From phase-field to sharp cracks: convergence of quasi-static evolutions in a special setting☆.

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Abstract

We consider the quasi-static evolution of a straight crack within the recent phase field approach and the classical sharp crack approach. We show a strong correlation between the two approaches: energy, minimizers, energy release rate and quasi-static evolutions converge as the internal length parameter of the phase-field approach tends to zero. A crucial point in the proof is a novel representation of the energy release rate, which allows to pass to the limit under weak convergence of the strains.

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1. Introduction

Consider a crack propagating on a straight line (or a regular path), denote by $\ell(t)$ its length at time $t \in [0,T]$ and by $G(t, \ell(t))$ the energy release; according to Griffith's criterion [11] and following [12, 15, 19] the evolution is given by a non-decreasing function ℓ which satisfies the following Kuhn-Tucker conditions (in weak form)

$$G(t, \ell^-(t)) \leq G \,, \qquad \text{for every } t \in [0,T], \tag{1}$$

$$\big(G(t, \ell^-(t)) - G_c\big)\, d\ell(t) = 0 \,, \qquad \text{in the sense of measures,} \tag{2}$$

together with the jump condition,

$$G(t, l) \geq G_c \,, \qquad \text{for every } l \in [\ell^-(t), \ell^+(t)] \text{ and every } t \in J(\ell). \tag{3}$$

Discontinuities (in time) are typical of rate independent evolutions and represent in the quasi-static picture the non-equilibrium regimes; abrupt evolutions of this type often occur in fracture, for instance in the case of short initial cracks. The above setting allows for a fine analysis and captures the main features of fracture propagation [14]; however, in real life cracks are often a collection of zig-zagging curves with kinks, bifurcations, self intersections etc. In this spirit a "natural" functional setting would be provided by the space SBD [1, 3], where the crack would be a countable collection of rectifiable sets; this is enough to represent any realistic fracture but generality leads to big technical issues, e.g. it is still not known a notion of energy release.

An effective alternative to the representation of cracks by paths or sets are the phase-field approaches, whose range of application now includes quasi-statics [6], dynamics [4, 13] and mixed-mode I+III [17]. Among the many declinations, we follow the one based on the Ambrosio-Tortorelli functional [2]. Denote

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by u the displacement and by $W(Du)$ the linear-elastic energy density. For $\varepsilon > 0$ (an internal length) and $\eta_\varepsilon = o(\varepsilon)$ (a regularization parameter) consider the energy

$$\mathcal{J}_\varepsilon(z,u) = \int_\Omega (z^2+\eta_\varepsilon) W(Du)\,dx + G_c \int_\Omega V^2(z)/\varepsilon + \varepsilon|\nabla z|^2\,dx\,; \tag{4}$$

the phase variable z takes values in $[0,1]$ and provides a "smeared" representation of the crack.

As phase-field models offer an alternative to a well established theory, it is natural to check whether they are consistent with the traditional sharp crack approach, at least in some representative examples. This question has been addressed numerically, for a straight crack under tension [5], and theoretically, for the energy release on a smooth path [18]. Our goal is to provide a rigorous connection between the phase-field and the sharp crack approach, showing convergence of energy, energy release and evolutions as the internal length vanishes. We consider the benchmark case of a straight crack together with an explicit choice of the phase variable z, suggested by convergence and numerical results.

Technically, the crucial point is the (uniform) convergence of the energy release, for which the volume integral representation, with the Eshelby tensor $\mathbb{E} = \boldsymbol{\sigma}(u)Du - W(Du)I$, seems the most convenient choice. Since $\mathbb{E}$ is quadratic and depends on the complete gradient Du (and not only on its symmetric part) it is continuous with respect to the strong convergence of gradients; this is not at hand in our context since the weight $(z^2+\eta_\varepsilon)$ in front of $W(Du)$ does not match with Korn's inequality. To by-pass the problem we provide a new formula for the energy release which is indeed linear in the symmetric gradient $\boldsymbol{\varepsilon}(u)$. Beside our purposes this formula is applicable in any fracture problem; all the known representations (with Eshelby tensor, J-integral and stress intensity factors) follow with some manipulations.

