STRUCTURE OF METRIC CYCLES AND NORMAL ONE-DIMENSIONAL CURRENTS

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Abstract. We prove that every one-dimensional real Ambrosio-Kirchheim normal current in a Polish (i.e. complete separable metric) space can be naturally represented as an integral of simpler currents associated to Lipschitz curves. As a consequence a representation of every such current with zero boundary (i.e. a cycle) as an integral of so-called elementary solenoids (which are, very roughly speaking, more or less the same as asymptotic cycles introduced by S. Schwartzman) is obtained. The latter result on cycles is in fact a generalization of the analogous result proven by S. Smirnov for classical Whitney currents in a Euclidean space. The same results are true for every complete metric space under suitable set-theoretic assumptions.

1. Introduction

In [10] it has been shown that every acyclic normal one-dimensional real current in a complete metric space can be naturally decomposed in curves, the decomposition preserving the mass and the boundary mass. Namely, roughly speaking, every such current T can be represented as an integral

$$T = \int_{\Theta(E)} [\![\theta]\!] \, d\eta(\theta)$$

of simple rectifiable currents $[\![\theta]\!]$ associated to injective Lipschitz curves $\theta\colon [0,1] \to E$ over some measure η defined on the latter set of curves $\Theta(E)$, the mass of the current $\mathbb{M}(T)$ being equal to the integral of the masses $\mathbb{M}([\![\theta]\!])$ (in this particular case equal to lengths $\ell(\theta)$) of the respective curves,

$$\mathbb{M}(T) = \int_{\Theta(E)} \mathbb{M}([\![\theta]\!]) \, d\eta(\theta) = \int_{\Theta(E)} \ell(\theta) \, d\eta(\theta),$$

with η-a.e. $\theta \in \Theta(E)$ belonging to the support of T, and a similar decomposition being valid also for boundary masses. This is a direct generalization to metric currents introduced first by E. De Giorgi and further studied by L. Ambrosio and B. Kirchheim in [1] of the analogous result for Whitney currents in a Euclidean space proven in [13].

The primary goal of this paper is to prove the analogous decomposition result for all (not only acyclic) real one-dimensional metric currents. This is accomplished in Corollary 3.3 based on Theorem 3.1 which fills the gap by providing an appropriate decomposition of *cycles*, i.e. real one-dimensional metric currents without boundary. It is curious to mention that the latter theorem is mainly based on the decomposition of acyclic currents.

Once the primary goal is accomplished, it becomes natural to ask whether any cycle can be decomposed as an integral of currents associated to closed curves. Unfortunately, as shown in [13] this is not true even for the Euclidean space, but at least in a Euclidean space every one-dimensional real Whitney currents with zero

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boundary (i.e. a cycle) can be decomposed in so-called elementary *solenoids* (called also solenoidal vector charges in [13]). Such solenoids, i.e. the natural "elementary" cycles, are strictly related to the *asymptotic cycles* introduced by S. Schwartzman in [11] and further studied in [12] (in fact, roughly speaking, up to technical details, and in particular up to the fact that Schwartzman asymptotic cycles are normally defined as elements of the space of homology classes [7], one may identify the two notions). It is worth remarking that these objects appear quite natural in the problem of representation of homology classes of manifolds (see [7, 9, 8, 6]). The decomposition of a one-dimensional cycle into such solenoids appeared to be quite helpful in the study of Mather's minimal measures [2, 5].

Here we prove the analogous result for Ambrosio-Kirchheim currents in an arbitrary complete metric space. Namely, we introduce the notion of a solenoid as a current S over a metric space E such that there exists a Lipschitz curve $\theta\colon \mathbb{R} \to E$ with $\operatorname{Lip}\theta \leq 1$ with the property

$$S = \lim_{t\to+\infty} \frac{1}{2t} [\![\theta \llcorner [-t,t]]\!]$$

in the appropriately weak sense, while the trace $\theta(\mathbb{R})$ of the curve θ is in the support of S, i.e. $\theta(\mathbb{R}) \subset \operatorname{supp} S$. We show then that, roughly speaking, for every cycle T with compact support there is a measure η concentrated over the set C of solenoids of unit mass such that

$$\begin{aligned} T &= \int_C S\, d\eta(S), \\ \mathbb{M}(T) &= \int_C \mathbb{M}(S)\, d\eta(S), \end{aligned}$$

and a similar result holds also for arbitrary cycles (not necessarily with compact support). The result we provide for cycles with compact support in an arbitrary metric space (Corollary 4.5) is the precise generalization of the result of [13] on decomposition of cycles in a Euclidean space restricted to cycles with compact support, since for Ambrosio-Kirchheim normal currents in compact subset of a Euclidean space the notion of mass coincides with that of the usual Whitney currents. The careful reader would observe that the result we provide for the general case of currents with possibly noncompact support (Theorem 4.4) is "almost like" the respective general result in a Euclidean space setting from [13], the difference standing in the different definitions of mass for metric currents and for Whitney currents in a Euclidean space.

It is curious to note that although the technique used to prove Theorem 3.1 which is the basis for all the results present in this paper resembles the basic idea of [13] of extending the space E by an "extra dimension" and considering the appropriate extension of the original current T, the main line of the proof is in a certain sense opposite to that used in [13]. Namely, here we use the representation result for acyclic currents from [10] as a starting point, while in [13] one does the contrary, i.e. first proves the decomposition result for cycles and then deduces the respective results for acyclic currents from the latter. Therefore, since the proofs in [10] do not depend on the results of [13], we may consider also the results present in this paper independent on that of [13] even in the Euclidean setting.

2. Notation and preliminaries

The metric spaces are always in the sequel assumed to be complete. The parametric length of a Lipschitz curve $\theta\colon [a,b] \to E$ will be denoted by $\ell(\theta)$. The space of Lipschitz functions $\theta\colon [0,1] \to E$ equipped with uniform distance and factorized by reparameterization will be denoted by $\Theta(E)$ (see [10]). Every element of $\Theta(E)$

therefore represents an oriented rectifiable curve. For a finite Borel measure η over $\Theta(E)$ we set $\eta(i) := e_{i\#}\eta$, where $e_i\colon \Theta(E) \to E$ are defined by $e_i(\theta) := \theta(i)$, $i = 0, 1$.

In the sequel we will always assume that the mass measures of the currents we are dealing with are all tight (in fact, Radon, since the underlying metric space is complete). This is not restrictive because, as mentioned in [1], the theory of metric currents remains valid under such a requirement. Thus, all our results hold in every complete metric space E for normal currents T when its mass measure μ_T (and the mass measure of its boundary $\mu_{\partial T}$, if appropriate) is tight, and hence, in particular, for normal currents in a Polish (i.e. complete separable metric) space. Equivalently, one could assume that the density character (i.e. the minimum cardinality of a dense subset) of every metric space is an Ulam number. This guarantees that every finite positive Borel measure is tight (even Radon when the space is complete), is concentrated on some σ-compact subset and the support of this measure is separable (see, e.g., proposition 7.2.10 from [4]), and is consistent with the Zermelo-Fraenkel set theory.

All the measures we will consider in the sequel are signed Borel measures with finite total variation over some metric space E. The narrow topology on measures is defined by duality with the space $C_b(E)$ of continuous bounded functions. The supremum norm over $C_b(E)$ is denoted by $\|\cdot\|_\infty$.

For metric spaces X and Y we denote by $\mathrm{Lip}(X,Y)$ (resp. $\mathrm{Lip}_k(X,Y)$ and $\mathrm{Lip}_b(X,Y)$) the set of all Lipschitz maps (resp. all Lipschitz maps with Lipschitz constant k, the set of bounded Lipschitz maps) $f\colon X \to Y$. If $Y = \mathbb{R}$ we write just $\mathrm{Lip}(X)$, $\mathrm{Lip}_k(X)$, $\mathrm{Lip}_b(X)$ respectively.

For the metric currents we use the notation from [10] which is almost completely taken from [1], except mainly the notation for the mass measure. In particular, $D^k(E) = \mathrm{Lip}_b(E) \times (\mathrm{Lip}(E))^k$ stands for the space of metric k-forms, its elements (i.e. k-forms) being denoted by $f\,d\pi$, where $f \in \mathrm{Lip}_b(E)$, $\pi \in (\mathrm{Lip}(E))^k$, $\mathcal{M}_k(E)$ stands for the space of k-dimensional metric currents, $\mathcal{N}_k(E)$ stands for the space of k-dimensional normal metric currents, $\mathbb{M}(T)$ stands for the mass of a current T, and μ_T stands for the mass measure associated to this current. The one-dimensional current associated to a Lipschitz curve $\theta\colon [a,b] \to E$ will be denoted by $[\![\theta]\!]$, namely,

$$[\![\theta]\!](f\,d\pi) := \int_a^b f(\theta(t))\,d\pi(\theta(t))$$

for every $f\,d\pi \in D^1(E)$. Recall that $\mathbb{M}([\![\theta]\!]) \le \ell(\theta)$. The weak topology in $\mathcal{M}_k(E)$ is defined by a family of seminorms $\{T \mapsto |T(\omega)| \,:\, \omega \in D^k(E)\}$. It is clearly a Hausdorff locally convex topology. The notation $S \le T$ means that S is a subcurrent of T in the sense that $\mathbb{M}(S) + \mathbb{M}(T - S) = \mathbb{M}(T)$.

3. Decomposition of normal currents in curves

The first important result of this paper is the following statement.

Theorem 3.1. *Let $T \in \mathcal{M}_1(E)$ satisfy $\partial T = 0$. Then there is a finite positive Borel measure $\bar\eta$ over $\Theta(E)$ such that*

$$T(\omega) = \int_{\Theta(E)} [\![\theta]\!](\omega)\,d\bar\eta(\theta),$$
$$\mathbb{M}(T) = \int_{\Theta(E)} \mathbb{M}([\![\theta]\!])\,d\bar\eta(\theta) = \int_{\Theta(E)} \ell(\theta)\,d\bar\eta(\theta),$$

for all $\omega \in D^1(E)$, while $\bar\eta(0) = \bar\eta(1) = \mu_T$ and $\bar\eta$-a.e. $\theta \in \Theta(E)$ belongs to $\operatorname{supp} T$ *and has $\mathbb{M}([\![\theta]\!]) = \ell(\theta) = 1$.*

To prove this theorem we need some preliminary constructions. Equip the space $E \times [0,1]$ with the distance

$$d_\infty((u_1,t_1),(u_2,t_2)) := d(u_1,u_2) \vee |t_1 - t_2|.$$

Let $T \in \mathcal{M}_1(E)$ satisfy $\partial T = 0$. Define

$$T' := T \times \mu_{[\![0,1]\!]} + \mu_T \times [\![0,1]\!] \in \mathcal{N}_1(E \times [0,1]).$$

Letting $P_E\colon (x,t) \in E \times [0,1] \mapsto x \in E$ and $P\colon (x,t) \in E \times [0,1] \to t \in [0,1]$, we get

$$P_{E\#}T' = T, \qquad P_\# T' = \mathbb{M}(T)[\![0,1]\!].$$

Further,

$$\partial T' = \mu_T \otimes (\delta_1 - \delta_0)$$

and $\mathbb{M}(T') = \mathbb{M}(T)$ by Lemma A.9. At last, we have the following statement.

