A variational model of interaction between continuum and discrete systems

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Abstract

We consider a variational model which describes a complex system composed, in its reference configuration, of a periodic distribution of 'small' interacting particles immersed in a continuous medium. We describe its macroscopic limit via Gamma-convergence, highlighting different regimes. In particular, we show how the interplay between the particles and the continuum leads, for a critical size of the particles, to a capacitary term. Eventually, we discuss how the presence of a continuum affects the properties of the ground states of the system of particles in terms of the validity or not of the so called 'Cauchy-Born' rule.

Keywords: Atomistic-to-continuum limit; Perforated domains; Complex fluids; Γ-convergence; Calculus of Variations.

1 Introduction

In this paper we study a variational model which describes a complex system composed, in its reference configuration, of a periodic distribution of 'small' interacting particles immersed in a continuous medium. The study of such systems has been an issue of increasing interest in the last years and it has been mainly focused on the analysis of complex fluids in the context of fluid mechanics. Such fluids can be observed in nature, as for examples blood or honey, or can be of artificial origin, such as magnetic fluids, colloidal suspensions, surfactant or polymer fluids, which turn out to be very useful in various applications (see for example [17] and [18]).

The main feature of such models is that the particles interact between themselves and with the surrounding medium. The total energy of the system must account for the deformation of the continuous medium and of the particles and for the interactions between the particles, which in turn are due to forces of various origin (such as Van-der-Waals forces, electrostatic potentials, etc.). We assume that the deformation of the continuum is governed by a local energy of integral type, as in the framework of continuum mechanics for hyper-elastic materials, and that the energy of the system of particles accounts only for pairwise interactions. For simplicity, we assume that the particles are ball shaped and are periodically distributed in their reference configuration, more precisely their centers occupy the nodes of a squared lattice $\varepsilon\mathbb{Z}^n$, $\varepsilon > 0$. Therefore, in mathematical terms, given an open set $\Omega \subset \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}^m$, we consider energies of the form

$$\begin{aligned} F_\varepsilon(u) &= \int_\Omega f_\varepsilon(x, Du)\, dx + \sum_{[\varepsilon i, \varepsilon j]\subset\Omega} g_\varepsilon(i, j, \overline{u}(\varepsilon j) - \overline{u}(\varepsilon i)) \\ &=: F^c_\varepsilon(u) + F^d_\varepsilon(u), \end{aligned} \tag{1.1}$$

where, for $i \in \mathbb{Z}^n$,

$$\overline{u}(\varepsilon i) := \fint_{B(\varepsilon i, r_\varepsilon)\cap\Omega} u(x)\, dx, \tag{1.2}$$

$B(\varepsilon i, r_\varepsilon)$ being the ball centered in εi and radius $r_\varepsilon \ll \varepsilon$. In the case $n = m = 3$, we can picture Ω as the reference configuration of the complex system, where $\cup_{i\in\mathbb{Z}^n} B(\varepsilon i, r_\varepsilon) \cap \Omega$ is the region occupied by the particles, and $u : \Omega \to \mathbb{R}^m$ as the displacement of the system (see Figure 1). The first term of the energy $F^c_\varepsilon(u)$ accounts for the deformations of the continuous media and of the particles, while the second term $F^d_\varepsilon(u)$ weighs the pairwise interactions through some suitable potentials g_ε. Our modeling assumption is that the potential, accounting for the interaction between the particles centered in εi, εj in their reference configuration, depends on the deformed configuration through the mean value of the displacement u on $B(\varepsilon i, r_\varepsilon)$ and $B(\varepsilon j, r_\varepsilon)$, respectively. Here the choice of the size r_ε of the particles plays a fundamental role, as in the context of perforated domains, where a suitable 'critical size' leads to the appearance of an additional potential in the limiting problem as ε goes to 0 (named 'strange term' in [13]).

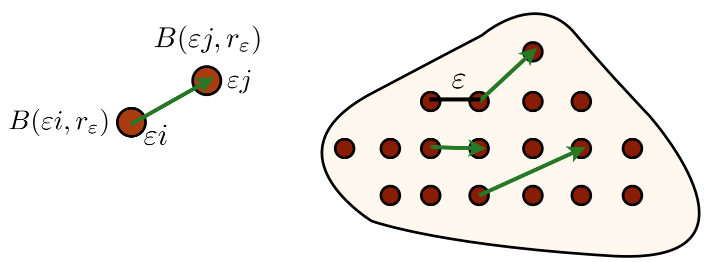


Figure 1: Reference configuration of the complex system. On the left a zoom on two interacting particles.

In [6], for $n = m = 3$ the authors consider the small non-stationary motion of a viscous incompressible fluid with a large number of small rigid interacting particles with radii of order ε^3, which is the critical size in their context. They formulate a stationary version of the problem and they reduce to a variational formulation involving energies of the type (1.1), where f_ε and g_ε are quadratic forms in the context of linearized elasticity. They show that the macroscopic homogenized model is described by two order parameters related to the limiting velocity of the fluid and of the system of particles, respectively. In the limiting energy the two order parameters are linked by an additional potential, which is qualitatively of the same type as that appearing in the homogenized perforated domain problem. In the previous paper [5], the author deals with a similar problem when the radii of the particles is much larger than ε^3, showing that in that case the complex fluid behaves in the limit as a one-phase medium.

In this paper we are interested in the macroscopic behavior of the ground states of complex systems governed by energies F_ε as in (1.1). To this end, we study, through a Γ-convergence procedure, the asymptotic behavior of F_ε as $\varepsilon \to 0$ for a general class of energy densities f_ε and interaction potentials g_ε and for every choice of $r_\varepsilon \ll \varepsilon$. Concerning the discrete term F^d_ε, we highlight the dependence of g_ε on discrete quotient and consider the ε-periodic and bulk scaled case $g_\varepsilon(i,j,\zeta) = \varepsilon^n g^{j-i}(i, \frac{\zeta}{\varepsilon|j-i|})$, with $g^\xi(\cdot, z)$ h-periodic for some $h \in \mathbb{Z}$ and for all $\xi \in \mathbb{Z}^n$ and $z \in \mathbb{R}^m$. We, then, find more convenient to rewrite the energies in (1) as

$$F_\varepsilon(u) \quad = \quad \int_\Omega f_\varepsilon(x, Du)\, dx + \sum_{\xi\in\mathbb{Z}^n} \sum_{\varepsilon[i,i+\xi]\subset\Omega} \varepsilon^n g^\xi \left(i, \frac{\overline{u}(\varepsilon i + \varepsilon\xi) - \overline{u}(\varepsilon i)}{\varepsilon|\xi|}\right).$$

Having in mind that in the physical models the continuous media and the particles may be composed of two different homogeneous materials, we assume that f_ε is of the form

$$f_\varepsilon(x, M) = \begin{cases} f^1(M) & \text{if } x \in \Omega \setminus \cup_{i\in\mathbb{Z}^n} B(\varepsilon i, r_\varepsilon) \\ f^2(M) & \text{otherwise,} \end{cases}$$

with f^1, $f^2 : \mathbb{M}^{m\times n} \to [0, +\infty)$ Borel functions satisfying standard p-growth conditions with $p > 1$ (see (3.4)). Concerning the discrete potentials, we assume that $g^\xi : \mathbb{Z}^n \times \mathbb{R}^m \to [0, +\infty)$, $\xi \in \mathbb{Z}^n$, satisfy suitable p-growth assumptions and decay hypotheses as $|\xi|$ tends to $+\infty$ (see (H2), (H3) in Section 2.1). Such conditions have been introduced in [1] to obtain an integral representation result for the Γ-limit of a wide class of discrete energies. More precisely, in [1] it has been proved that the functionals

$$E_\varepsilon(v) = \sum_{\xi\in\mathbb{Z}^n} \sum_i \varepsilon^n g^\xi \left(i, \frac{v(\varepsilon i + \varepsilon\xi) - v(\varepsilon i)}{\varepsilon|\xi|}\right),$$

defined on all discrete functions $v : \varepsilon\mathbb{Z}^n \cap \Omega \to \mathbb{R}^m$, Γ-converge to an integral functional E finite on $W^{1,p}(\Omega; \mathbb{R}^m)$ and defined as

$$E(v) = \int_\Omega g_{hom}(Dv)\, dx,$$

where $g_{hom}(M)$ is given by a suitable homogenization formula (see Theorem 2.1).

In this paper we perform an asymptotic analysis highlighting different phenomena depending on $p \leq n$ and $p > n$. For $p \leq n$ we distinguish three different cases according if r_ε is much smaller, much larger than or of the same order of a critical size $\overline{r}_\varepsilon$, which is the same appearing in the theory of perforated domains, namely $\overline{r}_\varepsilon = \varepsilon^{n/n-p}$ for $p < n$ and $\overline{r}_\varepsilon = \exp(-\varepsilon^{-n/n-1})$ for $p = n$. If $r_\varepsilon \ll \overline{r}_\varepsilon$ or $r_\varepsilon \sim \overline{r}_\varepsilon$, in analogy with [6], we show that the limiting energy may be described by two order parameters, representing in the physical case the limiting displacements of the continuous media and of the system of particles. Indeed, to any $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ we associate the 'discrete' function $v(u) : \varepsilon\mathbb{Z}^n \cap \Omega \to \mathbb{R}^m$ defined as

$$v(u)(\varepsilon i) = \bar{u}(\varepsilon i)\,,\ \varepsilon i \in \varepsilon\mathbb{Z}^n \cap \Omega,$$

with $\bar{u}$ given by (1.2). Hence, if we identify, as in [1], $v(u)$ with its piece-wise constant (or equivalently piece-wise affine) interpolation on the cell of the lattice $\varepsilon\mathbb{Z}^n$, then, thanks to the coercivity hypotheses, sequences $(u_\varepsilon, v(u_\varepsilon))$ with equibounded energy are compact in $[L^p(\Omega;\mathbb{R}^m)]^2$ (or weakly in $[W^{1,p}(\Omega;\mathbb{R}^m)]^2$)(see Theorem 3.1). As a consequence, we study the Γ-convergence of F_ε with respect to strong converging sequences $(u_\varepsilon, v(u_\varepsilon))$ in $[L^p(\Omega;\mathbb{R}^m)]^2$ and we show that we have an effective decoupling of continuous and discrete displacements in the limit. Indeed, for $r_\varepsilon \sim \overline{r}_\varepsilon$, the Γ-limit $F(u,v)$ is finite on $[W^{1,p}(\Omega;\mathbb{R}^m)]^2$ and of the form

$$F(u,v) = \int_\Omega Qf^1(Du)\,dx + \int_\Omega g_{hom}(Dv)\,dx + \int_\Omega \varphi(u-v)\,dx \tag{1.3}$$

(see Theorems 4.1 and 5.1). Here the first two terms are the Γ-limit of $F^c_\varepsilon(u)$ and $F^d_\varepsilon(v)$, respectively, while the third term accounts for the interplay between the continuum and the system of particles. The energy density φ is defined by a capacitary formula depending on f^1 and f^2 (see (4.3)). We remark that it is of the same type as that obtained in the Γ-convergence analysis of integral functionals defined on periodically perforated domains (see [3] and [19] for $p < n$ and $p = n$, respectively). The difference is that, due to our modeling assumptions, the test functions must satisfy a mean constraint weaker than the usual one. We consider also the case in which the particles are subject only to rigid movements, that corresponds to restrict the domain of F_ε to the set of functions $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ which are constant on $\cup_{i\in\mathbb{Z}^n} B(\varepsilon i, r_\varepsilon) \cap \Omega$. The Γ-limit is still of the form (1.3) with φ replaced by a function $\varphi^{\mathcal{R}} \geq \varphi$, which is defined by the same capacitary formula obtained in [3] and [19] for $p < n$ and $p = n$, respectively. In particular, if $f_1(M) = f_2(M) = |M|^p$, $p < n$, we get that $\varphi(z) = \overline{C}_p(B_1(0);\mathbb{R}^n)|z|^p$, $\varphi^{\mathcal{R}}(z) = C_p(B_1(0);\mathbb{R}^n)|z|^p$. Here $C_p(B_1;\mathbb{R}^n)$ is the usual notion of p-capacity of the unitary ball B_1 with respect to $\mathbb{R}^n$, while $\overline{C}_p(B_1;\mathbb{R}^n)$ is defined as

$$\overline{C}_p(B_1;\mathbb{R}^n) = \inf\Big\{\int_{\mathbb{R}^n} |D\zeta|^p\,dx : \zeta \in W^{1,p}(\mathbb{R}^n)\,,\ -\!\!\!\!\!\!\int_{B_1(0)} \zeta\,dx = 1\Big\}\,.$$

and gives rise to a new notion of capacity. In fact, we show that $\overline{C}_p(B_1;\mathbb{R}^n) < C_p(B_1;\mathbb{R}^n)$ (see Proposition 4.5) . Hence, in our model the limiting energy keeps memory of the deformability of the particles.

The case $r_\varepsilon \ll \overline{r}_\varepsilon$ looks less interesting, since the term accounting for the interaction between the continuum and the system of particles disappears. Indeed we show that the limiting energy is given by the functional (1.3) with $\varphi \equiv 0$ (see Theorems 4.11 and 5.4).

We easily derive from the previous results the Γ-limit of F_ε when these energies are regarded as function only of the continuous displacement u or the discrete displacement v(u) (see Theorems 7.1 and 7.3).

In the case $r_\varepsilon \gg \overline{r}_\varepsilon$, in analogy with [5], we show that the macroscopic energy is described by one order parameter, that is the displacements of the continuum and of the system of particles coincide in the limit. Despite the fact that the interaction between u and $v(u)$ is much stronger in this case, we show that we may optimize the energies by independently optimizing the continuous and discrete terms. Indeed we prove that the Γ-limit of F_ε is given by

$$F(u) = \int_\Omega Qf^1(Du)\,dx + \int_\Omega g_{hom}(Du)\,dx,$$

that is the sum of the Γ-limits of F^c_ε and F^d_ε (see Theorems 4.12 and 5.5).

All these results allow in particular to derive convergence results of minimum problems with boundary conditions prescribed on the order parameters u, $v(u)$ or both (see Corollaries 6.2, 7.2 and 7.4).

The case $p > n$ exhibits different phenomena. First of all, as in the case $p \le n$ and $r_\varepsilon \gg \overline{r}_\varepsilon$, we show that there is not an effective decoupling of variables, that is the displacements of the continuum and of the system of particles coincide in the limit. We then provide an integral representation result for the Γ-limit of F_ε which turns out to be defined on $W^{1,p}(\Omega;\mathbb{R}^m)$ and of the form

$$\int_\Omega f_{hom}(Du)\,dx,$$

with f_{hom} defined by a suitable homogenization formula (see Theorem 8.1). Moreover, conversely to the case $p \le n$, we show that, under some additional assumptions that cover many meaningful cases, the Γ-limit of F_ε does not depend on the choice of the size $r_\varepsilon \ll \varepsilon$ of the particles and then we may assume $r_\varepsilon = 0$ (see Proposition 8.3). The question of such dependence on r_ε in the general case remains open.

So far, a natural question is whether the Γ-limit can be obtained by independently optimizing the continuous and discrete terms as in the case $p \le n$ and $r_\varepsilon \gg \overline{r}_\varepsilon$, that is $f_{hom} = Qf_1 + g_{hom}$, or not. We answer this question negatively by noting that in general $f_{hom} \ge Qf_1 + g_{hom}$ and providing in Section 8.1 some one dimensional examples where the inequality is strict. Another important question we address is how the presence of a continuum affects the properties of the ground states of the system of particles in terms of the validity or not of the so called 'Cauchy-Born' rule (see Definition 8.6). We show that the continuum has a sort of 'regularizing' effect, in the sense that it may only increase the set of Cauchy-Born states. Moreover, the same one dimensional examples mentioned above show that such set may be strictly larger than the set of Cauchy-Born states when the interactions of the particles are governed only by the discrete energy. Eventually we show that, in the one dimensional case, this analysis can be extended to interactions governed by Lennard-Jones type potentials, which do not satisfy the growth hypotheses we have considered so far. Even in this case we show that the presence of a continuum increases the set of Cauchy-Born states.

Finally we underline that our analysis could be extended in many directions: first, having in mind variational model in fracture mechanics, one could investigate the case in which the density energies f_ε and the interaction potentials g^ξ satisfy linear or sub-linear growth assumptions and hence allow the limiting displacements to be discontinuous. Another interesting direction of investigation could be that of regarding the continuum as the macroscopic approximation of a system of particles interacting at another 'atomic' scale δ much smaller than ε. Hence, one could replace the energies F^c_ε by discrete functionals weighing the interactions at scale δ and study the asymptotic behavior of the energies as first δ and then ε tend to 0. It would be interesting to see if the introduction of the scale δ gives rise in the limit to some new scaling effects depending on the ratio between δ and ε.

2 Notation and preliminary results

We denote by $\{e_1, e_2, \ldots, e_n\}$ the standard basis in $\mathbb{R}^n$ and by $|\cdot|$ the usual euclidean norm in $\mathbb{R}^n$. Given $x \in \mathbb{R}^n$ and $r > 0$, $B_r(x)$ will denote the open ball with center x and radius r. If $x = 0$, we set $B_r := B_r(0)$. Moreover, we set $Q_r := (-r/2, r/2)^n$. We denote by $\mathbb{M}^{m\times n}$ the space of $m \times n$ matrices. For $x, y \in \mathbb{R}^n$, $[x, y]$ denotes the segment between x and y. Given an open set $\Omega \subset \mathbb{R}^n$, we denote by $\mathcal{O}(\Omega)$ the family of all open subsets of Ω. If $B \subset \mathbb{R}^n$ is a Borel set, we will denote by $|B|$ its Lebesgue measure. We use standard notation for L^p and Sobolev spaces.

2.1 Discrete energies

In this section we recall some Γ-convergence results about the discrete-to-continuum limits of discrete functionals. For the definition and the main properties of Γ-convergence we refer the reader to [7] and [14]. Given $\varepsilon > 0$ and $A \subset \mathbb{R}^n$, we identify the set of discrete functions $v : \varepsilon\mathbb{Z}^n \cap A \to \mathbb{R}^m$ with the set of piecewise-constant functions defined by

$$\mathcal{A}_\varepsilon(A) := \{v : \mathbb{R}^n \to \mathbb{R}^m,\ v \text{ constant on } \varepsilon i + [-\varepsilon/2, \varepsilon/2)^n,\ \forall\, i \in \mathbb{Z}^n \cap \frac{1}{\varepsilon}A\}\,. \quad (2.1)$$

Let Ω be a bounded open set of $\mathbb{R}^n$ with Lipschitz boundary and let $E_\varepsilon : L^p(\Omega;\mathbb{R}^m) \to [0,+\infty]$ be the family of functionals defined by

$$E_\varepsilon(v) = \begin{cases} \displaystyle\sum_{\xi\in\mathbb{Z}^n} \sum_{i\in R_\varepsilon^\xi(\Omega)} \varepsilon^n g^\xi(i, D_\varepsilon^\xi v(\varepsilon i)) & \text{if } v \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{2.2}$$

where

$$D_\varepsilon^\xi v(\varepsilon i) := \frac{v(\varepsilon i + \varepsilon\xi) - v(\varepsilon i)}{\varepsilon|\xi|}, \tag{2.3}$$

$$R_\varepsilon^\xi(\Omega) := \{i \in \mathbb{Z}^n : \ [\varepsilon i, \varepsilon(i+\xi)] \subset \Omega\}, \tag{2.4}$$

and there exist $h \in \mathbb{Z}$, $p > 1$, $c > 0$ and a family of positive constants $\{C^\xi\}_\xi$ such that the functions $g^\xi : \mathbb{Z}^n \times \mathbb{R}^m \to [0,+\infty)$, $\xi \in \mathbb{Z}^n$, satisfy:

(H1) (periodicity) $g^\xi(\cdot, z)$ is h-periodic for all $\xi \in \mathbb{Z}^n$ and $z \in \mathbb{R}^m$;

(H2) (coercivity) $g^{e_k}(i,z) \geq c(|z|^p - 1)$, $\forall (i,z) \in \mathbb{Z}^n \times \mathbb{R}^m$, $k = 1,\dots,n$;

(H3) (growth) $g^\xi(i,z) \leq C^\xi(|z|^p + 1)$ $\forall (i,z) \in \mathbb{Z}^n \times \mathbb{R}^m$, $\xi \in \mathbb{Z}^n$, with

$$\sum_{\xi\in\mathbb{Z}^n} C^\xi < +\infty.$$

For $A \in \mathcal{O}(\Omega)$ we denote by $E_\varepsilon(v, A)$ the 'localized version' of $E_\varepsilon(v)$ defined as in (2.2) replacing $R_\varepsilon^\xi(\Omega)$ by $R_\varepsilon^\xi(A)$. Eventually, given $\phi \in Lip(\mathbb{R}^n;\mathbb{R}^m)$, we set

$$\mathcal{A}_{\varepsilon,\phi}(A) := \{v \in \mathcal{A}_\varepsilon(\mathbb{R}^n) : \ v(\varepsilon i) = \phi(\varepsilon i) \text{ if } (\varepsilon i + [-\varepsilon,\varepsilon]^n) \cap A^c \neq \emptyset\}, \tag{2.5}$$

and, given $M \in \mathbb{M}^{m\times n}$, we denote by $\mathcal{A}_{\varepsilon,M}(A)$ the set defined in (2.5) with $\phi(x) = Mx$.

The following Theorem has been proved in [1].

Theorem 2.1 *Let E_ε be defined by (2.2) and let g^ξ satisfy (H1), (H2) and (H3). Then (E_ε) Γ-converges, with respect to the $L^p(\Omega;\mathbb{R}^m)$ topology, to the functional $E : L^p(\Omega;\mathbb{R}^m) \to [0,+\infty]$ defined by*

$$E(v) = \begin{cases} \displaystyle\int_\Omega g_{hom}(Dv)\,dx & \text{if } v \in W^{1,p}(\Omega;\mathbb{R}^m) \\ +\infty & \text{otherwise,} \end{cases}$$

where $g_{hom} : \mathbb{M}^{m\times n} \to [0,+\infty)$ is given by the following formula

$$g_{hom}(M) = \lim_{T\to+\infty} \frac{1}{T^n} \inf\{E_1(v, Q_T) : v \in \mathcal{A}_{1,M}(Q_T)\} \tag{2.6}$$

The formula defining g_{hom} in Theorem 2.1 can be simplified in some special case. In particular in [11], dealing with the one dimensional case and with energies accounting only for nearest and next-to-nearest neighbor interactions, it has been proven what follows.