2. Sharp crack setting

We restrict our analysis to in-plane elasticity. The reference configuration is represented by a bounded open, Lipschitz set Ω. We set a system of Cartesian coordinates in a way that the initial crack is the line segment $K_0 = [0,\ell_0]\times\{0\}$ with the first endpoint in $\partial\Omega$. Admissible cracks will be of the form $K_\ell = [0,\ell]\times\{0\}$ for $\ell \in [l_0, L]$. At the price of few technical difficulties our analysis will hold also for crack path of class C^2. Let $\partial_D\Omega$ be relatively open in $\partial\Omega$ and let $g \in H^{1/2}(\partial_D\Omega,\mathbb{R}^2)$. The set of admissible configurations is then given by

$$\mathcal{U}_\ell = \{u \in H^1(\Omega\setminus K_\ell,\mathbb{R}^2) : u = g\ \partial_D\Omega\}\,. \tag{5}$$

We employ the linear energy density $W(Du) = \frac12\,Du : \mathbf{C}[Du] = \frac12\,\boldsymbol{\varepsilon}(u) : \boldsymbol{\sigma}(u)$ for $\boldsymbol{\varepsilon}(u) = (Du + Du^T)/2$ and $\boldsymbol{\sigma}(u) = 2\mu\boldsymbol{\varepsilon}(u) + \lambda\mathrm{tr}(\boldsymbol{\varepsilon}(u))I$. The elastic energy is thus

$$\mathcal{E}(u) = \int_{\Omega\setminus K_\ell} W(Du)\,dx\,.$$

Given ℓ let $\{u_\ell\} = \mathrm{argmin}\{\mathcal{E}(u) : u \in \mathcal{U}_\ell\}$ and define the reduced energy $\mathcal{E}(\ell) = \mathcal{E}(u_\ell)$. Finally we introduce the total energy $\mathcal{F}(\ell) = \mathcal{E}(\ell) + G_c\ell$. The minimizer u_ℓ is characterized by the variational formulation

$$\int_{\Omega\setminus K_\ell} Dv : \mathbf{C}[Du_\ell]\,dx = 0 \qquad \text{for every } v \in \mathcal{V}_\ell, \tag{6}$$

where the set of admissible variations is

$$\mathcal{V}_\ell = \{v \in H^1(\Omega\setminus K_\ell,\mathbb{R}^2) : v = 0\ \partial_D\Omega\}\,. \tag{7}$$

To compute the energy release we borrow form [10] the idea of the expansion and then give a novel, very short proof. Let $\psi \in W^{1,\infty}(\Omega,\mathbb{R}^2)$ with $\|\psi\|_\infty \le 1$, $\psi(\ell,0) = \hat{e}_1$ and $\mathrm{supp}(\psi) \subset B_r(\ell,0)$ for $r \ll 1$. Let $\Psi_h(x) = x + h\psi(x)$. For $h \ll 1$ the map Ψ_h is a diffeomeorphism in Ω and $K_{\ell+h} = \Psi_h(K_\ell)$. Let $\bar{u}_{\ell+h} = u_{\ell+h}\circ\Psi_h$ (note it depends on Ψ) and write

$$\mathcal{E}(\ell+h) = \tfrac12 \int_{\Omega\setminus K_{\ell+h}} Du_{\ell+h} : \mathbf{C}[Du_{\ell+h}]\,dx = \tfrac12 \int_{\Omega\setminus K_\ell} D\bar{u}_{\ell+h} : \bar{\mathbf{C}}_h[D\bar{u}_{\ell+h}]\,dx\,,$$

where $\bar{\mathbf{C}}_h[F] = \mathbf{C}[FD\Psi_h^{-1}]D\Psi_h^{-T}\det D\Psi_h$. The variational formulation for $\bar{u}_{\ell+h}$ reads

$$\int_{\Omega\setminus K_\ell} Dv : \bar{\mathbf{C}}_h[D\bar{u}_{\ell+h}]\,dx = 0 \qquad \text{for every } v \in \mathcal{V}_\ell. \tag{8}$$