Lemma 3.2. *T' is acyclic.*

Proof. Let $C' \le T'$ be a cycle, and let $C := P_\# C'$. We have

$$\begin{aligned}\mathbb{M}(P_\# T' - C) + \mathbb{M}(C) &= \mathbb{M}(P_\#(T' - C')) + \mathbb{M}(P_\# C')\\ &\le \mathbb{M}(T' - C') + \mathbb{M}(C') = \mathbb{M}(T') = \mathbb{M}(P_\# T'),\end{aligned}$$

which means $C \le P_\# T' = \mathbb{M}(T)[\![0,1]\!]$, and hence $C = 0$. But $\mathbb{M}(C) = \mathbb{M}(C')$, since otherwise in the above relationship the inequality would be strict, which is impossible. Hence, $\mathbb{M}(C') = 0$, i.e. $C' = 0$. □

We are now ready to prove the announced result.

Proof of Theorem 3.1. By representation theorem for acyclic one-dimensional currents [10][theorem 5.1] one has

$$\begin{aligned}T'(\omega') &= \int_{\Theta(E\times[0,1])} [\![\theta]\!](\omega)\, d\eta'(\theta),\\ \mathbb{M}(T') &= \int_{\Theta(E\times[0,1])} \ell(\theta)\, d\eta'(\theta),\end{aligned}$$

for some finite positive Borel measure η' over $\Theta(E \times [0,1])$ and for all $\omega' \in D^1(E \times [0,1])$, while η'-a.e. $\theta \in E \times [0,1]$ is an arc belonging to $\operatorname{supp} T'$, and

$$\eta'(0) = \mu_T \otimes \delta_0, \qquad \eta'(1) = \mu_T \otimes \delta_1.$$

Denoting $\bar\eta := P_{E\#}\eta'$, we get

$$\begin{aligned}T(\omega) = (P_{E\#}T')(\omega) = T'(\omega \circ P_E) &= \int_{\Theta(E\times[0,1])} [\![\theta]\!](\omega \circ P_E)\, d\eta'(\theta),\\ &= \int_{\Theta(E\times[0,1])} [\![P_E(\theta)]\!](\omega)\, d\eta'(\theta) = \int_{\Theta(E)} [\![\theta]\!](\omega)\, d\bar\eta(\theta).\end{aligned} \tag{3.1}$$

This also implies

$$\mathbb{M}(T) \le \int_{\Theta(E)} \mathbb{M}([\![\theta]\!])\, d\bar\eta(\theta) \le \int_{\Theta(E)} \ell(\theta)\, d\bar\eta(\theta),$$

On the other hand,

$$\begin{aligned}\int_{\Theta(E)} \ell(\theta)\, d\bar\eta(\theta) &= \int_{\Theta(E\times[0,1])} \ell(P_E(\theta))\, d\eta'(\theta)\\ &\le \int_{\Theta(E\times[0,1])} \ell(\theta)\, d\eta'(\theta) = \mathbb{M}(T') = \mathbb{M}(T),\end{aligned} \tag{3.2}$$

and hence

$$\mathbb{M}(T) = \int_{\Theta(E)} \mathbb{M}([\![\theta]\!])\, d\bar\eta(\theta) = \int_{\Theta(E)} \ell(\theta)\, d\bar\eta(\theta). \tag{3.3}$$

Further, in (3.2) the inequality is in fact an equality, and hence

$$\ell(P_E(\theta)) = \ell(\theta) \geq d_\infty(\theta(0), \theta(1)) \geq 1$$

for η'-a.e. $\theta \in \Theta(E \times [0,1])$, the latter inequality being true because $P(\theta(0)) = 0$ and $P(\theta(1)) = 1$ for η'-a.e. $\theta \in \Theta(E \times [0,1])$. Thus $\ell(\theta) \geq 1$ for $\bar\eta$-a.e. $\theta \in \Theta(E)$. But then from (3.3) one has $\mathbb{M}(T) \geq \bar\eta(\Theta(E))$. Recall now that $\bar\eta(0) = \bar\eta(1) = \mu_T$. This implies $\mathbb{M}(T) \leq \bar\eta(\Theta(E))$, and therefore $\mathbb{M}(T) = \bar\eta(\Theta(E))$. Thus $\ell(\theta) = 1$ for $\bar\eta$-a.e. $\theta \in \Theta(E)$.

The relationship (3.1) implies also

$$\begin{aligned}(\partial T)(f) &= \int_{\Theta(E)} (f(\theta(1)) - f(\theta(0)))\, d\bar\eta(\theta) \\ &= \int_E f(x)\, d\bar\eta(1)(x) - \int_E f(x)\, d\bar\eta(0)(x) = \int_E f(x)\, d(\bar\eta(1) - \bar\eta(0))(x),\end{aligned}$$

so that $\partial T = \bar\eta(1) - \bar\eta(0)$, which gives $\bar\eta(0) = \bar\eta(1)$. Finally, $\bar\eta$-a.e. $\theta \in \Theta(E)$ belongs to $\operatorname{supp} P_{E\#}T' = \operatorname{supp} T$. □

The above theorem allows to formulate the following corollary on the structure of all one-dimensional real metric currents.

Corollary 3.3. *Let $T \in \mathcal{N}_1(E)$. Then there is a finite positive Borel measure $\bar\eta$ over $\Theta(E)$ with the total mass $\bar\eta(\Theta(E)) \leq \mathbb{M}(T) + \mathbb{M}(\partial T)$ such that*

$$\begin{aligned}T(\omega) &= \int_{\Theta(E)} [\![\theta]\!](\omega)\, d\bar\eta(\theta), \\ \mathbb{M}(T) &= \int_{\Theta(E)} \mathbb{M}([\![\theta]\!])\, d\bar\eta(\theta) = \int_{\Theta(E)} \ell(\theta)\, d\bar\eta(\theta),\end{aligned}$$

for all $\omega \in D^1(E)$, with $\bar\eta$-a.e. $\theta \in \Theta(E)$ belonging to $\operatorname{supp} T$.

Proof. Decompose (say, by proposition 3.8 from [10]) $T = S + C$ with $S \leq T$ acyclic and $C \leq T$ a cycle, i.e. $\partial C = 0$. Use theorem 5.1 from [10] to decompose S in curves and the above Theorem 3.1 to do the same for C. This gives the result. □

As a toy application we mention here for purely illustrative purposes the following immediate corollary on nonexistence of nontrivial normal currents in the space without rectifiable curves.

Corollary 3.4. *Let E be a metric space which has no nonconstant Lipschitz curves. Then $\mathcal{N}_k(E)$ contains only the zero current for all $k \geq 1$.*

Proof. For $k = 1$ this follows from the Corollary 3.3. For general k proceed by induction: suppose that the statement is true for $k-1$, i.e. $\mathcal{N}_{k-1}(E)$ contains only the zero current. Let $T \in \mathcal{N}_k(E)$, and consider an arbitrary $\pi_k \in \mathrm{Lip}(E)$. Then for every $t \in \mathbb{R}$ the slice $\langle T, \pi_k, t\rangle \in \mathcal{N}_{k-1}(E)$ and hence $\langle T, \pi_k, t\rangle = 0$ by induction assumption, which by slicing theorem 5.6 from [1] gives $T \llcorner d\pi_k = 0$. Consider now an arbitrary $f\, d\pi \in D^{k-1}(E)$. Then minding the alternating property of currents (theorem 3.5 from [1]), we get

$$|T(f\, d\pi_1 \wedge \ldots \wedge d\pi_{k-1} \wedge d\pi_k)| = |T \llcorner d\pi_k(f\, d\pi)| = 0,$$

so that $T = 0$. □

4. Decomposition of cycles in solenoids

This section is dedicated to another decomposition result for one-dimensional real metric currents without boundaries (i.e. cycles). In fact, given the validity of Theorem 3.1, it is natural to ask whether any cycle can be decomposed as an integral of (currents associated to) *closed* curves. As it is shown in [13] this is unfortunately not true even in the 3-dimensional Euclidean space $\mathbb{R}^3$ but at least in all Euclidean spaces there is a natural decomposition of cycles in so-called *solenoids* (called also solenoidal vector charges in [13]). We will extend this result to generic metric spaces.

We start with the following corollary of Theorem 3.1.

Corollary 4.1. *Let $T \in \mathcal{M}_1(E)$ satisfy $\partial T = 0$. There is a finite positive Borel measure $\tilde\eta$ over $X := C([0,1];E)$ (with the topology of uniform convergence) concentrated over* $\mathrm{Lip}_1([0,1];E)$ *such that*

$$\begin{aligned} T(\omega) &= \int_X [\![\theta]\!](\omega)\, d\tilde\eta(\theta),\\ \mathbb{M}(T) &= \int_X \ell(\theta)\, d\tilde\eta(\theta) = \tilde\eta(X), \end{aligned}$$

for all $\omega \in D^1(E)$, while $\tilde\eta$-a.e. $\theta \in X$ belongs to $\operatorname{supp} T$, *and $\tilde\eta(0) = \tilde\eta(1)$.*

Proof. Let $h\colon \Theta(E) \to X$ send every $\theta \in \Theta(E)$ in its parameterization with constant speed. It is enough to set then $\tilde\eta := h_\#\bar\eta$, where $\bar\eta$ is provided by Theorem 3.1. □

Now we will prove the following extension statement.

Proposition 4.2. *Let $\tilde\eta$ be a Borel measure over $C([0,1];E)$ satisfying the properties provided by Corollary 4.1. Then there is a Borel measure $\hat\eta$ over $C(\mathbb{R};E)$ (equipped with the topology of uniform convergence over bounded intervals) concentrated over* $\mathrm{Lip}_1(\mathbb{R};E)$ *such that*

(a) *$\pi_\#\hat\eta = \tilde\eta$, where $\pi\colon C(\mathbb{R};E) \to C([0,1];E)$ is the map defined by $\pi(\theta) := \theta\llcorner[0,1]$,*
(b) *$g^\pm_\#\hat\eta = \hat\eta$, where $g^\pm\colon C(\mathbb{R};E) \to C(\mathbb{R};E)$ are the shift maps defined by $g^\pm(\theta)(t) := \theta(t\pm 1)$,*
(c) *for $\hat\eta$-a.e. $\theta \in C(\mathbb{R};E)$ one has $\theta(\mathbb{R}) \subset \operatorname{supp} T$.*

Remark 4.3. The measure $\hat\eta$ provided by the above Proposition 4.2 satisfies

$$\begin{aligned} (m-n)T(\omega) &= \sum_{i=n}^{m-1} \int_{C([0,1];E)} [\![\theta]\!](\omega)\, d\tilde\eta(\theta) = \sum_{i=n}^{m-1} \int_{\mathrm{Lip}_1(\mathbb{R};E)} [\![\pi(\theta)]\!](\omega)\, d\hat\eta(\theta)\\ &= \sum_{i=n}^{m-1} \int_{\mathrm{Lip}_1(\mathbb{R};E)} [\![\theta\llcorner[0,1]]\!](\omega)\, d\hat\eta(\theta)\\ &= \sum_{i=n}^{m-1} \int_{\mathrm{Lip}_1(\mathbb{R};E)} [\![\theta\llcorner[i,i+1]]\!](\omega)\, d\hat\eta(\theta)\\ &= \int_{\mathrm{Lip}_1(\mathbb{R};E)} [\![\theta\llcorner[n,m]]\!](\omega)\, d\hat\eta(\theta) \end{aligned}$$

for all $\{m,n\} \subset \mathbb{Z}$ and $\omega \in D^1(E)$. Analogously,

$$(m-n)\mathbb{M}(T) = \int_{\mathrm{Lip}_1(\mathbb{R};E)} \mathbb{M}([\![\theta\llcorner[n,m]]\!])\, d\hat\eta(\theta).$$

Proof. The proof will be achieved in two steps.