Theorem 2.2 (nearest neighbors) *Let E_ε be defined by (2.2), with $n = 1$, $g^\xi \equiv 0$ if $\xi \neq e_1$ and $g^{e_1}(i, z) =: g(z)$ satisfying $g(z) \geq c(|z|^p - 1)$. Then the conclusions of Theorem 2.1 hold with*

$$g_{hom}(M) = g^{**}(M),$$

where g^{**} denotes the convex envelope of g.

Theorem 2.3 (next-to-nearest neighbors) *Let E_ε be defined by (2.2), with $n = 1$, $g^\xi \equiv 0$ if $\xi \neq e_1,\ e_2$ and $g^{e_1}(i, z) =: g_1(z)$, $g^{e_2}(i, z) =: g_2(z)$ satisfying $g_1(z) \geq c(|z|^p - 1)$. Then the conclusions of Theorem 2.1 hold with*

$$g_{hom}(z) = \hat{g}^{**}(z),$$

where

$$\hat{g}(z) := g_2(z) + \frac{1}{2}\inf\{g_1(z_1) + g_1(z_2) :\ z_1 + z_2 = 2z\}. \tag{2.7}$$

Let us give also the definition of 'Cauchy-Born' states, which, loosely speaking, are those deformations at macro level to which does not correspond any relaxation at microscopic scale (see for example [16]).

Definition 2.4 (Cauchy-Born states) *$M \in \mathbb{M}^{m\times n}$ is said to be a Cauchy-Born state of E_ε if*

$$g_{hom}(M) = \lim_{T\to+\infty} \frac{1}{T^n} E_1(M\cdot, Q_T).$$

2.2 Technical lemma

In this section we recall a technical result proved by Ansini-Braides in [3] which allows to modify sequences of functions near the balls $B_i^\varepsilon := B_{r_\varepsilon}(\varepsilon i)$, where $i \in \mathbb{Z}^n$ and $0 < r_\varepsilon \ll \varepsilon$. Its proof is close in spirit to the method introduced by De Giorgi to match boundary conditions for minimizing sequences.

Let ε_j be a sequence of positive numbers converging to 0, and let $f_j : \mathbb{R}^n \times \mathbb{M}^{m\times n} \to [0, +\infty)$ be Borel functions satisfying the growth conditions uniformly in j; *i.e.*,

$$c_1(|M|^p - 1) \leq f_j(x, M) \leq c_2(|M|^p + 1)$$

for every $M \in \mathbb{M}^{m\times n}$, $j \in \mathbb{N}$ and with c_1, c_2 strictly positive. Note that sometimes we use the notation $\varepsilon = \varepsilon_j$ not to overburden notation.

Lemma 2.5 (Lemma 3.1 [3]) *Let u_j converge weakly to u in $W^{1,p}(\Omega;\mathbb{R}^m)$, and let*

$$Z_j = \{i \in \mathbb{Z}^n : \ \mathrm{dist}\,(\varepsilon i, \mathbb{R}^n \setminus \Omega) > \varepsilon_j\}. \tag{2.8}$$

Let $k \in \mathbb{N}$ be fixed. Let ρ_j be a sequence of positive numbers with $\rho_j < \varepsilon_j/2$. For all $i \in Z_j$ there exists $k_i \in \{0, \dots, k-1\}$ such that, having set

$$C_i^j = \Big\{x \in \Omega : \ 2^{-k_i-1}\rho_j < |x - \varepsilon i| < 2^{-k_i}\rho_j\Big\},$$

$$u_j^i = |C_i^j|^{-1} \int_{C_i^j} u_j \, dx \quad (\textit{the } \text{mean value } \textit{of } u_j \textit{ on } C_i^j), \tag{2.9}$$

and

$$\rho_j^i = \frac{3}{4} 2^{-k_i} \rho_j \quad (\textit{the } \text{middle radius } \textit{of } C_i^j), \tag{2.10}$$

there exists a sequence (w_j), with $w_j \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ such that

$$w_j = u_j \ \textit{on } \Omega \setminus \bigcup_{i \in Z_j} C_i^j \tag{2.11}$$

$$w_j(x) = u_j^i \ \textit{if } |x - \varepsilon i| = \rho_j^i \tag{2.12}$$

and

$$\Big| \int_\Omega (f_j(x, Dw_j) - f_j(x, Du_j)) \, dx \Big| \le c \frac{1}{k}.$$

3 The model

In this section we introduce the class of energies we are going to consider in the rest of the paper. In what follows Ω will denote a bounded open set of $\mathbb{R}^n$ with Lipschitz boundary. Given $\varepsilon > 0$, let

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{i \in \mathbb{Z}^n} \overline{B_i^\varepsilon},$$

where B_i^ε denotes the open ball of center εi and radius $0 < r_\varepsilon \ll \varepsilon$. Given $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, we set for $i \in \mathbb{Z}^n \cap \frac{1}{\varepsilon}\Omega$

$$\bar{u}(\varepsilon i) = \fint_{B_i^\varepsilon \cap \Omega} u(x) \, dx. \tag{3.1}$$

We then consider the family of functionals $F_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \to [0, +\infty)$ defined as

$$F_\varepsilon(u) \quad := \quad \int_\Omega f_\varepsilon(x, Du) \, dx + \sum_{\xi \in \mathbb{Z}^n} \sum_{i \in R_\varepsilon^\xi(\Omega)} \varepsilon^n g^\xi(i, (D_\varepsilon^\xi \bar{u}(\varepsilon i))$$

$$=: \quad F^c_\varepsilon(u) + F^d_\varepsilon(u)\,, \tag{3.2}$$

where

$$f_\varepsilon(x, M) = \begin{cases} f^1(M) & \text{if } x \in \Omega_\varepsilon \\ f^2(M) & \text{otherwise,} \end{cases} \tag{3.3}$$

and $D^\xi_\varepsilon \bar{u}$ and $R^\xi_\varepsilon(\Omega)$ are defined by (2.3) and (2.4), respectively. Moreover, we assume that $f^1,\ f^2 : \mathbb{M}^{m\times n} \to [0, +\infty)$ satisfy

$$c_1(|M|^p - 1) \leq f^h(M) \leq c_2(|M|^p + 1) \quad \text{for every } M \in \mathbb{M}^{m\times n},\ h = 1, 2 \tag{3.4}$$

for some positive constants $c_1,\ c_2$ and $p > 1$, and the functions $g^\xi : \mathbb{R}^m \to [0, +\infty)$, $\xi \in \mathbb{Z}^n$ satisfy the assumptions (H1), (H2) and (H3) introduced in Section 2.1.

We may consider also a variant of the energies F_ε which allows only for 'rigid' displacements of the interacting particles. To this end, set

$$W^{1,p}_{\mathcal{R},\varepsilon}(\Omega; \mathbb{R}^m) = \{u \in W^{1,p}(\Omega; \mathbb{R}^m) \ :\ Du \equiv 0 \text{ on } \overline{B^\varepsilon_i} \cap \Omega\,,\ i \in \mathbb{Z}^n\}$$

we define $F^{\mathcal{R}}_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \to [0, +\infty]$ as

$$F^{\mathcal{R}}_\varepsilon(u) = \begin{cases} F_\varepsilon(u) & \text{if } u \in W^{1,p}_{\mathcal{R},\varepsilon}(\Omega; \mathbb{R}^m) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.5}$$

We may let the energy functionals of the system depend explicitly on the displacement of the interacting particles occupying the region $\cup_i B^i_\varepsilon \cap \Omega$. To this end, given $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ we define $v(u) \in \mathcal{A}_\varepsilon(\Omega)$ as follows

$$v(u)(\varepsilon i) = \bar{u}(\varepsilon i)\,,\ \varepsilon i \in \varepsilon\mathbb{Z}^n \cap \Omega. \tag{3.6}$$

Then, with a slight abuse of notation, we may extend $F_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \to [0, +\infty)$ to a functional $F_\varepsilon : [L^p(\Omega; \mathbb{R}^m)]^2 \to [0, +\infty]$ as

$$F_\varepsilon(u, v) = \begin{cases} F_\varepsilon(u) & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^m) \text{ and } v = v(u) \\ +\infty & \text{otherwise}\,. \end{cases} \tag{3.7}$$

Similarly, we extend $F^{\mathcal{R}}_\varepsilon$ to a functional $F^{\mathcal{R}}_\varepsilon : [L^p(\Omega; \mathbb{R}^m)]^2 \to [0, +\infty]$ as

$$F^{\mathcal{R}}_\varepsilon(u, v) = \begin{cases} F_\varepsilon(u) & \text{if } u \in W^{1,p}_{\mathcal{R},\varepsilon}(\Omega; \mathbb{R}^m) \text{ and } v = v(u) \\ +\infty & \text{otherwise}\,. \end{cases} \tag{3.8}$$

The family of functionals above enjoys the following compactness property with respect to the strong topology in $[L^p(\Omega; \mathbb{R}^m)]^2$.

Theorem 3.1 (Compactness) *Let $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^m)$ be such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$ and let $\phi, \psi \in W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^m)$. Then*

i) If $u_\varepsilon \equiv \phi$ on $\partial\Omega$, there exists a subsequence (not relabeled) and $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ such that

$$u_\varepsilon \to u \qquad \text{in } L^p(\Omega; \mathbb{R}^m)\,.$$

ii) if $v(u_\varepsilon) \in \mathcal{A}_{\varepsilon,\psi}(\Omega)$, there exists a subsequence (not relabeled) and $v \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that

$$v(u_\varepsilon) \to v \qquad \text{in } L^p(\Omega;\mathbb{R}^m)\,.$$

Proof. The compactness of u_ε holds true, since, by the coercivity assumption (3.4) and Poincaré inequality we get that $\|Du_\varepsilon\|_{1,p}$ is equi-bounded. The compactness of $v(u_\varepsilon)$ can be proved as in [1] and follows from Frechét-Kolmogorov Theorem, since there holds

$$\lim_{|h|\to 0}\sup_{\varepsilon}\|\tau_h\, v(u_\varepsilon) - v(u_\varepsilon)\|_{L^p(\mathbb{R}^n;\mathbb{R}^m)} = 0,$$

where we have set

$$(\tau_h\, v)(x) = v(x+h),\; x, h \in \mathbb{R}^n$$

(see the proof of Corollary 3.11 in [1]). □

In the following sections we will study the Γ-limit of F_ε with respect to strong converging sequences $(u_\varepsilon, v(u_\varepsilon))$ in $[L^p(\Omega;\mathbb{R}^m)]^2$. In Sections 4 and 5 we will see that for $p \le n$ and if the size r_ε of the particles is not greater than a suitable critical value, we may have an effective decoupling of continuous and discrete displacements in the limit. Such results may be employed to derive the Γ-limit of F_ε when we take into account only the strong convergence in $L^p(\Omega;\mathbb{R}^m)$ of the continuous displacements u_ε or the discrete displacements $v(u_\varepsilon)$ (see Section 7).

4 Growth conditions with $1 < p < n$

In this section we state and prove the main results when the growth conditions on f_ε and g^ξ are satisfied with $1 < p < n$. We distinguish three different cases: critical, subcritical and, supercritical, depending on the size of the particles.

4.1 Critical case

In the critical case we highlight the effects of the interactions between the continuum and the system of the particles through the appearance of an extra term in the limit energy. The size of the particles, which produces such phenomena, turns out to be of order $\varepsilon^{n/n-p}$ (critical size).

Theorem 4.1 *Let $1 < p < n$. Let ε_j be a sequence of strictly positive numbers converging to 0 and let F_j defined by the formula (3.7), with ε_j in place of ε and $r_{\varepsilon_j} := \varepsilon_j^{n/n-p}$. Then, upon possibly extracting a subsequence, for all $M \in \mathbb{M}^{m\times n}$ there exists the limit*

$$h(x,M) = \lim_{j\to+\infty} \varepsilon_j^{\frac{np}{n-p}} Q\tilde{f}\Big(x, \varepsilon_j^{-\frac{n}{n-p}} M\Big), \tag{4.1}$$

where $Q\tilde{f}$ denotes the quasiconvexification of $\tilde{f}$ defined by

$$\tilde{f}(x,M) = \begin{cases} f^1(M) & x \in B_1^c(0) \\ f^2(M) & x \in B_1(0)\,. \end{cases} \tag{4.2}$$

Moreover, the value

$$\varphi(z) = \inf\Big\{ \int_{\mathbb{R}^n} h(x, D\zeta)dx : \zeta \in W^{1,p}(\mathbb{R}^n;\mathbb{R}^m),\ \fint_{B_1(0)} \zeta = z \Big\} \tag{4.3}$$

is well defined for all $z \in \mathbb{R}^m$ and the sequence of functionals F_j Γ-converges, with respect to the $[L^p(\Omega;\mathbb{R}^m)]^2$-topology, to the functional $F : [L^p(\Omega;\mathbb{R}^m)]^2 \to [0,+\infty]$ defined by

$$F(u,v) = \begin{cases} \displaystyle\int_\Omega Qf^1(Du) + \int_\Omega g_{\rm hom}(Dv)\,dx + \int_\Omega \varphi(v-u)\,dx\,, & \\ & \text{if } (u,v) \in [W^{1,p}(\Omega;\mathbb{R}^m)]^2 \\ +\infty & \text{otherwise} \end{cases} \tag{4.4}$$

where $g_{\rm hom}$ is the homogenized integrand given by (2.6) with F_1^d in place of E_1.

The proof of Theorem 4.1 will be done in Section 4.2.

Theorem 4.2 (Rigid particles) *Under the same assumptions of Theorem 4.1, the functionals $F_j^{\mathcal{R}}$ defined by the formula (3.8), with ε_j in place of ε and $r_{\varepsilon_j} := \varepsilon_j^{n/n-p}$, Γ-converge, with respect to the $[L^p(\Omega;\mathbb{R}^m)]^2$-topology, to the functional $F^{\mathcal{R}} : [L^p(\Omega;\mathbb{R}^m)]^2 \to [0,+\infty]$ defined by*

$$F^{\mathcal{R}}(u,v) = \begin{cases} \displaystyle\int_\Omega Qf^1(Du) + \int_\Omega g_{\rm hom}(Dv)\,dx + \int_\Omega \varphi^{\mathcal{R}}(v-u)\,dx\,, & \\ & \text{if } (u,v) \in [W^{1,p}(\Omega;\mathbb{R}^m)]^2 \\ +\infty & \text{otherwise} \end{cases}$$

where

$$\varphi^{\mathcal{R}}(z) = \inf\Big\{ \int_{\mathbb{R}^n} h(x, D\zeta)\,dx : \zeta \in W^{1,p}(\mathbb{R}^n;\mathbb{R}^m)\,,\ \zeta = z \text{ on } B_1(0) \Big\}\,. \tag{4.5}$$

We omit the proof of Theorem 4.2 since it follows step by step the proof of Theorem 4.1 with natural changes due to the constraint on the domain of $F_\varepsilon^{\mathcal{R}}$.

Corollary 4.3 *If f^1, f^2 are positively homogeneous of degree p then the Γ-limits in Theorems 4.1, 4.2 are independent of the subsequence and*

$$\varphi(z) = \inf\Big\{ \int_{\mathbb{R}^n} \tilde{f}(x, D\zeta)dx : \zeta \in W^{1,p}(\mathbb{R}^n;\mathbb{R}^m)\,,\ \fint_{B_1(0)} \zeta = z \Big\}$$

$$\varphi^{\mathcal{R}}(z) = \inf\Big\{\int_{\mathbb{R}^n} \tilde{f}(x, D\zeta)\, dx : \zeta \in W^{1,p}(\mathbb{R}^n;\mathbb{R}^m)\,,\ \zeta = z \ \mathit{on}\ B_1(0)\Big\}\,. \tag{4.6}$$

for all $z \in \mathbb{R}^m$.

Proof. It suffices to remark that in this case formula (4.1) gives $h = Q\tilde{f}$ and that we may replace $Q\tilde{f}$ by $\tilde{f}$ in (4.3) and (4.5). □

Remark 4.4 If $f^1(M) = f^2(M) = |M|^p$ then

$$\varphi(z) = \overline{C}_p(B_1(0);\mathbb{R}^n)|z|^p\,,$$

$$\varphi^{\mathcal{R}}(z) = C_p(B_1(0);\mathbb{R}^n)|z|^p\,,$$

where

$$C_p(B_1;\mathbb{R}^n) = \inf\Big\{\int_{\mathbb{R}^n} |D\zeta|^p\, dx : \zeta \in W^{1,p}(\mathbb{R}^n)\,,\ \zeta = 1 \text{ on } B_1(0)\Big\}$$

and

$$\overline{C}_p(B_1;\mathbb{R}^n) = \inf\Big\{\int_{\mathbb{R}^n} |D\zeta|^p\, dx : \zeta \in W^{1,p}(\mathbb{R}^n)\,,\ \fint_{B_1(0)} \zeta\, dx = 1\Big\}\,.$$

Note that, $\overline{C}_p(B_1;\mathbb{R}^n) \le C_p(B_1;\mathbb{R}^n)$. In the next proposition we show that the strict inequality holds true. Hence, in our model the limiting energy keeps memory of the deformability of the particles.

Proposition 4.5 *Let* $1 < p < n$. *Then*

$$\overline{C}_p(B_1;\mathbb{R}^n) < C_p(B_1;\mathbb{R}^n)\,. \tag{4.7}$$

Proof. We can explicitly compute $\overline{C}_p(B_1;\mathbb{R}^n)$ by proceeding similarly to the computation of the classical capacity $C_p(B_1;\mathbb{R}^n)$. Indeed, one easily get that

$$\overline{C}_p(B_1;\mathbb{R}^n) = \lim_{R\to+\infty} \overline{C}_p(B_1;B_R) < +\infty, \tag{4.8}$$

where

$$\overline{C}_p(B_1;B_R) = \inf\Big\{\int_{B_R} |D\zeta|^p\, dx : \zeta \in W^{1,p}_0(B_R)\,,\ \fint_{B_1(0)} \zeta\, dx = 1\Big\}. \tag{4.9}$$

In order to compute $\overline{C}_p(B_1;B_R)$, by rotation invariance, one reduce to solve a one-dimensional minimum problem subjected to a mean constraint that can be solved by using a Lagrange multipliers method. Classical computations lead to

$$\overline{C}_p(B_1;\mathbb{R}^n) = \sigma_{n-1}\Big(\frac{n-p}{p-1}\Big)^{(p-1)}\Big(1+(p-1)^{1/(p-1)}(n-p)\frac{(n(p-1)+2)^{1/p-1}}{(n(p-1)+p)^{p/p-1}}\Big)^{(1-p)}\,.$$

Since

$$C_p(B_1(0);\mathbb{R}^n) = \sigma_{n-1}\Big(\frac{n-p}{p-1}\Big)^{(p-1)}\,,$$

then, (4.7) holds true. □

Remark 4.6 If $p = 2$ and $n \geq 3$, then

$$\overline{C}_2(B_1; \mathbb{R}^n) = \frac{2+n}{2n}\, C_2(B_1; \mathbb{R}^n).$$

Moreover, the capacitary potential for $\overline{C}_2(B_1; \mathbb{R}^n)$ is $\overline{\zeta}(x) = \overline{v}(|x|)$, where

$$\overline{v}(\rho) = \begin{cases} \left(\dfrac{4-n^2}{4n}\right)\rho^2 + \left(\dfrac{n+2}{4}\right) & \text{if } 0 \leq \rho \leq 1 \\ \left(\dfrac{n+2}{2n}\right)\dfrac{1}{\rho^{n-2}} & \text{if } \rho > 1, \end{cases}$$

while the capacitary potential for $C_2(B_1; \mathbb{R}^n)$ is $\zeta(x) = v(|x|)$, where

$$v(\rho) = \begin{cases} 1 & \text{if } 0 \leq \rho \leq 1 \\ \dfrac{1}{\rho^{n-2}} & \text{if } \rho > 1 \end{cases}$$

(see Figure 2).

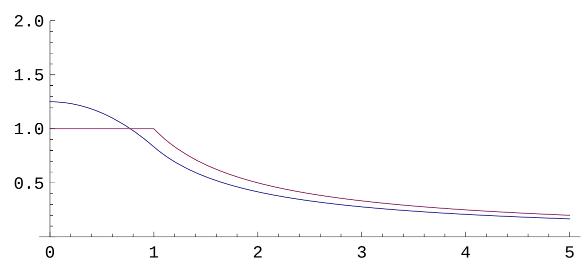


Figure 2: The capacitary potentials for $C_2(B_1; \mathbb{R}^n)$ (in red) and $\overline{C}_2(B_1; \mathbb{R}^n)$ (in blue).

4.2 Proof of Theorem 4.1

The main point in the proof of Theorem 4.1 is the computation of the energetic contribution due to the interaction between the system of particles and the continuos media. We will show that it is described by a capacitary type term which reminds that obtained in the theory of perforated domains (see for example [13],[3]) with the difference that our test functions should satisfy a mean constraint. Our analysis relies on the argument introduced in [3] to study the asymptotic behavior of integral functionals on perforated domains (see also [2]). Even if some steps of the proof closely follows those in [3], we believe that referring to that paper would require a huge effort of the reader to recover the whole proof, taking also into account the Erratum [4]. For this reason, we prefer to provide a self-contained and detailed proof. We start by some preliminary results contained in the next proposition.

