeomorphism is in the class $W^{1,\infty}$, but it is not necessarily Lipschitz continuous.

Given a countably piecewise affine homeomorphism u, for any $i \in I$ we define the numbers

$$\ell_i := \min \Big\{ \big|M_i(\omega)\big| : |\omega| = 1 \Big\}, \qquad L_i := \max \Big\{ \big|M_i(\omega)\big| : |\omega| = 1 \Big\}, \tag{1.2}$$

where $M_i : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear function coinciding with Du in the interior of the triangle T_i. Notice that L_i (resp., ℓ_i^{-1}) is exactly the Lipschitz constant of u in T_i (resp., of u^{-1} in $u(T_i)$). For any vertex a of the triangulation and any $x \in \Omega$, we define

$$\begin{aligned} R_a &:= \max\left\{ \frac{L_i}{\ell_i} : a \text{ is a vertex of } T_i \text{ for some } i \in I \right\}, \\ R(x) &:= \max\Big\{ R_a : a \text{ is a vertex of } T_i \ni x \text{ for some } i \in I \Big\}, \\ L_{\max}(a) &:= \max\Big\{ L_i : a \text{ is a vertex of } T_i \text{ for some } i \in I \Big\}, \\ \ell_{\min}(a) &:= \min\Big\{ \ell_i : a \text{ is a vertex of } T_i \text{ for some } i \in I \Big\}. \end{aligned} \tag{1.3}$$

Observe that $1 \le R_a \le L_{\max}(a)/\ell_{\min}(a)$. We are finally in a position to state our main result.

Theorem A (From piecewise affine to smooth). *Let $\Omega \subset \mathbb{R}^2$ be an open set, and let $u : \Omega \to \mathbb{R}^2$ be a countably piecewise affine homeomorphism. Then for any $1 \le p < \infty$ and any $\varepsilon > 0$, there exists a smooth diffeomorphism v from Ω onto $u(\Omega)$ such that*

$$\|v-u\|_{L^\infty(\Omega)} + \|Dv - Du\|_{L^p(\Omega)} + \|v^{-1} - u^{-1}\|_{L^\infty(u(\Omega))} + \|Dv^{-1} - Du^{-1}\|_{L^p(u(\Omega))} \le \varepsilon\,. \tag{1.4}$$

Moreover,

- *if u is continuous up to the boundary of Ω, then v can be chosen to be continuous up to the boundary of Ω and $u = v$ on $\partial\Omega$;*
- *for any $0 \le q \le 1$, the function v can be chosen to satisfy*

$$\big|Dv(x)\big| \le 13 R(x)^{1-q} \max_{J(x)} \big|Du\big|\,, \qquad \big|Dv^{-1}\big(u(x)\big)\big| \le 50 R(x)^{2q} \max_{u(J(x))} \big|Du^{-1}\big|\,, \tag{1.5}$$

  for all $x \in \Omega$, and if, in addition, u is orientation-preserving,

$$\det Dv(x) \ge \frac{1}{24} R(x)^{-2q} \det Du(x) \qquad \forall x \in \bigcup_{i \in I} \mathring{T}_i\,; \tag{1.6}$$

- *if u is L bi-Lipschitz, then v can be chosen to be $50L^{7/3}$ bi-Lipschitz, provided that the value of q in (1.5) and (1.6) is set to $1/3$.*

Notice the local estimate for Dv and Dv^{-1} in (1.5) but the pointwise estimate for $\det Dv$ in (1.6). For simplicity of notation, in (1.5) we have denoted $\max_{J(x)} |Du|$ what in reality is the maximum of $|Du|$ in $J(x) \cap \bigcup_{i \in I} \mathring{T}_i$, and analogously for $\max_{u(J(x))} |Du^{-1}|$; we will do so throughout the paper. We also remark that u is not assumed to be bounded or Sobolev, but still the estimate (1.4) holds.

2. Preliminary observations

In this section we introduce some notation and point out a couple of facts. Then, in the next section we will present our construction.

2.1. **Some notation and first observations.** Given a point $a \equiv (x_a, y_a) \in \mathbb{R}^2$, $r > 0$ and $\theta \in \mathbb{R}$ (or $\theta \in \mathbb{S}^1$), we will denote by $(r, \theta)_{P,a}$ the point in $\mathbb{R}^2$ whose polar coordinates with respect to a are r and θ, that is, the point $\big(x_a + r\cos\theta, y_a + r\sin\theta\big)$. The set $\mathbb{S}^1$ denotes the set of unit vectors in $\mathbb{R}^2$, and we will identify angles (measured in $\mathbb{R}$) with unit vectors (measured in $\mathbb{S}^1$). Given $a \in \mathbb{R}^2$ and $r > 0$, we denote by $B(a, r)$ the open ball of center a and radius r. For any matrix $M \in \mathbb{R}^{2\times 2}$, we consider the Frobenius norm $|M| = \sqrt{M_{11}^2 + M_{12}^2 + M_{21}^2 + M_{22}^2}$, and note that it is invariant under pre- and post-multiplication by rotations.

It is useful to observe a trivial formula to calculate the derivative of a function using polar coordinates: given two open sets $\Omega, \widetilde{\Omega} \subseteq \mathbb{R}^2$, two points $a \in \Omega$, $\tilde{a} \in \widetilde{\Omega}$, and a C^1 function $F : \Omega \to \widetilde{\Omega}$ that can be locally expressed as

$$F\big((\rho,\theta)_{P,a}\big) = \Big(F_1(\rho,\theta),\, F_2(\rho,\theta)\Big)_{P,\tilde{a}}\,,$$

for any $z \in \Omega \setminus \{a\}$ satisfying $F(z) \neq \tilde{a}$ one has that, in a suitable orthonormal system of coordinates,

$$DF(z) = \begin{pmatrix} \dfrac{\partial F_1}{\partial \rho} & \dfrac{1}{\rho}\,\dfrac{\partial F_1}{\partial \theta} \\ \tilde{\rho}\,\dfrac{\partial F_2}{\partial \rho} & \dfrac{\tilde{\rho}}{\rho}\,\dfrac{\partial F_2}{\partial \theta} \end{pmatrix}, \tag{2.1}$$

where we have called $\rho = |z-a|$ and $\tilde{\rho} = |F(z) - \tilde{a}|$. In particular, since the norm and the determinant are independent of the orthonormal basis,

$$\begin{aligned} |DF(z)|^2 &= \left(\frac{\partial F_1}{\partial \rho}\right)^2 + \left(\frac{1}{\rho}\,\frac{\partial F_1}{\partial \theta}\right)^2 + \left(\tilde{\rho}\,\frac{\partial F_2}{\partial \rho}\right)^2 + \left(\frac{\tilde{\rho}}{\rho}\,\frac{\partial F_2}{\partial \theta}\right)^2, \\ \det DF(z) &= \frac{\tilde{\rho}}{\rho}\left(\frac{\partial F_1}{\partial \rho}\,\frac{\partial F_2}{\partial \theta} - \frac{\partial F_1}{\partial \theta}\,\frac{\partial F_2}{\partial \rho}\right). \end{aligned} \tag{2.2}$$

A useful reduction in the proof of the theorem is as follows. An elementary geometric argument shows that, for any two triangles T_i, T_j sharing an edge whose interior is contained in Ω, any piecewise affine homeomorphism u must satisfy that the sign of $\det Du|_{\mathring{T}_i}$ equals the sign of $\det Du|_{\mathring{T}_j}$. Therefore, $\det Du > 0$ or $\det Du < 0$ almost everywhere in each connected component of Ω; in fact, more general and deeper results in this direction are known (see [15]). Hence without loss of generality, we can (and will, throughout the paper) assume u to satisfy $\det Du > 0$ almost everywhere.

Fix now a vertex a of the triangulation: up to a renumbering, we can assume that the triangles meeting at a are $T_1, \dots, T_N$, ordered in the counter-clockwise sense; we also call $T_{N+1} = T_1$ and $T_0 = T_N$. Choose $\delta = \delta_a$ much smaller than the inradius of each of the triangles $T_1, \dots, T_N$. Figure 1 depicts the image of $\partial B(a,\delta)$ under u. Observe that, since u is affine on each of the triangles $T_1, \dots, T_N$, then the image of $\partial B(a,\delta)$ is the union of N arcs of ellipses. Define now the functions $\tau_0 : \mathbb{S}^1 \to (0,\infty)$ and $\varphi_0 : \mathbb{S}^1 \to \mathbb{S}^1$ by the condition that the image of the point $\big(\delta,\theta\big)_{P,a}$ under u is $\big(\delta\tau_0(\theta), \varphi_0(\theta)\big)_{P,u(a)}$. Note that, keeping in mind (2.2), we have

$$|Du|^2 = \tau_0^2 + (\tau_0')^2 + (\varphi_0'\tau_0)^2\,, \qquad \det Du = \varphi_0'\tau_0^2\,. \tag{2.3}$$

Lemma 2.1. *For every $\theta \in \mathbb{S}^1$ such that $\big(\delta,\theta\big)_{P,a} \in \mathring{T}_i$, one has*

$$\ell_i \leq \tau_0(\theta) \leq L_i\,, \qquad \big|\tau_0'(\theta)\big| \leq L_i\,, \qquad \varphi_0'(\theta) = \frac{L_i\ell_i}{\tau_0(\theta)^2}\,, \qquad \frac{\ell_i}{L_i} \leq \varphi_0'(\theta) \leq \frac{L_i}{\ell_i}\,. \tag{2.4}$$

Moreover, $L_{\max}(a) \leq \max_{J(a)} |Du|$ and $\ell_{\min}(a)^{-1} \leq \max_{u(J(a))} |Du^{-1}|$.

Proof. Let $M \in \mathbb{R}^{2\times 2}$ be the matrix representation of Du in the triangle T_i. By the singular value decomposition, $M = RAQ$ for some rotation matrices R, Q and

$$A = \begin{pmatrix} L_i & 0 \\ 0 & \ell_i \end{pmatrix}.$$

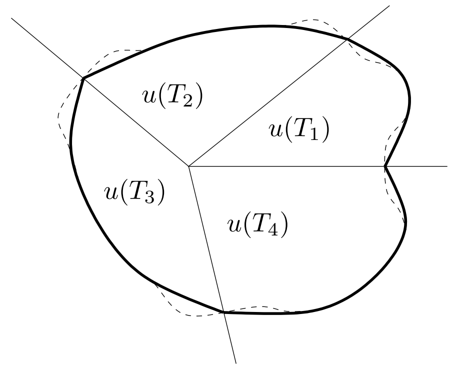

FIGURE 1. Image of $(\delta\tau_0,\varphi_0)_{P,u(a)}$ (solid lines) and of $(\delta\tau,\varphi)_{P,u(a)}$ (dotted lines).

Clearly, $\det Du = \det M = \det A = L_i\ell_i$, from which equality (2.3) yields $\varphi_0'(\theta)\tau_0(\theta)^2 = L_i\ell_i$. In addition,

$$\ell_i^2 + L_i^2 = |A|^2 = |M|^2 = |Du|^2 = \tau_0^2 + (\tau_0')^2 + (\varphi_0'\tau_0)^2 \geq \ell_i^2 + (\tau_0')^2,$$

so $\left|\tau_0'(\theta)\right| \leq L_i \leq |M|$, and, hence, $L_{\max}(a) \leq \max_{J(a)} |Du|$. Likewise, $\ell_i^{-1} \leq |M^{-1}|$ and $\ell_{\min}(a)^{-1} \leq \max_{u(J(a))} |Du^{-1}|$. On the other hand,

$$\tau_0(\theta) = \left|AQ(\cos\theta,\sin\theta)\right| \in \left[\min_{\nu\in\mathbb{S}^1} |A\nu|\,, \max_{\nu\in\mathbb{S}^1} |A\nu|\right] = \left[\ell_i\,, L_i\right].$$

This and all previous estimates yield (2.4). □

We note that a careful minimization analysis can show that in fact $\left|\tau_0'(\theta)\right| \leq L_i - \ell_i$.