As $D\Psi_h(x) = I + h\,D\psi(x)$ and $\det D\Psi_h = 1 + h\,\mathrm{tr}(D\psi) + h^2\det(D\psi)$, for $h \ll 1$ the inverse matrix can be written as

$$D\Psi_h^{-1} = \sum_{i=0}^{\infty}(-h\,D\psi)^n = I - h\,D\psi + o(h)\,.$$

The above identity holds in $L^\infty(\Omega,\mathbb{R}^{2\times 2})$, i.e. $\|D\Psi_h^{-1} - I + hD\psi\|_\infty = o(h)$. Then, define $\bar{\mathbf{C}}'$ and $\bar{\mathbf{C}}'_h$ as

$$\begin{aligned}\bar{\mathbf{C}}_h[F] &= \mathbf{C}\big[FD\Psi_h^{-1}\big]\,D\Psi_h^{-T}\det D\Psi_h\\ &= \mathbf{C}[F] + h\big(-\mathbf{C}[FD\psi] - \mathbf{C}[F]D\psi^T + \mathbf{C}[F]\mathrm{tr}D\psi\big) + o(h)\\ &= \mathbf{C}[F] + h\bar{\mathbf{C}}'[F] + o(h) = \mathbf{C}[F] + h\bar{\mathbf{C}}'_h[F]\,.\end{aligned} \tag{9}$$

Lemma 2.1. *The tensor $\bar{\mathbf{C}}_h$ is elliptic and coercive in $\mathcal{V}_\ell$, uniformly with respect to h. Moreover,* $\mathrm{supp}(\bar{\mathbf{C}}'_h) \subset \mathrm{supp}(\psi)$.

Lemma 2.2. *Let $\bar{u}'_{\ell+h}$ be defined by $\bar{u}_{\ell+h} = u_\ell + h\,\bar{u}'_{\ell+h}$. Then $\bar{u}'_{\ell+h} \rightharpoonup \bar{u}'_\ell$ in $H^1(\Omega\setminus K_\ell,\mathbb{R}^2)$ where $\bar{u}'_\ell$ solves the variational problem*

$$\int_{\Omega\setminus K_\ell} Dv : \mathbf{C}[D\bar{u}'_\ell]\,dx = -\int_{\Omega\setminus K_\ell} Dv : \bar{\mathbf{C}}'[Du_\ell]\,dx \qquad \textit{for every } v \in \mathcal{V}_\ell. \tag{10}$$

Proof. From (6) and (8) we get

$$\int_{\Omega\setminus K_\ell} Dv : \bar{\mathbf{C}}_h[D\bar{u}_{\ell+h} - Du_\ell]\,dx = -h\int_{\Omega\setminus K_\ell} Dv : \bar{\mathbf{C}}'_h[Du_\ell]\,dx \qquad \text{for every } v \in \mathcal{V}_\ell.$$

As $(\bar{u}_{\ell+h} - u_\ell) \in \mathcal{V}_\ell$ by Lax-Milgram Lemma and by Lemma 2.1 we get $\|\bar{u}_{\ell+h} - u_\ell\|_{H^1}/h \le c\,\|\bar{\mathbf{C}}'_h[Du_\ell]\|_{L^2}$. Since $\bar{\mathbf{C}}'_h[Du_\ell] \to \bar{\mathbf{C}}'[Du_\ell]$ strongly in $L^2(\Omega,\mathbb{R}^{2\times 2})$ the right hand side in the last inequality is uniformly bounded, hence (up to subsequences) $(\bar{u}_{\ell+h} - u_\ell)/h = \bar{u}'_{\ell+h} \rightharpoonup \bar{u}'_\ell$ in $H^1(\Omega\setminus K_\ell,\mathbb{R}^2)$. Hence $\bar{\mathbf{C}}_h[D\bar{u}'_{\ell+h}] \rightharpoonup \mathbf{C}[D\bar{u}'_\ell]$ in $L^2(\Omega,\mathbb{R}^{2\times 2})$. We can then pass to the limit above and get (10). □