Proposition 4.7 *Let $\varphi_{\gamma,j}$ be given by*

$$\varphi_{\gamma,j}(z) = \inf\Big\{\int_{B_{\gamma N_j}(0)} h_j(x, D\zeta)\, dx :\ \zeta \in W_0^{1,p}(B_{\gamma N_j}(0);\mathbb{R}^m),\ \fint_{B_1(0)} \zeta\, dx = z\Big\}\,, \tag{4.10}$$

with $N_j = \varepsilon_j^{-p/n-p}$ and $h_j : \mathbb{R}^n \times \mathbb{M}^{m\times n} \to [0,+\infty)$ defined by

$$h_j(x, M) := \varepsilon_j^{\frac{np}{n-p}} Q\tilde{f}\Big(x, \varepsilon_j^{-\frac{n}{n-p}} M\Big)\,.$$

Then

(i) there exists a constant $c > 0$ (independent of j and γ) such that

$$0 \leq \varphi_{\gamma,j}(z) \leq c(\gamma^n + |z|^p) \tag{4.11}$$

for every $z \in \mathbb{R}^m$, $j \in \mathbb{N}$ and $\gamma > 0$;

(ii) there exists a constant $c > 0$ (independent of j and γ) such that

$$|\varphi_{\gamma,j}(w) - \varphi_{\gamma,j}(z)| \leq c\,|w-z|\Big(\varepsilon_j^{\,n(p-1)/n-p} + \gamma^{n(p-1)/p} + |z|^{p-1} + |w|^{p-1}\Big) \tag{4.12}$$

for every $z, w \in \mathbb{R}^m$, $j \in \mathbb{N}$ and $\gamma > 0$;

(iii) up to subsequences, h_j is pointwise converging. Moreover, for the same subsequence (not relabeled), $\varphi_{\gamma,j}$ converges locally uniformly on $\mathbb{R}^m$, as $j \to +\infty$ and $\gamma \to 0^+$, to the function $\varphi : \mathbb{R}^m \to [0,+\infty)$ given by (4.3) and satisfying

$$0 \leq \varphi(z) \leq c|z|^p\,, \qquad |\varphi(z) - \varphi(w)| \leq c\,|z-w|\big(|z|^{p-1} + |w|^{p-1}\big) \tag{4.13}$$

for every $z, w \in \mathbb{R}^m$.

Proof. (i) By growth conditions we have that

$$\begin{aligned}\varphi_{\gamma,j}(z) &\leq c_2|B_{\gamma N_j}(0)|\varepsilon_j^{\,np/n-p} \\ &\quad + c_2 \inf\Big\{\int_{B_{\gamma N_j}(0)} |D\zeta|^p\, dx :\ \zeta \in W_0^{1,p}(B_{\gamma N_j}(0);\mathbb{R}^m),\ \fint_{B_1(0)} \zeta\, dx = z\Big\} \\ &\leq c\Big(\gamma^n + \overline{C}_p(B_1; B_{\gamma N_j})|z|^p\Big)\end{aligned}$$

By (4.8) we conclude the proof.

(ii) For every fixed $\eta > 0$, there exists $\zeta_j^\gamma \in W_0^{1,p}(B_{\gamma N_j}(0);\mathbb{R}^m)$ such that $\fint_{B_1(0)} \zeta_j^\gamma\, dx = z$ and

$$\int_{B_{\gamma N_j}(0)} h_j(x, D\zeta_j^\gamma)\, dx \leq \varphi_{\gamma,j}(z) + \eta\,. \tag{4.14}$$

Now we may reason as in Proposition 5.1 [2]. We want to modify ζ_j^γ in order to get an admissible test function for $\varphi_{\gamma,j}(w)$. More precisely, we just have to modify ζ_j^γ on a neighborhood of B_1 to change the mean value z into w on B_1. To this aim we introduce a cut-off function $\theta \in C_c^\infty(\mathbb{R}^n;[0,1])$ such that

$$\theta(x) = \begin{cases} 1 & \text{if } x \in B_1, \\ \\ 0 & \text{if } x \notin B_2. \end{cases} \qquad \text{and } |D\theta| \leq c$$

Hence, we define $\tilde{\zeta}_j^\gamma$ as follows

$$\tilde{\zeta}_j^\gamma(x) := \zeta_j^\gamma(x) + \theta(x)(w - z)$$

for every $x \in B_{\gamma N_j}(0)$. Note that $\tilde{\zeta}_j^\gamma \in W_0^{1,p}(B_{\gamma N_j}(0);\mathbb{R}^m)$ and $-\!\!\!\!\!\!\int_{B_1} \tilde{\zeta}_j^\gamma(x)\,dx = w$. Since $Q\tilde{f}(x,\cdot)$ is quasiconvex and satisfies uniformly a growth condition of order p, by definition h_j the following inequality holds

$$|h_j(x,A) - h_j(x,B)| \leq c\Big({\varepsilon_j}^{n(p-1)/n-p} + |A|^{p-1} + |B|^{p-1}\Big)|A - B|\,. \tag{4.15}$$

By Hölder inequality, we obtain that

$$\begin{aligned}
& \varphi_{\gamma,j}(w) - \varphi_{\gamma,j}(z) - \eta \\
\leq\ & \int_{B_{\gamma N_j}(0)} h_j(x, D\tilde{\zeta}_j^\gamma) - h_j(x, D\zeta_j^\gamma)\,dx \\
\leq\ & c\int_{B_{\gamma N_j}(0)} \Big({\varepsilon_j}^{n(p-1)/n-p} + |D\tilde{\zeta}_j^\gamma|^{p-1} + |D\zeta_j^\gamma|^{p-1}\Big)|D\tilde{\zeta}_j^\gamma - D\zeta_j^\gamma|\,dx \\
\leq\ & c\int_{B_{\gamma N_j}(0)} \Big({\varepsilon_j}^{n(p-1)/n-p} + |D\zeta_j^\gamma|^{p-1} + |w-z|^{p-1}|D\theta|^{p-1}\Big)|D\theta|\,|w-z|\,dx \\
\leq\ & c\Big(|w-z|^p \int_{B_{\gamma N_j}(0)} |D\theta|^p\,dx + {\varepsilon_j}^{n(p-1)/n-p}|w-z| \int_{B_{\gamma N_j}(0)} |D\theta|\,dx \\
& + |w-z| \int_{B_{\gamma N_j}(0)} |D\zeta_j^\gamma|^{p-1}|D\theta|\,dx\Big) \\
\leq\ & c\Big(|w-z|^p \int_{B_{\gamma N_j}(0)} |D\theta|^p\,dx + {\varepsilon_j}^{n(p-1)/n-p}|w-z| \int_{B_{\gamma N_j}(0)} |D\theta|\,dx + \\
& |w-z|\|D\theta\|_{L^p(B_{\gamma N_j};\mathbb{R}^m)}\, \|D\zeta_j^\gamma\|^{p-1}_{L^p(B_{\gamma N_j};\mathbb{R}^m)}\Big)\,.
\end{aligned}$$

Since $\gamma N_j > 2$ (for fixed γ and j large enough) and $\mathrm{Supp}(\theta) \subset B_2$, we obtain that

$$\begin{aligned}
& \varphi_{\gamma,j}(w) - \varphi_{\gamma,j}(z) - \eta \\
\leq\ & c\,|w-z|\Big(|w-z|^{p-1} + {\varepsilon_j}^{n(p-1)/n-p} + \|D\zeta_j^\gamma\|^{p-1}_{L^p(B_{\gamma N_j};\mathbb{R}^m)}\Big)\,.
\end{aligned}$$

By the growth conditions on h_j, (4.14) and (i)

$$\begin{aligned} c_1 \int_{B_{\gamma N_j}(0)} |D\zeta_j^\gamma|^p\,dx &\le \int_{B_{\gamma N_j}(0)} (h_j(x, D\zeta_j^\gamma) + c_1 {\varepsilon_j}^{np/n-p})\,dx \\ &\le \varphi_{\gamma,j}(z) + \eta + O(\gamma^n) \\ &\le c(|z|^p + \gamma^n) + \eta\,. \end{aligned} \tag{4.16}$$

Hence,

$$\varphi_{\gamma,j}(w) - \varphi_{\gamma,j}(z) \le c\,|w-z|\Big(|z|^{p-1} + |w|^{p-1} + {\varepsilon_j}^{n(p-1)/n-p} + \gamma^{n(p-1)/p} + \eta^{(p-1)/p}\Big) + \eta\,,$$

by the arbitrariness of η and symmetric argument we get the estimate in (ii).

(iii) By (4.15) and Ascoli-Arzela's Theorem we have that, up to subsequences, h_j is pointwise converging. In the following Steps 1 and 2 we prove that, for the same subsequence (not relabeled), $\varphi_{\gamma,j}$ pointwise converges to φ as $j \to +\infty$ and $\gamma \to 0^+$. Moreover, by (ii) and Ascoli-Arzela's Theorem, we deduce the uniform convergence of $\varphi_{\gamma,j}$ to φ on compact sets of $\mathbb{R}^m$. Finally, passing to the limit, as $j \to +\infty$ and $\gamma \to 0^+$, in (4.11) and (4.12), we get that $\varphi : \mathbb{R}^m \to [0, +\infty)$ satisfies (4.13).

Step 1: pointwise convergence of $\varphi_{\gamma,j}(z)$. In this step we prove that for every $z \in \mathbb{R}^m$ we have

$$\lim_{\gamma \to 0^+} \lim_{j \to +\infty} \varphi_{\gamma,j}(z) = \psi(z)$$

where

$$\psi(z) = \inf\Big\{ \int_{\mathbb{R}^n} h(x, D\zeta)dx : \zeta \in L^{p*}(\mathbb{R}^n;\mathbb{R}^m),\ D\zeta \in L^p(\mathbb{R}^n;\mathbb{M}^{m\times n})\,,\ \fint_{B_1(0)} \zeta = z \Big\}\,.$$

We first deal with the liminf inequality as j tends to $+\infty$. For a fixed $\eta > 0$, let $\zeta_j^\gamma \in W_0^{1,p}(B_{\gamma N_j}(0);\mathbb{R}^m)$ with $\fint_{B_1(0)} \zeta_j^\gamma\,dx = z$ such that

$$\int_{B_{\gamma N_j}(0)} h_j(x, D\zeta_j^\gamma)\,dx \le \varphi_{\gamma,j}(z) + \eta\,.$$

Reasoning as in (4.16) we get that

$$\int_{B_{\gamma N_j}(0)} |D\zeta_j^\gamma|^p\,dx \le c$$

uniformly in $\gamma > 0$ and $j \in \mathbb{N}$. We may extend ζ_j^γ to 0 on $\mathbb{R}^n$; hence,

$$\Big(\int_{\mathbb{R}^n} |\zeta_j^\gamma|^{p*}\,dx\Big)^{1/p*} \le c\Big(\int_{\mathbb{R}^n} |D\zeta_j^\gamma|^p\,dx\Big)^{1/p}$$

and the sequence ζ_j^γ converges, up to subsequences, weakly to ζ in $L^{p*}(\mathbb{R}^n;\mathbb{R}^m)$. In particular, $\zeta_j^\gamma \rightharpoonup \zeta$ in $W^{1,p}(A;\mathbb{R}^m)$ for every A bounded set of $\mathbb{R}^n$ with $D\zeta \in L^p(\mathbb{R}^n;\mathbb{M}^{m\times n})$. Since $h_j(x,\cdot) \to h(x,\cdot)$ pointwise , then

$$\int_A h(x, D\zeta)\,dx = \Gamma(L^p)\text{ - }\lim_{j\to+\infty}\int_A h_j(x,D\zeta)\,dx$$

for every $\zeta \in W^{1,p}(A;\mathbb{R}^m)$ and for all A bounded open sets of $\mathbb{R}^n$ (see e.g. Proposition 12.8 in [9]). Hence,

$$\begin{aligned}\liminf_{j\to+\infty}\int_{B_{\gamma N_j}(0)} h_j(x,D\zeta_j^\gamma)\,dx &\geq \liminf_{j\to+\infty}\int_{B_{\gamma N}(0)} h_j(x,D\zeta_j^\gamma)\,dx\\ &\geq \int_{B_{\gamma N}(0)} h(x,D\zeta)\,dx\end{aligned}$$

for every fixed N. Passing to the limit as $N\to+\infty$ we get

$$\begin{aligned}&\liminf_{j\to+\infty}\int_{B_{\gamma N_j}(0)} h_j(x,D\zeta_j^\gamma)\,dx\\ \geq\ &\inf\Big\{\int_{\mathbb{R}^n} h(x,D\zeta)dx : \zeta\in L^{p*}(\mathbb{R}^n;\mathbb{R}^m),\ D\zeta\in L^p(\mathbb{R}^n;\mathbb{M}^{m\times n})\,,\ \fint_{B_1(0)}\zeta = z\Big\}\,,\end{aligned}$$

which implies, by the arbitrariness of η, the liminf inequality

$$\liminf_{j\to+\infty}\varphi_{\gamma,j}(z) \geq \psi(z)\,,$$

for every $\gamma > 0$.

We now prove the limsup inequality. For a fixed $\eta > 0$, let $\zeta \in L^{p*}(\mathbb{R}^n;\mathbb{R}^m)$ with $D\zeta \in L^p(\mathbb{R}^n;\mathbb{M}^{m\times n})$ and $\fint_{B_1(0)}\zeta\,dx = z$ such that

$$\int_{\mathbb{R}^n} h(x,D\zeta)\,dx \leq \psi(z) + \eta\,.$$

Let $\phi_j^\gamma \in C_0^\infty(B_{\gamma N_j}(0))$ such that $\phi_j^\gamma \equiv 1$ on $B_{\gamma N_j/2}(0)$ and $|D\phi_j^\gamma| \leq (c/\gamma N_j)$. Then, defining $\zeta_j^\gamma = \phi_j^\gamma\zeta$ we get that $\zeta_j^\gamma \in W_0^{1,p}(B_{\gamma N_j}(0);\mathbb{R}^m)$ and $\fint_{B_1(0)}\zeta_j^\gamma\,dx = z$. Hence, ζ_j^γ is a test function for $\varphi_{\gamma,j}(z)$ and

$$\begin{aligned}\varphi_{\gamma,j}(z) &\leq \int_{B_{\gamma N_j}(0)} h_j(x,D\zeta_j^\gamma)\,dx\\ &\leq \int_{B_{\gamma N_j/2}(0)} h_j(x,D\zeta_j^\gamma)\,dx\end{aligned}$$

$$+c_2\Big(\varepsilon_j^{\frac{np}{n-p}}|B_{\gamma N_j}(0)\setminus B_{\gamma N_j/2}(0)| + \int_{B_{\gamma N_j}\setminus B_{\gamma N_j/2}} |D\zeta_j^\gamma|^p\,dx\Big)$$

$$\leq \int_{B_{\gamma N_j/2}(0)} h_j(x,D\zeta)\,dx + \tilde{c}\Big(\gamma^n + \int_{B_{\gamma N_j}\setminus B_{\gamma N_j/2}} |D\phi_j^\gamma|^p|\zeta|^p + |D\zeta|^p\,dx\Big)\,.$$

By Hölder inequality we have that

$$\int_{B_{\gamma N_j}\setminus B_{\gamma N_j/2}} |D\phi_j^\gamma|^p|\zeta|^p\,dx \leq c\Big(\int_{B_{\gamma N_j}\setminus B_{\gamma N_j/2}} |\zeta|^{p*}\,dx\Big)^{p/p*};$$

moreover, since $D\zeta\in L^p(\mathbb{R}^n;\mathbb{M}^{m\times n})$ we may conclude that

$$\lim_{j\to+\infty}\Big(\int_{B_{\gamma N_j}\setminus B_{\gamma N_j/2}} |D\phi_j^\gamma|^p|\zeta|^p + |D\zeta|^p\,dx\Big) = 0\,.$$

Passing to the limit as j tends to $+\infty$ we get

$$\limsup_{j\to+\infty}\varphi_{\gamma,j}(z) \leq \limsup_{j\to+\infty}\int_{B_{\gamma N_j/2}(0)} h_j(x,D\zeta)\,dx + O(\gamma^n)\,.$$

By Fatou's lemma we have that

$$\begin{aligned}\limsup_{j\to+\infty}\int_{B_{\gamma N_j/2}(0)} h_j(x,D\zeta)\,dx &\leq \limsup_{j\to+\infty}\int_{\mathbb{R}^n}\chi_{B_{\gamma N_j/2}(0)}h_j(x,D\zeta)\,dx + O(\gamma^n)\\ &= \int_{\mathbb{R}^n} h(x,D\zeta)dx + O(\gamma^n)\\ &\leq \psi(z)+\eta+O(\gamma^n)\,.\end{aligned}$$

Gathering liminf and limsup inequality, by the arbitrariness of η, we have that

$$\lim_{\gamma\to0^+}\lim_{j\to+\infty}\varphi_{\gamma,j}(z) = \psi(z)\,.$$

Step 2: $\varphi(z)$ is the pointwise limit. In this step we prove that the pointwise limit $\psi(z)$ of $\varphi_{\gamma,j}(z)$, computed in Step 1, coincides with $\varphi(z)$ given by the formula (4.3).

The inequality $\psi(z)\leq\varphi(z)$ is trivial. To prove the other inequality, we first observe that by (3.4), (4.1) and (4.2) we have that $c_1|M|^p\leq h(x,M)\leq c_2|M|^p$; hence, $h(x,0)\equiv 0$. Moreover, we may repeat the proof of the limsup inequality in Step 1 by replacing $\varphi_{\gamma,j}(z)$ with φ, γN_j with $N>2$, and, ϕ_j^γ with $\phi_N\in C_0^\infty(B_N(0))$ such that $\phi_N\equiv1$ on $B_{N/2}(0)$ and $|D\phi_N|\leq(c/N)$. Therefore,

$$\varphi(z) \leq \int_{\mathbb{R}^n} h(x,D(\phi_N\zeta))dx = \int_{B_N(0)} h(x,D(\phi_N\zeta))dx$$

$$\begin{aligned}
&\leq \int_{B_{N/2}(0)} h(x, D\zeta)dx + o(1) \\
&\leq \int_{\mathbb{R}^n} h(x, D\zeta)dx + o(1) \\
&\leq \psi(z) + \eta + o(1).
\end{aligned}$$

By the arbitrariness of η, passing to the limit as $N \to +\infty$, we get the inequality. □

Remark 4.8 Proposition 4.7 still holds true with $\varphi_{\gamma,j}$ replaced by

$$\varphi^{\mathcal{R}}_{\gamma,j}(z) = \inf\Big\{\int_{B_{\gamma N_j}(0)} h_j(x, D\zeta)\, dx : \ \zeta \in W^{1,p}_0(B_{\gamma N_j}(0); \mathbb{R}^m), \ \zeta = z \text{ on } B_1\Big\}, \tag{4.17}$$

and $\varphi^{\mathcal{R}}$, defined in (4.6), in place of φ.

Proposition 4.9 (Liminf inequality) *We have*

$$\Gamma\text{-}\liminf_{j\to+\infty} F_j(u, v) \geq \int_\Omega Qf^1(Du)\, dx + \int_\Omega g_{\text{hom}}(Dv)\, dx + \int_\Omega \varphi(v-u)\, dx\,, \tag{4.18}$$

for all $(u, v) \in [W^{1,p}(\Omega; \mathbb{R}^m)]^2$.

Proof. Let $(u_j, v(u_j))$ be a sequence strongly converging to $(u, v) \in [W^{1,p}(\Omega; \mathbb{R}^m)]^2$ with $v(u_j)$ as in (3.6) (with u_j in place of u). Since

$$\liminf_{j\to+\infty} F_j(u_j, v(u_j)) \geq \liminf_{j\to+\infty} F^c_j(u_j) + \liminf_{j\to+\infty} F^d_j(u_j)$$

we get (4.18) by proving separately the liminf inequality for $F^c_j(u_j)$ and $F^d_j(u_j)$, respectively. By the Theorem 2.1 in [1] we have that

$$\liminf_{j\to+\infty} F^d_j(u_j) \geq \int_\Omega g_{\text{hom}}(Dv)\, dx\,.$$

It remains to prove the liminf inequality for the continuous part of the functional, F^c_j, which is organized in 4 steps. In particular, to study the contributions of the particles, B^ε_i, we follow the approach in [3] (see also [4] and [2]).