It is clear by construction that the functions τ_0 and φ_0 are continuous and piecewise smooth, but not smooth. Since we aim to obtain a smooth approximation of u, we first have to modify the functions τ_0 and φ_0. In order to do so, let us start by calling $\theta_i \in \mathbb{S}^1$ the directions corresponding to the sides of the triangles T_i; in other words, for each $1 \leq i \leq N$, the two sides of the triangle T_i departing from a have directions θ_{i-1} and θ_i, in the counter-clockwise sense, again with respect to a. As before, call $\theta_{N+1} = \theta_1$ and $\theta_0 = \theta_N$. Let us also fix small positive constants $\lambda_i \ll |\theta_i - \theta_{i+1}|$. A simple mollification argument allows us to show the following approximation result.

Lemma 2.2. *There exist a C^∞ function $\tau : \mathbb{S}^1 \to (0,\infty)$ and a C^∞ increasing diffeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ such that*

i) $\tau(\theta_i) = \tau_0(\theta_i)$, $\varphi(\theta_i) = \varphi_0(\theta_i)$, $\tau'(\theta_i) = 0$ *and* $\varphi'(\theta_i) = \frac{\max\{L_i,L_{i+1}\}}{\tau_0(\theta_i)}$ *for each* $1 \leq i \leq N$ *;*

ii) $\tau \equiv \tau_0$ *and* $\varphi \equiv \varphi_0$ *in* $\left\{\theta \in \mathbb{S}^1 : |\theta - \theta_i| \geq \lambda_i \ \forall\, 1 \leq i \leq N\right\}$ *;*

iii) $\frac{1}{2}\varphi_0'(\theta) \leq \varphi'(\theta) \leq R_a$, $\frac{1}{2}\tau_0(\theta) \leq \tau(\theta) \leq 2\tau_0(\theta)$ *and* $\frac{\ell_{\min}}{2} \leq \tau(\theta)\varphi'(\theta) \leq 2L_{\max}$ *for all* $\theta \in \mathbb{S}^1$ *;*

iv) $\tau(\theta) \leq L_i$ *and* $|\tau'(\theta)| \leq 2L_i$ *for every* $\theta \in \mathbb{S}^1$ *for which* $\left(\delta,\theta\right)_{P,a} \in T_i$*, for some* $1 \leq i \leq N$ *.*

Proof. The result immediately follows by suitably modifying a regularization of τ_0 and φ_0, as soon as one checks that, for every $1 \leq i \leq N$, every $\theta \in \mathbb{S}^1$ and every $\vartheta \in \mathbb{S}^1$ such that $\left(\delta,\vartheta\right)_{P,a} \in T_i$,

$$\frac{\max\{L_i,L_{i+1}\}}{\tau_0(\theta_i)} \leq R_a\,, \quad \varphi_0'(\theta) \leq R_a\,, \quad \ell_{\min}(a) \leq \tau_0(\theta)\varphi_0'(\theta) \leq L_{\max}(a)\,, \quad \left|\tau_0'(\vartheta)\right| \leq L_i\,.$$

The first and the fourth property follow at once from (2.4). For the other two, we observe that, due again to (2.4), if $\theta \in \mathbb{S}^1$ is such that $\big(\delta,\theta\big)_{P,a} \in T_i$, then

$$\varphi_0'(\theta) = \frac{\ell_i L_i}{\tau_0(\theta)^2} \leq \frac{L_i}{\ell_i} \leq R_a \qquad \text{and} \qquad \tau_0(\theta)\varphi_0'(\theta) = \frac{\ell_i L_i}{\tau_0(\theta)} \in \big[\ell_i, L_i\big] \subseteq \big[\ell_{\min}(a), L_{\max}(a)\big]\,.$$

□

A sketch of the image of $(\tau_0,\varphi_0)_{P,u(a)}$ and of $(\tau,\varphi)_{P,u(a)}$ is shown in Figure 1.

2.2. **A useful auxiliary function.** Here we introduce an auxiliary smooth function ξ depending on six real parameters that will be extensively used in the sequel. Take any x_0, x_1, y_0, y_1, α, $\beta \in \mathbb{R}$ such that $x_0 < x_1$. We are going to define $\xi[x_0,\, x_1,\, y_0,\, y_1,\, \alpha,\, \beta] : [x_0, x_1] \to \mathbb{R}$ as a smooth function satisfying the requirements

$$\xi(x_0) = y_0\,, \qquad \xi(x_1) = y_1\,, \qquad \xi'(x_0) = \alpha\,, \qquad \xi'(x_1) = \beta\,. \tag{2.5}$$

When there is no risk of confusion, the value of the function at $t \in [x_0, x_1]$ will be simply denoted as $\xi(t)$, instead of $\xi[x_0,\, x_1,\, y_0,\, y_1,\, \alpha,\, \beta]\,(t)$. It is, of course, very simple to define such a function once the six parameters are fixed; however, we need a definition that also depends smoothly on the parameters. To give our definition, we fix a big constant $K \geq 6$; we will use the function ξ several times in the sequel, and will be allowed to take a different value of K each time: often, the choice $K = 6$ will suffice, but sometimes we will need to select a suitably large K. We thus consider the continuous function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned}
f(x_0) &= y_0\,, & f\Big(x_0 + \frac{x_1 - x_0}{K}\Big) &= y_0 + \alpha\,\frac{x_1 - x_0}{K}\,,\\
f\Big(x_0 + \frac{x_1 - x_0}{3}\Big) &= y_0 + \frac{y_1 - y_0}{3}\,, & f\Big(x_1 - \frac{x_1 - x_0}{3}\Big) &= y_1 - \frac{y_1 - y_0}{3}\,,\\
f\Big(x_1 - \frac{x_1 - x_0}{K}\Big) &= y_1 - \beta\,\frac{x_1 - x_0}{K}\,, & f(x_1) &= y_1\,,
\end{aligned}$$

and is affine on each of the intervals

$$\Big(-\infty, x_0 + \frac{x_1 - x_0}{K}\Big), \quad \Big(x_0 + \frac{x_1 - x_0}{K}, x_0 + \frac{x_1 - x_0}{3}\Big), \quad \Big(x_0 + \frac{x_1 - x_0}{3}, x_1 - \frac{x_1 - x_0}{3}\Big),$$
$$\Big(x_1 - \frac{x_1 - x_0}{3}, x_1 - \frac{x_1 - x_0}{K}\Big), \qquad \Big(x_1 - \frac{x_1 - x_0}{K}, \infty\Big).$$

Denoting then by ρ_ε the standard mollifier supported in $[-\varepsilon, \varepsilon]$, we set $\xi := \rho_{\frac{x_1 - x_0}{2K}} * f$. Figure 2 shows the graph of ξ. It is immediate to check that the function ξ satisfies the requirements (2.5). In fact, since the convolution of a linear function with the standard mollifier is the linear function itself, then the functions ξ and f coincide except close to the junction points of the piecewise affine function f.

We will use the notation $\xi_{,i}$ for the partial derivative of ξ with respect to the i-th variable, $i = 1, \ldots, 6$, and ξ' for the partial derivative of ξ with respect to the t variable. A quick direct calculation yields the following properties of ξ.

Lemma 2.3. *Let x_0, x_1, y_0, y_1, α, $\beta \in \mathbb{R}$ be such that $x_0 < x_1$. The following properties hold:*

i) $\big|\xi'\big| \leq 2\dfrac{\big|y_1 - y_0\big|}{x_1 - x_0} + \max\big\{\big|\alpha\big|,\, \big|\beta\big|\big\}\,.$

ii) If α, $\beta > 0$ and $\dfrac{\max\big\{\alpha,\, \beta\big\}(x_1 - x_0)}{K} \leq \dfrac{y_1 - y_0}{6}$, *then* $\xi' \geq \min\Big\{\alpha,\, \beta,\, \dfrac{y_1 - y_0}{2(x_1 - x_0)}\Big\}\,.$

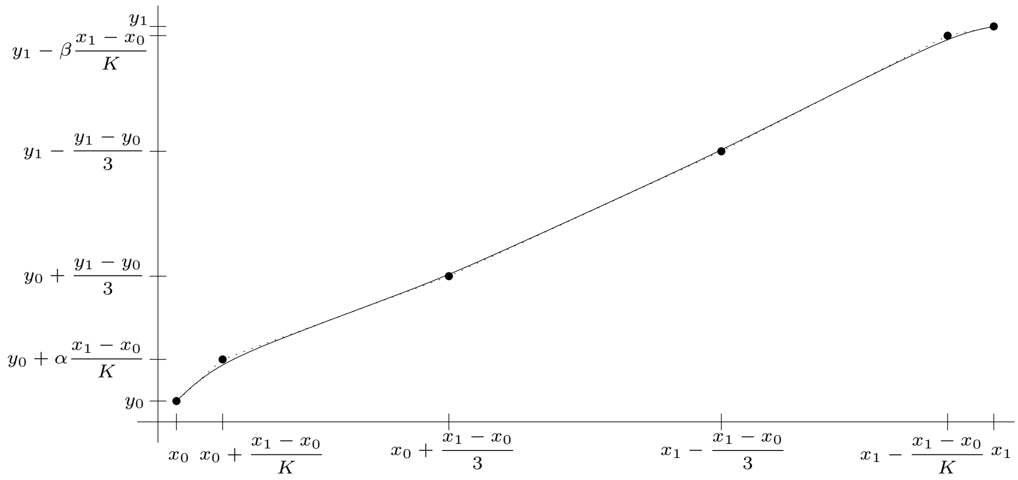

FIGURE 2. Graph of ξ (solid lines) and f (dotted lines).

iii) $0 \le \xi_{,3} \le 1$, $0 \le \xi_{,4} \le 1$ *and* $\xi_{,3} + \xi_{,4} = 1$. *Moreover,* $\xi_{,4}(t) \le \dfrac{t - x_0}{x_1 - x_0}$.

iv) $0 \le \xi_{,5} \le \dfrac{x_1 - x_0}{K}$ *and* $0 \ge \xi_{,6} \ge -\dfrac{x_1 - x_0}{K}$.

v) $|\xi_{,1}|, |\xi_{,2}| \le 2\left(|\alpha| + |\beta| + 2\,\dfrac{|y_1 - y_0|}{x_1 - x_0}\right)$.

3. CONSTRUCTION

This is the main section of the paper, and is devoted to present the construction of the diffeomorphism v approximating the piecewise affine homeomorphism u.

We first note that the following assumptions on the triangulation can be made:

(T1) $T_i \subset \Omega$ for each $i \in I$;
(T2) $\operatorname{diam} T_i \le \bar\varepsilon$ and $\operatorname{diam} u(T_i) \le \bar\varepsilon$ for all $i \in I$, where $\bar\varepsilon > 0$ is a number fixed beforehand, depending on the function u and on the number ε of Theorem A.
(T3) For any sequence $\{i_n\}_{n\in\mathbb{N}}$ of indices in I such that $\operatorname{dist}(T_{i_n}, \partial\Omega) \to 0$ as $n \to \infty$, we have $\operatorname{diam} T_{i_n} \to 0$ and $\operatorname{diam} u(T_{i_n}) \to 0$ as $n \to \infty$.

Condition (T1) tells us that every triangle of the triangulation is compactly contained in Ω; with this assumption one does not need to bother with the geometry of $\partial\Omega$ or the boundary values of u. Condition (T2) tells us that the triangles and their images under u can be made as small as we wish; this condition is crucial in order to ensure the approximation property (1.5). Condition (T3) tells us that the triangles and their images under u get smaller and smaller as they approach the boundary; this condition will ensure that if u is continuous up to the boundary then so will v be. There are several elementary ways of refining a given triangulation to obtain another satisfying conditions (T1)–(T3), while keeping the property of being locally finite. Of course, any triangulation satisfying (T1) or (T3) is countably infinite.