Step 1. Without loss of generality we may assume $\tilde\eta$ to be a probability measure. Let $X := C([0,1];E)$ be equipped with the usual uniform topology, and let $e_t\colon X \to E$ be defined by $e_t(\theta) := \theta(t)$. Consider the Borel probability measures $\eta^\pm_x$ over X defined by the disintegration formulae

$$\begin{aligned} \tilde\eta &= (e_{0\#}\tilde\eta) \otimes \eta^+_x = \tilde\eta(0) \otimes \eta^+_x,\\ \tilde\eta &= (e_{1\#}\tilde\eta) \otimes \eta^-_x = \tilde\eta(1) \otimes \eta^-_x, \end{aligned}$$

i.e.

$$\tilde\eta(e) = \int_E \eta_x^+(e)\, d\tilde\eta(0)(x) = \int_E \eta_x^-(e)\, d\tilde\eta(1)(x)$$

for every Borel $e \subset X$. It is worth remarking that since $\tilde\eta(0) = \tilde\eta(1)$, then $\eta_x^+ = \eta_x^-$, while both measures are defined for $\tilde\eta(0) = \tilde\eta(1)$-a.e. $x \in E$, so we may omit the superscripts writing just η_x instead of $\eta_x^\pm$.

Define now inductively the measures η_k over X^k by setting for all $k \in \mathbb{N}$ and Borel $e \subset X^k$,

$$\begin{aligned}
\eta_1 &:= \tilde\eta,\\
\eta_k(e) &:= \int_X \eta^+_{\theta_{k-1}(1)}(e_{(\theta_1,\dots,\theta_{k-1})})\, d\eta_{k-1}(\theta_1,\dots,\theta_{k-1}),
\end{aligned}$$

where $e_{(\theta_1,\dots,\theta_{k-1})} := \{\theta \in X \colon (\theta_1,\dots,\theta_{k-1},\theta) \in e\}$, so that in particular,

$$\begin{aligned}
\eta_k(e_1 \times \ldots \times e_k) &= \int_{e_1\times\ldots\times e_{k-1}} \eta^+_{\theta_{k-1}(1)}(e_k)\, d\eta_{k-1}(\theta_1,\dots,\theta_{k-1}),\\
\eta_2(e_1 \times e_2) &= \int_{e_1} \eta^+_{\theta_1(1)}(e_2)\, d\eta_1(\theta_1).
\end{aligned}$$

Let $\pi_{k-1}\colon X^k = X^{k-1}\times X \to X^{k-1}$ and $\pi^{k-1}\colon X^k = X \times X^{k-1} \to X^{k-1}$ be defined by

$$\begin{aligned}
\pi_{k-1}(x_1,\dots,x_{k-1},x_k) &:= (x_1,\dots,x_{k-1}),\\
\pi^{k-1}(x_1,x_2,\dots,x_k) &:= (x_2,\dots,x_k),
\end{aligned}$$

i.e. as in Lemma B.4. Note that

$$\begin{aligned}
\eta_k(e\times X) = \int_e \eta^+_{\theta_{k-1}(1)}(X)\, d\eta_{k-1}(\theta_1,\dots,\theta_{k-1}) &= \int_e d\eta_{k-1}(\theta_1,\dots,\theta_{k-1})\\
&= \eta_{k-1}(e)
\end{aligned}$$

for every Borel $e \subset X^{k-1}$, which means $\pi_{k-1\#}\eta_k = \eta_{k-1}$ for all $k \in \mathbb{N}$. On the other hand,

$$\eta_2(X\times e_2) = \int_{X\times e_2} \eta^+_{\theta(1)}(e_2)\, d\eta(\theta) = \int_E \eta_x^+(e_2)\, d\eta(1)(x) = \eta(e_2) = \eta_1(e_2).$$

Assuming inductively that $\pi^{k-1}_{\#}\eta_k = \eta_{k-1}$ for some $k \in \mathbb{N}$, $k \geq 2$, we get

$$\begin{aligned}
\eta_{k+1}(X\times e_2\times\ldots\times e_{k+1}) &= \int_{X\times e_2\times\ldots\times e_{k+1}} \eta^+_{\theta_k(1)}(e_{k+1})\, d\eta_k(\theta_1,\dots,\theta_k)\\
&= \int_{e_2\times\ldots\times e_{k+1}} \eta^+_{\theta_k(1)}(e_{k+1})\, d\eta_{k-1}(\theta_2,\dots,\theta_k)\\
&= \eta_k(e_2\times\ldots\times e_{k+1}),
\end{aligned}$$

and hence by induction $\pi^{k-1}_{\#}\eta_k = \eta_{k-1}$ for all $k \in \mathbb{N}$, $k \geq 2$.

By Lemma B.4 there is a Borel measure η_* over $X^{\mathbb{Z}}$ such that for $p_j(x) := (x)_j$ one has

$$\eta_*\left(\bigcap_{j=k}^{l} p_j^{-1}(e_j)\right) = \eta_{l-k}\left(\prod_{j=k}^{l} e_j\right) \tag{4.1}$$

for $p_j(x) = x_j$. In particular, one has

(i) η_* is concentrated over $(\mathrm{Lip}_1([0,1];E))^{\mathbb{Z}}$,
(ii) $\bar\pi_\#\eta_* = \tilde\eta$, where $\bar\pi\colon X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ is the map defined by $\bar\pi(\bar\theta) := \bar\theta_1$,
(iii) η_* is invariant with respect to the shift maps $\bar g^\pm\colon X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ defined by $(\bar g^\pm(\bar\theta))_k := \theta_{k\pm1}$,
(iv) for η_*-a.e. $\bar\theta = \{\theta_k\}_{k\in\mathbb{Z}}$ one has $\theta_k(1) = \theta_{k+1}(0)$ for all $k \in \mathbb{Z}$,

(v) for η_*-a.e. $\bar\theta = \{\theta_k\}_{k\in\mathbb{Z}}$ one has $\theta_k \subset \operatorname{supp} T$ for all $k \in \mathbb{Z}$.

In fact, (ii) and (iii) are immediate from (4.1), (i) follows from the fact that $\tilde\eta$ is concentrated over $\mathrm{Lip}_1([0,1];E)$, while to prove (iv) it is enough, in view of (iii), to prove that

$$\eta_2(\{(\theta_1,\theta_2)\in X^2\colon \theta_1(1)\neq\theta_2(0)\}) = 0.$$

The latter equality follows by a simple calculation

$$\begin{aligned}\eta_2(\{(\theta_1,\theta_2)\in X^2\colon \theta_1(1)\neq\theta_2(0)\}) &= \int_X \eta^+_{\theta_1(1)}(\{\theta_2\in X\colon \theta_1(1)\neq\theta_2(0)\})\,d\tilde\eta(\theta_1)\\ &= \int_E \eta^+_x(\{\theta_2\in X\colon x\neq\theta_2(0)\})\,d\tilde\eta(1)(x) = 0,\end{aligned}$$

the final equality being due to the fact that η^+_x is concentrated over $e_0^{-1}(x)$. Finally,

$$\begin{aligned}\eta_*(\{\bar\theta\colon \theta_k\not\subset\operatorname{supp}T\}) &= \eta_*(\{\bar\theta\colon \theta_1\not\subset\operatorname{supp}T\}) \text{ by (iii)}\\ &= \tilde\eta(\{\theta\colon \theta\not\subset\operatorname{supp}T\}) \text{ by (ii)}\\ &= 0,\end{aligned}$$

which proves (v).

STEP 2. Define the map $q\colon X^{\mathbb{Z}}\to E^{\mathbb{R}}$ by setting

$$q(\bar\theta)(t) := \theta_{\lfloor t\rfloor}(\{t\}),$$

and set $\hat\eta := q_\#\eta_*$. Clearly, for η_*-a.e. $\bar\theta = \{\theta_k\}_{k\in\mathbb{Z}}$ one has $q(\bar\theta)\in C(\mathbb{R};E)$, since for every $k\in\mathbb{Z}$ one has

$$\begin{aligned}\lim_{t\to k-0} q(\bar\theta)(t) &= \theta_{k-1}(1) = \theta_k(0), \qquad \text{by (iv)}\\ &= \lim_{t\to k+0} q(\bar\theta)(t),\end{aligned}$$

and hence, by (i), $q(\bar\theta)\in\mathrm{Lip}_1(\mathbb{R};E)$, so that $\hat\eta$ is concentrated over $\mathrm{Lip}_1(\mathbb{R};E)$. Finally, (a) follows from (ii), (b) follows from (iii) and (c) follows from (v). □

We are now in a position to prove our main results regarding decomposition of cycles in solenoids. Let E be a metric space with distance d and $X\subset E$ be a σ-compact set. Consider $\bar X$ to be equipped with the distance d and consider the new distance $\tilde d$ over $\bar X$ provided by Lemma B.1 (with $\bar X$ in place of E) and let $\tilde X$ stand for the completion of $\bar X$ with respect to $\tilde d$ (equipped with $\tilde d$). We may write, slightly abusing the notation, $\mathrm{Lip}(\tilde X)\subset\mathrm{Lip}(\bar X)$ identifying each $u\in\mathrm{Lip}(\tilde X)$ with its restriction to $\bar X$. Analogously, $C_b(\tilde X)\subset C_b(\bar X)$. Thus, $D^k(\tilde X)\subset D^k(\bar X)$, and hence, $\mathcal{M}_k(\bar X)\hookrightarrow\mathcal{M}_k(\tilde X)$ with continuous immersion (namely, for every $T\in\mathcal{M}_k(\bar X)$ one has that $\tilde{\mathbb{M}}(T)\le\mathbb{M}(T)$, where $\tilde{\mathbb{M}}$ stands for the mass norm in $\mathcal{M}_k(\tilde X)$). We start with the following general result.