Step 1: contribution of Du_j "far" from the particles. Let

$$\rho_j = \gamma\varepsilon_j, \ \gamma < 1/2\,,$$

and let $E_j = E^{k,\gamma}_j$ be given by

$$E_j = \bigcup_{i\in Z_j} B^j_i, \qquad \text{where} \qquad B^j_i = B_{\rho^i_j}(\varepsilon i)$$

for all $i \in Z_j$ (Z_j given by (2.8) and ρ_j^i by (2.10)). In this step we study the contribution of the part of Du_j outside the set E_j. More precisely, we prove that for any fixed $k \in \mathbb{N}$ (as in Lemma 2.5) we have

$$\liminf_{j \to +\infty} \int_{\Omega \setminus E_j} f_j(x, Du_j)\, dx \geq \int_\Omega Qf^1(Du)\, dx - \frac{c}{k}\,. \tag{4.19}$$

To this aim we observe that $f_j(x, Du_j) = f^1(Du_j)$ for every $x \in \Omega \setminus E_j$. Let

$$\tilde{w}_j(x) = \begin{cases} u_j^i & \text{if } x \in B_i^j \\ w_j(x) & \text{if } x \in \Omega \setminus E_j, \end{cases}$$

with u_j^i as in (2.9) and w_j as in (2.11), (2.12). Reasoning as in [2] we may prove that the modified sequence $(\tilde{w}_j)$ converges, up to subsequences, to u weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$. In fact, since $\rho_j^i < \rho_j < \varepsilon_j/2$ we have that

$$\begin{aligned} \int_\Omega |\tilde{w}_j - u|^p\, dx &= \sum_{i \in Z_j} \int_{B_i^j} |u_j^i - u|^p\, dx + \int_{\Omega \setminus E_j} |w_j - u|^p\, dx \\ &\leq \sum_{i \in Z_j} \int_{Q_i^j} |u_j^i - u|^p\, dx + \int_\Omega |w_j - u|^p\, dx\,, \end{aligned}$$

where $Q_i^j = \varepsilon_j i + (-\varepsilon_j/2, \varepsilon_j/2)^n$. By Poincarè inequality we have that

$$\int_{Q_i^j} |u_j - u_j^i|^p\, dx \leq c\, \varepsilon_j^p \int_{Q_i^j} |Du_j|^p\, dx\,. \tag{4.20}$$

Therefore,

$$\int_\Omega |\tilde{w}_j - u|^p\, dx \leq c\, \varepsilon_j^p \int_\Omega |Du_j|^p\, dx + \int_\Omega |u_j - u|^p\, dx + \int_\Omega |w_j - u|^p\, dx\,.$$

Since $u_j \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^m)$, by Lemma 2.5, passing to the limit in the previous inequality we get the weak convergence of $\tilde{w}_j$ to u in $W^{1,p}(\Omega; \mathbb{R}^m)$. So that

$$\begin{aligned} \liminf_j \int_{\Omega \setminus E_j} f^1(Du_j)\, dx + \frac{c}{k} &\geq \liminf_j \int_{\Omega \setminus E_j} f^1(Dw_j)\, dx \\ &= \liminf_j \int_\Omega f^1(D\tilde{w}_j)\, dx \geq \int_\Omega Qf^1(Du)\, dx\,. \end{aligned}$$

Step 2: contribution on E_j. We now turn to the estimate of the contribution on E_j. With fixed $j \in \mathbb{N}$ and $i \in Z_j$, let

$$\zeta(y) = w_j\Big(\varepsilon i + \varepsilon_j^{n/(n-p)} y\Big)$$

be defined on $B_{\gamma N_j^i}(0)$ where $N_j^i = (3/4)2^{-k_i}\varepsilon_j{}^{-p/n-p}$, and extended to u_j^i on $B_{\gamma N_j}(0) \setminus B_{\gamma N_j^i}(0)$ where $N_j = \varepsilon_j{}^{-p/n-p}$. Note that

$$\zeta - u_j^i \in W_0^{1,p}(B_{\gamma N_j}(0);\mathbb{R}^m) \qquad \text{and} \qquad \fint_{B_1(0)} (\zeta - u_j^i)\, dx = (\bar{u}_j(i) - u_j^i)\,,$$

with $\bar{u}_j(i)$ as in (3.1). By a change of variables we obtain

$$\int_{B_i^j} f_j(x, Dw_j)\, dx \quad = \quad \varepsilon_j{}^n \int_{B_{\gamma N_j^i}(0)} \varepsilon_j{}^{np/n-p} f_j(\varepsilon i + \varepsilon_j{}^{n/n-p} y, \varepsilon_j{}^{-n/n-p} D\zeta)\, dy$$

where

$$f_j(\varepsilon i + \varepsilon_j{}^{n/n-p} y, \varepsilon_j^{-n/n-p}\xi) = \begin{cases} f^2(\varepsilon_j^{-n/n-p}\xi), & y \in B_1(0) \\ f^1(\varepsilon_j^{-n/n-p}\xi), & y \in B_{\gamma N_j^i}(0) \setminus B_1(0)\,. \end{cases}$$

Therefore, by (4.2) and $\tilde{f}(x,0) = 0$ we have

$$\begin{aligned} & \int_{B_i^j} f_j(x, Dw_j)\, dx \\ = \quad & \varepsilon_j{}^n \int_{B_{\gamma N_j}(0)} \varepsilon_j{}^{np/n-p} \tilde{f}(x, \varepsilon_j{}^{-n/n-p} D\zeta)\, dx \\ \geq \quad & \varepsilon_j{}^n \inf\Big\{ \int_{B_{\gamma N_j}(0)} \varepsilon_j{}^{np/n-p} \tilde{f}(x, \varepsilon_j{}^{-n/n-p} D\zeta)\, dx : \ \zeta - u_j^i \in W_0^{1,p}(B_{\gamma N_j}(0);\mathbb{R}^m), \\ & \qquad \fint_{B_1(0)} (\zeta - u_j^i)\, dx = (\bar{u}_j(i) - u_j^i) \Big\} \\ \geq \quad & \varepsilon_j{}^n \inf\Big\{ \int_{B_{\gamma N_j}(0)} \varepsilon_j{}^{np/n-p} Q\tilde{f}(x, \varepsilon_j{}^{-n/n-p} D\zeta)\, dx : \ \zeta \in W_0^{1,p}(B_{\gamma N_j}(0);\mathbb{R}^m), \\ & \qquad \fint_{B_1(0)} \zeta\, dx = (\bar{u}_j(i) - u_j^i) \Big\} \\ = \quad & \varepsilon_j{}^n \varphi_{\gamma,j}(\bar{u}_j(i) - u_j^i)\,, \end{aligned} \tag{4.21}$$

where $\varphi_{\gamma,j}$ is defined by (4.10).

To conclude the estimate of the contribution on E_j we have to show that

$$\lim_{\gamma\to 0^+} \liminf_{j\to+\infty} \sum_{i\in Z_j} \varepsilon_j^n \varphi_{\gamma,j}(\bar{u}_j(i) - u_j^i) = \int_\Omega \varphi(v-u)\, dx\,. \tag{4.22}$$

Step 3: proof of (4.22) . Here we prove that

$$\lim_{\gamma\to 0^+} \limsup_{j\to+\infty} \int_\Omega \Big| \sum_{i\in Z_j} \varphi_{\gamma,j}(\bar{u}_j(i) - u_j^i)\chi_{Q_i^j} - \varphi(v-u) \Big|\, dx = 0\,,$$

from which (4.22) easily follows. To this aim we use the pointwise convergence of $\varphi_{\gamma,j}(z)$ to $\varphi(z)$ and the Lipschitz estimates stated in Proposition 4.7. More precisely, by (4.11) and Lesbegue Theorem we have that

$$\begin{aligned}
&\limsup_{j\to+\infty}\int_\Omega\Big|\sum_{i\in Z_j}\varphi_{\gamma,j}(\bar u_j(i)-u_j^i)\chi_{Q_i^j}-\varphi(v-u)\Big|\,dx\\
\le\ &\limsup_{j\to+\infty}\int_\Omega\Big|\sum_{i\in Z_j}\varphi_{\gamma,j}(\bar u_j(i)-u_j^i)\chi_{Q_i^j}-\varphi_{\gamma,j}(v-u)\Big|\,dx\\
&+\limsup_{j\to+\infty}\int_\Omega|\varphi_{\gamma,j}(v-u)-\varphi(v-u)|\,dx\\
\le\ &O(\gamma)+\limsup_{j\to+\infty}\int_\Omega\Big|\sum_{i\in Z_j}\varphi_{\gamma,j}(\bar u_j(i)-u_j^i)\chi_{Q_i^j}-\varphi_{\gamma,j}(v-u)\Big|\,dx.
\end{aligned}$$

By (4.12) and (4.11) we have that

$$\begin{aligned}
&\limsup_{j\to+\infty}\int_\Omega\Big|\sum_{i\in Z_j}\varphi_{\gamma,j}(\bar u_j(i)-u_j^i)\chi_{Q_i^j}-\varphi_{\gamma,j}(v-u)\Big|\,dx\\
\le\ &\limsup_{j\to+\infty}\Big(\sum_{i\in Z_j}\int_{Q_i^j}\Big|\varphi_{\gamma,j}(\bar u_j(i)-u_j^i)-\varphi_{\gamma,j}(v-u)\Big|\,dx+c\int_{\Omega\setminus\cup_{i\in Z_j}Q_j^i}(\gamma^n+|u-v|^p)\,dx\Big)\\
\le\ &c\limsup_{j\to+\infty}\sum_{i\in Z_j}\int_{Q_i^j}(|\bar u_j(i)-v|+|u-u_j^i|)(|\bar u_j(i)-u_j^i|^{p-1}+|v-u|^{p-1})\,dx+O(\gamma)\,.
\end{aligned}$$

By Hölder's Inequality it remains to prove that

$$\limsup_{j\to+\infty}\sum_{i\in Z_j}\int_{Q_i^j}(|\bar u_j(i)-v|^p+|u-u_j^i|^p)\,dx=0\,.$$

By assumption $(u_j,v(u_j))$ converges strongly in $L^p(\Omega;\mathbb{R}^m)$ to (u,v); hence, by (4.20) we get the thesis.

Step 4: liminf inequality for $F^c_{\varepsilon_j}$. Gathering (4.19) and (4.22) we get

$$\liminf_{j\to+\infty}F^c_{\varepsilon_j}(u_j)\ge\int_\Omega Qf^1(Du)\,dx+\int_\Omega\varphi(v-u)\,dx\,.$$

□

Proposition 4.10 (Limsup inequality) *We have*

$$\Gamma\text{-}\limsup_{j\to+\infty}F_j(u,v)\le\int_\Omega Qf^1(Du)\,dx+\int_\Omega g_{\rm hom}(Dv)\,dx+\int_\Omega\varphi(v-u)\,dx\,,$$

for all $(u,v)\in[W^{1,p}(\Omega;\mathbb{R}^m)]^2$.

Proof. Let $(u,v) \in [W^{1,p}(\Omega;\mathbb{R}^m)]^2$. Since Ω is a Lipschitz open set, there exists an extension of u, still denoted by u, in $W^{1,p}(\mathbb{R}^n;\mathbb{R}^m)$. Given an open set $\widetilde{\Omega}$ such that $\Omega \subset\subset \widetilde{\Omega}$, let u_j be a sequence weakly converging to u in $W^{1,p}(\widetilde{\Omega};\mathbb{R}^m)$ such that

$$\lim_{j\to+\infty} \int_{\widetilde{\Omega}} f^1(Du_j)\,dx = \int_{\widetilde{\Omega}} Qf^1(Du)\,dx.$$

We may also assume that $|Du_j|^p$ is equi-integrable on $\widetilde{\Omega}$ (see, e.g., [15], [9] Appendix C). Moreover, by Theorem 2.1, there exists a sequence $v_j \in \mathcal{A}_{\varepsilon_j}(\Omega)$ strongly converging to v in $L^p(\Omega;\mathbb{R}^m)$ such that

$$\lim_{j\to+\infty} F_j^d(v_j) = \int_\Omega g_{\hom}(Dv)\,dx\,. \tag{4.23}$$

By the equi-integrability of $|Du_j|^p$, Lemma 2.5 holds true with Ω replaced by $\widetilde{\Omega}$ and

$$C_i^j = \Big\{x \in \widetilde{\Omega}:\ \frac{2}{3}\rho_j < |x-\varepsilon i| < \frac{4}{3}\rho_j\Big\},$$

for every $i \in \mathbb{Z}^n \cap \frac{1}{\varepsilon_j}\Omega$. Therefore, there exists a sequence $\hat{w}_j$ weakly converging to u in $W^{1,p}(\widetilde{\Omega};\mathbb{R}^m)$ such that $\hat{w}_j = u_j^i$ on $\partial B_{\rho_j}(\varepsilon i)$, $\hat{w}_j = u_j$ on $\widetilde{\Omega} \setminus \displaystyle\bigcup_{i\in\mathbb{Z}^n\cap\frac{1}{\varepsilon_j}\Omega} C_i^j$ and

$$\limsup_{j\to+\infty} \int_{\widetilde{\Omega}} |f_j(x,D\hat{w}_j) - f_j(x,Du_j)|\,dx \le O(\gamma)\,. \tag{4.24}$$

Moreover, the sequence $|D\hat{w}_j|^p$ is also equi-integrable. Given $i \in Z_j$, with Z_j as in (2.8), by definition (4.10) we have that for a fixed $\eta > 0$ there exists $\zeta^i_{\gamma,j} \in W_0^{1,p}(B_{\gamma N_j}(0);\mathbb{R}^m)$ such that $\displaystyle⨍_{B_1(0)} \zeta^i_{\gamma,j}\,dx = (v_j(i) - u_j^i)$ and

$$\int_{B_{\gamma N_j}(0)} {\varepsilon_j}^{np/n-p} \tilde{f}(x, {\varepsilon_j}^{-n/n-p} D\zeta^i_{\gamma,j})\,dx \le \varphi_{\gamma,j}(v_j(i) - u_j^i) + \eta\,.$$

Then, we may define

$$w_j = u_j^i + \zeta^i_{\gamma,j}((x-\varepsilon i)\varepsilon_j^{-n/n-p}) \text{ on } \bigcup_{i\in Z_j} B_{\rho_j}(\varepsilon i)\,,$$

and

$$w_j = \hat{w}_j \text{ on } \widetilde{\Omega} \setminus \bigcup_{i\in\mathbb{Z}^n\cap\frac{1}{\varepsilon_j}\Omega} B_{\rho_j}(\varepsilon i)\,.$$

It remains to define w_j on $\bigcup_{i\in\mathbb{Z}^n\cap\frac{1}{\varepsilon_j}\Omega\setminus Z_j} B_{\rho_j}(\varepsilon i)$. Given $i \in \mathbb{Z}^n \cap \frac{1}{\varepsilon_j}\Omega \setminus Z_j$, by definition (4.17) we have that there exists $\bar{\zeta}^i_{\gamma,j} \in W_0^{1,p}(B_{\gamma N_j}(0);\mathbb{R}^m)$ such that

$\bar\zeta^i_{\gamma,j}\,dx = (v_j(i)-u^i_j)$ on B_1 and

$$\int_{B_{\gamma N_j}(0)} {\varepsilon_j}^{np/n-p}\tilde f(x,{\varepsilon_j}^{-n/n-p}D\bar\zeta^i_{\gamma,j})\,dx \le \varphi^{\mathcal R}_{\gamma,j}(v_j(i)-u^i_j)+\eta\,.$$

Then we define

$$w_j = u^i_j + \bar\zeta^i_{\gamma,j}((x-\varepsilon i)\varepsilon_j^{-n/n-p}) \text{ on } \bigcup_{i\in\mathbb{Z}^n\cap\frac{1}{\varepsilon_j}\Omega\setminus Z_j} B_{\rho_j}(\varepsilon i).$$

We now show that Dw_j is uniformly bounded in $L^p(\Omega;\mathbb{R}^m)$. In fact,

$$\begin{aligned}\int_\Omega |Dw_j|^p\,dx \;\le\; & \int_{\Omega\setminus\bigcup_{i\in\mathbb{Z}^n\cap\frac{1}{\varepsilon_j}\Omega} B_{\rho_j}(\varepsilon i)} |D\hat w_j|^p\,dx \\ & + \int_{\bigcup_{i\in Z_j} B_{\rho_j}(\varepsilon i)} |D\zeta^\gamma_j((x-\varepsilon i)\varepsilon_j^{-n/n-p})|^p\,dx \\ & + \int_{\bigcup_{i\in\mathbb{Z}^n\cap\frac{1}{\varepsilon_j}\Omega\setminus Z_j} B_{\rho_j}(\varepsilon i)} |D\bar\zeta^\gamma_j((x-\varepsilon i)\varepsilon_j^{-n/n-p})|^p\,dx\,.\end{aligned}$$

Reasoning as in (4.16) we have that

$$\begin{aligned} & \int_{\bigcup_{i\in Z_j} B_{\rho_j}(\varepsilon i)} |D\zeta^\gamma_j((x-\varepsilon i)\varepsilon_j^{-n/n-p})|^p\,dx \\ = \; & \varepsilon_j^n \sum_{i\in Z_j}\int_{B_{\gamma N_j}(0)} |D\zeta^\gamma_j|^p\,dx \\ \le \; & c\sum_{i\in Z_j}\varepsilon_j^n\varphi_{\gamma,j}(v_j(i)-u^i_j)+O(\gamma^n), \end{aligned} \tag{4.25}$$

and, similarly,

$$\begin{aligned} & \int_{\bigcup_{i\in\mathbb{Z}^n\cap\frac{1}{\varepsilon_j}\Omega\setminus Z_j} B_{\rho_j}(\varepsilon i)} |D\bar\zeta^\gamma_j((x-\varepsilon i)\varepsilon_j^{-n/n-p})|^p\,dx \\ & \qquad = \; \varepsilon_j^n \sum_{i\in\mathbb{Z}^n\cap\frac{1}{\varepsilon_j}\Omega\setminus Z_j}\int_{B_{\gamma N_j}(0)} |D\bar\zeta^\gamma_j|^p\,dx \\ & \qquad \le \; c\sum_{i\in\mathbb{Z}^n\cap\frac{1}{\varepsilon_j}\Omega\setminus Z_j}\varepsilon_j^n\varphi^{\mathcal R}_{\gamma,j}(v_j(i)-u^i_j)+o(1)\,. \end{aligned}$$

The boundness of the right-hand side of (4.25) follows from (4.22). Arguing as in the proof of (4.22) we can also show that

$$\lim_{j\to\infty}\sum_{i\in\mathbb{Z}^n\cap\frac{1}{\varepsilon_j}\Omega\setminus Z_j}\varepsilon_j^n\varphi^{\mathcal R}_{\gamma,j}(v_j(i)-u^i_j)=0. \tag{4.26}$$

Therefore, we conclude that Dw_j is uniformly bounded in L^p. We now prove that w_j converges to u strongly in $L^p(\Omega;\mathbb{R}^m)$. Indeed, let χ_j be defined as the characteristic function of the complement of $\bigcup_{i\in\mathbb{Z}^n} B_{\rho_j}(\varepsilon i)$. We have that the weak*-L^∞ limit of χ_j is a strictly positive constant K and hence the weak-L^p limits of $w_j\chi_j$ and of $\hat{w}_j\chi_j$ are Kw and Ku, respectively. Since $w_j\chi_j = \hat{w}_j\chi_j$ we deduce that $w = u$.

Note that, by construction, $⨍_{B_i^\varepsilon\cap\Omega} w_j(x)\,dx = v_j(i)$ for any $i\in\mathbb{Z}^n\cap\frac{1}{\varepsilon_j}\Omega$. Therefore, according to (3.1), (3.6) we get that $v(w_j)\equiv v_j$ and we conclude that the sequence $(w_j, v(w_j))$ converges to $(u,v)\in[W^{1,p}(\Omega;\mathbb{R}^m)]^2$ strongly in $[L^p(\Omega;\mathbb{R}^m)]^2$.

We now study the asymptotic behavior of $F_j(w_j)$. We get

$$\begin{aligned} F_j^c(w_j) &= \int_\Omega f_j(x, Dw_j)\,dx \\ &= \int_{\Omega\setminus\bigcup_{i\in\mathbb{Z}^n} B_{\rho_j}(\varepsilon i)} f^1(D\hat{w}_j)\,dx \\ &\quad + \int_{\bigcup_{i\in\mathbb{Z}^n} B_{\rho_j}(\varepsilon i)\cap\Omega} f_j(x, Dw_j)\,dx\,. \end{aligned}$$

By (4.24) we have that

$$\begin{aligned} \limsup_{j\to+\infty} \int_{\Omega\setminus\bigcup_{i\in\mathbb{Z}^n} B_{\rho_j}(\varepsilon i)} f^1(D\hat{w}_j)\,dx &\le \limsup_{j\to+\infty} \int_{\widetilde{\Omega}} f^1(Du_j)\,dx + O(\gamma) \\ &= \int_{\widetilde{\Omega}} Qf^1(Du)\,dx + O(\gamma). \end{aligned} \tag{4.27}$$

By construction we have that for $i\in Z_j$

$$\begin{aligned} \int_{B_{\rho_j}(\varepsilon i)} f_j(x, Dw_j)\,dx &= \int_{B_{\rho_j}(\varepsilon i)} f_j(x, D\zeta_j^\gamma((x-\varepsilon i)\varepsilon_j^{-n/n-p}))\,dx \\ &\le {\varepsilon_j}^n\,\varphi_{\gamma,j}(v_j(i)-u_j^i) + {\varepsilon_j}^n\,\eta\,, \end{aligned}$$

and, for $i\in\mathbb{Z}^n\cap\frac{1}{\varepsilon_j}\Omega\setminus Z_j$

$$\begin{aligned} \int_{B_{\rho_j}(\varepsilon i)\cap\Omega} f_j(x, Dw_j)\,dx &\le \int_{B_{\rho_j}(\varepsilon i)} f_j(x, D\bar{\zeta}_j^\gamma((x-\varepsilon i)\varepsilon_j^{-n/n-p}))\,dx \\ &\le {\varepsilon_j}^n\,\varphi_{\gamma,j}^{\mathcal{R}}(v_j(i)-u_j^i) + {\varepsilon_j}^n\,\eta\,. \end{aligned}$$

By (4.22) and (4.26) we have that

$$\limsup_{j\to+\infty} \int_{\bigcup_{i\in\mathbb{Z}^n} B_{\rho_j}(\varepsilon i)\cap\Omega} f_j(x, Dw_j)\,dx \le \int_\Omega \varphi(v-u)\,dx + |\Omega|\eta\,. \tag{4.28}$$

Gathering (4.27), (4.28) together with (4.23) we get that

$$\begin{aligned}\Gamma\text{-}\limsup_{j\to\infty} F_j(u,v) &\le \limsup_{j\to\infty} F_j(w_j)\\ &\le \int_{\widetilde\Omega} f^1(Du_j)\,dx + \int_\Omega \varphi(v-u)\,dx + \int_\Omega g_{\rm hom}(Dv)\,dx\\ &\quad +|\Omega|\eta + O(\gamma).\end{aligned}$$

By the arbitrariness of η, letting γ tend to zero and $\widetilde\Omega$ tend to Ω we get the conclusion. □

4.3 Subcritical an supercritical case

In this section we study the case where the size of the particles r_ε is smaller or bigger than the critical size (but always very small compared to the lattice size ε). More precisely, we start by considering the subcritical cases; *i.e.,* $r_\varepsilon \ll \varepsilon^{n/n-p}$. We show that there is no interaction between the deformation of the continuous media and that of the system of particles, being the Γ-limit of F_ε the sum of the Γ-limits of F^c_ε and F^d_ε.

Theorem 4.11 (subcritical size) *Let $1 < p < n$ and $r_\varepsilon \ll \varepsilon^{n/n-p}$. Then the functionals F_ε defined by the formula (3.7) Γ-converge, with respect to the $[L^p(\Omega;\mathbb{R}^m)]^2$-topology, to the functional $F : [L^p(\Omega;\mathbb{R}^m)]^2 \to [0,+\infty]$ defined by*

$$F(u,v) = \begin{cases} \displaystyle\int_\Omega Qf^1(Du) + \int_\Omega g_{\rm hom}(Dv)\,dx\,, & \text{if } (u,v)\in [W^{1,p}(\Omega;\mathbb{R}^m)]^2\\ +\infty & \text{otherwise}\end{cases} \tag{4.29}$$

where $g_{\rm hom}$ is the homogenized integrand given by (2.6) with F^d_1 in place of E_1.