Lemma 2.3. *Let $\tilde{g}$ be a lifting of the boundary datum with* $\mathrm{supp}(\tilde{g}) \cap \mathrm{supp}(\psi) = \emptyset$. *Then*

$$G(\ell) = -\lim_{h\to 0}\big(\mathcal{E}(\ell+h) - \mathcal{E}(\ell)\big)/h = -\tfrac{1}{2}\int_{\Omega\setminus K_\ell} D\tilde{g} : \mathbf{C}[D\bar{u}'_\ell]\,dx\,. \tag{11}$$

Proof. Write $u_\ell = w_0 + \tilde{g}$ and $\bar{u}_{\ell+h} = \bar{w}_h + \tilde{g}$ for $w_0, \bar{w}_h \in \mathcal{V}_\ell$. Then by (6) we get

$$\mathcal{E}(\ell) = \tfrac{1}{2}\int_{\Omega\setminus K_\ell} Du_\ell : \mathbf{C}[Du_\ell]\,dx = \tfrac{1}{2}\int_{\Omega\setminus K_\ell} D\tilde{g} : \mathbf{C}[Du_\ell]\,dx\,.$$

By (8) and (9) we also have

$$\begin{aligned}\mathcal{E}(\ell+h) &= \tfrac{1}{2}\int_{\Omega\setminus K_\ell} D\bar{u}_{\ell+h} : \bar{\mathbf{C}}_h[D\bar{u}_{\ell+h}]\,dx\\ &= \tfrac{1}{2}\int_{\Omega\setminus K_\ell} D\tilde{g} : \mathbf{C}[D\bar{u}_{\ell+h}]\,dx + \tfrac{1}{2}h\int_{\Omega\setminus K_\ell} D\tilde{g} : \bar{\mathbf{C}}'_h[D\bar{u}_{\ell+h}]\,dx\\ &= \tfrac{1}{2}\int_{\Omega\setminus K_\ell} D\tilde{g} : \mathbf{C}[D\bar{u}_{\ell+h}]\,dx\,,\end{aligned}$$

where last equality holds for the hypothesis on the supports. Then by Lemma 2.2

$$\big(\mathcal{E}(\ell+h) - \mathcal{E}(\ell)\big)/h = \int_{\Omega\setminus K_\ell} D\tilde{g} : \mathbf{C}[D\bar{u}'_{\ell+h}]\,dx \to \int_{\Omega\setminus K_\ell} D\tilde{g} : \mathbf{C}[D\bar{u}'_\ell]\,dx\,,$$

which is (11). □

3. Phase field setting

Let $V : [0,1] \to [0,1]$ be non-increasing, continuous in $[0,1]$, with $V(z) > 0$ if $0 \le z < 1$, $V(1) = 0$ and $\int_0^1 V(z)\,dz = 1/4$. For $0 < \eta_\varepsilon = o(\varepsilon)$ and $0 < s_\varepsilon = o(1)$, let $z_\varepsilon \in W^{1,\infty}((0,+\infty),[0,1])$ such that $z_\varepsilon(0) = 0$, $z_\varepsilon(s) = 1$ in $[s_\varepsilon, +\infty)$ and such that

$$\int_0^{\delta_\varepsilon} z_\varepsilon^2\,ds = O(\delta_\varepsilon \eta_\varepsilon) \qquad \text{for } \delta_\varepsilon = (\varepsilon\eta_\varepsilon)^{1/2},$$

$$\int_0^{s_\varepsilon} \varepsilon|z_\varepsilon'|^2 + V^2(z_\varepsilon)/\varepsilon\,ds = (1-c_\varepsilon)/2 \to 1/2\,.$$