Theorem 4.4. *For every $T\in\mathcal{N}_1(E)$ having $\partial T = 0$ and supported over a σ-compact set $X\subset E$ there is a finite positive Borel measure $\hat\eta$ over $\mathrm{Lip}_1(\mathbb{R};E)$ such that for $\hat\eta$-a.e. θ there is a limit*

$$S_\theta = \lim_{t\to+\infty}\frac{1}{2t}[\![\theta\llcorner[-t,t]]\!]\in\mathcal{M}_1(\tilde X) \tag{4.2}$$

in the weak sense of currents in $\mathcal{M}_1(\tilde X)$, while

$$\begin{aligned}T(\omega) &= \int_{\mathrm{Lip}_1(\mathbb{R};E)} S_\theta(\omega)\,d\hat\eta(\theta) \text{ for all } \omega\in D^1(\tilde X),\\ \mathbb{M}(T) &= \hat\eta(\mathrm{Lip}_1(\mathbb{R};E))\end{aligned}\tag{4.3}$$

and the trace $\theta(\mathbb{R})\subset\operatorname{supp}T$, *Proof.* For every $\omega \in D^1(E)$ and every $\theta \in \mathrm{Lip}_1(\mathbb{R}; E)$ we define

$$f_\omega(\theta) := [\![\theta \llcorner [0,1]]\!](\omega).$$

By Remark 4.3 (with $m := 1$, $n := 0$) one has that $f_\omega \in L^1(C(\mathbb{R}; E); \hat{\eta})$ (so that in particular f_ω is finite on $C(\mathbb{R}; E)$ for $\hat{\eta}$-a.e. $\theta \in C(\mathbb{R}; E)$). By the ergodic theorem one has the existence for $\hat{\eta}$-a.e. $\theta \in C(\mathbb{R}; E)$ of a limit

$$\begin{aligned} \bar{f}_\omega(\theta) &:= \lim_{k\to+\infty} \frac{1}{2k} \sum_{j=-k}^{k} f_\omega((g^+)^j(\theta)) \\ &= \lim_{k\to+\infty, k\in\mathbb{N}} \frac{1}{2k} [\![\theta \llcorner [-k,k]]\!](\omega), \end{aligned} \tag{4.4}$$

and the validity of the relationship

$$\int_{C(\mathbb{R};E)} f_\omega(\theta)\, d\hat{\eta}(\theta) = \int_{C(\mathbb{R};E)} \bar{f}_\omega(\theta)\, d\hat{\eta}(\theta). \tag{4.5}$$

Let $\{\omega^j\} \subset D^1(\tilde{X})$ be as in the proof of Lemma A.1, and let $C_j \subset C(\mathbb{R}; E)$ be such a set of curves that (4.4) is valid for $\omega = \omega^j$ and all $\theta \in C(\mathbb{R}; E) \setminus C_j$, so that $\hat{\eta}(C_j) = 0$. Set $C := \cup_j C_j$. Minding that $\mu_{\partial \frac{1}{2k}[\![\theta\llcorner[-k,k]]\!]}(E) = 1/2k \to 0$ as $k \to \infty$ and $\mathbb{M}\left(\frac{1}{2k}[\![\theta \llcorner [-k,k]]\!]\right) \leq 1$, for all $\theta \in \mathrm{Lip}_1(\mathbb{R}; E)$, hence for $\hat{\eta}$-a.e. θ, while the $\tilde{X}$ is compact, we get that the sequence of currents $\{\frac{1}{2k}[\![\theta \llcorner [-k,k]]\!]\}$ is precompact in the weak topology of currents in $\mathcal{M}_1(\tilde{X})$. On the other hand, by the choice of C one has that the latter sequence of currents is convergent in the distance d_w, and thus, by Lemma A.1, also in the weak sense of currents in $\mathcal{M}_1(\tilde{X})$ for all $\theta \in C(\mathbb{R}; E) \setminus C$.

We have proven therefore the existence for $\hat{\eta}$-a.e. $\theta \in C(\mathbb{R}; E)$ of a limit

$$S_\theta = \lim_{k\to+\infty, k\in\mathbb{N}} \frac{1}{2k} [\![\theta \llcorner [-k,k]]\!]$$

in the weak sense of currents $\mathcal{M}_1(\tilde{X})$ with

$$\begin{aligned} \int_{C(\mathbb{R};E)} S_\theta(\omega)\, d\hat{\eta}(\theta) &= \int_{C(\mathbb{R};E)} f_\omega(\theta)\, d\hat{\eta}(\theta) = \int_{C(\mathbb{R};E)} [\![\pi(\theta)]\!](\omega)\, d\hat{\eta}(\theta) \\ &= \int_{C([0,1];E)} [\![\sigma]\!](\omega)\, d\tilde{\eta}(\sigma) = T(\omega) \end{aligned}$$

for all $\omega \in D^1(\tilde{X})$. We show now that S_θ is in fact as in the statement being proven, i.e.

$$S_\theta = \lim_{k\to+\infty} \frac{1}{2t_k} [\![\theta \llcorner [-t_k, t_k]]\!]$$

in the weak sense of currents for every sequence $t_k \to +\infty$ as $k \to +\infty$, because

$$\begin{aligned} &\frac{1}{2t_k} [\![\theta \llcorner [-t_k, t_k]]\!] - \frac{1}{2\lfloor t_k \rfloor} [\![\theta \llcorner [-\lfloor t_k \rfloor, \lfloor t_k \rfloor]]\!] \\ &\quad = \frac{1}{2t_k} \left([\![\theta \llcorner [-t_k, -\lfloor t_k \rfloor]]\!] + [\![\theta \llcorner [\lfloor t_k \rfloor, t_k]]\!] \right) + \frac{1}{2\lfloor t_k \rfloor} \left(1 - \frac{\lfloor t_k \rfloor}{t_k}\right) [\![\theta \llcorner [-\lfloor t_k \rfloor, \lfloor t_k \rfloor]]\!], \end{aligned}$$

which implies

$$\mathbb{M}\left(\frac{1}{2t_k} [\![\theta \llcorner [-t_k, t_k]]\!] - \frac{1}{2\lfloor t_k \rfloor} [\![\theta \llcorner [-\lfloor t_k \rfloor, \lfloor t_k \rfloor]]\!] \right) \leq \frac{2}{2t_k} + \left(1 - \frac{\lfloor t_k \rfloor}{t_k}\right) \to 0$$

as $k \to \infty$.

Minding that

$$\mathbb{M}(T) = \hat{\eta}(C(\mathbb{R}; E)),$$

and that $\hat{\eta}$ is concentrated over $\mathrm{Lip}_1(\mathbb{R}; E)$, we conclude the proof. $\square$

We may now formulate the following important corollaries to the above statement.

Corollary 4.5. *Let E be a metric space. Then for every $T \in \mathcal{N}_1(E)$ with compact support having $\partial T = 0$ there is a finite positive Borel measure η over $\mathrm{Lip}_1(\mathbb{R}; E)$ such that for η-a.e. θ there is a limit*

$$S_\theta = \lim_{t\to+\infty} \frac{1}{2t} [\![\theta \llcorner [-t,t]]\!] \in \mathcal{M}_1(E)$$

in the weak sense of currents in $\mathcal{M}_1(E)$, and the trace $\theta(\mathbb{R}) \subset \operatorname{supp} T$, while

$$\begin{aligned} T &= \int_{\mathrm{Lip}_1(\mathbb{R};E)} S_\theta \, d\eta(\theta), \\ \mathbb{M}(T) &= \int_{\mathrm{Lip}_1(\mathbb{R};E)} \mathbb{M}(S_\theta)\, d\eta(\theta) = \eta(\mathrm{Lip}_1(\mathbb{R};E)), \end{aligned} \tag{4.6}$$

so that in particular, $S_\theta \subset \mathcal{M}_1(E)$ has unit mass for η-a.e. $\theta \in \mathrm{Lip}_1(\mathbb{R}; E)$. Finally, we may assume $\theta(\mathbb{R}) \subset \operatorname{supp} S_\theta$ for η-a.e. θ.

Proof. Without loss of generality we assume E to be compact. We now repeat the proof of Theorem 4.4 with the original space E instead of the compactification $\tilde{X}$, getting the existence for $\hat\eta$-a.e. θ of a limit

$$S_\theta = \lim_{t\to+\infty} \frac{1}{2t} [\![\theta \llcorner [-t,t]]\!] \in \mathcal{M}_1(E)$$

in the weak sense of currents in $\mathcal{M}_1(E)$, such that

$$T(\omega) = \int_{\mathrm{Lip}_1(\mathbb{R};E)} S_\theta(\omega)\, d\hat\eta(\theta)$$

for all $\omega \in D^1(E)$, which implies

$$\mathbb{M}(T) \le \int_{C(\mathbb{R};E)} \mathbb{M}(S_\theta)\, d\hat\eta(\theta) \le \hat\eta(C(\mathbb{R};E)) = \mathbb{M}(T),$$

and hence in particular $\mathbb{M}(S_\theta) = 1$ for $\hat\eta$-a.e. $\theta \in C(\mathbb{R}; E)$.

The trace $\theta(\mathbb{R}) \subset \operatorname{supp} T$ for $\hat\eta$-a.e. $\theta \in C(\mathbb{R}; E)$ by Corollary 4.1. This gives all the claims of the theorem being proven but the last one for $\eta := \hat\eta$. Finally, to prove the last claim, consider the set

$$\Sigma := \{S \in \mathcal{M}_1(E) \,:\, \partial S = 0, \mathbb{M}(S) \le 1\}.$$

Clearly Σ is a convex compact subset of $\mathcal{M}_1(E)$, and Σ equipped with the weak topology of currents is compact and metrizable by Lemma A.1. We claim now that if $S \in \Sigma$ is extremal, then $S = S_\theta$ for some $\theta \in \mathrm{Lip}_1(\mathbb{R}; E)$. In fact, consider the representation

$$\begin{aligned} S(\omega) &= \int_{C(\mathbb{R};E)} S_\theta(\omega)\, d\eta(\theta), \\ \mathbb{M}(S) &= \int_{C(\mathbb{R};E)} \mathbb{M}(S_\theta)\, d\eta(\theta) = \eta(C(\mathbb{R};E)) \end{aligned} \tag{4.7}$$

for all $\omega \in D^1(E)$. Note that (4.7) implies that for every Borel $e \subset C(\mathbb{R}; E)$, defined

$$S_1(\omega) := \int_e S_\theta(\omega)\, d\eta(\theta),$$

for all $\omega \in D^1(E)$, one has $S_1 \le S$, because

$$(S - S_1)(\omega) := \int_{C(\mathbb{R};E)\setminus e} S_\theta(\omega)\, d\eta(\theta),$$

and hence

$$\begin{aligned} \mathbb{M}(S_1) &\le \eta(e) \\ \mathbb{M}(S - S_1) &\le \eta(C(\mathbb{R};E) \setminus e), \end{aligned}$$

so that $\mathbb{M}(S_1) + \mathbb{M}(S - S_1) \leq \eta(C(\mathbb{R}; E)) = \mathbb{M}(S)$. Since S is extremal, then $S_1 = \lambda S$ for some $\lambda \in [0,1]$ and thus $\mathbb{M}(S_1) = \eta(e) = \lambda$, so that we can write

$$S(\omega) = \frac{1}{\eta(e)} \int_e S_\theta(\omega)\, d\eta(\theta). \tag{4.8}$$

If $S \neq S_\theta$, then there are two different curves $\{\theta_1, \theta_2\} \subset \mathrm{Lip}_1(\mathbb{R}; E)$ such that $R_1 := S_{\theta_1} \neq R_2 := S_{\theta_2}$ and for every $\varepsilon > 0$ one has $\eta(\hat{B}_\varepsilon(R_i)) > 0$, where $\hat{B}_\varepsilon(R_i)$ stands for the set of $\theta \in C(\mathbb{R}; E)$ in the support of η such that $S_\theta \in B_\varepsilon(R_i)$, the notation $B_\varepsilon(R_i)$ standing for the ball of radius ε and center R_i in the space of cycles (with respect to the distance d_w provided by Lemma A.1), $i = 1, 2$. Choose an $\omega \in D^1(E)$ such that

$$\alpha := |R_1(\omega) - R_2(\omega)| > 0,$$

and an $\varepsilon > 0$ such that

$$|R(\omega) - R_i(\omega)| < \frac{\alpha}{4} \text{ for all } R \in B_\varepsilon(R_i), \qquad i = 1, 2.$$

Then

$$\begin{aligned} &\Big| \frac{1}{\eta(\hat{B}_\varepsilon(R_i))} \int_{\hat{B}_\varepsilon(R_i)} S_\theta(\omega)\, d\eta(\theta) - R_i(\omega) \Big| \\ &\qquad \leq \frac{1}{\eta(\hat{B}_\varepsilon(R_i))} \int_{\hat{B}_\varepsilon(R_i)} |S_\theta(\omega) - R_i(\omega)|\, d\eta(\theta) < \frac{\alpha}{4}, \qquad i = 1, 2, \end{aligned}$$

so that

$$\Big| \frac{1}{\eta(\hat{B}_\varepsilon(R_1))} \int_{\hat{B}_\varepsilon(R_1)} S_\theta(\omega)\, d\eta(\theta) - \frac{1}{\eta(\hat{B}_\varepsilon(R_2))} \int_{\hat{B}_\varepsilon(R_2)} S_\theta(\omega)\, d\eta(\theta) \Big| \geq \frac{\alpha}{2}.$$

This contradicts the equality

$$\frac{1}{\eta(\hat{B}_\varepsilon(R_1))} \int_{\hat{B}_\varepsilon(R_1)} S_\theta(\omega)\, d\eta(\theta) = \frac{1}{\eta(\hat{B}_\varepsilon(R_2))} \int_{\hat{B}_\varepsilon(R_2)} S_\theta(\omega)\, d\eta(\theta) = S(\omega)$$

valid in view of (4.8), and thus shows the claim.