Proof. The liminf inequality is straightforward. For the limsup inequality, let $(u_\varepsilon, v_\varepsilon)$ converging to (u,v) strongly in $[L^p(\Omega;\mathbb{R}^m)]^2$ such that

$$\lim_{\varepsilon\to 0} F^c_\varepsilon(u_\varepsilon) = \int_\Omega Qf^1(Du)\,dx$$

and

$$\lim_{\varepsilon\to 0} F^d_\varepsilon(v_\varepsilon) = \int_\Omega g_{\rm hom}(Dv)\,dx\,.$$

Then, reasoning as in Proposition 4.10 with r_ε in place of $\varepsilon^{n/n-p}$ and by Proposition 4.7 (i) we get that there exists a sequence w_ε converging to u strongly in $L^p(\Omega;\mathbb{R}^m)$ such that

$$\begin{aligned}\limsup_{\varepsilon\to 0} F_\varepsilon(w_\varepsilon) &\le \int_\Omega Qf^1(Du)\,dx + \int_\Omega g_{\rm hom}(Dv)\,dx\\ &\quad +c\,\limsup_{\varepsilon\to 0}\frac{r_\varepsilon^{n-p}}{\varepsilon^n}\sum_{i\in Z_\varepsilon}\varepsilon^n|v_\varepsilon(\varepsilon i)-u^i_\varepsilon|^p + c(\gamma)\,,\end{aligned}$$

where Z_ε and u^i_ε are defined as in Lemma 2.5 with ε in place of ε_j and $c(\gamma) \to 0$ as $\gamma \to 0^+$. Since $\lim_{\varepsilon\to 0} r_\varepsilon^{n-p}/\varepsilon^n = 0$ and

$$\lim_{\varepsilon\to 0} \sum_{i\in Z_\varepsilon} \varepsilon^n |v_\varepsilon(\varepsilon i) - u^i_\varepsilon|^p = \int_\Omega |u-v|^p\, dx\,,$$

letting first ε and then γ goes to zero, we get the limsup inequality. □

In the next theorem we consider the supercritical case; *i.e.*, $r_\varepsilon \gg \varepsilon^{n/n-p}$. We show that the Γ-limit of F_ε is still the sum of the Γ-limits of F^c_ε and F^d_ε but in this case there is no decoupling of variables, that is, the dlsplacements of the continuum and of the system of particles coincide in the limit.

Theorem 4.12 (supercritical size) *Let $1 < p < n$ and $r_\varepsilon \gg \varepsilon^{n/n-p}$. Then the functionals F_ε defined by the formula (3.7) Γ-converge, with respect to the $[L^p(\Omega;\mathbb{R}^m)]^2$-topology, to the functional $F : [L^p(\Omega;\mathbb{R}^m)]^2 \to [0,+\infty]$ defined by*

$$F(u,v) = \begin{cases} \displaystyle\int_\Omega Qf^1(Du) + \int_\Omega g_{\rm hom}(Du)\, dx\,, & \textit{if } u = v \in W^{1,p}(\Omega;\mathbb{R}^m) \\ +\infty & \textit{otherwise} \end{cases}$$

where $g_{\rm hom}$ is the homogenized integrand given by (2.6) with F^d_1 in place of E_1.

Proof. Let u_ε be any sequence with bounded energy and such $(u_\varepsilon, v(u_\varepsilon))$ converges to (u,v) strongly in $[L^p(\Omega;\mathbb{R}^m)]^2$. Reasoning as in the proof of Proposition 4.9 (see, e.g., (4.21)), by the growth conditions (3.4) we have that

$$\frac{r_\varepsilon^{n-p}}{\varepsilon^n} \sum_{i\in Z_\varepsilon} \varepsilon^n |v(u_\varepsilon)(\varepsilon i) - u^i_\varepsilon|^p < C\,.$$

Since, $\lim_{\varepsilon\to 0} r_\varepsilon^{n-p}/\varepsilon^n = +\infty$ and

$$\lim_{\varepsilon\to 0} \sum_{i\in Z_\varepsilon} \varepsilon^n |v_\varepsilon(\varepsilon i) - u^i_\varepsilon|^p = \int_\Omega |u-v|^p\, dx\,,$$

we get that $u = v$ a.e. $x \in \Omega$ and the liminf inequality easily follows.

We now deal with the limsup inequality. By a density argument, we may restrict ourself to prove the inequality for pieciewise affine functions on Ω. By a 'localizing' argument, we may further restrict to the case $u(x) = Mx$, $M \in \mathbb{M}^{m\times n}$. By the formula (2.6) we have that for a fixed $\eta > 0$ there exists $T \in \mathbb{N}$ and $\psi \in \mathcal{A}_{1,0}(Q_T)$ such that

$$\frac{1}{T^n} F^d_1(M\cdot + \psi, Q_T) \le g_{hom}(M) + \eta\,.$$

We set $v_\varepsilon(\varepsilon i) = Mi + \varepsilon\psi(i)$, where ψ is extended on $\mathbb{Z}^n$ by periodicity. Note that, $\lim_{\varepsilon\to 0} F^d_\varepsilon(v_\varepsilon) \le (g_{hom}(M) + \eta)|\Omega|$. Without loss of generality we may assume

f^1 quasiconvex; hence, the equality $\lim_{\varepsilon\to 0} F^c_\varepsilon(u_\varepsilon) = \displaystyle\int_\Omega f^1(Du)\,dx$ is trivially satisfied by $u_\varepsilon(x) = Mx$. Therefore, reasoning as in Proposition 4.10 with r_ε in place of $\varepsilon^{n/n-p}$ and by Proposition 4.7 (i) we get that there exists a sequence w_ε converging to Mx strongly in L^p such that

$$\begin{aligned}\limsup_{\varepsilon\to 0} F_\varepsilon(w_\varepsilon) &\leq |\Omega|(f^1(M) + g_{\rm hom}(M) + \eta)\\ &\quad + c\,\limsup_{\varepsilon\to 0} \frac{r_\varepsilon^{n-p}}{\varepsilon^n} \sum_{i\in Z_\varepsilon} \varepsilon^n |v_\varepsilon(\varepsilon i) - u^i_\varepsilon|^p + c(\gamma)\\ &= |\Omega|(f^1(M) + g_{\rm hom}(M) + \eta)\\ &\quad + c\,\limsup_{\varepsilon\to 0} \Big(\frac{r_\varepsilon}{\varepsilon}\Big)^{n-p} \sum_{i\in Z_\varepsilon} \varepsilon^n |\psi(i)|^p + c(\gamma)\end{aligned}$$

where $c(\gamma) \to 0$ as $\gamma \to 0^+$. Since $r_\varepsilon \ll \varepsilon$ and ψ is bounded we get the conclusion letting $\gamma \to 0^+$ and by the arbitrariness of η. □

5 $p = n$

In this section we analyze the case $1 < p = n$, where, as well known in the theory of perforated domains (see [13], [19]), scaling argument leads to a critical size with exponential decay, namely $r_\varepsilon = \exp(-\varepsilon^{-n/n-1})$. Moreover, in this critical regime the energy concentrates on a scale larger than that of the perforations, giving rise to a rescaled formula of capacitary type.

We state the analogue of Theorems 4.1, 4.11 and 4.12. The proofs are closely related to the analysis of integral functionals on perforated domains performed in [19] and rely on the same argument exploited in Section 4.2. For this reason, we just highlight the main difference due to the different scaling of the energies and we refer to the proofs of Section 4.2 when no significative changes are needed.

Theorem 5.1 (Critical size) *Let $1 < p = n$. Let (ε_j) be a sequence of strictly positive numbers converging to 0 and $r_{\varepsilon_j} = \exp(-\varepsilon_j^{-n/n-1})$. For all $j \in \mathbb{N}$ and $\gamma > 0$ and $z \in \mathbb{R}^m$ we set*

$$\begin{aligned}\varphi_{\gamma,j}(z) &= (\log r_{\varepsilon_j}^{-1})^{n-1} \inf\Big\{\int_{B_{\gamma N_j}(0)} h_j(x, D\zeta)\,dx :\ \zeta \in W^{1,p}_0(B_{\gamma N_j}(0);\mathbb{R}^m),\\ &\qquad\qquad \fint_{B_1(0)} \zeta\,dx = z\Big\}\end{aligned} \tag{5.1}$$

where

$$h_j(x, M) = r^n_{\varepsilon_j}\, Q\tilde{f}(x, r^{-1}_{\varepsilon_j} M)\,, \tag{5.2}$$

with $Q\tilde{f}$ denotes the quasiconvexification of $\tilde{f}$ defined by

$$\tilde{f}(x,M) = \begin{cases} f^1(M) & x \in B_1^c(0) \\ f^2(M) & x \in B_1(0)\,, \end{cases}$$

and $N_j = \varepsilon_j / r_{\varepsilon_j}$. Then, upon possibly extracting a subsequence, for all $z \in \mathbb{R}^m$ there exists the limit

$$\varphi(z) = \lim_{\gamma \to 0^+} \lim_{j \to +\infty} \varphi_{\gamma,j},$$

uniformly on the compact sets of $\mathbb{R}^m$.

Moreover, the functionals F_j, defined by the formula (3.7) with ε_j in place of ε, Γ-converge, with respect to the $[L^n(\Omega;\mathbb{R}^m)]^2$-topology, to the functional $F : [L^n(\Omega;\mathbb{R}^m)]^2 \to [0,+\infty]$ defined by

$$F(u,v) = \begin{cases} \displaystyle\int_\Omega Qf^1(Du) + \int_\Omega g_{\mathrm{hom}}(Dv)\,dx + \int_\Omega \varphi(v-u)\,dx\,, & \\ & \text{if } (u,v) \in [W^{1,n}(\Omega;\mathbb{R}^m)]^2 \\ +\infty & \text{otherwise} \end{cases} \tag{5.3}$$

where g_{hom} is the homogenized integrand given by (2.6) with F_1^d in place of E_1.

Proof. We may reasoning as in Section 4.2 with r_{ε_j} in place of $\varepsilon_j^{n/n-p}$. Here we just provide the scaling argument leading to the critical size $r_{\varepsilon_j} = \exp(-\varepsilon_j^{-n/n-1})$ (see also [19]). Proceeding as in Step 2 Proposition 4.9, we get that the contribution of the energy near one particle can be estimated from below by

$$I_j := \inf\Big\{ \int_{B_{\gamma N_j}(0)} h_j(x, D\zeta)\,dx : \ \zeta \in W_0^{1,p}(B_{\gamma N_j}(0);\mathbb{R}^m),\ \fint_{B_1(0)} \zeta\,dx = z \Big\},$$

(which turn out to be the optimal bound) with h_j defined by (5.2) and $N_j = \varepsilon_j / r_{\varepsilon_j}$. By the growth conditions we get that

$$c\Big(\overline{C}_n(B_1; B_{\gamma N_j})|z|^p - \gamma^n \varepsilon_j^n\Big) \le I_j \le C\Big(\gamma^n \varepsilon_j^n + \overline{C}_n(B_1; B_{\gamma N_j})|z|^p\Big)\,,$$

where $\overline{C}_n(B_1; B_{\gamma N_j})$ is defined by (4.9) with $p = n$ and $R = \gamma N_j$. Applying Proposition 5.3 below, we have that I_j is of the same order of $(\log \gamma N_j)^{(-n+1)}$ while the number of particles is of order ε_j^{-n}. Therefore, in order to get the total energy near the particles finite and not vanishing, we have to choose r_{ε_j} such that

$$\lim_{j\to+\infty} \varepsilon_j^{-n} (\log \frac{\gamma \varepsilon_j}{r_{\varepsilon_j}})^{(-n+1)} = \lim_{j\to+\infty} [\varepsilon_j^{n/n-1} \log(\frac{\gamma\varepsilon_j}{r_{\varepsilon_j}})]^{(-n+1)} = a\,,$$

for some constant $a > 0$, that is

$$\lim_{j\to+\infty} [\varepsilon_j^{n/n-1} (\log \frac{\gamma\varepsilon_j}{r_{\varepsilon_j}})] = \lim_{j\to+\infty} [\varepsilon_j^{n/n-1} \log(\gamma\varepsilon_j) - \varepsilon_j^{n/n-1} \log(r_{\varepsilon_j})] = a\,.$$

Choosing for simplicity $a = 1$, we get that $r_{\varepsilon_j} = \exp(-\varepsilon_j^{-n/n-1})$. □

Remark 5.2 (Rigid particles) By a similar argument it can be proved the analogue of Theorem 4.2. More precisely, the Γ-limit of the functionals $F_j^{\mathcal{R}}$ defined by (3.8) with $p = n$ and ε_j in place of ε, is given by the functional defined as in (5.3), by replacing in (5.1) the constraint $\fint_{B_1(0)} \zeta\, dx = z$ with $\zeta \equiv z$ on B_1.

Proposition 5.3 *Let $1 < p = n$. Then*

$$\lim_{R\to+\infty} (\log R)^{(n-1)} \overline{C}_n(B_1; B_R) = \lim_{R\to+\infty} (\log R)^{(n-1)} C_n(B_1; B_R) = \sigma_{n-1}\,. \quad (5.4)$$

Proof. We can explicitly compute $\overline{C}_n(B_1; B_R)$, by proceeding as in the proof of Proposition 4.5 for the computation of the capacity $\overline{C}_p(B_1(0); B_R)$ with $1 < p < n$. In this case we get that

$$(\log R)^{n-1}\, \overline{C}_n(B_1; B_R) = \sigma_{n-1} \frac{(\log R)^n + c_n}{(\log R + c_n)^n}$$

with $c_n = (n^2/(n-1)^{n+1})^{1/(n-1)}$ and $\sigma_{n-1} = |\partial B_1|_{n-1}$. We recall that

$$(\log R)^{(n-1)} C_n(B_1; B_R) = \sigma_{n-1}$$

Therefore, one easily get (5.4). □

Theorem 5.4 (Subcritical size) *Let $1 < p = n$ and $\log(r_\varepsilon^{-1}) \gg \varepsilon^{-n/n-1}$. Then the functionals F_ε defined by the formula (3.7) Γ-converge, with respect to the $[L^n(\Omega;\mathbb{R}^m)]^2$-topology, to the functional $F : [L^n(\Omega;\mathbb{R}^m)]^2 \to [0, +\infty]$ defined by*

$$F(u,v) = \begin{cases} \displaystyle\int_\Omega Qf^1(Du) + \int_\Omega g_{\text{hom}}(Dv)\, dx\,, & \text{if } (u,v) \in [W^{1,n}(\Omega;\mathbb{R}^m)]^2 \\ +\infty & \text{otherwise} \end{cases} \quad (5.5)$$

where g_{hom} is the homogenized integrand given by (2.6) with F_1^d in place of E_1.

Proof. We may proceed as in the proof of Theorem 4.11, observing that, by the growth conditions, for any sequence u_ε such that $(u_\varepsilon, v(u_\varepsilon)) \to (u, v)$ strongly in $[L^n(\Omega;\mathbb{R}^m)]^2$, the contribution of $F_\varepsilon(u_\varepsilon)$ 'near' the particles can be bounded from below and above by

$$\frac{1}{\varepsilon^n (\log(r_\varepsilon^{-1}))^{n-1}} \sum_{i\in Z_\varepsilon} \varepsilon^n |v(u_\varepsilon)(i) - u_\varepsilon^i|^p\,,$$

where Z_ε and u_ε^i are defined as in Lemma 2.5 with ε in place of ε_j. □

Theorem 5.5 (Supercritical size) *Let* $1 < p = n$ *and* $\log(r_\varepsilon^{-1}) \ll \varepsilon^{-n/n-1}$. *Then the functionals* F_ε *defined by the formula (3.7)* Γ*-converge, with respect to the* $[L^n(\Omega;\mathbb{R}^m)]^2$*-topology, to the functional* $F : [L^n(\Omega;\mathbb{R}^m)]^2 \to [0,+\infty]$ *defined by*

$$F(u,v) = \begin{cases} \displaystyle\int_\Omega Qf^1(Du) + \int_\Omega g_{\rm hom}(Du)\,dx\,, & \textit{if } u = v \in W^{1,n}(\Omega;\mathbb{R}^m) \\ +\infty & \textit{otherwise} \end{cases}$$

where $g_{\rm hom}$ *is the homogenized integrand given by (2.6) with* F_1^d *in place of* E_1.

Proof. Proceed as in the proof of Theorem 4.12. □

6 Convergence of minimum problems

In this section we study the convergence of minimum problems with boundary data. To this end, we derive a Γ-convergence result for functionals with Dirichlet boundary conditions. We will focus only on the critical case, since the subcritical and the supercritical cases can be treated similarly.

Let us fix r'_ε such that $0 < r_\varepsilon < r'_\varepsilon << \varepsilon$ and define

$$\partial^\varepsilon\Omega := \partial\Omega \setminus \bigcup_{i\in\mathbb{Z}^n} B_{r'_\varepsilon}(\varepsilon i). \tag{6.6}$$

Given $\phi,\psi \in W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^m)$ we define $F_\varepsilon^{\phi,\psi} : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty]$ as

$$F_\varepsilon^{\phi,\psi}(u,v) = \begin{cases} F_\varepsilon(u) & \text{if } u = \phi \text{ on } \partial^\varepsilon\Omega,\ v = v(u) \in \mathcal{A}_{\varepsilon,\psi}(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{6.7}$$

where F_ε is given by (3.2) and we identify u with its trace on $\partial\Omega$.

Theorem 6.1 *Let* $1 < p \le n$. *Let* ε_j *be a sequence of strictly positive numbers converging to* 0 *such that Theorem 4.1 or Theorem 5.1 holds true. Moreover let* $F_j^{\phi,\psi}$ *defined by (6.7), with* ε_j *in place of* ε *and* $r_{\varepsilon_j} = \varepsilon_j^{n/n-p}$ *for* $p < n$ *and* $r_{\varepsilon_j} = \exp(-\varepsilon_j^{-n/n-1})$ *for* $p = n$. *Then the functionals* $F_j^{\phi,\psi}$ Γ*-converge, with respect to the* $[L^p(\Omega;\mathbb{R}^m)]^2$*-topology, to the functional* $F^{\phi,\psi} : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty]$ *defined by*

$$F^{\phi,\psi}(u,v) = \begin{cases} F(u,v) & \textit{if } (u-\phi, v-\psi) \in [W_0^{1,p}(\Omega;\mathbb{R}^m)]^2 \\ +\infty & \textit{otherwise,} \end{cases}$$

where F *is defined by (4.4) for* $p < n$ *and by (5.3) for* $p = n$.

PROOF. The proof follows those of Theorem 4.1 for $p < n$ and Theorem 5.1 for $p = n$, up to minor changes. For the Γ-lim inf inequality, we just note that if $u_j \to u$, $v(u_j) \to v$ strongly in $L^p(\Omega;\mathbb{R}^m)$ with $u_j = \phi$ on $\partial^{\varepsilon_j}\Omega$, $v(u_j) \in \mathcal{A}_{\varepsilon_j,\psi}(\Omega)$ and $\sup F_j(u_j) < +\infty$, then we easily derive that $u - \phi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ and $v - \psi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ (see Theorem 3.10 in [1]). Moreover, given u, v such that $(u - \phi, v - \psi) \in [W_0^{1,p}(\Omega;\mathbb{R}^m)]^2$ we can easily modify the construction of the optimal sequence w_j, provided in the proof of Proposition 4.10, so that $w_j = \phi$ on $\partial^{\varepsilon_j}\Omega$ and $v(w_j) \in \mathcal{A}_{\varepsilon_j,\psi}(\Omega)$ (see Theorem 3.10 in [1]). □

As a consequence of the previous theorem and Theorem 3.1, we derive the following result about the convergence of minimum problems with boundary data.

Corollary 6.2 *Under the hypotheses of Theorem 6.1 we get that*

$$\begin{aligned}&\lim_{j\to+\infty} \inf\{F_j(u),\ u - \phi \text{ on } \partial^{\varepsilon_j}\Omega,\ v(u) \in \mathcal{A}_{\varepsilon_j,\psi}(\Omega)\}\\ &= \min\{F(u,v):\ (u-\phi, v-\psi) \in [W_0^{1,p}(\Omega;\mathbb{R}^m)]^2\}.\end{aligned}$$

Moreover, if $(u_j, v(u_j))$ *is a converging sequence such that*

$$\lim_{j\to+\infty} F_j(u_j) = \lim_{j\to+\infty} \inf\{F_j(u),\ u - \phi \text{ on } \partial^{\varepsilon_j}\Omega, v(u) \in \mathcal{A}_{\varepsilon_j,\psi}(\Omega)\},$$

then its limit is a minimizer for $\min\{F(u,v):\ (u-\phi, v-\psi) \in [W_0^{1,p}(\Omega;\mathbb{R}^m)]^2\}$.

7 Coupled case

In this section we derive, making use of the results proved in Sections 4 and 5, the Γ-limit of F_ε in the coupled case, that is, when the energies are regarded as function only of the continuous displacement u or the discrete displacement $v(u)$. These results will allow in particular to derive convergence results of minimum problems when boundary conditions are prescribed on the order parameters u, $v(u)$ or both.

7.1 Limit of the continuous displacement

Theorem 7.1 *Let* $1 < p \le n$. *The following* Γ*-convergence results hold.*

(i) (critical size) Let ε_j *be a sequence of strictly positive numbers converging to* 0 *such that Theorem 4.1 or Theorem 5.1 holds true. Moreover let* F_j *be defined by (3.2), with* ε_j *in place of* ε *and* $r_{\varepsilon_j} = \varepsilon_j^{n/n-p}$ *for* $p < n$ *and* $r_{\varepsilon_j} = \exp(-\varepsilon_j^{-n/n-1})$ *for* $p = n$. *Then the functionals* F_j Γ*-converge, with respect to the* $L^p(\Omega;\mathbb{R}^m)$*-topology, to the functional* $\overline{F} : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty)$ *defined by*

$$\overline{F}(u) = \min\{F(u,v):\ v \in W^{1,p}(\Omega;\mathbb{R}^m)\}, \tag{7.8}$$

where F *is defined by (4.4) for* $p < n$ *and by (5.3) for* $p = n$.