To perform our construction, we fix a generic vertex a of the triangulation, and focus on the triangles having a as one of their vertices. The union of these triangles will be subdivided into four zones, called Z_1, Z_2, Z_3 and Z_4, and we will define the approximating function v separately in the four zones; we will call $\widetilde{Z}_i = v(Z_i)$ for $i = 1, \ldots, 4$, and $\tilde a = u(a)$. Of course, we will have to check not only that $v|_{Z_i}$ is smooth for $i = 1, \ldots, 4$, but also that v remains smooth around the intersections of the adjacent zones. In truth, the zones Z_i depend of a, and should be called $Z_i(a)$, but since in most of the construction the vertex a is fixed, they will be simply called Z_i.

We fix a parameter $q \in [0,1]$, corresponding to that of the statement of Theorem A.

3.1. **Construction in zone Z_1.** Let us start by recalling from Section 2.1 that $\delta = \delta_a$ has been chosen as a length much smaller than the inradius of each of the finitely many triangles meeting at a. For simplicity, the numbers $\ell_{\min}(a)$ and $L_{\max}(a)$ of (1.3) will just be called $\ell_{\min}$ and $L_{\max}$, respectively. Let us now fix another constant $\eta = \eta_a$, much smaller than $\ell_{\min}/L_{\max} \leq R_a^{-1}$. We call then $Z_1 = B(a, \eta\delta)$ and $\widetilde{Z}_1 = B(\tilde{a}, \eta\delta R_a^{-q} L_{\max})$. We define $v : Z_1 \to \widetilde{Z}_1$, using polar coordinates as in Section 2.1, as

$$v\big((\rho,\theta)_{P,a}\big) := \Big(R_a^{-q} L_{\max}\,\rho, \xi[0,\eta\delta,\theta,\varphi(\theta),0,0](\rho)\Big)_{P,\tilde{a}}\,.$$

The function $\xi[0,\eta\delta,\theta,\varphi(\theta),0,0]$ in this section will be just called ξ. The following property holds.

Lemma 3.1. *The function $v : Z_1 \to \widetilde{Z}_1$ is a smooth bijection, and for every point $x \in Z_1$ one has*

$$\big|Dv\big| \leq 13 R_a^{1-q} \max_{J(a)} |Du|\,, \quad |Dv^{-1}| \leq 26 R_a^q \max_{u(J(a))} |Du^{-1}|\,, \quad \det Dv(x) \geq \frac{\det Du(x)}{2R_a^{2q}}\,. \tag{3.1}$$

where the natural neighborhood $J(a)$ has been defined in (1.1).

Proof. The smoothness of v follows from the definition of ξ; indeed, v is clearly smooth outside a, and in a neigborhood of a, the map v coincides with the map $(\rho,\theta)_{P,a} \mapsto \big(R_a^{-q} L_{\max}\,\rho, \theta\big)_{P,\tilde{a}}$, which is smooth. Hence, to conclude that v is a bijection between Z_1 and $\widetilde{Z}_1$ it suffices to prove that, for each $\rho \in (0,\eta\delta)$, the smooth function $F(\theta) := \xi[0,\eta\delta,\theta,\varphi(\theta),0,0](\rho)$ is a bijection from $\mathbb{S}^1$ to itself. In fact, one has $F' = \xi_{,3} + \xi_{,4}\varphi'$ by definition, $\varphi' > 0$ and $\int_0^{2\pi} \varphi' = 2\pi$ by Lemma 2.2, and $0 \leq \xi_{,3},\, \xi_{,4} \leq 1$, $\xi_{,3} + \xi_{,4} = 1$ by (iii) of Lemma 2.3. This yields that $F' > 0$ and $\int_0^{2\pi} F' < 4\pi$, thus it must be $\int_0^{2\pi} F' = 2\pi$ and then the first part of the thesis is concluded.

Let us now pass to consider Dv and Dv^{-1}: first of all, recalling formula (2.1), we calculate

$$Dv = R_a^{-q} L_{\max} \begin{pmatrix} 1 & 0 \\ \rho\xi' & \xi_{,3} + \xi_{,4}\varphi' \end{pmatrix} \tag{3.2}$$

in a suitable basis. Concerning ξ', by (i) of Lemma 2.3 we know that

$$\big|\xi'\big| \leq 2\,\frac{|\varphi(\theta) - \theta|}{\eta\delta} \leq \frac{4\pi}{\eta\delta}\,,$$

and then $\big|\rho\xi'\big| \leq 4\pi$. Moreover, by (iii) of Lemma 2.3, Lemma 2.2 and (2.4) we get

$$\xi_{,3} + \xi_{,4}\varphi' \leq \max\{1,\varphi'\} \leq R_a\,, \qquad\qquad \xi_{,3} + \xi_{,4}\varphi' \geq \min\{1,\varphi'\} \geq \frac{\ell_i}{2L_i}\,. \tag{3.3}$$

Using in addition Lemma 2.1, (2.3) and (2.4), we get that for any $x \in \mathring{T}_i$,

$$|Dv| \leq R_a^{-q} L_{\max} \Big(1 + 16\pi^2 + R_a^2\Big)^{1/2} \leq 13 R_a^{1-q} L_{\max} \leq 13 R_a^{1-q} \max_{J(a)} |Du|\,, \tag{3.4}$$

$$\det Dv(x) = R_a^{-2q} L_{\max}^2 \big(\xi_{,3} + \xi_{,4}\varphi'\big) \geq R_a^{-2q} L_{\max}^2 \,\frac{\ell_i}{2L_i} \geq \frac{R_a^{-2q}}{2}\,\ell_i L_i = \frac{R_a^{-2q}}{2}\,\det Du(x)\,. \tag{3.5}$$

Finally, thanks to (3.2) we have

$$Dv^{-1}\Big(v\big(\rho,\theta\big)_{P,a}\Big)=\frac{R_a^q}{L_{\max}}\begin{pmatrix}1 & 0\\ -\frac{\rho\xi'}{\xi_{,3}+\xi_{,4}\varphi'} & \frac{1}{\xi_{,3}+\xi_{,4}\varphi'}\end{pmatrix}$$

in a suitable basis, so, again by (3.3) and Lemma 2.1,

$$\begin{aligned}|Dv^{-1}|&\leq\frac{R_a^q}{L_{\max}}\bigg(1+\Big(\frac{2L_i}{\ell_i}\Big)^2\Big(1+16\pi^2\Big)\bigg)^{1/2}\leq 26\,\frac{R_a^q}{L_{\max}}\,\frac{L_i}{\ell_i}\leq\frac{26R_a^q}{\ell_i}\leq\frac{26R_a^q}{\ell_{\min}}\\&\leq 26R_a^q\max_{u(J(a))}|Du^{-1}|.\end{aligned}\tag{3.6}$$

Putting together (3.4), (3.5) and (3.6) we find (3.1), thus the thesis is concluded. □

Remark 3.2. *For future reference, we observe that, by the definition of the function ξ, for $\rho\leq\eta\delta$ close to $\eta\delta$ one has*

$$v\big((\rho,\theta)_{P,a}\big)=\Big(R_a^{-q}L_{\max}\,\rho,\varphi(\theta)\Big)_{P,\tilde a}.$$

3.2. **Construction in zone** Z_2**.** Let us now pass to the zone Z_2, defined as the annulus $B(a,\delta)\setminus B(a,\eta\delta)$, while the zone $\widetilde Z_2$ is given by

$$\widetilde Z_2:=\Big\{(\rho,\theta)_{P,\tilde a}:\eta\delta R_a^{-q}L_{\max}\leq\rho<\delta\tau(\varphi^{-1}(\theta))\Big\}.$$

Notice that, since $\tau\geq\ell_{\min}/2$ by Lemma 2.2 and (2.4), while $\eta\ll\ell_{\min}/L_{\max}$, the set $\widetilde Z_2$ has a non-empty intersection with every half-line $\big\{(\rho,\theta)_{P,\tilde a}:\ \rho>0\big\}$ for $\theta\in\mathbb{S}^1$. The union $\widetilde Z_1\cup\widetilde Z_2$ is the region enclosed by the dotted line of Figure 1. We can now define $v:Z_2\to\widetilde Z_2$ as

$$v\big((\rho,\theta)_{P,a}\big):=\Big(\xi[\eta\delta,\delta,\eta\delta R_a^{-q}L_{\max},\tau(\theta)\delta,R_a^{-q}L_{\max},\tau(\theta)](\rho)\ ,\ \varphi(\theta)\Big)_{P,\tilde a}.$$

In this section, the function $\xi[\eta\delta,\delta,\eta\delta R_a^{-q}L_{\max},\tau(\theta)\delta,R_a^{-q}L_{\max},\tau(\theta)]$ will be just called ξ. As in Remark 3.2, we immediately notice the following formulas around ∂Z_2.

Remark 3.3. *For $\rho\geq\eta\delta$ close to $\eta\delta$ one has*

$$v\big((\rho,\theta)_{P,a}\big)=\Big(R_a^{-q}L_{\max}\,\rho,\varphi(\theta)\Big)_{P,\tilde a},$$

while for $\rho\leq\delta$ close to δ one has

$$v\big((\rho,\theta)_{P,a}\big)=\Big(\rho\,\tau(\theta),\varphi(\theta)\Big)_{P,\tilde a}.$$

As in Lemma 3.1, we show now the estimates for Dv in zone Z_2.

Lemma 3.4. *The function $v:Z_2\to\widetilde Z_2$ is a smooth bijection, smoothly matching with the function v defined in zone Z_1, and for every point $x\in Z_2$ one has*

$$\big|Dv\big|\leq 5R_a^{1-q}\max_{J(a)}|Du|\,,\quad|Dv^{-1}|\leq 50R_a^{2q}\max_{u(J(a))}|Du^{-1}|\,,\quad\det Dv(x)\geq\frac{\det Du(x)}{24R_a^{2q}}.\tag{3.7}$$

Proof. The smoothness of v comes from that of ξ and φ, while the smooth matching around ∂Z_1 follows by comparing the formulas of Remarks 3.2 and 3.3. Moreover, by the definition of v and since the image of ∂Z_2 is $\partial\widetilde Z_2$, and recalling also that φ is a bijection from $\mathbb{S}^1$ onto itself, checking that v is a bijection from Z_2 onto $\widetilde Z_2$ reduces to checking that $\xi'>0$ on Z_2.