Clearly also for every extremal point S of Σ one has $\mathbb{M}(S) = 1$, hence

$$\eta(C(\mathbb{R}; E)) = \mathbb{M}(S) = 1,$$

and therefore we have proven that for such S one has the representation (4.7) with $S = S_\theta$ for η-a.e. $\theta \in C(\mathbb{R}; E)$. Since it has already been proven that one may assume in (4.7) that $\theta(\mathbb{R}) \subset \operatorname{supp} S$ for η-a.e. $\theta \in C(\mathbb{R}; E)$, then one has $\theta(\mathbb{R}) \subset \operatorname{supp} S_\theta$. It remains now to refer to Choquet theorem [3, theorem 4.2] to show the existence of a representation (3.2) with $\theta(\mathbb{R}) \subset \operatorname{supp} S_\theta$ for η-a.e. $\theta \in \mathrm{Lip}_1(\mathbb{R}; E)$. □

Another corollary refers to the noncompact case.

Corollary 4.6. *For every $T \in \mathcal{N}_1(E)$ having $\partial T = 0$ and supported over a σ-compact set $X \subset E$ there is a finite positive Borel measure η over $\mathrm{Lip}_1(\mathbb{R}; \tilde{X})$ such that for η-a.e. θ there is a limit*

$$S_\theta = \lim_{t \to +\infty} \frac{1}{2t} [\![\theta \llcorner [-t, t]]\!] \in \mathcal{M}_1(\tilde{X})$$

in the weak sense of currents in $\mathcal{M}_1(\tilde{X})$, and the trace $\theta(\mathbb{R}) \subset \operatorname{supp} T$, while

$$\begin{aligned} T &= \int_{\mathrm{Lip}_1(\mathbb{R}; \tilde{X})} S_\theta\, d\eta(\theta), \\ \tilde{\mathbb{M}}(T) &= \int_{\mathrm{Lip}_1(\mathbb{R}; \tilde{X})} \tilde{\mathbb{M}}(S_\theta)\, d\eta(\theta) = \eta(\mathrm{Lip}_1(\mathbb{R}; \tilde{X})), \end{aligned} \tag{4.9}$$

so that in particular, $S_\theta \subset \mathcal{M}_1(\tilde{X})$ has unit mass for η-a.e. $\theta \in \mathrm{Lip}_1(\mathbb{R}; \tilde{X})$. Finally, μ_{S_θ} are concentrated over $\theta(\mathbb{R})$ for η-a.e. θ.

Proof. It is enough to apply Corollary 4.5 with $\tilde{X}$ instead of E. □

Appendix A. Some statements regarding currents

Here we collect some statements regarding currents which are used in this paper. We start with the following statement regarding metrizability of the weak topology of currents.

Lemma A.1. *Let $X \subset E$ be a σ-compact set. Then there is a distance d_w over $\mathcal{M}_k(\bar{X})$ which generates a topology coarser than the weak topology of currents, such that for every $\Sigma \subset \mathcal{M}_k(\bar{X})$ weakly sequentially precompact, the topology generated by d_w over Σ coincides with the weak one.*

In particular, if $\Sigma \subset \mathcal{M}_k(E)$ is such that the family of measures $\{\mu_T + \mu_{\partial T}\}_{T\in\Sigma}$ is uniformly tight and there is a $C > 0$ such that $\mathbb{M}(T) + \mathbb{M}(\partial T) \le C$ for all $T \in \Sigma$, then weak topology of currents is metrizable over Σ.

Proof. Let $\{K_\nu\}$ be an increasing sequence of compact subsets of E such that $X = \cup_\nu K_\nu$. Notice that $\mathrm{Lip}_m(K_\nu)$ is separable with respect to the norm $\|\cdot\|_\infty$. Recall that every function in $\mathrm{Lip}_m(X)$ can be uniquely extended to a function in $\mathrm{Lip}_m(\bar{X})$. Hence it is possible to endow $\mathrm{Lip}_m(\bar{X})$ with a separable metric inducing uniform convergence on each K_ν. Let $Z^m \subset \mathrm{Lip}_m(\bar{X})$ and

$$Z_b^{m,n} \subset \{u \in \mathrm{Lip}_m(\bar{X}) \,:\, \|u\|_\infty \le n\}$$

be countable dense subsets with respect to this metric. Set $Z := \cup_{m=1}^\infty Z^m$ and $Z_b := \cup_{m=1,n=1}^\infty Z_b^{m,n}$. We let then

$$d_w(T, T') := \sum_{j=1}^\infty 2^{-j}(|T(\omega^j) - T'(\omega^j)| \wedge 1),$$

where $\{\omega^j\} = Z_b \times (Z)^k \subset D^k(\bar{X})$, i.e. $\omega^j = f^j\, d\pi_1^j \wedge \ldots \wedge d\pi_k^j$ with $f^j \in Z_b$, $\pi_i^j \in Z$ for all $j \in \mathbb{N}$ and all $i = 1, \ldots, k$. To show that this is a distance, assume $d_w(T, T') = 0$ for some $T \in \mathcal{N}_k(\bar{X})$, $T' \in \mathcal{N}_k(\bar{X})$. This means $T(\omega^j) = T'(\omega^j)$ for all ω^j. But for any $\omega = f\, d\pi_1 \wedge \ldots \wedge d\pi_k \in D^k(\bar{X})$, letting $m \in \mathbb{Z}$ be such that $\mathrm{Lip}\,\pi_i \le m$, $\mathrm{Lip} f \le m$, we may find a sequence of $\omega^j = f^j\, d\pi_1^j \wedge \ldots \wedge d\pi_k^j \in Z_b \times (Z)^k$ with $f^j \to f$, $\pi_i^j \to \pi_i$, $i = 1, \ldots, k$, pointwise over X (in fact, even uniformly over each K_ν) as $j \to \infty$ and $\mathrm{Lip}\,\pi_i^j \le m$, $\mathrm{Lip} f^j \le m$ for all $j \in \mathbb{N}$ and $i = 1, \ldots, k$. Then for every $\bar{x} \in \bar{X}$ and and arbitrary $x \in X$ one has

$$\begin{aligned} |\pi_i^j(\bar{x}) - \pi_i(\bar{x})| &\le |\pi_i^j(\bar{x}) - \pi_i^j(x)| + |\pi_i^j(x) - \pi_i(x)| + |\pi_i(\bar{x}) - \pi_i(x)| \\ &\le 2md(\bar{x}, x) + |\pi_i^j(x) - \pi_i(x)|, \end{aligned}$$

which, minding the convergence $\pi_i^j(x) \to \pi_i(x)$, gives the convergence $\pi_i^j(\bar{x}) \to \pi_i(\bar{x})$ as $j \to \infty$. Hence, $\pi_i^j \to \pi_i$, $i = 1, \ldots, k$ (and analogously $f^j \to f$) pointwise over $\bar{X}$, and thus by the continuity property of currents we get $T(\omega) = T'(\omega)$ for all $\omega \in D^k(\bar{X})$, which means $T = T'$. Clearly, the topology induced by d_w is coarser than the weak topology of currents.

Let now $T_\nu \in \Sigma$, where Σ be as in the statement of the lemma being proven, and assume that $d_w(T_\nu, \bar{T}) \to 0$ for some $\bar{T} \in \mathcal{M}_k(\bar{X})$, so that $T_\nu(\omega^j) \to \bar{T}(\omega^j)$ for each $\omega^j \in Z_b \times (Z)^k$ as $\nu \to \infty$. Under the assumptions on Σ, every subsequence of T_ν has a further subsequence (all subsequences not relabeled) such that $T_\nu \rightharpoonup T$ in the weak sense of currents, hence also $d_w(T_\nu, T) \to 0$ as $\nu \to \infty$. Hence $d_w(\bar{T}, T) = 0$, i.e. $\bar{T} = T$, and therefore the whole sequence $\{T_\nu\}$ converges to T in the weak sense of currents.

In the particular case indicated in the statement we let $\{K_\nu\}$ be an increasing sequence of compact subsets of E such that $\mu_T(K_\nu) + \mu_{\partial T}(K_\nu) \le 1/\nu$, $X := \cup_\nu K_\nu$, and refer to the fact that Σ is sequentially precompact in the weak topology of currents by theorem 5.2 from [1]. $\square$

Remark A.2. In the above Lemma A.1 it is possible to choose the distance d_w over $\mathcal{M}_k(\bar{X})$ so as to have additionally the semicontinuity property for masses

$$\mathbb{M}(T \llcorner U) \leq \liminf_{\nu} \mathbb{M}(T_\nu \llcorner U) \tag{1.1}$$

for every open $U \subset \bar{X}$, and in particular

$$\mathbb{M}(T) \leq \liminf_{\nu} \mathbb{M}(T_\nu)$$

whenever $d_w(T_\nu, T) \to 0$. In fact, for this purpose let $\{x_i\} \subset \bar{X}$ stand for a countable dense subset of $\bar{X}$, and consider the countable family of open sets $\mathcal{F} = \{U_j\}$ consisting of all finite unions of open balls $B_{r_j}(x_i)$, where $\{r_j\} = \mathbb{Q}$ is the enumeration of rational numbers. Let also p_k stand for the projection map from $\mathbb{R}^k$ to the Euclidean unit ball $B_1(0) \subset \mathbb{R}^k$. Now, in the proof of the above Lemma A.3 when constructing Z_b one should first add to $\tilde{Z}_b := \cup_{m=1,n=1}^{\infty} Z_b^{m,n}$ to each k-uple of functions $(f_1, \ldots, f_k) \subset \tilde{Z}_b$ also the function $p_k(f_1(\cdot), \ldots, f_k(\cdot))$, and then add to the obtained set of functions (let us call it Z_b') all functions of the form $u1_{U_j}$ for all $u \in Z_b'$ and $U_j \in \mathcal{F}$ thus forming the set Z_b. Now, to prove (1.1) it is enough to prove

$$\sum_{i=1}^{k} T(f_i \, d\pi_i) \leq \liminf_{\nu} \mathbb{M}(T_\nu \llcorner U) \tag{1.2}$$

whenever $\sum_{i=1}^k f_i \leq 1_U$ and $\mathrm{Lip}\pi_i \leq 1$. One can then find sequences $\{f_i^j\}_{j=1}^\infty \subset Z_b$ and $\{\pi_i^\nu\}_{j=1}^\infty \subset Z$ such that $\sum_{i=1}^k f_i^j \leq 1_U$, $\mathrm{Lip}\pi_i^j \leq 1$ and

$$f_i^j \to f_i, \qquad \pi_i^j \to \pi_i$$

pointwise as $j \to \infty$. Then

$$\begin{aligned}\liminf_{\nu} \mathbb{M}(T_\nu \llcorner U) &\geq \liminf_{\nu} \sum_{i=1}^{k} T_\nu(f_i^j \, d\pi_i^j) \\ &\geq \sum_{i=1}^{k} \liminf_{\nu} T_\nu(f_i^j \, d\pi_i^j) = \sum_{i=1}^{k} T(f_i^j \, d\pi_i^j),\end{aligned}$$

and taking a limit in the above inequality as $j \to \infty$ we get (1.2), and hence (1.1).