(ii) (subcritical size) Let $r_\varepsilon \ll \varepsilon^{n/n-p}$ *for* $p < n$ *and* $\log(r_\varepsilon^{-1}) \gg \varepsilon^{-n/n-1}$ *for* $p = n$*. Then the functionals* F_ε*, defined by (3.2),* Γ*-converge, with respect to the* $L^p(\Omega;\mathbb{R}^m)$*-topology, to the functional* $\overline{F} : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty)$ *defined as in (7.8) with* F *given by (4.29) for* $p < n$ *and by (5.5) for* $p = n$*.*

Proof.
Proof of (i). Let us prove first the Γ-lim inf inequality. Let $u_j \to u$ strongly in $L^p(\Omega;\mathbb{R}^m)$ be such that $\sup_j F_j(u_j) < +\infty$ and let $v_j := v(u_j)$ be defined by (3.6) and identified with their affine interpolations on compact sets of Ω. By the equiboundedness of $F_j^d(u_j)$ we deduce (see [1]) that, for any fixed open set Ω' with Lipschitz boundary and compactly contained in Ω, we have

$$\|Dv_j\|_{L^p(\Omega';\mathbb{R}^m)} \leq C.$$

Moreover, the argument used in the proof of Proposition 4.9 (see e.g. (4.21) in Step 2) together with the growth conditions show that

$$\int_{\Omega'} |v_j - u_j|^p \leq C. \tag{7.9}$$

Hence, v_j is bounded in $W^{1,p}(\Omega';\mathbb{R}^m)$ and then, up to extracting a subsequence (not relabeled), we get that $v_j \to v$ in $L^p(\Omega';\mathbb{R}^m)$ for some $v \in W^{1,p}(\Omega';\mathbb{R}^m)$. Thus, by using Theorem 4.1 for $p < n$ or Theorem 5.1 for $p = n$, with Ω' in place of Ω, we get

$$\liminf_{j\to+\infty} F_j(u_j) \geq F(u,v,\Omega') \geq \overline{F}(u,\Omega'),$$

where $F(u,v,\Omega')$ and $\overline{F}(u,\Omega')$ are the 'localized' version of the functionals defined in (4.4) and (7.8), respectively, obtaining by replacing Ω with Ω'. The conclusion follows noting that

$$\sup\{\overline{F}(u,\Omega') : \ \Omega' \subset\subset \Omega\} = \overline{F}(u).$$

The Γ-lim sup inequality is straightforward. Indeed, given $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, let $\overline{v} \in W^{1,p}(\Omega;\mathbb{R}^m)$ be such that $F(u,\overline{v}) = \overline{F}(u)$. Then, by Theorem 4.1 for $p < n$ or Theorem 5.1 for $p = n$, there exist $u_j \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that $(u_j, v(u_j)) \to (u,\overline{v})$ in $[L^p(\Omega;\mathbb{R}^m)]^2$ and $\lim_j F(u_j) = F(u,\overline{v}) = \overline{F}(u)$. Hence the inequality is proved.
Proof of (ii). The only difference with respect to the case (i) is in the proof of Γ-lim inf inequality, since (7.9) could not be satisfied in the subcritical case. In fact, we have shown that in the computation of the Γ-limit of F_ε the term accounting for the interaction between the continuous and the discrete part disappears (see (4.29) and (5.5)). We can argue as follows: given $u_\varepsilon \to u$ strongly in $L^p(\Omega;\mathbb{R}^m)$ and such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$, set $w_\varepsilon := v_\varepsilon - \fint_{\Omega'} v_\varepsilon \, dx$. By Poincare' inequality we have that w_ε is bounded in $W^{1,p}(\Omega';\mathbb{R}^m)$, thus, up to passing to a subsequence (not relabeled), we get that $w_\varepsilon \to w$ strongly in $L^p(\Omega';\mathbb{R}^m)$ for some $w \in W^{1,p}(\Omega';\mathbb{R}^m)$.

Since $F^d_\varepsilon(u_\varepsilon,\Omega')=F^d_\varepsilon(w_\varepsilon,\Omega')$, by using separately the Γ-convergence result for F^c_ε and F^d_ε, we have

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon)\geq \liminf_{\varepsilon\to 0} F^c_\varepsilon(u_\varepsilon,\Omega')+\liminf_{\varepsilon\to 0} F^d_\varepsilon(w_\varepsilon,\Omega')\geq F(u,w,\Omega').$$

Hence we may conclude by proceeding as above. □

As a consequence of the previous theorem and Theorem 3.1, by reasoning as in Section 6, we may derive the following result about the convergence of minimum problems with boundary data.

Corollary 7.2 *Let $\phi\in W^{1,\infty}(\Omega;\mathbb{R}^m)$. Then, under the hypotheses of Theorem 7.1 we get that*

$$\lim_{j\to+\infty}\inf\{F_j(u),\ u=\phi \text{ on } \partial^{\varepsilon_j}\Omega\}=\min\{\overline{F}(u):\ u-\phi\in W^{1,p}_0(\Omega;\mathbb{R}^m)\}.$$

Moreover, if u_j is a converging sequence such that

$$\lim_{j\to+\infty} F_j(u_j)=\lim_{j\to+\infty}\inf\{F_j(u),\ u=\phi \text{ on } \partial^{\varepsilon_j}\Omega\},$$

then its limit is a minimizer for $\min\{\overline{F}(u):\ u-\phi\in W^{1,p}_0(\Omega;\mathbb{R}^m)\}$.

7.2 Limit of the discrete displacement

We may regard the family of energy functionals as functions only of the discrete displacement $v(u)$. To this end, since we are interested in minimum problems involving the energies F_ε, given $v\in\mathcal{A}_\varepsilon(\Omega)$ we defined the energy of v by minimizing $F_\varepsilon(u)$ among all $u\in W^{1,p}(\Omega;\mathbb{R}^m)$ such that $v=v(u)$. We set

$$\widehat{F}_\varepsilon(v)=\begin{cases}\inf\{F_\varepsilon(u):\ u\in W^{1,p}(\Omega;\mathbb{R}^m),\ v=v(u)\} & \text{if } v\in\mathcal{A}_\varepsilon(\Omega)\\ \\ +\infty & \text{otherwise,}\end{cases}\tag{7.10}$$

where F_ε is defined by (3.2). The following theorem holds.

Theorem 7.3 *Let $1<p\leq n$. The following Γ-convergence results hold.*

(i) (critical size) Let ε_j be a sequence of strictly positive numbers converging to 0 such that Theorem 4.1 or Theorem 5.1 holds true. Moreover, let $\widehat{F}_j$ be defined by (7.10), with ε_j in place of ε and $r_{\varepsilon_j}:=\varepsilon_j^{n/n-p}$ for $p<n$ and $r_{\varepsilon_j}=\exp(-\varepsilon_j^{-n/n-1})$ for $p=n$. Then the functionals $\widehat{F}_j$ Γ-converge, with respect to the $L^p(\Omega;\mathbb{R}^m)$- strong topology, to the functional $\widehat{F}: L^p(\Omega;\mathbb{R}^m)\to[0,+\infty)$ defined by

$$\widehat{F}(v)=\begin{cases}\min\{F(u,v):\ u\in W^{1,p}(\Omega;\mathbb{R}^m)\} & \text{if } v\in W^{1,p}(\Omega;\mathbb{R}^m)\\ \\ +\infty & \text{otherwise,}\end{cases}\tag{7.11}$$

where F is defined by (4.4) for $p<n$ and by (5.3) for $p=n$.

(ii) (subcritical size) Let $r_\varepsilon \ll \varepsilon^{n/n-p}$ for $p < n$ and $\log(r_\varepsilon^{-1}) \gg \varepsilon^{-n/n-1}$ for $p = n$. Then the functionals $\widehat{F}_\varepsilon$, defined by (7.10), Γ-converge, with respect to the $L^p(\Omega;\mathbb{R}^m)$-topology, to the functional $\widehat{F} : L^p(\Omega;\mathbb{R}^m) \to [0,+\infty)$ defined as in (7.11) with F given by (4.29) for $p < n$ and by (5.5) for $p = n$.

Proof. We argue as in the proof of Theorem 7.1 by reversing the role of u_j and v_j. □

As a consequence of the previous theorem and Theorem 3.1, by reasoning as in Section 6, we may derive the following result about the convergence of minimum problems with boundary data.

Corollary 7.4 *Let $\psi \in W^{1,\infty}(\Omega;\mathbb{R}^m)$. Then, under the hypotheses of Theorem 7.3 we get that*

$$\lim_{j\to+\infty} \inf\{F_j(u) : \ u \in W^{1,p}(\Omega;\mathbb{R}^m), v(u) \in \mathcal{A}_{\varepsilon_j,\psi}(\Omega)\} = \min\{\widehat{F}(v) : \ v-\psi \in W_0^{1,p}(\Omega;\mathbb{R}^m)\}.$$

Moreover, if u_j is a sequence such that $v(u_j)$ converges and

$$\lim_{j\to+\infty} F_j(u_j) = \lim_{j\to+\infty} \inf\{F_j(u) : \ u \in W^{1,p}(\Omega;\mathbb{R}^m), v(u) \in \mathcal{A}_{\varepsilon_j,\psi}(\Omega)\},$$

then the limit of $v(u_j)$ is a minimizer for $\min\{\widehat{F}(v) : \ v - \psi \in W_0^{1,p}(\Omega;\mathbb{R}^m)\}$.

8 $p > n$

In this section we study the Γ-convergence of the family of functionals F_ε and $F_\varepsilon^{\mathcal{R}}$ defined by (3.2) and (3.5), respectively, with $p > n$. In this case there is no decoupling of variables. Indeed, let u_ε be a sequence with bounded energy. Then, by Sobolev immersion and the growth conditions, we easily get

$$\|u_\varepsilon - v(u_\varepsilon)\|_\infty \le C\varepsilon^{1-\frac{n}{p}} \|Du_\varepsilon\|_p^p \le C\varepsilon^{1-\frac{n}{p}} \to 0$$

Hence, we will perform the Γ-convergence analysis with respect to the $L^p(\Omega;\mathbb{R}^m)$-strong topology and the Γ-limits will depend only on the variable u, which is the limit displacement both of the continuous and of the discrete part of the system.

For any $A \in \mathcal{O}(\Omega)$ we denote by $F_\varepsilon(u, A)$ the 'localized' version of $F_\varepsilon(u)$ obtained by replacing Ω with A in (3.2).

In the next theorem we state the integral representation result for the Γ-limits of the functionals F_ε and $F_\varepsilon^{\mathcal{R}}$. We omit the proof, since it relies on the same 'localization' argument exploited in Theorem 2.1 and in the theory of homogenization for integral functionals with densities periodic in the spatial variable. Loosely speaking, it consists in proving that, for any $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, $\Gamma\text{-}\lim_{\varepsilon\to0} F_\varepsilon(u,\cdot)$ is the restriction to $\mathcal{O}(\Omega)$ of a Radon measure and that $\Gamma\text{-}\lim_{\varepsilon\to0} F_\varepsilon(\cdot,\cdot)$ satisfies all the hypotheses of the integral representation result for functionals defined on pair function-sets stated in [12].

Theorem 8.1 *Let $p > n$. Then, the functionals F_ε, defined by (3.2), Γ-converge, with respect to the $L^p(\Omega;\mathbb{R}^m)$-strong topology, to the functional $F : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty)$ defined by*

$$F(u) = \int_\Omega f_{\rm hom}(Du)\,dx\,, \tag{8.1}$$

where for every $f_{hom} : \mathbb{M}^{m\times n} \to [0,+\infty)$ is a quasiconvex function defined by

$$f_{\rm hom}(M) = \lim_{T\to+\infty} \frac{1}{T^n} \inf\Big\{ \int_{Q_T} \hat f_T(x,Du)\,dx + F_1^d(u,Q_T) \;\; u \in W^{1,p}(Q_T;\mathbb{R}^m), \\ u(x) = Mx \text{ on } \partial^{\frac{1}{T}}(Q_T) \Big\}, \tag{8.2}$$

where $\hat f_T(\cdot,\cdot) := f_{\frac{1}{T}}(\frac{\cdot}{T},\cdot)$, with $f_{\frac{1}{T}}(\cdot,\cdot)$ defined by (3.3), and $\partial^{\frac{1}{T}}(Q_T)$ is defined by (6.6) with $\frac{1}{T}$ and Q_T in place of ε and Ω, respectively.

An analogous result holds for the Γ-limit of the functionals $F_\varepsilon^{\mathcal{R}}$, defined by (3.5), where in this case the density f_{hom} is obtained by replacing $W^{1,p}(Q_T;\mathbb{R}^m)$ with $W^{1,p}_{\mathcal{R},\frac{1}{T}}(\Omega;\mathbb{R}^m)$ in (8.2).

As in Section 7.2, also in this case we may regard the family of energies as functions only of the discrete displacement $v(u)$, by considering the functionals $\widehat{F}_\varepsilon$ defined in (7.10). By the way, since there is no decoupling of variables, it is easy to show that $\widehat{F}_\varepsilon$ share the same Γ-limit of F_ε given by (8.1), as stated in the following theorem.

Theorem 8.2 *Let $p > n$ and $\widehat{F}_\varepsilon$ be defined by (7.10). Then, the functionals $\widehat{F}_\varepsilon$ Γ-converge, with respect to the $L^p(\Omega;\mathbb{R}^m)$-strong topology , to the functional $F : L^p(\Omega;\mathbb{R}^m) \to [0,+\infty)$ defined by*

$$F(u) = \begin{cases} \int_\Omega f_{\rm hom}(Du)\,dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m) \\ \\ +\infty & \text{otherwise,} \end{cases}$$

where, for every $M \in \mathbb{M}^{m\times n}$, $f_{\rm hom}(M)$ is defined by (8.2).

The question we address now is whether the Γ-limits of F_ε and $F_\varepsilon^{\mathcal{R}}$ depend on the size r_ε of the particles or not. In the next proposition we show that there is no such dependence under some additional assumptions that cover many meaningful cases.

Proposition 8.3 *Let $p > n$, $0 \le r_\varepsilon \ll \varepsilon$ and let F_ε and $F_\varepsilon^{\mathcal{R}}$ be given by (3.2) and (3.5), respectively. Moreover assume that one of the following hypotheses is satisfied*

(i) (regularity of the discrete potentials) there exists $L > 0$ such that for any $i \in \mathbb{Z}^n$, $z_1, z_2 \in \mathbb{R}^m$ and $\xi \in \mathbb{Z}^n$

$$|g^\xi(i,z_1) - g^\xi(i,z_2)| \le L|\xi|(|z_1|^{p-1} + |z_2|^{p-1})|z_1 - z_2|; \tag{8.3}$$

(ii) (scalar case) $m = 1$;

(iii) (isotropic case) $f^1(M) = f^1(|M|)$, with f^1 non decreasing;

(iv) (one dimensional case) $n = 1$.

Then

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon = \Gamma\text{-}\lim_{\varepsilon\to 0} F^0_\varepsilon = \Gamma\text{-}\lim_{\varepsilon\to 0} F^{\mathcal{R}}_\varepsilon$$

where

$$F^0_\varepsilon(u) = \int_\Omega f^1(Du)\,dx + \sum_{\xi\in\mathbb{Z}^n}\sum_{i\in R^\xi_\varepsilon(\Omega)} \varepsilon^n g^\xi(i, D^\xi_\varepsilon u(i)). \tag{8.4}$$

Proof.

First of all, note that without loss of generality we may assume that f^1 is quasiconvex (since the Γ-limits are the same if we replace the approximating functionals by their lower semicontinuous envelopes). We prove the thesis only for the Γ-limit of $F^{\mathcal{R}}_\varepsilon$, since all the other cases can be treated by the same argument. Since $F^{\mathcal{R}}_\varepsilon \ge F^0_\varepsilon$, we need only to prove that for all $u \in W^{1,p}(\Omega;\mathbb{R}^m)$

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F^{\mathcal{R}}_\varepsilon(u) \le \Gamma\text{-}\lim_{\varepsilon\to 0} F^0_\varepsilon(u). \tag{8.5}$$

For any open set $A \subseteq \mathbb{R}^n$ let us denote by $F^0_\varepsilon(u, A)$ the functional defined as in (8.4) with A in place of Ω and set

$$F^0(u, A) := \Gamma\text{-}\lim_{\varepsilon\to 0} F^0_\varepsilon(u, A).$$

Let, moreover, $\tilde\Omega \supset\supset \Omega$ and, given $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, let $\tilde u \in W^{1,p}(\tilde\Omega;\mathbb{R}^m)$ be an extension of u on $\tilde\Omega$. By Theorem 8.1, one easily shows that

$$F^0(u, \Omega) = F^0(\tilde u, \Omega) = \inf_{\Omega'\supset\supset\Omega} F^0(\tilde u, \Omega').$$

Hence, in order to prove (8.5), it is sufficient to prove that for any $\Omega' \supset\supset \Omega$ and every $u_\varepsilon \to \tilde u$ strongly in $L^p(\Omega';\mathbb{R}^m)$ such that $\sup_\varepsilon F^0_\varepsilon(u_\varepsilon, \Omega') < +\infty$ there exists a sequence v_ε converging to u strongly in $L^p(\Omega;\mathbb{R}^m)$ such that

$$F^{\mathcal{R}}_\varepsilon(v_\varepsilon) \le F^0_\varepsilon(u_\varepsilon, \Omega') + o(1). \tag{8.6}$$

Let u_ε be as above. Note that, by the coercivity assumption (3.4), we have in particular that $\sup_\varepsilon ||Du_\varepsilon\|_{L^p(\Omega';\mathbb{R}^m)} < +\infty$. By Lemma 2.5, up to adding a small error, we may assume that $u_\varepsilon(x) \equiv u^i_\varepsilon$ on $\partial B_{\rho^i_\varepsilon}(\varepsilon i)$, for any $i \in \mathbb{Z}^n \cap \frac{1}{\varepsilon}\Omega$, with $r_\varepsilon \ll \rho^i_\varepsilon < \rho_\varepsilon \ll \varepsilon$.

We define v_ε in two different ways, according if assumption (i) or one of the assumptions (ii)-(iv) is satisfied.

In the case (i), we define v_ε as follows

$$v_\varepsilon(x) := \begin{cases} u_\varepsilon^i & \text{if } x \in B_{\rho_\varepsilon^i}(\varepsilon i),\ i \in \mathbb{Z}^n \cap \frac{1}{\varepsilon}\Omega \\ u_\varepsilon(x) & \text{otherwise.} \end{cases}$$

We easily get that

$$\int_\Omega f_\varepsilon(x, Dv_\varepsilon) \leq \int_\Omega f^1(Du_\varepsilon) + o(1),$$

thus, in order to prove (8.6), we have just to show that

$$(F_\varepsilon^{\mathcal{R}})^d(v_\varepsilon) \leq (F_\varepsilon^{\mathcal{R}})^d(u_\varepsilon) + o(1).$$

Note that, by Sobolev immersion, we get that for any $i \in \mathbb{Z}^n \cap \frac{1}{\varepsilon}\Omega$

$$|u_\varepsilon^i - u_\varepsilon(\varepsilon i)| \leq c\, \rho_\varepsilon^{1-\frac{n}{p}} \left(\int_{Q_\varepsilon^i} |Du_\varepsilon|^p \, dx \right)^{\frac{1}{p}}. \tag{8.7}$$

By the coercivity assumption (H2), we have

$$\sum_{k=1}^n \sum_{i \in R_\varepsilon^{e_k}(\Omega')} \varepsilon^n |D_\varepsilon^{e_k} u(\varepsilon i)|^p \leq C.$$

Hence, by (8.3) and (8.7), given $\Omega \subset\subset \Omega'' \subset\subset \Omega'$, we infer

$$\begin{aligned} \sum_{k=1}^n \sum_{i \in R_\varepsilon^{e_k}(\Omega'')} \varepsilon^n |D_\varepsilon^{e_k} u_\varepsilon^i|^p \quad &\leq \quad CL \sum_{k=1}^n \sum_{i \in R_\varepsilon^{e_k}(\Omega'')} \Big(\frac{\rho_\varepsilon}{\varepsilon}\Big)^{p-n} \Big(\int_{Q_\varepsilon^i} |Du_\varepsilon|^p \, dx \\ &+ \quad \int_{Q_\varepsilon^{i+e_k}} |Du_\varepsilon|^p \, dx \Big) + \sum_{k=1}^n \sum_{i \in R_\varepsilon^{e_k}(\Omega')} \varepsilon^n |D_\varepsilon^{e_k} u(\varepsilon i)|^p \leq C \end{aligned}$$

By Lemma 3.6 in [1], we get that

$$\sup_{\xi \in \mathbb{Z}^n} \sum_{i \in R_\varepsilon^\xi(\Omega)} \varepsilon^n |D_\varepsilon^\xi u_\varepsilon^i|^p \leq \sum_{k=1}^n \sum_{i \in R_\varepsilon^{e_k}(\Omega'')} \varepsilon^n |D_\varepsilon^{e_k} u_\varepsilon^i|^p \leq C$$

and, analogously,

$$\sup_{\xi \in \mathbb{Z}^n} \sum_{i \in R_\varepsilon^\xi(\Omega)} \varepsilon^n |D_\varepsilon^\xi u(\varepsilon i)|^p \leq \sum_{k=1}^n \sum_{i \in R_\varepsilon^{e_k}(\Omega'')} \varepsilon^n |D_\varepsilon^{e_k} u(\varepsilon i)|^p \leq C.$$

By the growth assumption (H3), we then have that for any $\delta > 0$ there exist R_δ such that

$$\sum_{\xi \in \mathbb{Z}^n : |\xi| > R_\delta} \sum_{i \in R_\varepsilon^\xi(\Omega)} \varepsilon^n g^\xi(i, D_\varepsilon^\xi u_\varepsilon^i) < \delta.$$

Hence, by (8.3) and (8.7)we get

$$(F_\varepsilon^{\mathcal{R}})^d(v_\varepsilon) - (F_\varepsilon^{\mathcal{R}})^d(u_\varepsilon) \leq \sum_{\xi\in\mathbb{Z}^n:|\xi|\leq R_\delta} \sum_{i\in R_\varepsilon^\xi(\Omega)} \varepsilon^n (g^\xi(i, D_\varepsilon^\xi u_\varepsilon^i) - g^\xi(i, D_\varepsilon^\xi u(\varepsilon i)))$$

$$+ \delta \leq CL\varepsilon^{n-1}\rho_\varepsilon^{1-\frac{n}{p}} \sum_{\xi\in\mathbb{Z}^n:|\xi|\leq R_\delta} \sum_{i\in R_\varepsilon^\xi(\Omega)} (|D_\varepsilon^\xi u_\varepsilon^i|^{p-1} + |D_\varepsilon^\xi u(\varepsilon i)|^{p-1})$$

$$\Big(\Big(\int_{Q_\varepsilon^i} |Du_\varepsilon|^p\,dx\Big)^{\frac{1}{p}} + \Big(\int_{Q_\varepsilon^{i+\xi}} |Du_\varepsilon|^p\,dx\Big)^{\frac{1}{p}}\Big) + \delta$$

$$\leq CL\Big(\frac{\rho_\varepsilon}{\varepsilon}\Big)^{1-\frac{n}{p}} \sum_{\xi\in\mathbb{Z}^n:|\xi|\leq R_\delta} \frac{1}{|\xi|}\Big(\Big(\sum_{i\in R_\varepsilon^\xi(\Omega)} \varepsilon^n(|D_\varepsilon^\xi u_\varepsilon^i|^p + |D_\varepsilon^\xi u(\varepsilon i)|^p)\Big)^{1-\frac{1}{p}}$$

$$\Big(\sum_{i\in R_\varepsilon^\xi(\Omega)} \int_{Q_\varepsilon^i} |Du_\varepsilon|^p\,dx + \int_{Q_\varepsilon^{i+\xi}} |Du_\varepsilon|^p\,dx\Big)^{\frac{1}{p}}\Big) + \delta \leq CLM_\delta\Big(\frac{\rho_\varepsilon}{\varepsilon}\Big)^{1-\frac{n}{p}} + \delta,$$

where M_δ is a positive constant depending on δ. Letting first ε and then δ go to 0, we get the conclusion.