Let us now use (2.1) to calculate

$$Dv = \begin{pmatrix} \xi' & \frac{1}{\rho}\,(\delta\xi_{,4}+\xi_{,6})\tau'(\theta) \\ 0 & \frac{\xi}{\rho}\,\varphi'(\theta) \end{pmatrix}, \qquad Dv^{-1} = \begin{pmatrix} \frac{1}{\xi'} & -\frac{(\delta\xi_{,4}+\xi_{,6})\tau'(\theta)}{\xi'\xi\varphi'(\theta)} \\ 0 & \frac{\rho}{\xi\varphi'(\theta)} \end{pmatrix} \tag{3.8}$$

in a suitable basis. Hence, in order to estimate $|Dv|$, $|Dv^{-1}|$ and $\det Dv(x)$, we need bounds for ξ', ξ/ρ, $\xi_{,4}$ and $\xi_{,6}$. First of all, property (ii) of Lemma 2.3 ensures that

$$\xi' \geq \min\left\{R_a^{-q}L_{\max},\, \tau(\theta),\, \frac{\tau(\theta)-\eta R_a^{-q}L_{\max}}{2(1-\eta)}\right\} \tag{3.9}$$

as soon as

$$\frac{\max\left\{R_a^{-q}L_{\max},\, \tau(\theta)\right\}(1-\eta)}{K} \leq \frac{\tau(\theta)-\eta R_a^{-q}L_{\max}}{6}\,.$$

In fact, the latter inequality holds true when choosing $K \geq 48L_{\max}/\ell_{\min}$, and in turn this is clearly an admissible choice (notice that the constant K used within Z_2 can be chosen independently of the constants used in other zones, since by Remark 3.3 the value of v close to ∂Z_2 does not depend on K). As a consequence, we have the validity of (3.9), which easily implies

$$\frac{\ell_{\min}}{6R_a^q} \leq \frac{\tau(\theta)}{3R_a^q} \leq \xi' \leq 4L_{\max} \tag{3.10}$$

also recalling again (2.4), Lemma 2.2, (i) of Lemma 2.3 and the fact that $\eta \ll \ell_{\min}/L_{\max}$. In particular, (3.10) implies that $\xi' > 0$, hence, as noticed above, it follows that $v : Z_2 \to \widetilde{Z}_2$ is a bijection.

Passing to ξ/ρ, we observe that the function f used in Section 2.2 in the construction of ξ satisfies

$$\min\left\{L_{\max}R_a^{-q},\, \tau(\theta)\right\} \leq \frac{f}{\rho} \leq \max\left\{L_{\max}R_a^{-q},\, \tau(\theta)\right\};$$

therefore, by the properties of the convolution, and using Lemma 2.2 and (2.4) as well,

$$\tau(\theta)R_a^{-q} \leq \min\left\{L_{\max}R_a^{-q},\, \tau(\theta)\right\} \leq \frac{\xi}{\rho} \leq \max\left\{L_{\max}R_a^{-q},\, \tau(\theta)\right\}.$$

These inequalities, together with the following ones

$$\frac{\ell_{\min}}{2} \leq \tau(\theta)\varphi'(\theta) \leq 2L_{\max} \leq 2L_{\max}R_a^{1-q}\,, \qquad L_{\max}R_a^{-q}\varphi'(\theta) \leq L_{\max}R_a^{1-q}\,,$$

which are consequences of (iii) of Lemma 2.2, imply that

$$\frac{\ell_{\min}}{2}\,R_a^{-q} \leq \tau(\theta)R_a^{-q}\varphi'(\theta) \leq \frac{\xi}{\rho}\,\varphi'(\theta) \leq 2L_{\max}R_a^{1-q}\,. \tag{3.11}$$

In addition, applying (iii) and (iv) of Lemma 2.3 one concludes

$$\Big|\delta\xi_{,4}(\rho)+\xi_{,6}(\rho)\Big| \leq \rho\,, \tag{3.12}$$

as can be seen by considering separately the cases $\delta\xi_{,4}+\xi_{,6} \geq 0$ and $\delta\xi_{,4}+\xi_{,6} \leq 0$. Finally, from (3.12), (3.11), Lemma 2.2, (3.10) and (2.4) one can easily obtain

$$\begin{aligned}\left|\frac{(\delta\xi_{,4}+\xi_{,6})\tau'(\theta)}{\xi'\xi\varphi'(\theta)}\right| &\leq \frac{\rho|\tau'(\theta)|}{\xi'\xi\varphi'(\theta)} \leq \frac{|\tau'(\theta)|R_a^q}{\xi'\varphi'(\theta)\tau(\theta)} \leq \frac{2L_iR_a^q}{\xi'\varphi'(\theta)\tau(\theta)} \leq \frac{6L_iR_a^{2q}}{\varphi'(\theta)\tau^2(\theta)} \leq \frac{48L_iR_a^{2q}}{\varphi_0'(\theta)\tau_0^2(\theta)} \\ &= \frac{48R_a^{2q}}{\ell_i} \leq \frac{48R_a^{2q}}{\ell_{\min}}\,.\end{aligned} \tag{3.13}$$

Keeping in mind (3.8), and putting together (3.10), (3.11), (3.12), (3.13), (2.3), and Lemmas 2.1 and 2.2, we obtain then

$$|Dv| \leq \sqrt{16L_{\max}^2 + \tau'(\theta)^2 + \Big(2L_{\max}R_a^{1-q}\Big)^2} \leq 5L_{\max}R_a^{1-q} \leq 5R_a^{1-q}\max_{J(a)}|Du|\,,$$

$$|Dv^{-1}| \leq \sqrt{\left(\frac{6R_a^q}{\ell_{\min}}\right)^2 + \left(\frac{48R_a^{2q}}{\ell_{\min}}\right)^2 + \left(\frac{2R_a^q}{\ell_{\min}}\right)^2} \leq 50\,\frac{R_a^{2q}}{\ell_{\min}} \leq 50\,R_a^{2q}\max_{u(J(a))}|Du^{-1}|\,,$$

$$\det Dv(x) = \frac{\xi\xi'\varphi'(\theta)}{\rho} \geq \frac{\tau^2(\theta)\varphi'(\theta)}{3R_a^{2q}} \geq \frac{\tau_0^2(\theta)\varphi_0'(\theta)}{24R_a^{2q}} = \frac{L_i\ell_i}{24R_a^{2q}} = \frac{\det Du(x)}{24R_a^{2q}}\,.$$

This yields (3.7) and then concludes the thesis. □

Observe that, thanks to Remark 3.3, the value of v around $\partial B(a,\delta)$ does not depend on the choice of q. In fact, the definition of v in the third and fourth zone will not depend on q either.

3.3. **Construction in zone Z_3.** While in zones Z_1 and Z_2 (and in their images $\widetilde{Z}_1$ and $\widetilde{Z}_2$) we have worked close to the point a (respectively, $\tilde{a}$), we need now to deal with the remaining part of the triangles having a as a vertex. We aim to further subdivide this region into two parts: a part Z_3 made by narrow neigborhoods of the sides of the triangles T_i, and the remaining part Z_4. The zone Z_4, which will be much bigger than the other three, is made by N disjoint subzones, each of them inside a triangle T_i. Our strategy will be to define $v \equiv u$ in Z_4, and this will be smooth because in the part inside T_i the function u is affine, and these parts have been disconnected from each other by zone Z_3. Instead, in Z_3 we need to be very careful in order to define a function v correctly matching the definition in zones Z_2 and Z_4. Figure 3 shows a rough picture of how zones Z_3 and Z_4 look in a triangle T_i of vertices a, b and c. The figure is only intended to express the overall idea, since, as we will see during the construction (see in particular Figure 6), the boundary of Z_4 will in fact have curves where the picture shows straight lines.

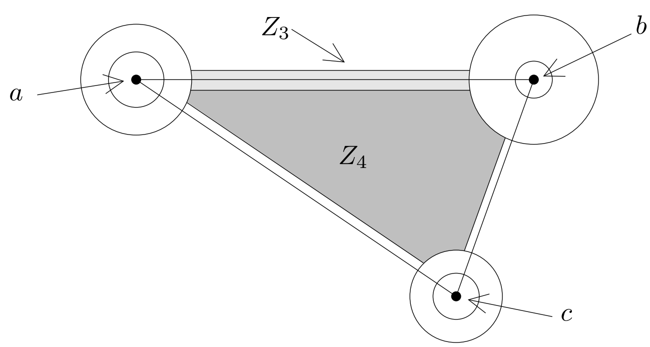

FIGURE 3. A rough idea of zones Z_3 and Z_4 for a triangle T_i.

Since zone Z_3 will be the union of N disjoint narrow neigborhoods of the sides of the triangulation departing from a, and the construction in each zone is completely independent of the rest, let us focus on a single side, say ab. We can assume that our construction has been already done in the zones Z_1 and Z_2 corresponding to a, and also in the analogous zones corresponding to b. To distinguish the functions and constants involved, we will use the subscripts a and b, thus writing δ_a, δ_b, η_a, η_b, $\lambda_{i,a}$, $\lambda_{i,b}$ and so on. To fix ideas, let us assume that the sides ab and $\tilde{a}\tilde{b}$ are horizontal, as in Figure 4, and let us fix $1 \leq i \leq N_a$ and $1 \leq j \leq N_b$ so that the lower (resp., upper) triangle having ab as a side is $T_{i,a} = T_{j+1,b}$ (resp., $T_{i+1,a} = T_{j,b}$): notice that asking ab to be horizontal means $\theta_{i,a} = 0$ and $\theta_{j,b} = \pi$, while asking $\tilde{a}\tilde{b}$ to be horizontal means $\varphi_a(0) = 0$ and $\varphi_b(\pi) = \pi$. Let us fix now a small constant $h = h(a, b) \ll \min\{\delta_a, \delta_b\}$; by choosing the constants λ_i's small enough (see Lemma 2.2 and the paragraph before, and notice that the value of each constant λ_i can be chosen independently of the others), it is admissible to assume that

$$\delta_a \sin\lambda_{i,a} < h\,, \qquad \delta_a \sin\lambda_{i+1,a} < h\,, \qquad \delta_b \sin\lambda_{j,b} < h\,, \qquad \delta_b \sin\lambda_{j+1,b} < h\,. \tag{3.14}$$

Let us then consider the circles centered at a and b, with radii δ_a and δ_b respectively; these two circles are well separated thanks to the choice of δ_a and δ_b. Let us call, as in Figure 4 (left), P, Q, R and S the points in the two circles at a distance h from the segment ab. Let us also call $\widetilde{P}$, $\widetilde{Q}$, $\widetilde{R}$ and $\widetilde{S}$ the images of P, Q, R and S under the map u. Observe that, since u is affine in both $T_{i,a} = T_{j+1,b}$ and $T_{i+1,a} = T_{j,b}$, and since the segment ab is sent onto $\tilde{a}\tilde{b}$ by both affine maps, then the distance of $\widetilde{P}$ and $\widetilde{Q}$ from the segment $\tilde{a}\tilde{b}$ is the same, say h^+, and analogously the distance of $\widetilde{R}$ and $\widetilde{S}$ is the same, say h^-. Consider now Figure 4 (right): the image under u of the arc of the circle centered at a and with radius δ_a (given in polar coordinates by $\big(\delta_a\tau_{0,a}, \varphi_{0,a}\big)_{P,\tilde{a}}$ by definition) close to ab is the union of two arcs of ellipse, which meet in a continuous but not differentiable way on $\tilde{a}\tilde{b}$. The image of the same circle under v, according to the construction of Section 3.2 and thanks to Remark 3.3, is the smooth curve $\big(\delta_a\tau_a, \varphi_a\big)_{P,\tilde{a}}$, shown with a dotted curve in the figure. Thanks to (3.14), the smooth curve and the two arcs of ellipse differ only in the regions between $\widetilde{P}$ and $\widetilde{R}$, and between $\widetilde{Q}$ and $\widetilde{S}$.

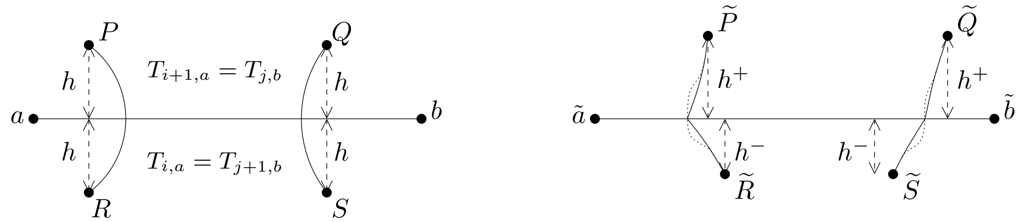

FIGURE 4. First step in the construction of zones Z_3 and $\widetilde{Z}_3$.