For the existence of the family z_ε see for instance [7]. Given ℓ let $d(x,K_\ell)$ be the distance function to the crack set K_ℓ. We assume that the transition profile is given by $z_{\varepsilon,\ell}(x) = z_\varepsilon(d(x,K_\ell))$. Note that $z_{\varepsilon,\ell} \in W^{1,\infty}(\Omega)$ and that the support of $(1 - z_{\varepsilon,\ell})$ is contained in the s_ε-neighborhood of K_ℓ.

In the phase-field framework the spaces of admissible deformation and admissible variations are

$$\mathcal{U} = \{u \in H^1(\Omega,\mathbb{R}^2) : u = g\ \partial_D\Omega\} \quad \text{and} \quad \mathcal{V} = \{v \in H^1(\Omega,\mathbb{R}^2) : v = 0\ \partial_D\Omega\}\,.$$

Let $u_{\varepsilon,\ell} \in \mathcal{U}$ be the unique minimizer of the strictly convex energy

$$\mathcal{E}_\varepsilon(z_{\varepsilon,\ell},u) = \int_\Omega (z_{\varepsilon,\ell}^2 + \eta_\varepsilon) W(Du)\,dx$$

and, by abuse notation, denote $\mathcal{E}_\varepsilon(\ell) = \mathcal{E}_\varepsilon(z_{\varepsilon,\ell},u_{\varepsilon,\ell})$. Let

$$\mathcal{L}_\varepsilon(\ell) = \mathcal{L}_\varepsilon(z_{\varepsilon,\ell}) = \int_\Omega \varepsilon|\nabla z_{\varepsilon,\ell}|^2 + V^2(z_{\varepsilon,\ell})/\varepsilon\,dx\,.$$

Then, the phase field approx of $\mathcal{F}$ will be $\mathcal{F}_\varepsilon(\ell) = \mathcal{E}_\varepsilon(\ell) + G_c\mathcal{L}_\varepsilon(\ell) = \mathcal{J}_\varepsilon(z_{\varepsilon,\ell},u_{\varepsilon,\ell})$.

Remark 3.1. *Note that in general neither the phase-field energy $\mathcal{F}_\varepsilon$ nor the sharp crack energy $\mathcal{F}$ are convex. Moreover, the profile $z_{\varepsilon,\ell}$ is qualitatively consistent with the one obtained in numerical experiments, where the diffusive effect (due to the Dirichlet energy) is almost negligible.*

Let Ψ_h be as above and note that $z_{\varepsilon,\ell} = z_{\varepsilon,\ell+h} \circ \Psi_h$. Denote $\bar{u}_{\varepsilon,\ell+h} = u_{\varepsilon,\ell+h} \circ \Psi_h$. Arguing as in the previous section we can prove the following Lemmas.

Lemma 3.2. *Let $\bar{u}'_{\varepsilon,\ell+h}$ be defined by $\bar{u}_{\varepsilon,\ell+h} = u_{\varepsilon,\ell} + h\,\bar{u}'_{\varepsilon,\ell+h}$. Then $\bar{u}'_{\varepsilon,\ell+h} \rightharpoonup \bar{u}'_{\varepsilon,\ell}$ in $H^1(\Omega,\mathbb{R}^2)$ where $\bar{u}'_{\varepsilon,\ell}$ solves*

$$\int_\Omega (z_{\varepsilon,\ell}^2 + \eta_\varepsilon) Dv : \mathbf{C}[D\bar{u}'_{\varepsilon,\ell}]\,dx = -\int_\Omega (z_{\varepsilon,\ell}^2 + \eta_\varepsilon) Dv : \bar{\mathbf{C}}'[Du_{\varepsilon,\ell}]\,dx \qquad \text{for every } v \in \mathcal{V}\,. \tag{12}$$