Let, now, one of the assumptions (ii)-(iv) be satisfied. In this case, since we do not assume any regularity on the discrete potentials, in order to construct v_e we modify u_ε without changing its discrete energy. So, let v_ε be defined as follows

$$v_\varepsilon(x) := \begin{cases} u_\varepsilon(\varepsilon i) & \text{if } x\in B_i^\varepsilon,\ i\in\mathbb{Z}^n\cap\frac{1}{\varepsilon}\Omega \\ w_\varepsilon^i(x) & \text{if } x\in S_\varepsilon^i,\ i\in\mathbb{Z}^n\cap\frac{1}{\varepsilon}\Omega \\ u_\varepsilon(x) & \text{otherwise,} \end{cases}$$

where we have set

$$S_\varepsilon^i := B_{\rho_\varepsilon^i}(\varepsilon i)\setminus B_i^\varepsilon,$$

and w_ε^i minimizes

$$M_\varepsilon^i := \min\left\{\int_{S_\varepsilon^i} f^1(Dw):\ w = u_\varepsilon(\varepsilon i) \textit{ on } \partial B_i^\varepsilon,\ w = u_\varepsilon^i \textit{ on } \partial B_{\rho_\varepsilon^i}(\varepsilon i)\right\} \tag{8.8}$$

We easily get that $(F_\varepsilon^{\mathcal{R}})^d(u_\varepsilon) = (F_\varepsilon^{\mathcal{R}})^d(v_\varepsilon)$. Moreover, since $v_\varepsilon = u_\varepsilon$ on $\Omega\setminus\cup_i B_{\rho_\varepsilon^i}(\varepsilon i)$, we have

$$F_\varepsilon^{\mathcal{R}}(v_\varepsilon) \leq F_\varepsilon^0(u_\varepsilon) + \sum_i \left(\int_{S_\varepsilon^i} f^1(Dv_\varepsilon)\,dx - \int_{S_\varepsilon^i} f^1(Du_\varepsilon)\,dx\right) + o(1).$$

Thus, in order to get the conclusion, it suffices to show that

$$\limsup_{\varepsilon\to 0}\sum_i \left(\int_{S_\varepsilon^i} f^1(Dv_\varepsilon)\,dx - \int_{S_\varepsilon^i} f^1(Du_\varepsilon)\,dx\right) \leq 0. \tag{8.9}$$

We will provide a different proof of (8.9), according if (ii), (iii) or (iv) is satisfied.

Step 1: the scalar case $m = 1$. Note that, by Sobolev immersion, we get

$$\sup_{x \in B_i^\varepsilon} |u_\varepsilon(x) - u_\varepsilon(\varepsilon i)| \leq c\, r_\varepsilon^{1-\frac{n}{p}} \left(\int_{Q_\varepsilon^i} |Du_\varepsilon|^p \right)^{\frac{1}{p}} =: \overline{z}_\varepsilon^i. \tag{8.10}$$

Set, moreover,

$$z_\varepsilon^i := |u_\varepsilon^i - u_\varepsilon(\varepsilon i)|. \tag{8.11}$$

In order to prove (8.9), fixed $k \in \mathbb{N}$, we provide a different estimate for $\int_{S_\varepsilon^i} f^1(Dv_\varepsilon)$ according if $z_\varepsilon^i \leq k\overline{z}_\varepsilon^i$ or not. Let then $i \in \mathbb{Z}^n \cap \frac{1}{\varepsilon}\Omega$ such that $z_\varepsilon^i \leq k\overline{z}_\varepsilon^i$ and define $\tilde{w}_\varepsilon^i : S_\varepsilon^i \to \mathbb{R}$ as

$$\tilde{w}_\varepsilon^i(x) = u_\varepsilon(\varepsilon i) + (u_\varepsilon^i - u_\varepsilon(\varepsilon i)) \frac{|x - \varepsilon i| - r_\varepsilon}{\rho_\varepsilon^i - r_\varepsilon}.$$

Note that $\tilde{w}_\varepsilon^i$ is an admissible function for the minimum problem defining M_ε^i in (8.8) and

$$D\tilde{w}_\varepsilon^i(x) = \frac{u_\varepsilon^i - u_\varepsilon(\varepsilon i)}{\rho_\varepsilon^i - r_\varepsilon} \frac{x - \varepsilon i}{|x - \varepsilon i|}.$$

Hence, by the p-growth assumption on f^1 and by (8.10) and (8.11), we get

$$\begin{aligned}
\int_{S_\varepsilon^i} f^1(Dv_\varepsilon)\, dx &\leq C(|S_\varepsilon^i| + \int_{S_\varepsilon^i} |D\tilde{w}_\varepsilon^i|^p\, dx \leq C|S_\varepsilon^i| \left(1 + \left| \frac{k\overline{z}_\varepsilon^i}{\rho_\varepsilon^i - r_\varepsilon} \right|^p \right) \\
&\leq C \left((\rho_\varepsilon^i)^n + k^p \frac{r_\varepsilon^{p-n}}{|\rho_\varepsilon^i - r_\varepsilon|^p} (\rho_\varepsilon^i)^n \int_{Q_\varepsilon^i} |Du_\varepsilon|^p\, dx \right),
\end{aligned}$$

and, summing up on all i such that $z_\varepsilon^i \leq k\overline{z}_\varepsilon^i$, we obtain

$$\sum_{i:\ z_\varepsilon^i \leq k\overline{z}_\varepsilon^i} \int_{S_\varepsilon^i} f^1(Dv_\varepsilon)\, dx \leq C \left(\left(\frac{\rho_\varepsilon}{\varepsilon} \right)^n + k^p \left(\frac{r_\varepsilon}{\rho_\varepsilon} \right)^{p-n} \right) = o(1). \tag{8.12}$$

Let now $i \in \mathbb{Z}^n \cap \frac{1}{\varepsilon}\Omega$ such that $z_\varepsilon^i > k\overline{z}_\varepsilon^i$ and let $\hat{w}_\varepsilon^i : S_\varepsilon^i \to \mathbb{R}$ the admissible function for M_ε^i defined by

$$\hat{w}_\varepsilon^i = \phi_\varepsilon^i(u_\varepsilon), \tag{8.13}$$

where $\phi_\varepsilon^i : \mathbb{R} \to \mathbb{R}$ is given by

$$\phi_\varepsilon^i(z) := \begin{cases} u_\varepsilon(\varepsilon i) & \text{if } |z - u_\varepsilon(\varepsilon i)| \leq \overline{z}_\varepsilon^i \\ \dfrac{z_\varepsilon^i}{z_\varepsilon^i - \overline{z}_\varepsilon^i}(z - u_\varepsilon(\varepsilon i)) \left(1 - \dfrac{\overline{z}_\varepsilon^i}{|z - u_\varepsilon(\varepsilon i)|} \right) + u_\varepsilon(\varepsilon i) & \text{if } \overline{z}_\varepsilon^i < |z - u_\varepsilon(\varepsilon i)| < z_\varepsilon^i \\ z & \text{if } |z - u_\varepsilon(\varepsilon i)| \geq z_\varepsilon^i. \end{cases}$$

Note that

$$D\phi_\varepsilon^i(z) = \begin{cases} 0, & \text{if } |z - u_\varepsilon(\varepsilon i)| \le \overline{z}_\varepsilon^i \\ \\ 1, & \text{if } |z - u_\varepsilon(\varepsilon i)| \ge z_\varepsilon^i, \end{cases}$$

and for every z such that $\overline{z}_\varepsilon^i < |z - u_\varepsilon(\varepsilon i)| < z_\varepsilon^i$

$$D\phi_\varepsilon^i(z) = \frac{z_\varepsilon^i}{z_\varepsilon^i - \overline{z}_\varepsilon^i}.$$

In particular $\|D\phi_\varepsilon^i\|_\infty \le C$ and

$$|1 - D\phi_\varepsilon^i(z)| = \frac{\overline{z}_\varepsilon^i}{z_\varepsilon^i - \overline{z}_\varepsilon^i}$$

for $z \in \mathbb{R}$ such that $\overline{z}_\varepsilon^i < |z - u_\varepsilon(\varepsilon i)| < z_\varepsilon^i$. Hence, setting

$$\hat{S}_\varepsilon^i := \{x \in S_\varepsilon^i : \ \overline{z}_\varepsilon^i < |\ u_\varepsilon(x) - u_\varepsilon(\varepsilon i)| < z_\varepsilon^i\},$$

we have

$$\begin{aligned} \int_{\hat{S}_\varepsilon^i} |Du_\varepsilon - D\hat{w}_\varepsilon^i|^p dx &= \int_{\hat{S}_\varepsilon^i} |I - D\phi_\varepsilon^i(u_\varepsilon)|^p |Du_\varepsilon|^p \, dx \\ &= \left(\frac{\overline{z}_\varepsilon^i}{z_\varepsilon^i - \overline{z}_\varepsilon^i} \right)^p \int_{\hat{S}_\varepsilon^i} |Du_\varepsilon|^p dx \\ &\le \frac{1}{(k-1)^p} \int_{\hat{S}_\varepsilon^i} |Du_\varepsilon|^p \, dx. \end{aligned} \tag{8.14}$$

We are now in a position to provide an upper estimate for $\int_{S_\varepsilon^i} f^1(Dv_\varepsilon)\, dx$ which, together with (8.12), will allow to prove (8.9). We easily get that

$$\int_{S_\varepsilon^i} f^1(Dv_\varepsilon)\, dx \le C\rho_\varepsilon^n + \int_{S_\varepsilon^i} f^1(Du_\varepsilon)\, dx + \int_{\hat{S}_\varepsilon^i} \left(f^1(D\hat{w}_\varepsilon^i)\, dx - f^1(Du_\varepsilon) \right) dx. \tag{8.15}$$

Set

$$S_{\varepsilon,k} = \bigcup_{i:\ z_\varepsilon^i > k\overline{z}_\varepsilon^i} \hat{S}_\varepsilon^i$$

and let $w_\varepsilon^k : S_{\varepsilon,k} \to \mathbb{R}$ be defined as

$$w_\varepsilon^k(x) = \hat{w}_\varepsilon^i(x), \quad x \in \hat{S}_\varepsilon^i.$$

Hence, from inequality (8.15) and summing up on all i such that $z_\varepsilon^i > k\overline{z}_\varepsilon^i$, we obtain

$$\sum_{i:\ z_\varepsilon^i > k\overline{z}_\varepsilon^i} \int_{S_\varepsilon^i} f^1(Dv_\varepsilon)\,dx \leq \sum_{i:\ z_\varepsilon^i > k\overline{z}_\varepsilon^i} \int_{S_\varepsilon^i} f^1(Du_\varepsilon)\,dx + \int_{S_{\varepsilon,k}} \big(f^1(Dw_\varepsilon^k)\,dx - f^1(Du_\varepsilon)\big)\,dx + o(1).\quad (8.16)$$

Now we prove that

$$\int_{S_{\varepsilon,k}} (f^1(Dw_\varepsilon^k) - f^1(Du_\varepsilon))\,dx \leq O(\frac{1}{k}) + o(1).$$

To this end, by the quasiconvexity of f_1 and by Hölder inequality, we have

$$\begin{aligned} & \int_{S_{\varepsilon,k}} |f^1(Du_\varepsilon) - f^1(Dw_\varepsilon^k)|\,dx \\ \leq\ & C\int_{S_{\varepsilon,k}} (1 + |Du_\varepsilon|^{p-1} + |Dw_\varepsilon^k|^{p-1})|Du_\varepsilon - Dw_\varepsilon^k|\,dx \\ \leq\ & C\left(\int_{S_{\varepsilon,k}} |Du_\varepsilon - Dw_\varepsilon^k|^p\,dx\right)^{\frac{1}{p}} \\ & \times\left(|S_{\varepsilon,k}|^{\frac{p-1}{p}} + \left(\int_{S_{\varepsilon,k}} |Du_\varepsilon|^p\,dx\right)^{\frac{p-1}{p}} + \left(\int_{S_{\varepsilon,k}} |Dw_\varepsilon^k|^p\,dx\right)^{\frac{p-1}{p}}\right). \end{aligned}$$

Hence, taking into account that $Dw_\varepsilon^k = D\phi_\varepsilon^i(u_\varepsilon)Du_\varepsilon$ on S_ε^i, for every i such that $z_\varepsilon^i > k\overline{z}_\varepsilon^i$, and that $\|Du_\varepsilon\|_p$ and $\|D\phi_\varepsilon^i\|_\infty$ are equibounded, we get

$$\int_{S_{\varepsilon,k}} |f^1(Du_\varepsilon) - f^1(Dw_\varepsilon^k)|\,dx \leq C\left(\int_{S_{\varepsilon,k}} |Du_\varepsilon - Dw_\varepsilon^k|^p\,dx\right)^{\frac{1}{p}}.$$

and, by (8.14), we may deduce that

$$\int_{S_{\varepsilon,k}} |f^1(Du_\varepsilon) - f^1(Dw_\varepsilon^k)|\,dx \leq C\frac{1}{k-1}. \quad (8.17)$$

Eventually, (8.9) follows from (8.12), (8.16), (8.17) and the arbitrariness of $k \in \mathbb{N}$.

Step 2: the isotropic case $f^1(M) = f^1(|M|)$, with f^1 non decreasing. In this case we will show that we may reduce to the scalar case, by using the isotropy and monotonicity of f^1. Let $\hat{u}_\varepsilon$ be defined by

$$\hat{u}_\varepsilon(x) := \begin{cases} \hat{u}_\varepsilon^i(x) & \text{if } x \in B_{\rho_\varepsilon^i}(\varepsilon i) \cap \Omega \\ u_\varepsilon(x) & \text{if } x \in \Omega \setminus \cup_i B_{\rho_\varepsilon^i}(\varepsilon i), \end{cases}$$

where $\hat{u}_\varepsilon^i$ minimizes

$$\hat{M}_\varepsilon^i := \min\left\{\int_{B_{\rho_\varepsilon^i}(\varepsilon i)} f^1(|Dw|):\ w(\varepsilon i)=u_\varepsilon(\varepsilon i),\ w=u_\varepsilon^i \text{ on } \partial B_{\rho_\varepsilon^i}(\varepsilon i)\right\} \tag{8.18}$$

Note that $\hat{u}_\varepsilon$ still converge to u strongly in $L^p(\Omega;\mathbb{R}^m)$ and

$$F_\varepsilon^0(\hat{u}_\varepsilon) \le F_\varepsilon^0(u_\varepsilon).$$

Hence we easily deduced that (8.9) is implied by

$$\limsup_{\varepsilon\to 0}\sum_i\left(\int_{B_{\rho_\varepsilon^i}(\varepsilon i)} f^1(Dv_\varepsilon)\,dx - \int_{B_{\rho_\varepsilon^i}(\varepsilon i)} f^1(D\hat{u}_\varepsilon)\,dx\right) \le 0. \tag{8.19}$$

Note now that a minimizer of the minimum problem (8.18) is given by

$$\psi_\varepsilon^i(x)\frac{u_\varepsilon^i - u_\varepsilon(\varepsilon i)}{|u_\varepsilon^i - u_\varepsilon(\varepsilon i)|} + u_\varepsilon(\varepsilon i),$$

where ψ_ε^i is a suitable scalar function . Indeed, by isotropy, we easily get

$$\begin{aligned}\hat{M}_\varepsilon^i &= \min\left\{\int_{B_{\rho_\varepsilon^i}(\varepsilon i)} f^1(|Dw|):\ w(\varepsilon i)=0,\ w=u_\varepsilon^i - u_\varepsilon(\varepsilon i) \text{ on } \partial B_{\rho_\varepsilon^i}(\varepsilon i)\right\}\\ &= \min\left\{\int_{B_{\rho_\varepsilon^i}(\varepsilon i)} f^1(|Dw|):\ w(\varepsilon i)=0,\ w=|u_\varepsilon^i - u_\varepsilon(\varepsilon i)|e_1 \text{ on } \partial B_{\rho_\varepsilon^i}(\varepsilon i)\right\}.\end{aligned}$$

Then, using the monotonicity of f^1, we infer that $\hat{M}_\varepsilon^i$ is also equal to the following scalar minimum problem

$$\hat{M}_\varepsilon^i = \min\left\{\int_{B_{\rho_\varepsilon^i}(\varepsilon i)} f^1(|D\psi|):\ \psi(\varepsilon i)=0,\ \psi=|u_\varepsilon^i - u_\varepsilon(\varepsilon i)| \text{ on } \partial B_{\rho_\varepsilon^i}(\varepsilon i)\right\}.$$

In order to prove (8.19), we may now proceed as in Step 1, with the only significant change consisting in replacing u_ε by ψ_ε^i in (8.13).

Step 3: the one dimensional case $n=1$. Let $i \in \mathbb{Z}\cap\frac{1}{\varepsilon}\Omega$ and let $\overline{w}_\varepsilon^i : S_\varepsilon^i \to \mathbb{R}$ the admissible function for M_ε^i defined by

$$\overline{w}_\varepsilon^i = u_\varepsilon(\overline{\phi}_\varepsilon^i),$$

where $\overline{\phi}_\varepsilon^i : S_\varepsilon^i \to \mathbb{R}$ is given by

$$\overline{\phi}_\varepsilon^i(x) := \frac{\rho_\varepsilon^i}{\rho_\varepsilon^i - r_\varepsilon}(x-\varepsilon i)\left(1-\frac{r_\varepsilon}{|x-\varepsilon i|}\right)+\varepsilon i.$$

Note that

$$D\overline{\phi}^i_\varepsilon(x) = \frac{\rho^i_\varepsilon}{\rho^i_\varepsilon - r_\varepsilon}.$$

and in particular

$$|1 - D\overline{\phi}^i_\varepsilon(x)| = \frac{r_\varepsilon}{\rho^i_\varepsilon - r_\varepsilon}.$$

Hence we have

$$\begin{aligned}\int_{S^i_\varepsilon} |Du_\varepsilon - D\overline{w}^i_\varepsilon|^p dx &= \int_{S^i_\varepsilon} |I - D\overline{\phi}^i_\varepsilon(u_\varepsilon)|^p |Du_\varepsilon|^p\, dx \\ &= \left(\frac{r_\varepsilon}{\rho^i_\varepsilon - r_\varepsilon}\right)^p \int_{S^i_\varepsilon} |Du_\varepsilon|^p dx.\end{aligned}$$

By using the same argument leading to (8.17), we get that

$$\sum_i \int_{S^i_\varepsilon} |f^1(Du_\varepsilon) - f^1(D\overline{w}^i_\varepsilon)|\, dx \le C\frac{r_\varepsilon}{\rho^i_\varepsilon - r_\varepsilon} \to 0. \tag{8.20}$$

Therefore, since we have

$$\sum_i \int_{S^i_\varepsilon} (f^1(Dv_\varepsilon) - f^1(Du_\varepsilon))\, dx \le C\rho^n_\varepsilon + \sum_i \int_{S^i_\varepsilon} \big(f^1(D\overline{w}^i_\varepsilon)\, dx - f^1(Du_\varepsilon)\big)\, dx,$$

we easily derive (8.9) from (8.20). □

As a consequence of Theorem 8.1 and Proposition 8.3, we easily derive the following result.

Corollary 8.4 *Let $p > n$ and let one of the assumptions (i)-(iv) of Proposition 8.3 be satisfied. Then, the results of Theorem 8.1 holds with f_{hom} given by*

$$f_{\hom}(M) = \lim_{T\to+\infty} \frac{1}{T^n} \inf\Big\{\int_{Q_T} f^1(Du)\, dx + F^d_1(u, Q_T)\,,\ u - Mx \in W^{1,p}_0(Q_T; \mathbb{R}^m)\Big\}. \tag{8.21}$$

Arguing as in Section 6, we may derive a result about the convergence of minimum problems with boundary data, that we state for simplicity only in the case one of the assumptions (i)-(iv) of Proposition 8.3 is satisfied. Since there is no decoupling of variables, it is equivalent to prescribe boundary data on u or $v(u)$. Hence we get the following result.