We make now a fundamental observation: as depicted in Figure 5, a simple trigonometric argument shows that the quantity $\tau_a(\theta_{i,a})\varphi_a'(\theta_{i,a})$ represents the (vertical) speed at which the curve $\theta \mapsto v\big((\delta_a, \theta)_{P,a}\big)$ departs from the segment ab at the value $\theta = \theta_{i,a}$; note that this speed is vertical by the choice $\tau_a'(\theta_{i,a}) = 0$ made in (i) of Lemma 2.2. Analogously, the speed of the curve $\theta \mapsto v\big((\delta_b, \theta)_{P,b}\big)$ is $\tau_b(\theta_{j,b})\varphi_b'(\theta_{j,b})$. In turn, these two speeds coincide, since by (i) of Lemma 2.2 one has

$$\tau_a(\theta_{i,a})\varphi_a'(\theta_{i,a}) = \max\big\{L_{i,a},\, L_{i+1,a}\big\} = \max\big\{L_{j+1,b},\, L_{j,b}\big\} = \tau_b(\theta_{j,b})\varphi_b'(\theta_{j,b})\,. \tag{3.15}$$

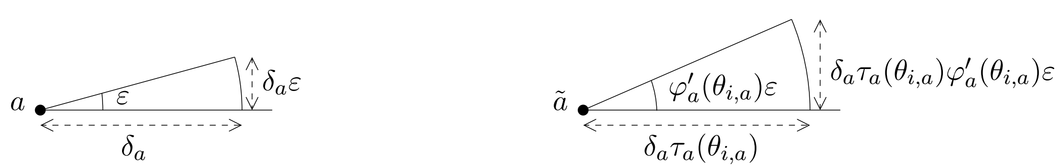

FIGURE 5. Departing speed of the curve $\theta \mapsto v(\delta, \theta)$.

For any $-h \le t \le h$, let us call $P(t)$ the point on $\partial B(a, \delta_a)$ having height t with respect to ab, so that $P = P(h)$ and $R = P(-h)$. We call $Q(t)$ the analogous point on $\partial B(b, \delta_b)$, so that $Q = Q(h)$ and $S = Q(-h)$. Moreover, set $\widetilde{P}(t) := v\big(P(t)\big)$ and $\widetilde{Q}(t) := v\big(Q(t)\big)$; note that v is already defined because the points $P(t)$ and $Q(t)$ belong to the boundaries of the zones Z_2 corresponding to a and b, respectively. We make the following claim: in the construction of τ and φ done in Section 2.1, one can impose the additional requirement that for every $t \in (-h, h)$, the heights of $\widetilde{P}(t)$ and $\widetilde{Q}(t)$ with respect to $\tilde{a}\tilde{b}$ are the same, say $\tilde{t} = \tilde{t}(t)$. Formally, this means that, calling θ_a, θ_b the smooth functions defined in a neigborhood of $[-h, h]$ such that

$$\delta_a \sin\big(\theta_a(t)\big) = t = \delta_b \sin\big(\theta_b(t)\big)\,, \tag{3.16}$$

and $\theta_a(0) = 0$, $\theta_b(0) = \pi$, we are requiring

$$\delta_a\, \tau_a\big(\theta_a(t)\big) \sin\big(\varphi_a(\theta_a(t))\big) = \tilde{t} = \delta_b\, \tau_b\big(\theta_b(t)\big) \sin\big(\varphi_b(\theta_b(t))\big)\,. \tag{3.17}$$

This requirement is indeed admissible, because: (a) equality (3.17) is trivially true at $t = 0$; (b) equality (3.17) is true around $t = \pm h$, since, as noticed above, $u = v$ around P, Q, R and S; and (c) the necessary condition coming from differentiating (3.17) at $t = 0$, namely,

$$\delta_a\, \tau_a(0)\, \varphi_a'(0)\, \theta_a'(0) = \delta_b\, \tau_b(\pi)\, \varphi_b'(\pi)\, \theta_b'(0)\,,$$

is true by (3.15) and by the equalities $\delta_a\theta_a'(0) = \delta_b\theta_b'(0) = 1$, which are consequences of (3.16). For future reference, we note that differentiating (3.17) and (3.16), for every $-h \le t \le h$ one has

$$\begin{aligned}\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t} &= \delta_a\, \theta_a'\Big(\tau_a'(\theta_a)\sin(\varphi_a(\theta_a)) + \tau_a(\theta_a)\cos(\varphi_a(\theta_a))\varphi_a'(\theta_a)\Big)\\ &= \frac{1}{\cos\theta_a}\Big(\tau_a'(\theta_a)\sin(\varphi_a(\theta_a)) + \tau_a(\theta_a)\cos(\varphi_a(\theta_a))\varphi_a'(\theta_a)\Big) \approx \tau_a(\theta_a)\,\varphi_a'(\theta_a)\,,\end{aligned} \tag{3.18}$$

where we have used that the angles $\varphi_a(\theta_a)$ can be made arbitrarily small by choosing a small $h(a, b)$. Using the estimates of Lemmas 2.1 and 2.2, we obtain in particular that

$$\frac{\ell_{i,a}}{4} \le \tau_a(\theta_a)\,\varphi_a'(\theta_a) \le 2L_{\max}(a)\,, \qquad\qquad \frac{\ell_{i,a}}{5} \le \frac{\mathrm{d}\tilde{t}}{\mathrm{d}t} \le 3L_{\max}(a)\,. \tag{3.19}$$

As a consequence, the map $t \mapsto \tilde{t}$ is a diffeomorphism from $(-h, h)$ onto $(-h^-, h^+)$.

For simplicity of notation, we assume that both a and $\tilde{a}$ coincide with the origin of $\mathbb{R}^2$. In order to define zone Z_3, we need a couple of definitions more: first, for every $-h \le t \le h$ we call

$$\chi_0(t) := \delta_a \cos\theta_a(t)\,, \qquad\qquad \chi_1(t) := \overline{ab} + \delta_b \cos\theta_b(t)\,, \tag{3.20}$$

where by $\overline{ab}$ we denote the length of the segment ab. Observe that by (3.16) one has

$$\chi_0(t) = \sqrt{\delta_a^2 - t^2}\,, \quad \chi_1(t) = \overline{ab} - \sqrt{\delta_b^2 - t^2}\,, \quad P(t) \equiv \big(\chi_0(t), t\big)\,, \quad Q(t) \equiv \big(\chi_1(t), t\big)\,. \tag{3.21}$$

Notice that, in writing the position of $P(t)$ and $Q(t)$, we are using Cartesian coordinates, and no more polar coordinates. Define also the rectangoloid

$$\Gamma := \Big\{(\sigma, t) \in \mathbb{R}^2 :\ t \in (-h, h)\,, \sigma \in \big(\chi_0(t), \chi_1(t)\big)\Big\}\,,$$

which has two long horizontal straight sides and two small circular lateral sides, and the map

$$\gamma : \Gamma \to \mathbb{R}^2\,, \qquad \gamma(\sigma, t) := \Big(\sigma, \xi[\chi_0(t), \chi_1(t), t, t, \tan\theta_a(t), \tan\theta_b(t)](\sigma)\Big)\,.$$

We will denote $\bar{\xi} = \xi[\chi_0(t), \chi_1(t), t, t, \tan\theta_a(t), \tan\theta_b(t)]$, since we are using different functions ξ in this section. Observe that, for every $-h < t < h$, the smooth curve $\gamma(\cdot, t)$ (shown in Figure 6, left) connects $P(t)$ with $Q(t)$. Moreover, the first and the last part of the curve smoothly connect with the segments $aP(t)$ and $Q(t)b$, respectively, and most of the curve lies at height t with respect to ab. We can now show the following property.

Lemma 3.5. *The map $\gamma : \Gamma \to \mathbb{R}^2$ is smooth and injective. Moreover,*

$$\big|D\gamma\big| \approx 1\,, \qquad\qquad \big|D\gamma^{-1}\big| \approx 1\,, \qquad\qquad \det D\gamma \approx 1\,. \tag{3.22}$$

Proof. By construction, to show the injectivity of γ it is enough to prove that $\dfrac{\mathrm{d}\overline{\xi}}{\mathrm{d}t} > 0$. One has

$$\frac{\mathrm{d}\overline{\xi}}{\mathrm{d}t} = \overline{\xi}_{,1}\chi_0' + \overline{\xi}_{,2}\chi_1' + \overline{\xi}_{,3} + \overline{\xi}_{,4} + \overline{\xi}_{,5}\big(\tan\theta_a(t)\big)' + \overline{\xi}_{,6}\big(\tan\theta_b(t)\big)'\,, \tag{3.23}$$

and since $\overline{\xi}_{,3}+\overline{\xi}_{,4} = 1$ by (iii) of Lemma 2.3 it is enough to show that the other terms of the right hand side of (3.23) can be made arbitrarily small. Concerning $\big|\overline{\xi}_{,1}\big|$ and $\big|\overline{\xi}_{,2}\big|$, they can be made arbitrarily small by (v) of Lemma 2.3, just by choosing a sufficiently small $h(a,b)$. Moreover, differentiating (3.20) and (3.16) we get

$$\chi_0'(t) = -\delta_a \sin\big(\theta_a(t)\big)\theta_a'(t) = -\tan\theta_a(t)\,, \qquad\qquad \chi_1'(t) = -\tan\theta_b(t)\,, \tag{3.24}$$

hence $|\chi_0'|$ and $|\chi_1'|$ are arbitrarily small as well; thus the term $\big|\overline{\xi}_{,1}\chi_0' + \overline{\xi}_{,2}\chi_1'\big|$ is very small. Let us now consider the other terms; if $h(a,b)$ is small enough, then $\big(\tan\theta_a(t)\big)' \leq 2\theta_a' \leq 3/\delta_a$, thus also $\overline{\xi}_{,5}\big(\tan\theta_a(t)\big)'$ is arbitrarily small by (iv) of Lemma 2.3 if K is chosen sufficiently large (keep in mind that the constant K can be different in each zone, since as pointed out in Remarks 3.2 and 3.3, the values of v close to the boundary of the zones do not depend on K). Analogously, $\Big|\overline{\xi}_{,6}\big(\tan\theta_b(t)\big)'\Big|$ can be assumed to be arbitrarily small. Recalling (3.23), we deduce that $\mathrm{d}\overline{\xi}/\mathrm{d}t \approx 1$, and, hence, the injectivity of γ. Since

$$D\gamma = \begin{pmatrix} 1 & 0 \\ \overline{\xi}' & \dfrac{\mathrm{d}\overline{\xi}}{\mathrm{d}t} \end{pmatrix}\,, \qquad\qquad D\gamma^{-1} = \begin{pmatrix} 1 & 0 \\ -\dfrac{\overline{\xi}'}{\mathrm{d}\overline{\xi}/\mathrm{d}t} & \dfrac{1}{\mathrm{d}\overline{\xi}/\mathrm{d}t} \end{pmatrix} \tag{3.25}$$

and, by (i) of Lemma 2.3, $\big|\overline{\xi}'\big| \leq \max\big\{|\tan\theta_a(t)|,\, |\tan\theta_b(t)|\big\}$, which is small as seen before, we thus obtain (3.22). ☐

After these preliminaries, we define zone Z_3 as $Z_3 = \gamma(\Gamma)$. In a similar way, we pass to define zone $\widetilde{Z}_3$. Analogously to (3.20), we set

$$\widetilde{\chi}_0(t) := \delta_a\,\tau_a(\theta_a(t))\cos\Big(\varphi_a\big(\theta_a(t)\big)\Big)\,, \qquad \widetilde{\chi}_1(t) := \overline{\tilde{a}\tilde{b}} + \delta_b\,\tau_b(\theta_b(t))\cos\Big(\varphi_b\big(\theta_b(t)\big)\Big)\,, \tag{3.26}$$

so that

$$\widetilde{P}(t) \equiv \big(\widetilde{\chi}_0(t), \tilde{t}\,\big)\,, \qquad\qquad \widetilde{Q}(t) \equiv \big(\widetilde{\chi}_1(t), \tilde{t}\,\big)\,, \tag{3.27}$$

and we introduce the rectangoloid

$$\widetilde{\Gamma} := \Big\{(\sigma,\tilde{t}) \in \mathbb{R}^2 :\ \tilde{t} \in (-h^-, h^+)\,,\, \sigma \in \big(\widetilde{\chi}_0(t), \widetilde{\chi}_1(t)\big)\Big\}\,,$$

where for every $\tilde{t} \in (-h^-, h^+)$ we call $t \in (-h, h)$ the only number such that $\tilde{t} = \tilde{t}(t)$. We then define the map $\tilde{\gamma} : \widetilde{\Gamma} \to \mathbb{R}^2$ as

$$\tilde{\gamma}(\sigma,\tilde{t}) := \Big(\sigma, \xi\big[\widetilde{\chi}_0(t), \widetilde{\chi}_1(t), \tilde{t}, \tilde{t}, \tan\big(\varphi_a\big(\theta_a(t)\big)\big), \tan\big(\varphi_b\big(\theta_b(t)\big)\big)\big](\sigma)\Big) = \Big(\sigma, \tilde{\xi}(\sigma)\Big)\,. \tag{3.28}$$

Actually, we will eventually slightly modify the definition of $\tilde{\xi}$: the reason will become clear later, but we just point out now that the modification will not affect the coming proofs. Figure 6, right, shows the curve $\tilde{\gamma}(\cdot,\tilde{t})$. We now adapt Lemma 3.5 to the case of $\tilde{\gamma}$.