Lemma 3.3. *Let $\tilde{g}$ be a lifting of the boundary datum with* $\mathrm{supp}(\tilde{g}) \cap \mathrm{supp}(\psi) = \emptyset$. *Then*

$$G_\varepsilon(\ell) = -\lim_{h\to 0} \big(\mathcal{E}_\varepsilon(\ell+h) - \mathcal{E}_\varepsilon(\ell)\big)/h = -\tfrac{1}{2}\int_\Omega (z_{\varepsilon,\ell}^2 + \eta_\varepsilon) D\tilde{g} : \mathbf{C}[D\bar{u}'_{\varepsilon,\ell}]\,.$$

4. Convergence of energy and minimizers

Lemma 4.1. *$\mathcal{F}_\varepsilon \to \mathcal{F}$ uniformly in $[l_0, L]$. Moreover if $\ell_\varepsilon \to \ell$ then $(z_{\varepsilon,\ell_\varepsilon}^2 + \eta_\varepsilon)^{1/2}\boldsymbol{\varepsilon}(u_{\varepsilon,\ell_\varepsilon}) \rightharpoonup \boldsymbol{\varepsilon}(u_\ell)$ in $L^2(\Omega,\mathbb{R}^{2\times 2})$ while $u_{\varepsilon,\ell_\varepsilon} \to u_\ell$ in $L^2_{loc}(\Omega \setminus K_\ell,\mathbb{R}^2)$.*

Proof. Since $\mathcal{F}$ is continuous in $[l_0, L]$ it follows (for instance by contradiction, see e.g. [16]) that $\mathcal{F}_\varepsilon \to \mathcal{F}$ uniformly in $[l_0, L]$ if and only if $\mathcal{F}_\varepsilon(\ell_\varepsilon) \to \mathcal{F}(\ell)$ for every $\ell_\varepsilon \to \ell$. Since $z_{\varepsilon,\ell}(x) = z_\varepsilon(d(x, K_\ell))$ the explicit calculation of $\mathcal{L}_\varepsilon(\ell)$ shows that $G_c\mathcal{L}_\varepsilon(\ell_\varepsilon) \to G_c\ell$. For the convergence of $\mathcal{E}_\varepsilon(\ell_\varepsilon)$ to $\mathcal{E}(\ell)$ it is sufficient to use the Γ-convergence [9] proof of [8] together with the properties of z_ε and the following compactness argument.

Consider u_ε such that $\mathcal{E}_\varepsilon(z_{\varepsilon,\ell_\varepsilon}, u_\varepsilon) \le C$. Then $(z^2_{\varepsilon,\ell_\varepsilon} + \eta_\varepsilon)^{1/2}\boldsymbol{\varepsilon}(u_\varepsilon)$ is bounded in $L^2(\Omega, \mathbb{R}^{2\times2})$. Thus, up to subsequences, $(z^2_{\varepsilon,\ell_\varepsilon} + \eta_\varepsilon)^{1/2}\boldsymbol{\varepsilon}(u_\varepsilon) \rightharpoonup \xi$ for some $\xi \in L^2(\Omega, \mathbb{R}^{2\times2})$. For every δ let $K^\delta_\ell = \{d(x, K_\ell) < \delta\}$. For $s_\varepsilon < \delta$ we have that $\boldsymbol{\varepsilon}(u_\varepsilon)$ is bounded in $L^2(\Omega \setminus K^\delta_\ell, \mathbb{R}^{2\times2})$, hence by Korn's inequality $u_\varepsilon \rightharpoonup u$ (up to subsequences) in $H^1(\Omega \setminus K^\delta_\ell, \mathbb{R}^2)$. By a diagonal argument we then get $u_\varepsilon \rightharpoonup u$ (up to subsequences) in $H^1(\Omega\setminus K^\delta_\ell, \mathbb{R}^2)$ for every δ. Therefore $\xi = \boldsymbol{\varepsilon}(u)$ and $(z^2_{\varepsilon,\ell_\varepsilon} + \eta_\varepsilon)^{1/2}\boldsymbol{\varepsilon}(u_\varepsilon) \rightharpoonup \boldsymbol{\varepsilon}(u)$ while $u_\varepsilon \to u$ in $L^2(\Omega\setminus K^\delta_\ell, \mathbb{R}^2)$ for every δ. □