Corollary 8.5 *Let $\phi \in W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^m)$. Under the hypotheses of Corollary 8.4 we get that*

$$\lim_{\varepsilon\to0} \inf\{F_\varepsilon(u),\ u - \phi \in W^{1,p}_0(\Omega; \mathbb{R}^m)\} = \lim_{\varepsilon\to0} \inf\{F_\varepsilon(u),\ v(u) \in \mathcal{A}_{\varepsilon,\phi}(\Omega)\}$$

$$= \min\{F(u) : \ u - \phi \in W_0^{1,p}(\Omega;\mathbb{R}^m)\},$$

where F is defined by (8.1), with f_{hom} given by (8.21). Moreover, if (u_ε) is a converging sequence such that

$$\begin{aligned}\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) &= \lim_{\varepsilon\to 0} \inf\{F_\varepsilon(u), \ u - \phi \in W_0^{1,p}(\Omega;\mathbb{R}^m)\} \\ &= \lim_{\varepsilon\to 0} \inf\{F_\varepsilon(u), v(u) \in \mathcal{A}_{\varepsilon,\phi}(\Omega)\},\end{aligned}$$

then its limit is a minimizer for $\min\{F(u) : \ u - \phi \in W_0^{1,p}(\Omega;\mathbb{R}^m)\}$.

Remark 8.6 (Cauchy-Born states) Let f_{hom} be defined by (8.21) and let g_{hom} the limit energy density of the discrete energies F_ε^d, given by (2.6) with F_1^d in place of E_1. If we assume f^1 quasiconvex (which is not a restriction since we obtain the same Γ-limit if in (8.4) we replace f^1 by Qf^1), then we note that:

(1) $f_{hom}(M) \geq f^1(M) + g_{hom}(M)$ for all $M \in \mathbb{M}^{m\times n}$;

(2) if $g_{hom}(M) = \lim_{T\to\infty} \frac{1}{T^n} F_1^d(Mx, Q_T)$, then the equality holds true in (1) and $f_{hom}(M) = \lim_{T\to\infty} \frac{1}{T^n} F_1(Mx, Q_T)$.

Hence the set of *Cauchy-Born states* of F_ε^d is contained in the set of *Cauchy-Born states* of F_ε (or equivalently $\widehat{F}_\varepsilon$), according to Definition 8.6. In the next section we consider some one-dimensional examples to show that in (1) the inequality can be strict and the set of *Cauchy-Born states* of F_ε can be strictly larger than that of F_ε^d.

8.1 One-dimensional examples

In this section we frame our analysis in the one dimensional setting. We have already shown in Proposition 8.3 that in this case we may restrict to the study of the functionals defined by (8.4), corresponding to $r_\varepsilon = 0$. Without loss of generality, we may assume $\Omega = [0,1]$, $\varepsilon_j = \frac{1}{j}$, $j \in \mathbb{N}$, and $p > 1$. Hence, the functionals in (8.4), defined on $W^{1,p}((0,1);\mathbb{R}^m)$, can be written as follows

$$\begin{aligned}F_j(u) &= \int_0^1 f^1(u'(x))\,dx + \sum_{k=1}^{j}\sum_{i=0}^{j-k} \frac{1}{j} g_k\left(i, \frac{j}{k}\Big(u\big(\frac{i+k}{j}\big) - u\big(\frac{i}{j}\big)\Big)\right) \\ &= F_j^c(u) + F_j^d(u) \qquad (8.22)\end{aligned}$$

with $g_k = g^{ke_1}$ satisfying hypotheses (H1), (H2), and (H3) introduced in Section 2.1. Moreover, let $\widehat{F}_j : L^p((0,1);\mathbb{R}^m) \to [0,+\infty]$ be the functionals defined by (7.10) with F_j in place of F_ε.

It is easy to show, by Jensen inequality, that $\widehat{F}_j$ equals to

$$\widehat{F}_j(v) = \begin{cases} \displaystyle\sum_{k=1}^{j} \sum_{i=0}^{j-k} \frac{1}{j}\, \tilde{g}_k \left(i, \frac{j}{k}\Big(v(\frac{i+k}{j}) - v(\frac{i}{j}) \Big) \right) & \text{if } v \in \mathcal{A}_{\frac{1}{j}}(0,1) \\ +\infty & \text{otherwise} \end{cases} \tag{8.23}$$

with

$$\tilde{g}_1 = (f^1)^{**} + g_1 \tag{8.24}$$

and $\tilde{g}_k = g_k$ for $k > 1$.

We have already showed in Theorems 8.1 and 8.2 that F_j and $\widehat{F}_j$ share the same Γ-limit. Hence, by applying Theorem 2.1 to the functionals $\widehat{F}_j$ we immediately get the following result.

Proposition 8.7 *The functionals F_j and $\widehat{F}_j$ defined by (8.22) and (8.23), respectively, Γ-converge, with respect to the $L^p((0,1);\mathbb{R}^m)$-strong topology, to the functional $F : L^p((0,1);\mathbb{R}^m) \to [0,+\infty]$ defined by*

$$F(u) = \begin{cases} \displaystyle\int_0^1 g_{hom}(u')\,dx & \textit{if } u \in W^{1,p}((0,1);\mathbb{R}^m) \\ +\infty & \textit{otherwise,} \end{cases} \tag{8.25}$$

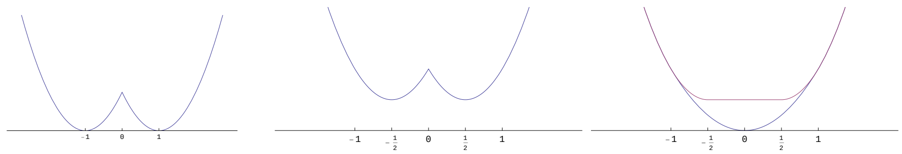
where g_{hom} is defined by (2.6) with $\widehat{F}_1$ in place of E_1.

Remark 8.8 (Cauchy-Born states: n = 1) As a consequence of Proposition 8.7 and Theorem 2.2, if F_j^d in (8.22) accounts only for nearest neighbor interactions, that is $g_k \equiv 0$ if $k > 1$, then the limit energy density g_{hom} in (8.25) reduces to

$$g_{hom}(z) = ((f^1)^{**} + g_1)^{**}(z).$$

Therefore, in this case the Cauchy-Born states for F_j (or $\widehat{F}_j$) are all $z \in \mathbf{R}^m$ such that $\tilde{g}_1(z) = (\tilde{g}_1)^{**}(z)$, where $\tilde{g}_1$ is defined by (8.24), while the Cauchy-Born states for F_j^d are all $z \in \mathbf{R}^m$ such that $g_1(z) = g_1^{**}(z)$.

Note, moreover, that $\{z \in \mathbf{R}^m : g_1(z) = g_1^{**}(z)\} \subseteq \{z \in \mathbf{R}^m : \tilde{g}_1(z) = (\tilde{g}_1)^{**}(z)\}$ and it is not difficult to provide examples where the inclusion is strict. Take $f^1(z) = z^2$, $g_1(z) = (|z|-1)^2$ and $m = 1$. Easy computations show that in this case $\{z \in \mathbf{R} : g_1(z) = g_1^{**}(z)\} = \mathbf{R} \setminus (-1,1)$, while $\{z \in \mathbf{R} : \tilde{g}_1(z) = (\tilde{g}_1)^{**}(z)\} = \mathbf{R} \setminus (-1/2, 1/2)$ (see Figure 3). The same example shows that in formula (1) of Remark 8.6 the inequality can be strict. Notice, however, that the lack of validity of the Cauchy-Born rule corresponds in this case to a convexification process that takes place at a mesoscale greater than ε, that is there is no relaxation at 'atomic scale'. Indeed one can obtained the same Γ-limit by first computing the point-wise limit of F_j, that is assuming the Cauchy-Born rule, and then relaxing the limit functional.



(a) (b) (c)

Figure 3: (a) $g_1(z) = (|z|-1)^2$; (b) $g_1(z)+z^2$; (c) $(g_1+z^2)^{**}$(in red) and $g_1^{**}(z)+z^2$ (in blue).

The phenomenon is instead more significant if F_j^d accounts also for next-to-nearest neighbor interactions, since the inf-convolution process defined by (2.7) corresponds to an oscillation at 'atomic scale' (if the minimizers z_1 and z_2 do not coincide with z). In this case in (8.22) $g_k \equiv 0$ if $k > 2$, and, by Theorems 8.7 and 2.3, the limit energy density g_{hom} in (8.25) reduces to

$$g_{hom}(z) = h^{**}(z),$$

where

$$h(z) := g_2(z) + \frac{1}{2}\inf\{\tilde{g}_1(z_1) + \tilde{g}_1(z_2) : \ z_1 + z_2 = 2z\}. \tag{8.26}$$

Hence, the Cauchy-Born states for F_j^d (resp. F_j) are all $z \in \mathbf{R}^m$ such that $\hat{g}(z) = \hat{g}^{**}(z) = g_2(z) + g_1(z)$ (resp. $h(z) = h^{**}(z) = g_2(z) + \tilde{g}_1(z)$), that is in (2.7) (resp. (8.26)) $z_1 = z_2 = z$. Note that this last condition is in particular satisfied if $g_1(z) = g_1^{**}(z)$ (resp. $\tilde{g}_1(z) = (\tilde{g}_1)^{**}(z)$). So, taking, as before, $f^1(z) = z^2$ and $g_1(z) = (|z|-1)^2$, we get that to each $z \in (-1,1) \setminus (-1/2, 1/2)$ corresponds a relaxation at 'atomic scale' for F_j^d but not for F_j, highlighting the 'regularizing effect' of adding a continuous term to the discrete energy.

8.1.1 Lennard-Jones type potentials

The results of the previous section can be extended also to interactions governed by Lennard-Jones type potentials, which do not satisfy growth hypotheses of the type (H2) and (H3), under Dirichlet boundary conditions. We will focus on energies accounting for nearest and next-to-nearest neighbor interactions and on the scalar case $m = 1$. We start by recalling some Γ-convergence results about this type of discrete energies.

Let $H_j : L^1(0,1) \to [0, +\infty]$ be a sequence of functional energies accounting only for nearest and next-to-nearest neighbor interactions defined by

$$H_j(u) := \begin{cases} \displaystyle\sum_{i=0}^{j-1} \frac{1}{j} J_1\left(j\Big(u\big(\frac{i+1}{j}\big) - u\big(\frac{i}{j}\big)\Big)\right) + \sum_{i=0}^{j-2} \frac{1}{j} J_2\left(\frac{j}{2}\Big(u\big(\frac{i+2}{j}\big) - u\big(\frac{i}{j}\big)\Big)\right) & \text{if } u \in \mathcal{A}_{\frac{1}{j}}(0,1) \\ +\infty & \text{otherwise.} \end{cases} \tag{8.27}$$

Here $J_k : \mathbf{R} \to (-\infty, +\infty]$, $k = 1, 2$, are Borel functions bounded from below and satisfying the following assumption: there exists a convex function $\psi : \mathbf{R} \to [0, +\infty]$ such that

$$\lim_{z \to -\infty} \frac{\psi(z)}{|z|} = +\infty, \qquad \lim_{z \to +\infty} \frac{\psi(z)}{|z|} = 0$$

and there exist two constants $c_1,\ c_2$ such that

$$c_1(\psi(z) - 1) \le J_k(z) \le c_2\, \psi(z) \quad \forall z \in \mathbf{R},\ k = 1, 2. \tag{8.28}$$

This hypothesis is designed to cover the case of Lennard-Jones potentials, where $J_1(z) = J(z),\ J_2(z) = J(2z)$ with

$$J(z) = \begin{cases} +\infty & \text{if } z \le 0 \\ \frac{k_1}{z^{12}} - \frac{k_2}{z^6} & \text{if } z > 0, \end{cases} \tag{8.29}$$

for some $k_1,\ k_2 > 0$. Note that this kind of potentials allows the deformation to be discontinuous in the limit, hence the limiting energy is finite on BV functions, but the growth assumption from below forces the singular part of the derivative to be positive.

Given $l \in \mathbf{R}$, we also define $H_j^l : L^1(0,1) \to [0, +\infty]$ as follows

$$H_j^l(u) = \begin{cases} H_j(u) & \text{if } u(0) = 0,\ u(1) = l \\ +\infty & \text{otherwise.} \end{cases} \tag{8.30}$$

In the following theorem we recall the analogue of Theorem 2.2 for this type of discrete energies, under Dirichlet boundary conditions (see Theorem 3.2 in [10] for more general interaction potentials) and the result of Theorem 4.2 in [8].

Theorem 8.9 *(a)* **(nearest neighbors)** *Let $J_1 : \mathbf{R} \to (-\infty, +\infty]$, be a Borel function bounded below satisfying (8.28) and $J_2 \equiv 0$. Then the Γ-limit of the sequence of functionals H_j^l, defined by 8.30, with respect to the L^1_{loc}-topology is given by*

$$H^l(u) = \begin{cases} \int_0^1 J_1^{**}(u')\,dx & \text{if } u \in BV^l(0,1),\ D_s u > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

where we have set

$$BV^l(0,1) := \{u \in BV(0,1) :\ u(0+) = 0,\ u(1-) = l\}$$

and $D_s u$ denotes the singular part of the measure Du with respect to the Lebesgue measure.

(b) **(next-to-nearest neighbors)** *Let $J_k : \mathbf{R} \to (-\infty, +\infty]$, $k = 1, 2$, be Borel functions bounded below satisfying (8.28). Then the Γ-limit of the sequence of functionals H_j^l, defined by 8.30, with respect to the L^1_{loc}-topology is given by*

$$L^l(u) = \begin{cases} \int_0^1 J_0^{**}(u')\,dx & \text{if } u \in BV^l(0,1),\ D_s u > 0 \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$J_0(z) := J_2(z) + \frac{1}{2}\inf\{J_1(z_1) + J_1(z_2) : z_1 + z_2 = 2z\}.$$

Let us consider now functionals as in (8.22) with F_j^d of the type (8.27), that is

$$\begin{aligned} F_j(u) &= \int_0^1 f^1(u'(x))\,dx + \sum_{k=1}^{2}\sum_{i=0}^{j-k}\frac{1}{j}J_k\left(\frac{j}{k}\Big(u(\frac{i+k}{j}) - u(\frac{i}{j})\Big)\right) \\ &= F_j^c(u) + F_j^d(u) \end{aligned} \tag{8.31}$$

with $J_k : \mathbf{R} \to (-\infty, +\infty]$ satisfying (8.28). It is easy to show that even in this case there is no decoupling of variables and that the functionals F_j share the same Γ-limit of the functionals $\widehat{F}_j : L^p((0,1);\mathbf{R}^m) \to [0,+\infty]$ defined by (7.10) with F_j as in (8.31) in place of F_ε. Moreover, as in the previous case, one can easily show, by applying Jensen inequality, that $\widehat{F}_j$ equals to

$$\widehat{F}_j(v) = \begin{cases} \displaystyle\sum_{k=1}^{2}\sum_{i=0}^{j-k}\frac{1}{j}\,\tilde{J}_k\left(\frac{j}{k}\Big(v(\frac{i+k}{j}) - v(\frac{i}{j})\Big)\right) & \text{if } v \in \mathcal{A}_{\frac{1}{j}}(0,1) \\ +\infty & \text{otherwise} \end{cases} \tag{8.32}$$

with

$$\tilde{J}_1 = (f^1)^{**} + J_1 \tag{8.33}$$

and $\tilde{J}_2 = J_2$. Therefore $\widehat{F}_j$ satisfies the assumptions of Theorem 2.3 or, if $J_2 = 0$, Theorem 2.2 (see Theorem).

Given $l \in \mathbf{N}$, let us consider also the functionals $F_j^l : W^{1,p}(0,1) \to [0,+\infty]$, $\widehat{F}_j^l : L^p(0,1) \to [0,+\infty]$ defined by

$$F_j^l(u) = \begin{cases} F_j(u) & \text{if } u(0) = 0,\ u(1) = l \\ +\infty & \text{otherwise,} \end{cases} \tag{8.34}$$

$$\widehat{F}_j^l(u) = \begin{cases} \widehat{F}_j(u) & \text{if } u(0) = 0,\ u(1) = l \\ +\infty & \text{otherwise.} \end{cases} \tag{8.35}$$

Hence we derive the following result.

Proposition 8.10 *(i) Let J_1 satisfy (8.28) and $J_2 \equiv 0$. Then the functionals F_j^l and $\widehat{F}_j^l$ defined by (8.34) and (8.35), respectively, Γ-converge, with respect to the $L^p((0,1);\mathbf{R}^m)$ strong topology, to the functional $F : L^p((0,1);\mathbf{R}^m) \to [0,+\infty]$ defined by*

$$F(u) = \begin{cases} \displaystyle\int_0^1 (\tilde{J}_1)^{**}(u')\,dx & \text{if } u \in W^{1,p}((0,1);\mathbf{R}^m) \\ +\infty & \text{otherwise,} \end{cases}$$

where $\tilde{J}_1$ is defined by (8.33).

(ii) *Let $J_1,\ J_2$ satisfy (8.28). Then the functionals F_j^l and $\widehat{F}_j^l$ defined by (8.34) and (8.35), respectively, Γ-converge, with respect to the $L^p((0,1);\mathbf{R}^m)$ strong topology, to the functional $F : L^p((0,1);\mathbf{R}^m) \to [0,+\infty]$ defined by*

$$F(u) = \begin{cases} \displaystyle\int_0^1 (\tilde{J}_0)^{**}(u')\,dx & \text{if } u \in W^{1,p}((0,1);\mathbf{R}^m) \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\tilde{J}_0(z) := J_2(z) + \frac{1}{2}\inf\{\tilde{J}_1(z_1) + \tilde{J}_1(z_2) : z_1 + z_2 = 2z\}.$$

Remark 8.11 (Cauchy-Born states: L-J potentials) As shown in Proposition 8.10, in the case of a system of particles whose interactions are governed by Lennard-Jones type potentials the presence of a continuous media deformed according to potentials satisfying superlinear growth assumptions forbids the limiting deformation to be discontinuous (the material 'does note break'). But, beside this fact, we want to highlight that even in this case the continuos media has the regularizing effect of increasing the set of Cauchy-Born states. Let, for example, $J_1 = J$ be defined by (8.29), $J_2 \equiv 0$ and $f^1(z) = cz^2$ for some constant $c > 0$. It is easy to see that, while $J_1(z) = J_1^{**}(z)$ if and only if $z \leq (\frac{2k_1}{k_2})^{\frac{1}{6}}$, we may choose $c > 0$ large enough so that $\tilde{J}_1(z)$ is convex for all $z \in \mathbf{R}$.

The last example we want to consider is the reverse case when the density f^1, governing the deformation of the continuous media, is of Lennard-Jones type and the potentials accounting for the interactions of the particles satisfy superlinear growth assumptions. We show how this last assumption suffices to prevent discontinuities of the limiting deformation, although the energy density f^1 allows it.

Let, then, $G_j : W^{1,p}(0,1) \to [0,+\infty]$ be defined as follows

$$\begin{aligned} G_j(u) &= \int_0^1 J(u'(x))\,dx + \sum_{k=1}^{2}\sum_{i=0}^{j-k} \frac{1}{j} g_k\left(\frac{j}{k}\Big(u(\frac{i+k}{j}) - u(\frac{i}{j})\Big)\right) \\ &= G_j^c(u) + G_j^d(u) \end{aligned} \tag{8.36}$$

where $J : \mathbf{R} \to (-\infty,+\infty]$ is defined by (8.29) and $g_k = g^{ke_1}$, $k = 1,2$, satisfy hypotheses (H2) and (H3). Although J does not satisfy (3.4), we still may deduce that there is no decoupling of variables. Indeed, let $u_j \in W^{1,p}(0,1)$ be such that $\sup_j G_j(u_j) < +\infty$ and set $v_j = v(u_j)$, according to (3.6). Observe that if $v_j \to v$ in $L^p(0,1)$, then also $v_j(\cdot + \frac{1}{j}) \to v$ in $L^p(0,1)$. Hence, since by the definition of J, u_j must be increasing, we get that $v_j \leq u_j \leq v_j(\cdot + \frac{1}{j})$ and so also $u_j \to v$ in $L^p(0,1)$.

Therefore the functionals G_j share the same Γ-limit of the functionals $\widehat{G}_j : L^p((0,1);\mathbf{R}^m) \to [0,+\infty]$ defined by (7.10) with G_j as in (8.31) in place of F_ε. Moreover, as before, we get that $\widehat{G}_j$ equals to

$$\widehat{G}_j(v) = \begin{cases} \displaystyle\sum_{k=1}^{2}\sum_{i=0}^{j-k} \frac{1}{j}\,\tilde{g}_k\left(\frac{j}{k}\Big(v(\frac{i+k}{j}) - v(\frac{i}{j})\Big)\right) & \text{if } v \in \mathcal{A}_{\frac{1}{j}}(0,1) \\ +\infty & \text{otherwise} \end{cases} \tag{8.37}$$

with

$$\tilde{g}_1 = (J)^{**} + g_1 \tag{8.38}$$

and $\tilde{g}_k = g_k$ for $k > 1$. Thus, we infer that the analogue of Proposition 8.10 holds true for the family of functionals G_j and $\widehat{G}_j$, as stated in the following proposition.

Proposition 8.12 *(i) Let g_1 satisfy (H2), (H3) and let $g_2 \equiv 0$. Then the functionals G_j and $\widehat{G}_j$ defined by (8.36) and (8.37), respectively, Γ-converge, with respect to the $L^p((0,1);\mathbf{R}^m)$ strong topology, to the functional $G : L^p((0,1);\mathbf{R}^m) \to [0,+\infty]$ defined by*

$$G(u) = \begin{cases} \displaystyle\int_0^1 (\tilde{g}_1)^{**}(u')\,dx & \text{if } u \in W^{1,p}((0,1);\mathbf{R}^m) \\ +\infty & \text{otherwise,} \end{cases}$$

where $\tilde{g}_1$ is defined by (8.38).

(ii) Let g_1, g_2 satisfy (H2), (H3). Then the functionals G_j and $\widehat{G}_j$ defined by (8.36) and (8.37), respectively, Γ-converge, with respect to the $L^p((0,1);\mathbf{R}^m)$ strong topology, to the functional $G : L^p((0,1);\mathbf{R}^m) \to [0,+\infty]$ defined by

$$G(u) = \begin{cases} \displaystyle\int_0^1 (\tilde{g}_0)^{**}(u')\,dx & \text{if } u \in W^{1,p}((0,1);\mathbf{R}^m) \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\tilde{g}_0(z) := g_2(z) + \frac{1}{2}\inf\{\tilde{g}_1(z_1) + \tilde{g}_1(z_2) : z_1 + z_2 = 2z\}.$$

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