FIGURE 6. Curves $\gamma(\cdot,t)$ and $\tilde{\gamma}(\cdot,\tilde{t})$. The following equalities of angles hold: $\widehat{baP(t)} = \theta_a(t)$, $\widehat{abQ(t)} = \pi - \theta_b(t)$, $\widehat{\tilde{b}\tilde{a}\widetilde{P}(t)} = \varphi_a(\theta_a(t))$ and $\widehat{\tilde{a}\tilde{b}\widetilde{Q}(t)} = \pi - \varphi_b(\theta_b(t))$.

Lemma 3.6. *The map $\tilde{\gamma} : \widetilde{\Gamma} \to \mathbb{R}^2$ is smooth and injective, and*

$$\big|D\tilde{\gamma}\big| \approx 1\,, \qquad \big|D\tilde{\gamma}^{-1}\big| \approx 1\,, \qquad \det D\tilde{\gamma} \approx 1\,. \tag{3.29}$$

Proof. As in the proof of Lemma 3.5, showing the injectivity of $\tilde{\gamma}$ reduces to showing that $\dfrac{\mathrm{d}\tilde{\xi}}{\mathrm{d}\tilde{t}} > 0$, and this time we have

$$\frac{\mathrm{d}\tilde{\xi}}{\mathrm{d}\tilde{t}} = \bigg(\tilde{\xi}_{,1}\widetilde{\chi}_0' + \tilde{\xi}_{,2}\widetilde{\chi}_1' + \tilde{\xi}_{,5}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\tan\varphi_a\big(\theta_a(t)\big)\Big) + \tilde{\xi}_{,6}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\tan\varphi_b\big(\theta_b(t)\big)\Big)\bigg)\frac{\mathrm{d}t}{\mathrm{d}\tilde{t}} + \tilde{\xi}_{,3} + \tilde{\xi}_{,4}\,. \tag{3.30}$$

The proof of the positivity of $\frac{\mathrm{d}\tilde{\xi}}{\mathrm{d}\tilde{t}}$ follows the lines of Lemma 3.5: the only difference is that, while for Lemma 3.5 one had χ_0' and χ_1' arbitrarily small, in the present case one can only say that $\widetilde{\chi}_0'$ and $\widetilde{\chi}_1'$ are bounded. More precisely, differentiating (3.16) and (3.26) we find

$$\begin{aligned}\widetilde{\chi}_0' &= \delta_a\theta_a'\Big[\tau_a'(\theta_a)\cos\big(\varphi_a(\theta_a)\big) - \tau_a(\theta_a)\sin\big(\varphi_a(\theta_a)\big)\varphi_a'(\theta_a)\Big] \\ &= \frac{1}{\cos(\theta_a)}\Big[\tau_a'(\theta_a)\cos\big(\varphi_a(\theta_a)\big) - \tau_a(\theta_a)\sin\big(\varphi_a(\theta_a)\big)\varphi_a'(\theta_a)\Big],\end{aligned}$$

and using the estimates of Lemma 2.2 and the fact that h is small, we obtain that

$$\big|\widetilde{\chi}_0'(t)\big| \le \frac{5}{2}\,L_{i(t),a}\,, \tag{3.31}$$

where $i(t) = i$ for $t < 0$, and $i(t) = i+1$ for $t > 0$. Similarly,

$$\big|\widetilde{\chi}_1'(t)\big| \le \frac{5}{2}\,L_{i(t),a}\,. \tag{3.32}$$

Since, as in the proof of Lemma 3.5, $\Big|\tilde{\xi}_{,1}\Big|$ and $\Big|\tilde{\xi}_{,2}\Big|$ are again arbitrarily small, we conclude that the quantity $\Big|\tilde{\xi}_{,1}\widetilde{\chi}_0' + \tilde{\xi}_{,2}\widetilde{\chi}_1'\Big|$ can be made as small as we wish. As in Lemma 3.5, and also using the estimates of Lemma 2.2, we find that the term

$$\left|\tilde{\xi}_{,5}\frac{\mathrm{d}}{\mathrm{d}t}\left(\tan\varphi_a\left(\theta_a(t)\right)\right) + \tilde{\xi}_{,6}\frac{\mathrm{d}}{\mathrm{d}t}\left(\tan\varphi_b\left(\theta_b(t)\right)\right)\right|$$

is small provided the constant K is big enough. Using (3.19), we find that the first term of the right hand side of (3.30) is much smaller than $1 = \tilde{\xi}_{,3} + \tilde{\xi}_{,4}$, and this implies that $\mathrm{d}\tilde{\xi}/\mathrm{d}\tilde{t} \approx 1$ and, hence, $\tilde{\gamma}$ is injective.

Now, the formulas for $D\tilde{\gamma}$ and $D\tilde{\gamma}^{-1}$ are given by the same formulas as in (3.25), but replacing $\overline{\xi}'$ with $\tilde{\xi}'$, and $\mathrm{d}\overline{\xi}/\mathrm{d}t$ with $\mathrm{d}\tilde{\xi}/\mathrm{d}\tilde{t}$. Moreover, as in Lemma 3.5, we have that the quantity $\Big|\tilde{\xi}'\Big|$ is small as long as so is $h(a,b)$. As shown before, $\mathrm{d}\tilde{\xi}/\mathrm{d}\tilde{t} \approx 1$, which implies at once estimates (3.29). $\square$

Analogously as before, we finally set $\widetilde{Z}_3 := \tilde{\gamma}(\widetilde{\Gamma})$, and $\Phi : \Gamma \to \mathbb{R}^2$ as

$$\Phi(\sigma,t) := \Big(\xi\big[\chi_0(t),\chi_1(t),\widetilde{\chi}_0(t),\widetilde{\chi}_1(t),\alpha(t),\beta(t)\big](\sigma),\tilde{t}\Big) = \Big(\hat{\xi}(\sigma),\tilde{t}\Big)\,, \tag{3.33}$$

where the functions α and β are defined as

$$\alpha(t) := \tau_a\big(\theta_a(t)\big)\,\frac{\cos\big(\varphi_a(\theta_a(t))\big)}{\cos\big(\theta_a(t)\big)}\,, \qquad \beta(t) := \tau_b\big(\theta_b(t)\big)\,\frac{\cos\big(\varphi_b(\theta_b(t))\big)}{\cos\big(\theta_b(t)\big)}\,. \tag{3.34}$$

Lemma 3.7. *The map Φ is a diffeomorphism from Γ onto $\widetilde{\Gamma}$. Moreover,*

$$\big|D\Phi\big| \leq 5\,\max_{J(a)}|Du|\,, \qquad \big|D\Phi^{-1}\big| \leq 33\,\max_{u(J(a))}|Du^{-1}|\,, \qquad \det D\Phi(x) \geq \frac{1}{16}\,\det Du(x)\,. \tag{3.35}$$

Proof. As the map $t \mapsto \tilde{t}$ is a diffeomorphism from $(-h,h)$ onto $(-h^-,h^+)$, in order to ensure that Φ is a diffeomorphism from Γ onto $\widetilde{\Gamma}$, it is enough to check that $\hat{\xi}' > 0$ (notice that $\hat{\xi}'$ refers to the derivative with respect to σ, not to t).

Observe first that, by (3.34), for h very small one has $\alpha(t) \approx \tau_a(0)$ and $\beta(t) \approx \tau_b(\pi)$. On the other hand, by Lemma 2.2 and the definition of τ_0 we have

$$\tau_a(0) = \tau_{0,a}(0) = \frac{\overline{\tilde{a}\tilde{b}}}{\overline{ab}} = \tau_{0,b}(\pi) = \tau_b(\pi)\,.$$

Thus, choosing h small we have

$$\alpha(t) \approx \beta(t) \approx \tau_a(\theta_a(t)) \approx \frac{\overline{\tilde{a}\tilde{b}}}{\overline{ab}} \approx \frac{\widetilde{\chi}_1(t)-\widetilde{\chi}_0(t)}{\chi_1(t)-\chi_0(t)}\,, \tag{3.36}$$

which by Lemmas 2.1 and 2.2 readily implies

$$\frac{1}{3}\min\{\ell_{i,a},\ell_{i+1,a}\} \leq \frac{2}{3}\,\tau_a(\theta_a) \leq \hat{\xi}' \leq 3\,\tau_a(\theta_a) \leq 3\max\{L_{i,a},L_{i+1,a}\}\,, \tag{3.37}$$

as one can observe by the definition of $\hat{\xi}$ (specifically, by the same argument that leads to property i) of Lemma 2.3). In particular, $\hat{\xi}' > 0$ and then Φ is a bijection. To find the estimates (3.35), let us observe that

$$D\Phi = \begin{pmatrix} \hat{\xi}' & 0 \\ \dfrac{\mathrm{d}\hat{\xi}}{\mathrm{d}t} & \dfrac{\mathrm{d}\tilde{t}}{\mathrm{d}t} \end{pmatrix}\,, \qquad D\Phi^{-1} = \begin{pmatrix} \dfrac{1}{\hat{\xi}'} & 0 \\ -\dfrac{\mathrm{d}\hat{\xi}/\mathrm{d}t}{\hat{\xi}'\cdot\mathrm{d}\tilde{t}/\mathrm{d}t} & \dfrac{1}{\mathrm{d}\tilde{t}/\mathrm{d}t} \end{pmatrix}\,. \tag{3.38}$$

Since we have both upper and lower estimates for $\hat{\xi}'$ thanks to (3.37), and for $\mathrm{d}\tilde{t}/\mathrm{d}t$ thanks to (3.18) and (3.19), we need to take care of $\mathrm{d}\hat{\xi}/\mathrm{d}t$, which is given by

$$\frac{\mathrm{d}\hat{\xi}}{\mathrm{d}t} = \hat{\xi}_{,1}\chi_0' + \hat{\xi}_{,2}\chi_1' + \hat{\xi}_{,3}\widetilde{\chi}_0' + \hat{\xi}_{,4}\widetilde{\chi}_1' + \hat{\xi}_{,5}\alpha' + \hat{\xi}_{,6}\beta'\,. \tag{3.39}$$