5. Convergence of energy release

Proposition 5.1. *$G_\varepsilon \to G$ uniformly in $[l_0, L]$.*

Proof. Since G is continuous in $[l_0, L]$ it is enough to show that $G_\varepsilon(\ell_\varepsilon) \to G(\ell)$ whenever $\ell_\varepsilon \to \ell$. Thanks to the representation Lemmas 2.3 and 3.3 it is sufficient to show that

$$\int_\Omega (z^2_{\varepsilon,\ell_\varepsilon} + \eta_\varepsilon) D\tilde{g} : \mathbf{C}[D\bar{u}'_{\varepsilon,\ell_\varepsilon}] \to \int_\Omega D\tilde{g} : \mathbf{C}[D\bar{u}'_\ell] \, .$$

Note that $(z^2_{\varepsilon,\ell_\varepsilon} + \eta_\varepsilon)^{1/2} D\tilde{g} \to D\tilde{g}$ strongly in $L^2(\Omega, \mathbb{R}^{2\times2})$, hence it is sufficient to show that $(z^2_{\varepsilon,\ell_\varepsilon} + \eta_\varepsilon)^{1/2}\mathbf{C}[D\bar{u}'_{\varepsilon,\ell_\varepsilon}] \rightharpoonup \mathbf{C}[D\bar{u}'_\ell]$ in $L^2(\Omega, \mathbb{R}^{2\times2})$. Recall that $\bar{u}'_{\varepsilon,\ell_\varepsilon}$ solves

$$\int_\Omega (z^2_{\varepsilon,\ell_\varepsilon} + \eta_\varepsilon) Dv : \mathbf{C}[D\bar{u}'_{\varepsilon,\ell_\varepsilon}] \, dx = -\int_\Omega (z^2_{\varepsilon,\ell_\varepsilon} + \eta_\varepsilon) Dv : \bar{\mathbf{C}}'[Du_{\varepsilon,\ell_\varepsilon}] \, dx \qquad \text{for every } v \in \mathcal{V}.$$

By Lax-Milgram Lemma and Lemma 4.1 it follows that

$$\|(z_{\varepsilon,\ell_\varepsilon} + \eta_\varepsilon)^{1/2}\boldsymbol{\varepsilon}(\bar{u}'_{\varepsilon,\ell_\varepsilon})\|_{L^2} \le C\|(z_{\varepsilon,\ell_\varepsilon} + \eta_\varepsilon)^{1/2}\boldsymbol{\varepsilon}(u_{\varepsilon,\ell_\varepsilon})\|_{L^2} \le C' \, .$$

Hence $(z_\varepsilon + \eta_\varepsilon)^{1/2}\boldsymbol{\varepsilon}(\bar{u}'_{\varepsilon,\ell_\varepsilon}) \rightharpoonup \xi$ in $L^2(\Omega, \mathbb{R}^{2\times2})$. Then, by Korn's inequality $\bar{u}'_{\varepsilon,\ell_\varepsilon} \rightharpoonup w$ in $H^1(\Omega \setminus K^\delta_\ell, \mathbb{R}^2)$ for every $\delta > 0$ and thus $\xi = \boldsymbol{\varepsilon}(w)$. By Lemma 4.1 we can pass to the limit for $v \in \mathcal{V}$ and get

$$\int_\Omega Dv : \mathbf{C}[Dw] \, dx = -\int_\Omega Dv : \bar{\mathbf{C}}'[Du_\ell] \, dx \qquad \text{for every } v \in \mathcal{V}.$$