Lemma A.3. *If $T_\nu \in \mathcal{M}_k(E)$ and $T \in \mathcal{M}_k(E)$ be such that $T_\nu \rightharpoonup T$ in the weak sense of currents and $\mathbb{M}(T_\nu) \to \mathbb{M}(T)$ as $\nu \to \infty$, then $\mu_{T_\nu} \rightharpoonup \mu_T$ in the narrow sense of measures.*

Proof. One has $\mu_{T_\nu}(E) \to \mu(E)$ and

$$\mu_T(U) \leq \liminf_{\nu} \mu_{T_\nu}(U)$$

for every open $U \subset E$, and therefore $\mu_{T_\nu} \rightharpoonup \mu_T$ in the narrow sense of measures by theorem 8.2.3 from [4]. $\square$

Remark A.4. It is easy to observe that the result of the above Lemma A.3 remains true if the condition $T_j \rightharpoonup T$ in the weak sense of currents is substituted by the weaker one $d_w(T_j, T) \to 0$ once the distance d_w satisfies the semicontinuity property (1.1) (the proof is word-to-word identical to the above one).

The following lemma allows to pass to diagonal subsequences in the weak convergence of currents.

Lemma A.5. *Let E be a metric space, $T \in \mathcal{N}_k(E)$, $T_j \in \mathcal{N}_k(E)$ and $T_j^m \in \mathcal{N}_k(E)$ be such that*

$$T_j \rightharpoonup T \text{ as } j \to \infty,$$
$$T_j^m \rightharpoonup T_j \text{ as } m \to \infty,$$

in the weak sense of currents, and

$$\mathbb{M}(T_j) \to \mathbb{M}(T), \qquad \mathbb{M}(\partial T_j) \to \mathbb{M}(\partial T) \text{ as } j \to \infty,$$
$$\mathbb{M}(T_j^m) \to \mathbb{M}(T_j), \qquad \mathbb{M}(\partial T_j^m) \to \mathbb{M}(\partial T_j) \text{ as } k \to \infty.$$

Then there is a subsequence of $m = m(j)$ such that $T_j^{m(j)} \rightharpoonup T$ in the weak sense of currents, $\mu_{T_j^{m(j)}} \rightharpoonup \mu_T$ and $\mu_{\partial T_j^{m(j)}} \rightharpoonup \mu_{\partial T}$ in the narrow sense of measures as $j \to \infty$.

Proof. Note that under the conditions of the statement being proven

$$\mu_{T_j} \rightharpoonup \mu_T, \qquad \mu_{\partial T_j} \rightharpoonup \mu_{\partial T} \text{ as } j \to \infty,$$
$$\mu_{T_j^m} \rightharpoonup \mu_{T_j}, \qquad \mu_{\partial T_j^m} \rightharpoonup \mu_{\partial T_j} \text{ as } m \to \infty,$$

in the narrow sense of measures by Lemma A.3.

Let $K_\nu \subset E$ and $K_\nu^j \subset E$ be such compact sets that

$$\mu_{T_j}(K_\nu^c) + \mu_{\partial T_j}(K_\nu^c) \le 1/\nu \text{ for all } j \in \mathbb{N},$$
$$\mu_{T_j^m}((K_\nu^j)^c) + \mu_{\partial T_j^m}((K_\nu^j)^c) \le 1/\nu, \text{ for all } m \in \mathbb{N}.$$

Note that setting

$$X := \bigcup_{j,\nu} K_\nu^j \cup \bigcup_\nu K_\nu,$$

we have that all T_j, T_j^m and T are concentrated over $\bar{X}$. Let d_w stand for the distance over $\mathcal{N}_k(\bar{X})$ provided by Lemma A.1, and denote by $\|\cdot\|_0$ the Kantorovich-Rubinstein norm metrizing the narrow topology on positive finite Borel measures over $\bar{X}$ (see [4][theorem 8.3.2]). For every $n \in \mathbb{N}$ choose a $j = j(n)$ and $m = m(n)$ such that

$$d_w(T_j, T) \le \frac{1}{n}, d_w(T_j^m, T_j) \le \frac{1}{n},$$
$$\left\|\mu_{T_j} - \mu_T\right\|_0 \le \frac{1}{n}, \left\|\mu_{\partial T_j} - \mu_{\partial T}\right\|_0 \le \frac{1}{n},$$
$$\left\|\mu_{T_j^m} - \mu_{T_j}\right\|_0 \le \frac{1}{n}, \left\|\mu_{\partial T_j^m} - \mu_{\partial T_j}\right\|_0 \le \frac{1}{n}.$$

Clearly, with this construction

$$T = \lim_{j\to\infty} T_j = \lim_{j\to\infty} T_j^{m(j)} \tag{1.3}$$

in distance d_w. But the sequences $\{\mu_{T_j^{m(j)}}\}$ and $\{\mu_{\partial T_j^{m(j)}}\}$ converge in the norm $\|\cdot\|_0$ (hence also in the narrow sense of measures), and therefore, they are uniformly tight by the Prokhorov theorem for nonnegative measures (theorem 8.6.4 from [4]). Thus, by Lemma A.1, the convergence in (1.3) is also in the weak topology of currents. □

The following lemma is in fact implicitly contained in [10] in the sense that its arguments are widely used in that paper. We make it explicit here for the readers' convenience.

Lemma A.6. *Let E be a Banach space with metric approximation property, $T \in \mathcal{N}_k(E)$ with μ_T and $\mu_{\partial T}$ concentrated over a σ-compact set. Then there is a sequence of currents $T_n \in \mathcal{N}_k(E_n)$ supported over some finite dimensional subspaces $E_n \subset E$, such that $T_n \rightharpoonup T$ weakly as currents in $\mathcal{M}_k(E)$, $\mu_{T_n} \rightharpoonup \mu_T$ and $\mu_{\partial T_n} \rightharpoonup \mu_{\partial T}$ in the* *narrow sense of measures as $n \to \infty$. In particular, if $k = 1$, then identifying the zero-dimensional currents with measures one has $(\partial T_n)^{\pm} \rightharpoonup (\partial T)^{\pm}$ in the narrow sense of measures as $n \to \infty$.*

Remark A.7. From the proof of the above Lemma it is clear that when T is a cycle (i.e. $\partial T = 0$) with bounded support, then T_n are cycles as well.

Proof. Let $\{K_\nu\}$ be an increasing sequence of compact subsets of E such that μ_T and $\mu_{\partial T}$ are concentrated on $\cup_\nu K_\nu$, and let P_ν be a finite rank projection of norm one such that $\|P_\nu x - x\| \leq 1/\nu$ for all $x \in K_\nu$. Thus $P_\nu x \to x$ as $\nu \to \infty$ for all $x \in \cup_\nu K_\nu$.

Consider first the case when $\operatorname{supp} T$ is bounded. Let $T_n := P_{n\#}T$. Then $T_n \rightharpoonup T$ in the weak sense of currents. In fact, for every $f\, d\pi \in D^k(E)$ with $\mathrm{Lip}\pi_i \leq 1$ for all $i = 1, \dots, k$ we have

$$\begin{aligned}
|T(f\circ P_n\, d\pi \circ P_n) - T(f\, d\pi)| &\leq |T(f\circ P_n\, d\pi\circ P_n) - T(f\circ P_n\, d\pi)| + \\
&\quad |T(f\circ P_n\, d\pi) - T(f\, d\pi)| \\
&\leq \sum_{i=1}^k \int_E |f\circ P_n|\cdot|\pi_i\circ P_n - \pi_i|\, d\mu_{\partial T} + \\
&\quad \mathrm{Lip}\, f \sum_{i=1}^k \int_E |\pi_i\circ P_n - \pi_i|\, d\mu_T + \\
&\quad |T(f\circ P_n\, d\pi) - T(f\, d\pi)| \qquad \text{by proposition 5.1 of [1]} \\
&\leq (\|f\|_\infty + \mathrm{Lip}\, f)k \int_E \|P_n x - x\|\, d(\mu_{\partial T} + \mu_T) + \\
&\quad |T(f\circ P_n\, d\pi) - T(f\, d\pi)|,
\end{aligned}$$

all the terms in the right-hand side tending to zero as $n \to \infty$ by the choice of P_n (the first one by Lebesgue theorem, recalling that $\|P_n x - x\| \leq 2\|x\|$ and the support of T, and hence of ∂T, is bounded, while the last term because $f(P_n(x)) \to f(x)$ for μ_T-a.e. $x \in E$).

Further, we have $\mathbb{M}(T_n) \leq \mathbb{M}(T)$ which together with lower semicontinuity of the mass with respect to weak convergence gives $\mathbb{M}(T_n) \to \mathbb{M}(T)$, and the latter implies $\mu_{T_n} \rightharpoonup \mu_T$ in the narrow sense of measures as $n \to \infty$. In the same way one shows that $\mu_{\partial T_n} \rightharpoonup \mu_{\partial T}$.

For the general case of a current T with possibly unbounded support, we approximate T by a sequence $\{T_\nu\} \subset \mathcal{M}_k(E)$, such that each T_ν has bounded support and $\mathbb{M}(T_\nu - T) + \mathbb{M}(\partial T_\nu - \partial T) \to 0$ as $\nu \to \infty$ (for this purpose just take $T_\nu := T \llcorner g_\nu$ for a $g_\nu \in \mathrm{Lip}_1(E)$ with bounded support having $0 \leq g_\nu \leq 1$ and $g_\nu = 1$ on $B_\nu(0)$). Approximating now each T_ν by the currents T_ν^n as above, and choosing a diagonal subsequence provided by Lemma A.5, we get the result. □

Lemma A.8. *Let E be a finite-dimensional normed space endowed with the norm $\|\cdot\|$, and $T \in \mathcal{M}_1(E)$. Then*

$$T(f\, d\pi) = \int_E f(x)(\nabla\pi(x), l(x))\, d\mu_T(x), \tag{1.4}$$

when $\pi \in C^1(E)$, for some Borel measurable vector field $l\colon E \to E$ satisfying $\|l(x)\| = 1$ for μ_T-a.e. $x \in \Sigma$, where $(\cdot,\cdot)$ stands for the scalar product of vectors.