Thanks to (3.24), we know that χ_0' and χ_1' are arbitrarily small, while $\hat{\xi}_{,1}$ and $\hat{\xi}_{,2}$ are bounded by (v) of Lemma 2.3 and (3.36); hence, the first two terms of the right hand side of (3.39) are arbitrarily small. We can observe that the last two terms are arbitrarily small as well; indeed, differentiating (3.16) and (3.34),

$$\alpha' = \frac{1}{\delta_a\cos^2\theta_a}\bigg(\tau_a'(\theta_a)\,\cos\big(\varphi_a(\theta_a)\big) - \tau_a(\theta_a)\sin\big(\varphi_a(\theta_a)\big)\varphi_a'(\theta_a) - \tau_a(\theta_a)\cos\big(\varphi_a(\theta_a)\big)\tan\theta_a\bigg)\,,$$

and thanks to the bounds of Lemmas 2.1 and 2.2 we infer that $|\alpha'| \leq \frac{3L_{\max}(a)}{\delta_a}$. On the other hand, $\hat{\xi}_{,5}$ is arbitrarily small by (iv) of Lemma 2.3, provided the constant K is big enough. Hence $\left|\hat{\xi}_{,5}\alpha'\right|$ is small, and, analogously, so is $\left|\hat{\xi}_{,6}\beta'\right|$. We are then left to consider $\left|\hat{\xi}_{,3}\widetilde{\chi}_0' + \hat{\xi}_{,4}\widetilde{\chi}_1'\right|$, which, thanks to (iii) of Lemma 2.3, estimates (3.31) and (3.32), can be bounded by $\frac{5}{2}\, L_{i(t),a}$. Therefore, (3.39) allows us to conclude

$$\left|\frac{\mathrm{d}\hat{\xi}}{\mathrm{d}t}\right| \leq \frac{31}{12}\, L_{i(t),a}\,. \tag{3.40}$$

Then by (3.18), (3.37), (3.40), and Lemmas 2.1 and 2.2,

$$\left|\frac{\mathrm{d}\hat{\xi}/\mathrm{d}t}{\hat{\xi}'\cdot \mathrm{d}\tilde{t}/\mathrm{d}t}\right| \leq 4\left|\frac{L_{i(t),a}}{\tau_a(\theta_a)^2\varphi_a'(\theta_a)}\right| \leq 32\,\frac{L_{i(t),a}}{\tau_{0,a}(\theta_a)^2\varphi_{0,a}(\theta_a)} = \frac{32}{\ell_{i(t),a}}\,. \tag{3.41}$$

Using in (3.38) the estimates (3.37), (3.19), (3.40) and (3.41), and Lemmas 2.1 and 2.2, we obtain (3.35) and we are done. □

We are now in a position to define $v : Z_3 \to \widetilde{Z}_3$ as $v := \tilde{\gamma}\circ\Phi\circ\gamma^{-1}$.

Lemma 3.8. *The function $v : Z_3 \to \widetilde{Z}_3$ is a diffeomorphism, smoothly matching with the function v defined in zone Z_2, and for every point $x \in Z_3$ one has*

$$\big|Dv\big| \leq 6\max_{J(a)}|Du|\,, \qquad |Dv^{-1}| \leq 34\max_{u(J(a))}|Du^{-1}|\,, \qquad \det Dv(x) \geq \frac{1}{17}\det Du(x)\,. \tag{3.42}$$

Proof. Both the fact that v is a diffeomorphism from Z_3 onto $\widetilde{Z}_3$, and estimates (3.42) are immediate consequences of Lemmas 3.5, 3.6 and 3.7. Thus, we have to check that the definitions of v in Z_2 and Z_3 match smoothly: we do this around the boundary of the zone Z_2 corresponding to a, the situation corresponding to b being identical.

We observe that, by the definition of ξ, for each $-h < t < h$, the maps $\gamma(\cdot,t)$, $\tilde{\gamma}(\cdot,\tilde{t})$ and $\Phi(\cdot,t)$ are affine close to their end points. Therefore, for a sufficiently small $\varepsilon > 0$, we can calculate, using (3.16) and (3.20),

$$\gamma\big(\chi_0(t)+\varepsilon,t\big) = \big(\chi_0(t)+\varepsilon, t+\varepsilon\tan\theta_a(t)\big) = \bigg(\delta_a + \frac{\varepsilon}{\cos\theta_a(t)}, \theta_a(t)\bigg)_{P,a}\,,$$

which can be rewritten as

$$\gamma^{-1}\bigg(\Big(\delta_a+\varepsilon,\theta_a(t)\Big)_{P,a}\bigg) = \Big(\chi_0(t)+\varepsilon\cos\theta_a(t),t\Big)\,.$$

Applying Φ, we find then, thanks to definition (3.33) and by (3.34),

$$\Phi\Big(\chi_0(t)+\varepsilon\cos\theta_a(t),t\Big) = \big(\widetilde{\chi}_0(t)+\varepsilon\,\alpha(t)\cos\theta_a(t),\tilde{t}\big) = \Big(\widetilde{\chi}_0(t)+\varepsilon\,\tau_a\big(\theta_a(t)\big)\cos\big(\varphi_a\big(\theta_a(t)\big)\big),\tilde{t}\Big)\,.$$

Finally, applying $\tilde{\gamma}$ one has, recalling (3.28), (3.17) and (3.26),

$$\begin{aligned}
v\Big(\big(\delta_a+\varepsilon,\theta_a(t)\big)_{P,a}\Big) &= \tilde{\gamma}\Big(\widetilde{\chi}_0(t)+\varepsilon\,\tau_a\big(\theta_a(t)\big)\cos\big(\varphi_a\big(\theta_a(t)\big)\big),\tilde{t}\Big)\\
&= \Big(\widetilde{\chi}_0(t)+\varepsilon\,\tau_a\big(\theta_a(t)\big)\cos\big(\varphi_a\big(\theta_a(t)\big)\big),\tilde{t}+\varepsilon\,\tau_a\big(\theta_a(t)\big)\sin\big(\varphi_a\big(\theta_a(t)\big)\big)\Big)\\
&= (\delta_a+\varepsilon)\tau_a\big(\theta_a(t)\big)\Big(\cos\big(\varphi_a\big(\theta_a(t)\big),\sin\big(\varphi_a\big(\theta_a(t)\big)\Big)\\
&= (\delta_a+\varepsilon)\Big(\tau_a\big(\theta_a(t)\big),\varphi_a\big(\theta_a(t)\big)\Big)_{P,\tilde{a}}\,.
\end{aligned}$$

We thus see that the two expressions of v close to ∂Z_2 (the one in Z_2 of Remark 3.3 and the one in Z_3 just calculated) coincide. Therefore, v remains smooth around ∂Z_2 and the proof is concluded. $\square$

To conclude, we have to check the behaviour of v in the "upper boundary" of Z_3, that is, for points $\gamma(\sigma,t)$ with $t<h$ and $t\approx h$.

Lemma 3.9. *For $t<h$ sufficiently close to h and for every $\sigma\in\big(\chi_0(t),\chi_1(t)\big)$ one has $v\big(\gamma(\sigma,t)\big)=u\big(\gamma(\sigma,t)\big)$.*

Proof. If t is close enough to h, thanks to (3.14) and (iii) of Lemma 2.2 we have that $\tau_a\big(\theta_a(t)\big)=\tau_{0,a}\big(\theta_a(t)\big)$ and $\varphi_a\big(\theta_a(t)\big)=\varphi_{0,a}\big(\theta_a(t)\big)$. In addition, recall from Remark 3.3 that $v=u$ around $\partial B(a,\delta_a)$ whenever $\tau_a=\tau_{0,a}$ and $\varphi_a=\varphi_{0,a}$. As a consequence, by Lemma 3.8 we get that $v=u$ at every point $\gamma(\sigma,t)$ if $t<h$ is close enough to h and $\sigma>\chi_0(t)$ is close enough to $\chi_0(t)$. Take now a generic point $(\sigma,t)\in\Gamma$, with $t<h$ close enough to h. If we express the affine map u as the matrix $\begin{pmatrix}A & B\\ 0 & C\end{pmatrix}$ for some reals A,B,C, then

$$u\big(\gamma(\sigma,t)\big)=u\big(\sigma,\bar\xi(\sigma)\big)=\Big(A\sigma+B\bar\xi(\sigma),C\bar\xi(\sigma)\Big)\,. \tag{3.43}$$

Let us now consider $v\big(\gamma(\sigma,t)\big)$, which can be expressed as

$$v\big(\gamma(\sigma,t)\big)=\tilde\gamma\big(\Phi(\sigma,t)\big)=\tilde\gamma\Big(\hat\xi(\sigma),\tilde t\Big)=\Big(\hat\xi(\sigma),\tilde\xi\big(\hat\xi(\sigma)\big)\Big)\,. \tag{3.44}$$

Comparing (3.43) and (3.44), we have to check that

$$\hat\xi(\sigma)=A\sigma+B\bar\xi(\sigma)\,,\qquad\qquad \tilde\xi\big(\hat\xi(\sigma)\big)=C\bar\xi(\sigma)\,. \tag{3.45}$$

Let us start with the first equality: since

$$\begin{aligned}A\sigma+B\bar\xi(\sigma)&=A\sigma+B\xi[\chi_0(t),\chi_1(t),t,t,\tan\theta_a(t),\tan\theta_b(t)](\sigma)\\&=A\sigma+\xi[\chi_0(t),\chi_1(t),Bt,Bt,B\tan\theta_a(t),B\tan\theta_b(t)](\sigma)\\&=\xi[\chi_0(t),\chi_1(t),A\chi_0(t)+Bt,A\chi_1(t)+Bt,A+B\tan\theta_a(t),A+B\tan\theta_b(t)](\sigma)\,,\end{aligned} \tag{3.46}$$

and recalling the definition (3.33) of $\hat\xi$, we have to show that

$$\widetilde\chi_0(t)=A\chi_0(t)+Bt\,,\quad \widetilde\chi_1(t)=A\chi_1(t)+Bt\,,\quad \alpha(t)=A+B\tan\theta_a(t)\,,\quad \beta(t)=A+B\tan\theta_b(t)\,.$$

In fact, the first two equalities are true because $u\big(P(t)\big)=\widetilde P(t)$ and $u\big(Q(t)\big)=\widetilde Q(t)$, and by the matrix expression of u, having in mind (3.21) and (3.27); for future reference, note that these equalities also show that $\tilde t=Ct$. On the other hand, as noticed above, $v=u$ at $\gamma(\sigma,t)$ for $t\approx h$ and $\sigma\approx\chi_0(t)$, so $\widetilde\chi_0(t)=A\chi_0(t)+Bt$ for this range of (σ,t), and comparing (3.46) with the definition of $\hat\xi$, we obtain that $\alpha(t)=A+B\tan\theta_a(t)$ for $t<h$ with $t\approx h$. Analogously, $\beta(t)=A+B\tan\theta_b(t)$, and, thus, the first identity in (3.45) is established.

Let us then consider $\tilde\xi\circ\hat\xi$; recalling the definition of $\tilde\xi$ and of $\hat\xi$, one may be tempted to think that

$$\tilde\xi\circ\hat\xi=\xi\Big[\chi_0(t),\chi_1(t),\tilde t,\tilde t,\alpha(t)\tan\big(\varphi_a(\theta_a(t))\big),\beta(t)\tan\big(\varphi_b(\theta_b(t))\big)\Big]\,. \tag{3.47}$$

One may easily check that the above equality is not true; nevertheless, it is possible to use this equality as a *definition* for $\tilde\xi$, in place of the one given in (3.28). In fact, one can verify that also this modified definition of $\tilde\xi$ satisfies all the properties given by (2.5), as well as by Lemma 2.3; and since these properties are actually all we have used of the function $\tilde\xi$, the results of Lemmas 3.7 and 3.8 are still valid (we will give a formal and more detailed proof of this assert in Remark 3.10 below).