If $v \in \mathcal{V}_\ell \setminus \mathcal{V}$ and $v \in W^{1,\infty}(\Omega \setminus K_\ell)$ it is sufficient to provide $v_\varepsilon \in \mathcal{V}$ such that $(z^2_{\varepsilon,\ell_\varepsilon} + \eta_\varepsilon)^{1/2} Dv_\varepsilon \to Dv$ strongly in $L^2(\Omega, \mathbb{R}^{2\times2})$, in this way we can write the above variational problem for v_ε and then pass to the limit. The approximation v_ε is usually done, e.g. [7], choosing $v_\varepsilon = v$ in $\Omega \setminus K^{\delta_\varepsilon}_\ell$ with $\|\nabla v_\varepsilon\|_{L^\infty} \le C/\delta_\varepsilon$. In conclusion, w solves the variational problem

$$\int_{\Omega\setminus K_\ell} Dv : \mathbf{C}[Dw] \, dx = -\int_{\Omega\setminus K_\ell} Dv : \bar{\mathbf{C}}'[Du_\ell] \, dx \qquad \text{for every } v \in \mathcal{V}_\ell.$$

Hence, w coincides with $\bar{u}'$, the unique solution of (10). □

By the definition of $\mathcal{L}_\varepsilon$ it follows easily that $\mathcal{L}'_\varepsilon = 1$; thus the derivatives $\mathcal{F}'_\varepsilon$ converge uniformly to $\mathcal{F}'$ in $[l_0, L]$.

6. Convergence of evolutions

For $\alpha \in W^{1,1}(0, T)$ consider the sets of admissible configurations

$$\mathcal{U}_{t,\ell} = \{u \in H^1(\Omega \setminus K_\ell, \mathbb{R}^2) : u = \alpha(t) g \ \partial_D\Omega\} \, .$$

For $\{u_{t,\ell}\} = \text{argmin}\{\mathcal{E}(u) : u \in \mathcal{U}_{t,\ell}\}$ we define the reduced energy $\mathcal{E}(t,\ell) = \mathcal{E}(u_{t,\ell})$ and the energy release $G(t,\ell) = -\partial_\ell \mathcal{E}(t,\ell)$. A quasi-static evolution $\ell : [0,T] \to [l_0, L]$ is a non-decreasing function, with $\ell(0) = 0$, characterized by (1)-(3). Existence of an evolution can be proven both by incremental problems [15] and vanishing viscosity [12].

For the phase field approach, since we are choosing the "phase" $z_{\varepsilon,\ell}$ to be parametrized by ℓ we will write again the evolution in terms of crack length. Denoting by $\mathcal{E}_\varepsilon(t,\ell)$ the reduced energy and by $G_\varepsilon(t,\ell) = -\partial_\ell \mathcal{E}_\varepsilon(t,\ell)$ the energy release, the evolution $\ell_\varepsilon : [0,T] \to [l_0, L]$ will be given by a non-decreasing function, with $\ell_\varepsilon(0) = 0$, such that (1)-(3) holds for ℓ_ε.

Then we have the following Theorem.

Theorem 6.1. *The phase field evolution ℓ_ε converge, pointwise in $[0,T]$, to the sharp crack evolution ℓ.*

Proof. By linearity we can separate space and time variables to get $G(t,\ell) = \alpha^2(t)G(1,\ell)$. We already known by Proposition 5.1 that $G_\varepsilon(1,\ell)$ converge to $G(1,\ell)$ uniformly, then

$$|G_\varepsilon(t,\ell) - G(t,\ell)| = \alpha^2(t)|G_\varepsilon(1,\ell) - G(1,\ell)| \, .$$

It follows that $G_\varepsilon(t,\ell)$ converge to $G(t,\ell)$ uniformly in $[0,T] \times [\ell_0, L]$. Invoking Theorem 5.1 in [15] follows the pointwise convergence of the evolutions. □

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