Proof. The representation of T in the form (1.4) with $l \in L^\infty(E; \mu_T)$ is due to theorem 1.3 from [16] when $\mu_T \ll \mathcal{L}^n$; the general case follows by approximating T by a sequence of $T_k \in \mathcal{M}_1(E)$ with $T_k \rightharpoonup T$, $\mu_{T_k} \rightharpoonup \mu_T$ as $k \to +\infty$, and $\mu_{T_k} \ll \mathcal{L}^n$

for all $k \in \mathbb{N}$. Further, minding that $\|\nabla\pi(x)\|' \le \operatorname{Lip}\pi$ for all $x \in E$, where $\|\cdot\|'$ stands for the norm in the space E' dual to E, the representation (1.4) implies

$$|T(f\,d\pi)| \le \int_E f(x)\|\nabla\pi(x)\|' \cdot \|l(x)\|\,d\mu_T(x) \le \operatorname{Lip}\pi \int_E f(x)\|l(x)\|\,d\mu_T(x),$$

so that, by the definition of the mass measure of a metric current one has $\mu_T \le \|l\|\mu_T$. This implies $\|l(x)\| \ge 1$ for μ_T-a.e. $x \in \Sigma$. To prove the opposite inequality, let $a\colon \mathbb{R}^n \to \mathbb{R}^n$ be a Borel measurable vector field with $\|a(x)\|' = 1$ such that $(a(x), l(x)) = \|l(x)\|$ (such a vector field exists, say, in view of corollary A.2.1 of [15]). Denote for the sake of brevity $\mu := \|l\|\mu_T$. For a given $\varepsilon > 0$, we choose a finite δ-net $\{c_i\}_{i=1}^k$ of the unit sphere $\{\|x\|' = 1\}$, where $\delta = \varepsilon/\mu(E)$, and set

$$\begin{aligned} &E_i := \{\|a(x) - c_i\|' \le \delta\} \\ &D_1 := E_1, \qquad D_i := E_i \setminus \cup_{j=1}^{i-1} D_i, \end{aligned}$$

so that for $a_\varepsilon := \sum_{i=1}^k \mathbf{1}_{D_i} c_i$ one has

$$\begin{aligned} \int_E \|a(x) - a_\varepsilon(x)\|\,d\mu &= \sum_{i=1}^k \int_{D_i} \|a(x) - c_i\|\,d\mu \\ &\le \delta \sum_{i=1}^k \mu(D_i) = \delta\mu(E) \le \varepsilon. \end{aligned}$$

Letting $\pi_i\colon E \to \mathbb{R}$ be a Lipschitz function with $\operatorname{Lip}\pi_i = 1$ and $\nabla\pi_i = c_i$, one gets

$$\begin{aligned} \mu_T(\{l > 1+\alpha\}) &\ge \sum_{i=1}^k T\left(\mathbf{1}_{\{l>1+\alpha\}}\mathbf{1}_{D_i}\,d\pi_i\right) \\ &= \int_{\{l>1+\alpha\}} (a_\varepsilon, l(x))\,d\mu_T(x) \\ &\ge \int_{\{l>1+\alpha\}} (a, l(x))\,d\mu_T(x) - \int_E \|a - a_\varepsilon\|' \cdot \|l(x)\|\,d\mu_T(x) \\ &\ge (1+\alpha)\mu_T(\{l > 1+\alpha\}) - \varepsilon. \end{aligned}$$

Sending $\varepsilon \to 0^+$, we get $\mu_T(\{l > 1+\alpha\}) \ge (1+\alpha)\mu_T(\{l > 1+\alpha\})$, which can be only true when $\mu_T(\{l > 1+\alpha\}) = 0$. Since $\alpha > 0$ can be taken arbitrary, we get $\|l\| \le 1$ which concludes the proof. $\square$

Now we consider another construction which is used in the paper. Let (E_i, d_i) be metric spaces, $i = 1, 2$, $T_1 \in \mathcal{M}_1(E_1)$ and $\mu_2 \in \mathcal{M}_0(E_2)$. We define the current $T_1 \times \mu_2 \in \mathcal{M}_1(E_1 \times E_2)$ by setting

$$(T_1 \times \mu_2)(\omega) := \int_{E_2} T_1(\omega(\cdot, x_2))\,d\mu_2(x_2) \tag{1.5}$$

for every $\omega = f\,d\pi \in D^1(E_1 \times E_2)$. Analogously we define $\mu_1 \times T_2 \in \mathcal{M}_1(E_1 \times E_2)$ for $T_2 \in \mathcal{M}_1(E_2)$ and $\mu_1 \in \mathcal{M}_0(E_1)$.

Lemma A.9. *Let (E_i, d_i) be complete spaces, $T_i \in \mathcal{N}_1(E_i)$, $i = 1, 2$, and*

$$T := T_1 \times \mu_{T_2} + \mu_{T_1} \times T_2 \in \mathcal{N}_1(E_1 \times E_2).$$

Then $\mathbb{M}(T) = \mathbb{M}(T_1)\mathbb{M}(T_2)$, if the distance d in $E_1 \times E_2$ is defined by

$$d((x_1, x_2), (x_1', x_2')) := d_1(x_1, x_1') \vee d_2(x_2, x_2').$$

Proof. Let us first observe that it is enough to show

$$\mathbb{M}(T) \le \mathbb{M}(T_1)\mathbb{M}(T_2). \tag{1.6}$$

In fact, denoting by $P_1\colon E_1\times E_2\to E_1$ the projection map $P_1(x_1,x_2):=x_1$, we have

$$P_{1\#}(T_1\times\mu_{T_2})=\mathbb{M}(T_2)T_1,\qquad P_{1\#}(\mu_{T_1}\times T_2)=0,$$

so that

$$\mathbb{M}(T)\geq\mathbb{M}(P_{1\#}T)=\mathbb{M}(T_1)\mathbb{M}(T_2).$$

We divide the proof of the remaining claim (1.6) in three steps.

Step 1. Consider the case when E_i are finite-dimensional normed spaces with norms $\|\cdot\|_i$. Then $E_1\times E_2$ is equipped with the norm $\|(x_1,x_2)\|:=\|x_1\|_1\vee\|x_2\|_2$. By Lemma A.8 we may assume

$$T_i(f_i\,d\pi_i)=\int_{E_i}f_i(x_i)(\nabla\pi_i(x_i),l_i)\,d\mu_{T_i}(x_i),\qquad \|l_i\|_{E_i}=1,$$

for every $f_i\,d\pi_i\in D^1(E_i)$. Then, for $f\,d\pi\in D^1(E_1\times E_2)$ one has

$$\begin{aligned}T(f\,d\pi)&=\int_{E_2}\left(\int_{E_1}f(x_1,x_2)(\nabla\pi(x_1,x_2),l_1)\,d\mu_{T_1}(x_1)\right)d\mu_{T_2}(x_2)\\&\quad+\int_{E_1}\left(\int_{E_2}f(x_1,x_2)(\nabla\pi(x_1,x_2),l_2)\,d\mu_{T_2}(x_2)\right)d\mu_{T_1}(x_1)\\&=\int_{E_1\times E_2}f(x_1,x_2)(\nabla\pi(x_1,x_2),l)\,d\mu_{T_1}\otimes\mu_{T_2}(x_1,x_2),\end{aligned}$$

where $l:=(l_1,0)+(0,l_2)=(l_1,l_2)\in E_1\times E_2$. Since $\|l\|=1$, we have by Lemma A.8 that $\mu_T=\mu_{T_1}\otimes\mu_{T_2}$, so that in particular $\mathbb{M}(T)=\mathbb{M}(T_1)\mathbb{M}(T_2)$.

Step 2. We now show this result for the case when both E_i are Banach spaces with metric approximation property. Let $T_i^n\in\mathcal{M}_1(E_i)$ be normal currents supported over some finite-dimensional subspaces of E_i such that $T_i^n\rightharpoonup T_i$ in the weak sense of currents, while $\mu_{T_n^i}\rightharpoonup\mu_{T_i}$ as $n\to\infty$ (such sequences of currents exist due to Lemma A.6, with $T_n^i:=P_{n\#}T_i$ in the notation of its proof). We claim that for

$$T^n:=T_1^n\times\mu_{T_2^n}+\mu_{T_1^n}\times T_2^n\in\mathcal{N}_1(E_1\times E_2)$$

one has $T^n\rightharpoonup T$ in the weak sense of currents as $n\to\infty$. This would complete the proof of this step since then

$$\begin{aligned}\mathbb{M}(T)&\leq\liminf_n\mathbb{M}(T^n)\\&=\liminf_n\mathbb{M}(T_1^n)\mathbb{M}(T_2^n)&&\text{(by Step 1)}\\&=\lim_n\mathbb{M}(T_1^n)\lim_n\mathbb{M}(T_2^n)=\mathbb{M}(T_1)\mathbb{M}(T_2).\end{aligned}$$

To show the claim consider an arbitrary $\omega=f\,d\pi\in D^1(E_1\times E_2)$. One has

$$\begin{aligned}\int_{E_2}T_1^n(\omega(\cdot,x_2))\,d\mu_{T_2^n}(x_2)&=\int_{E_2}(T_1^n-T_1)(\omega(\cdot,x_2))\,d\mu_{T_2^n}(x_2)\\&\quad+\int_{E_2}T_1(\omega(\cdot,x_2))\,d\mu_{T_2^n}(x_2).\end{aligned}\tag{1.7}$$

But $(T_1^n-T_1)(\omega(\cdot,x_2))\to 0$ since $T_1^n\to T_1$, and moreover, the above convergence is uniform over compact subsets of E_2. In fact, $x_2^n\to x_2$ in E_2 implies $f(P_n(\cdot),x_2^n)\to f(\cdot,x_2)$, pointwise, and hence also in μ_{T_1} (because $\|f(\cdot,x_2^n)\|_\infty\leq\|f\|_\infty$), and $\pi(\cdot,x_2^n)\to\pi(\cdot,x_2)$, pointwise with $\operatorname{Lip}\pi(\cdot,x_2^n)\leq\operatorname{Lip}\pi$, so that

$$T_1^n(\omega(\cdot,x_2^n))=T_1(\omega(P_n(\cdot),x_2^n))\to T_1(\omega(\cdot,x_2))$$

as $n \to \infty$. Therefore,

$$\left|\int_{E_2} (T_1^n - T_1)(\omega(\cdot, x_2))\, d\mu_{T_2^n}(x_2)\right| \le \int_{E_2} |(T_1^n - T_1)(\omega(\cdot, x_2))|\, d(P_{n\#}\mu_{T_2})(x_2)$$
$$= \int_{E_2} |(T_1^n - T_1)(\omega(\cdot, P_n(x_2)))|\, d\mu_{T_2}(x_2) \to 0$$

as $n \to +\infty$ by Lebesgue dominated convergence theorem, minding that

$$|(T_1^n - T_1)(\omega(\cdot, P_n(x_2)))| \le (\mathbb{M}(T_1^n) + \mathbb{M}(T_1))\|f\|_\infty \mathrm{Lip}\pi \le 3\mathbb{M}(T_1)\|f\|_\infty \mathrm{Lip}\pi$$

when n is sufficiently large. On the other hand, the map $x_2 \subset E_2 \mapsto T_1(\omega(\cdot, x_2))$ is bounded by $\mathbb{M}(T_1)\|f\|_\infty \mathrm{Lip}\,\pi$ and continuous by the basic properties of currents, because, as just shown, $x_2^k \to x_2$ in E_2 implies $f(\cdot, x_2^k) \to f(\cdot, x_2)$ in μ_{T_1} and $\pi(\cdot, x_2^k) \to \pi(\cdot, x_2)$ pointwise with uniformly bounded Lipschitz constants. Therefore,

$$\int_{E_2} T_1(\omega(\cdot, x_2))\, d\mu_{T_2^n}(x_2) \to \int_{E_2} T_1(\omega(\cdot, x_2))\, d\mu_{T_2}(x_2),$$

since $\mu_{T_n^2} \rightharpoonup \mu_{T_2}$ as $n \to \infty$. Thus, from (1.7) we get

$$\int_{E_2} T_1^n(\omega(\cdot, x_2))\, d\mu_{T_2^n}(x_2) \to \int_{E_2} T_1(\omega(\cdot, x_2))\, d\mu_{T_2}(x_2).$$

Analogously we obtain

$$\int_{E_1} T_2^n(\omega(x_1, \cdot))\, d\mu_{T_1^n}(x_1) \to \int_{E_1} T_2(\omega(x_1, \cdot))\, d\mu_{T_1}(x_2),$$

and hence the claim.