To sum up, we are allowed to assume that (3.47) holds true, and in order to conclude our proof we have just to establish the equality

$$\xi\left[\chi_0(t),\chi_1(t),\tilde t,\tilde t,\alpha(t)\tan\left(\varphi_a(\theta_a(t))\right),\beta(t)\tan\left(\varphi_b(\theta_b(t))\right)\right] \\ = C\,\xi\left[\chi_0(t),\chi_1(t),t,t,\tan\theta_a(t),\tan\theta_b(t)\right],$$

which amounts to checking that

$$\tilde t = Ct\,,\qquad \alpha(t)\tan\left(\varphi_a(\theta_a(t))\right) = C\tan\theta_a(t)\,,\qquad \beta(t)\tan\left(\varphi_b(\theta_b(t))\right) = C\tan\theta_b(t)\,. \tag{3.48}$$

The first equality was observed earlier in the proof. On the other hand, recalling the definitions (3.34), (3.17) and (3.16) of $\alpha(t)$, $\tilde t$ and $\theta_a(t)$, we have

$$\alpha(t)\tan\big(\varphi_a(\theta_a(t))\big) = \tau_a\big(\theta_a(t)\big)\frac{\sin\big(\varphi_a(\theta_a(t))\big)}{\cos\theta_a(t)} = \frac{\tilde t}{\delta_a\cos\theta_a(t)} = \frac{Ct}{\delta_a\cos\theta_a(t)} = C\tan\theta_a(t)\,,$$

so the second equality in (3.48) holds true as well. The third one being completely analogous, we have concluded our proof. ☐

Remark 3.10. *Let us explain more in detail why using* (3.47) *as a definition for $\tilde\xi$ is admissible (that is, keeps the validity of Lemmas 3.7 and 3.8). First of all, let us call $\check\xi$ the function $\xi[\chi_0(t),\chi_1(t),\tilde t,\tilde t,\alpha(t)\tan\varphi_a(\theta_a(t)),\beta(t)\tan\varphi_b(\theta_b(t))]$. As the maps Φ and $t\mapsto\tilde t$ are diffeomorphisms, then so is the map $\Gamma\ni(\sigma,t)\mapsto(\hat\xi(\sigma),t)$. Hence there exists a unique smooth map $\tilde\zeta$ defined on the set $\{(\sigma,t):\,-h<t<h,\ \widetilde\chi_0(t)<\sigma<\widetilde\chi_1(t)\}$ such that $\tilde\zeta\circ\hat\xi=\check\xi$, i.e.,*

$$\tilde\zeta\Big(\xi\big[\chi_0(t),\chi_1(t),\widetilde\chi_0(t),\widetilde\chi_1(t),\alpha(t),\beta(t)\big](\sigma),t\Big) \\ = \xi\Big[\chi_0(t),\chi_1(t),\tilde t,\tilde t,\alpha(t)\tan\big(\varphi_a(\theta_a(t))\big),\beta(t)\tan\big(\varphi_b(\theta_b(t))\big)\Big](\sigma)$$

for all $(\sigma,t)\in\Gamma$. The chain rule provides the equalities

$$\frac{\mathrm d\tilde\zeta}{\mathrm d\tilde t} = \frac{\mathrm d\check\xi}{\mathrm d\tilde t} - \frac{\check\xi'\cdot\mathrm d\hat\xi/\mathrm dt}{\hat\xi'}\frac{\mathrm dt}{\mathrm d\tilde t}\,,\qquad\qquad \tilde\zeta' = \frac{\check\xi'}{\hat\xi'}\,. \tag{3.49}$$

Moreover,

$$\frac{\mathrm d\check\xi}{\mathrm d\tilde t} = \bigg(\check\xi_{,1}\chi_0' + \check\xi_{,2}\chi_1' + \check\xi_{,5}\frac{\mathrm d}{\mathrm dt}\Big(\alpha(t)\tan\big(\varphi_a(\theta_a(t))\big)\Big) + \check\xi_{,6}\frac{\mathrm d}{\mathrm dt}\Big(\beta(t)\tan\big(\varphi_b(\theta_b(t))\big)\Big)\bigg)\frac{\mathrm dt}{\mathrm d\tilde t} + 1\,.$$

Just as Lemma 3.6 showed that $\mathrm d\tilde\xi/\mathrm d\tilde t\approx 1$ we can show now that $\mathrm d\check\xi/\mathrm d\tilde t\approx 1$. Now, by (i) of Lemma 2.3, $\left|\check\xi'\right|$ is small. Together with (3.19), (3.37), (3.40) *and* (3.49) *we obtain that $\mathrm d\tilde\zeta/\mathrm d\tilde t\approx 1$ and $\left|\tilde\zeta'\right|$ is small, which are the two essential estimates that Lemma 3.6 uses in its proof. Hence the conclusion of Lemma 3.6 holds if one replaces $\tilde\xi$ with $\tilde\zeta$. Now let us show that the same happens with Lemma 3.8. A quick inspection of its proof tells us that the only property to be checked is that, for $\varepsilon>0$ small, one has*

$$\tilde\zeta\left(\widetilde\chi_0(t)+\varepsilon\right) = \tilde t + \varepsilon\tan\left(\varphi_a(\theta_a(t))\right) \qquad\qquad \tilde\zeta\left(\widetilde\chi_1(t)-\varepsilon\right) = \tilde t - \varepsilon\tan\left(\varphi_b(\theta_b(t))\right)\,.$$

As a matter of fact, these two properties are immediate consequences of the definition of ξ and the identity $\tilde\zeta\circ\hat\xi=\check\xi$ applied to $\chi_0(t)+\varepsilon/\alpha(t)$ and $\chi_1(t)-\varepsilon/\beta(t)$.

3.4. **Construction in zone Z_4 and proof of Theorem A.** Until now, we have fixed two adjacent vertices a and b and defined the zones Z_i and $\widetilde{Z}_i$ for $i = 1, 2, 3$; these zones were in fact depending on a and b, hence we should have written $Z_1(a)$, $Z_2(a)$ and $Z_3(a, b)$. We then redefine Z_i (resp., $\widetilde{Z}_i$) as the union of the corresponding zones for all the adjacent vertices a and b of the triangulation. Finally, we define Z_4 as the remaining part of Ω, and, analogously, $\widetilde{Z}_4$ as the remaining part of the image space $u(\Omega)$. As noted earlier, Z_4 and $\widetilde{Z}_4$ are made by disconnected pieces, consisting of interior parts of every triangle of the triangulations in Ω and in $u(\Omega)$: a rough picture of zone Z_4 corresponding to the triangle abc was depicted in Figure 3. Notice that Z_4 contains most of Ω, since $\Omega \setminus Z_4$ is just a narrow neighborhood of the sides of the triangulation of Ω; the same is true for $\widetilde{Z}_4$. We complete our definition of v by setting $v \equiv u$ in Z_4, and noticing that it attaches smoothly with the function v defined in Z_2 thanks to Remark 3.3, and with that defined in Z_3 thanks to Lemma 3.9.

Since every affine map is smooth, the original function u was already smooth except around the sides of the triangulation. The basic idea of our construction consisted in leaving $v \equiv u$ everywhere except close to the sides of the triangles, and finding a way to connect smoothly the different pieces. We are now ready to prove our main result.

Proof of Theorem A. Let u be a piecewise affine function as in the assumptions of the theorem, and take a triangulation of Ω such that u is affine on each triangle of the triangulation. As observed at the beginning of Section 3, we can assume that the triangulation satisfies properties (T1)–(T3), and as mentioned in Section 2.1, we can assume that u is orientation preserving. Let moreover v be defined as explained through this section: by our results, we know that v is smooth, and by construction it is a bijection between Ω and $u(\Omega)$. Putting together Lemma 3.1, Lemma 3.4 and Lemma 3.8, we also directly obtain (1.5) and (1.6).

An easy but crucial observation is that for any $x \in \Omega$ one has by construction that $v(x)$ belongs to the natural neighborhood of $u(x)$, so $|v(x) - u(x)| \le \operatorname{diam} J(u(x))$, and, similarly, $\left|v^{-1}(x') - u^{-1}(x')\right| \le \operatorname{diam} J(u^{-1}(x'))$ for any $x' \in u(\Omega)$. By property (T2), this implies that the terms $\|u - v\|_{L^\infty(\Omega)}$ and $\|v^{-1} - u^{-1}\|_{L^\infty(u(\Omega))}$ can be made as small as we wish. Moreover, property (T3) implies that if u (resp., u^{-1}) is continuous up to the boundary, then so is v (resp., v^{-1}), and $v = u$ (resp., $v^{-1} = u^{-1}$) on the boundary.

We pass to the bi-Lipschitz property, so assume that u and u^{-1} are Lipschitz of constant $L \ge 1$. By the definition (1.2), we find that $L_i \le L$ and $\ell_i^{-1} \le L$. Therefore, $R(x) \le L^2$ for every x, hence applying (1.5) with $q = 1/3$ we find that $|Dv| \le 13L^{7/3}$ and $|Dv^{-1}| \le 50L^{7/3}$. Now take two different points x, y in Ω. If the segment xy is contained in Ω then clearly $|v(x) - v(y)| \le 13L^{7/3}|x - y|$; if not, there exist $x_1, y_1 \in \partial\Omega$ such that the points x, x_1, y_1, y are aligned, and the interior of the segments xx_1 and y_1y are contained in Ω. As seen before, $v(x_1) = u(x_1)$ and $v(y_1) = u(y_1)$, since u, being Lipschitz, is continuous up to the boundary. Then,

$$
\begin{aligned}
|v(x) - v(y)| &\le |v(x) - v(x_1)| + |u(x_1) - u(y_1)| + |v(y_1) - v(y)| \\
&\le 13L^{7/3}\,|x - x_1| + L\,|x_1 - y_1| + 13L^{7/3}\,|y_1 - y| \\
&\le 13L^{7/3}\,(|x - x_1| + |x_1 - y_1| + |y_1 - y|) = 13L^{7/3}\,|x - y|\,.
\end{aligned}
$$

An analogous argument shows that $\left|v^{-1}(x') - v^{-1}(y')\right| \le 50L^{7/3}\,|x - y|$, for any $x', y' \in u(\Omega)$, which proves that v is bi-Lipschitz.

Let us now consider the term

$$\|Dv - Du\|_{L^p(\Omega)} = \left(\sum_{i \in I} \|Dv - Du\|_{L^p(T_i)}^p \right)^{1/p},$$

and focus on a particular triangle T_i of vertices a, b and c. By construction, $\|Du\|_{L^\infty(T_i)}$ and $\|Dv\|_{L^\infty(T_i)}$ are bounded by some constant M_i; indeed, Du is constant on T_i, and, on the other hand, the bound on $|Dv|$ in T_i comes from (1.5) and from the fact that Du is bounded in the natural neighborhood of each point in T_i. By the triangular inequality,

$$\|Dv - Du\|_{L^p(T_i)} \leq \big(2M_i\big) \left| \Big\{ y \in T_i : \, u(y) \neq v(y) \Big\} \right|^{1/p}.$$

Finally, the set $\big\{ y \in T_i : \, u(y) \neq v(y) \big\}$ can be made as small as we wish, by decreasing the constants δ_a, δ_b and δ_c as needed. Hence, we can assume $\|Dv - Du\|_{L^p(T_i)}$ to be as small as we wish. Arguing in the same way for every triangle, we can make $\|Dv - Du\|_{L^p(\Omega)}$ small. Repeating the same argument for $\|Dv^{-1} - Du^{-1}\|_{L^p(u(\Omega))}$, we finally conclude (1.4). □

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