Step 3. In view of lemma 5.5 from [10] and of the previous step of the proof the result is proven in the case $E_1 = E_2 = \ell^\infty$. If E_i are arbitrary complete metric spaces, we may assume without loss of generality that they be Polish (otherwise just take $\operatorname{supp} T_i$ in place of E_i). Denoting then by $j_i\colon E_i \to \ell^\infty$ the isometric imbeddings, and minding that $\mu_{j_{i\#}T_i} = j_{i\#}\mu_{T_i}$, we get that

$$j_{1\#}T_1 \times \mu_{j_{2\#}T_2} + \mu_{j_{1\#}T_1} \times j_{2\#}T_2 = (j_1, j_2)_\# T.$$

But then $\mathbb{M}((j_1, j_2)_\# T) \le \mathbb{M}(j_{1\#}T_1)\mathbb{M}(j_{2\#}T_2) = \mathbb{M}(T_1)\mathbb{M}(T_2)$, but since the map $(j_1, j_2)\colon E_1 \times E_2 \to \ell^\infty \times \ell^\infty$ is an isometry, then $\mathbb{M}(T) = \mathbb{M}((j_1, j_2)_\# T)$, and the proof is completed. $\square$

Appendix B. Auxiliary lemmata from probability theory

Here we collect some more or less folkloric statements (or something "around" mathematical folklore) from abstract probability theory which are used in the paper. We start with the following compactification result which is a variation on the theme of lemma 3.1.4 from [14].

Lemma B.1. *Let (E, d) be a separable metric space. Then there is a new distance $\tilde d \le d$ over E topologically equivalent to d such that $(E, \tilde d)$ is totally bounded. In particular, denoting by $\tilde E$ the completion of E with respect to $\tilde d$ we have that $\tilde E$ is compact, while the space $C(\tilde E) = C_b(\tilde E)$ is separable. Thus, letting $C_u(E, \tilde d)$ to stand for the set of bounded functions uniformly continuous over E with respect to $\tilde d$, we get the existence of a countable set $\{f_k\} \subset C_u(E, \tilde d)$ dense in $C_u(E, \tilde d)$ in the uniform norm $\|\cdot\|_\infty$.*

Proof. Let $\{x_k\} \subset E$ stand for a countable dense set in E, and consider the map $g\colon E \to [0,1]^{\mathbb{N}}$ defined by

$$g_n(x) := \frac{d(x, x_n)}{1 + d(x, x_n)}$$

for all $x \in E$. Note that $[0,1]^{\mathbb{N}}$ is compact when equipped with the product topology, while the latter may be metrized, say, by the distance

$$\hat{d}(x,y) := \sum_{k=1}^{\infty} \frac{|x_k - y_k|}{2^k}.$$

Thus, defining

$$\tilde{d}(x,y) := \hat{d}(g(x), g(y))$$

for all $\{x,y\} \subset E \times E$, we get that $(E, \tilde{d})$ is totally bounded.

Now, clearly,

$$\begin{aligned}
\tilde{d}(x,y) &= \sum_{k=1}^{\infty} \frac{1}{2^k} \left| \frac{d(x,x_k)}{1+d(x,x_k)} - \frac{d(y,x_k)}{1+d(y,x_k)} \right| \\
&\leq \sum_{k=1}^{\infty} \frac{1}{2^k} \left| d(x,x_k) - d(y,x_k) \right| \leq \sum_{k=1}^{\infty} \frac{1}{2^k} d(x,y) = d(x,y).
\end{aligned}$$

Vice versa, $\tilde{d}(y_j, y) \to 0$ for some $y \in E$ implies $g_n(y_j) \to g_n(y)$, and hence $d(y_j, x_n) \to d(y, x_n)$ for all $n \in \mathbb{N}$ as $j \to \infty$. Then

$$d(y_j, y) \leq d(y_j, x_n) + d(y, x_n) \to 2d(y, x_n),$$

and hence, since x_n is an arbitrary element of a dense set in E, we get $d(y_j, y) \to 0$ as $j \to \infty$.

Hence, denoting by $\tilde{E}$ the completion of E with respect to $\tilde{d}$ we have that $\tilde{E}$ is compact, and therefore the space $C(\tilde{E}) = C_b(\tilde{E})$ is separable. Further, every $f \in C_b(\tilde{E})$ is clearly uniformly continuous. Vice versa, if $f \in C_u(E, \tilde{d})$, then for every fundamental sequence $\{y_j\} \subset E$ one has that $\{f(y_j)\} \subset \mathbb{R}$ is fundamental, and hence f can be extended by continuity to a function from $C(\tilde{E})$. Thus we may identify $C_u(E, \tilde{d})$ with $C(\tilde{E})$, so that the last claim of the lemma being proven is just separability of $C(\tilde{E})$. □

Remark B.2. For the case of a Euclidean space $E := \mathbb{R}^n$ (or, more generally, for a space with Heine-Borel property, i.e. a space where closed balls are compact) the above Lemma B.1 gives just the ordinary Alexandrov one-point compactification $\tilde{E}$. In fact, if a sequence $\{y_k\} \subset E$ is fundamental with respect to the new distance $\tilde{d}$, then so is the sequence $\{d(y_k, x_n)/(1 + d(y_k, x_n))\} \subset \mathbb{R}$ for each $n \in \mathbb{N}$. Then either of the following two separate cases may happen.

(i) $d(y_k, x_n)/(1 + d(y_k, x_n)) \to 1$, which means $d(y_k, x_n) \to \infty$ for some $n \in \mathbb{N}$, which happens if and only if $y_k \to \infty$ (i.e. $d(y_k, y) \to \infty$ for all $y \in E$) as $k \to \infty$. This is the case when the $\{y_k\}$ determines the point $\infty \in \tilde{E}$.

(ii) The sequence $\{y_k\}$ is uniformly bounded (note that the case of $y_k \to \infty$ as $k \to \infty$ only for a subsequence of $\{y_k\}$ is excluded since otherwise one would have $d(y_k, x_n)/(1+d(y_k, x_n)) \to 1$ along this subsequence, and hence for the whole sequence, since the latter sequence of numbers is fundamental). Then up to a subsequence (not relabeled) $y_k \to y \in E$, hence $d(y_k, x_n) \to d(y, x_n)$, and therefore also $d(y_k, x_n)/(1 + d(y_k, x_n)) \to d(y, x_n)/(1 + d(y, x_n))$ for all $n \in \mathbb{N}$ as $k \to \infty$. Again, the latter convergence must be now valid for the whole original sequence, which means that the same must be true also for convergence $d(y_k, x_n) \to d(y, x_n)$. Now, for any other convergent subsequence of $\{y_k\}$ (again not relabeled), say, $y_k \to z \in E$, one would have $d(y_k, x_n) \to d(z, x_n)$, which implies $d(y, x_n) = d(z, x_n)$ for all $n \in \mathbb{N}$. This means $y = z$ and hence the whole sequence $\{y_k\}$ is convergent to $y \in E$.

Summing up, we have $\tilde{E} = E \cup \{\infty\}$. Therefore, $C_u(E, \tilde{d})$ consists of continuous (with respect to d) functions having (finite) limits at infinity.

Remark B.3. In the case of a Euclidean space $E := \mathbb{R}^n$ for every compact $K \subset E$ there is a $C > 0$ such that

$$d(y,z) \leq C\tilde{d}(y,z) \text{ for all } (y,z) \in K \times K.$$

To show this suppose the contrary, i.e. the existence of $\{(y_k, z_k)\} \subset K \times K$ such that

$$\lim_k \frac{\tilde{d}(y_k, z_k)}{d(y_k, z_k)} \to 0,$$

and thus

$$\lim_k \left| \frac{d(y_k, x_n)}{1 + d(y_k, x_n)} - \frac{d(z_k, x_n)}{1 + d(z_k, x_n)} \right| \frac{1}{d(y_k, z_k)} = 0,$$

for all $n \in \mathbb{N}$, which is only possible when

$$\lim_k \frac{|d(y_k, x_n) - d(z_k, x_n)|}{d(y_k, z_k)} = 0, \tag{2.1}$$

for all $n \in \mathbb{N}$. Since by compactness of K we may assume without loss of generality that $z_k \to z$ and $y_k \to y$ as $k \to \infty$, then the above equality is only possible once and in particular $y = z$. But for $x_n \neq y$ the relationship

$$\frac{d(y_k, x_n) - d(z_k, x_n)}{d(y_k, z_k)} = \frac{y - x_n}{|y - x_n|} \cdot \frac{y_k - z_k}{|y_k - z_k|} + o(1)$$

for $k \to \infty$ holds. Since up to a subsequence (not relabeled) $(y_k - z_k)/|y_k - z_k| \to e$ as $k \to \infty$ for some unit vector e, then choosing an $n \in \mathbb{N}$ such that

$$\frac{y - x_n}{|y - x_n|} \cdot e > 0,$$

we get a contradiction with (2.1).

We also use in the paper the following easy consequence of corollary 7.7.2 from [4] (i.e. of the Kolmogorov extension theorem).

Lemma B.4. *Let (X, Σ) be a measure space (X being a metric space and Σ being its Borel σ-algebra) and η_k be (Borel) tight probability measures over X^k satisfying the following compatibility conditions:*

$$\pi_{k-1\#}\eta_k = \eta_{k-1},$$
$$\pi^{k-1}_{\#}\eta_k = \eta_{k-1},$$

where $\pi_{k-1} \colon X^k = X^{k-1} \times X \to X^{k-1}$ and $\pi^{k-1} \colon X^k = X \times X^{k-1} \to X^{k-1}$ are defined by

$$\pi_{k-1}(x_1, \ldots, x_{k-1}, x_k) := (x_1, \ldots, x_{k-1}),$$
$$\pi^{k-1}(x_1, x_2, \ldots, x_k) := (x_2, \ldots, x_k).$$

Then there is a probability measure η_* over $X^{\mathbb{Z}}$ such that

$$\eta_* \left(\bigcap_{j=k}^{l} p_j^{-1}(e_j) \right) = \eta_{l-k} \left(\prod_{j=k}^{l} e_j \right), \qquad e_j \in \Sigma,$$

where $p_j \colon X^{\mathbb{Z}} \to X$ is defined by $p_j(x) := (x)_j$.

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