Minimising convex combinations of low eigenvalues

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Abstract

We consider the variational problem

$$\inf\{\alpha\lambda_1(\Omega) + \beta\lambda_2(\Omega) + (1-\alpha-\beta)\lambda_3(\Omega) \mid \Omega \text{ open in } \mathbb{R}^n,\ |\Omega| \leq 1\},$$

for $\alpha, \beta \in [0, 1]$, $\alpha+\beta \leq 1$, where $\lambda_k(\Omega)$ is the k'th eigenvalue of the Dirichlet Laplacian acting in $L^2(\Omega)$ and $|\Omega|$ is the Lebesgue measure of Ω. We investigate for which values of α, β a minimiser is connected.

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1 Introduction

If $\Omega \subset \mathbb{R}^n$ is an open set with finite Lebesgue measure $|\Omega|$, and $-\Delta$ is the Dirichlet Laplacian acting in $L^2(\Omega)$, then the spectrum of $-\Delta$ consists of an increasing sequence of eigenvalues $\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots$ (see for example [9]). In this paper we are interested in minimising convex combinations of the first three eigenvalues among open sets of fixed Lebesgue measure. A min-max formula holds for eigenvalues (see [10, Section 1.3]),

$$\lambda_k(\Omega) = \min_{\substack{E_k \subset H_0^1(\Omega) \\ k-\text{dimensional subspace}}} \max_{u \in E_k} \frac{\|Du\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}, \quad k \in \mathbb{N}, \tag{1}$$

and the optimal functions u_k are called eigenfunctions.

The following two inequalities are important to our analysis. The Faber-Krahn inequality (Theorem 3.2.1 in [10]) asserts that

$$\lambda_1(\Omega) \geq \lambda_1(B) \left(\frac{|B|}{|\Omega|}\right)^{2/n}, \tag{2}$$

where B is the ball of unit measure in $\mathbb{R}^n$, and with equality if and only if Ω is any ball (up to sets of capacity zero). The latter follows as for an open set $\Omega \subseteq \mathbb{R}^n$ we have $|t\Omega| = t^n|\Omega|$, and

$$\lambda_k(t\Omega) = t^{-2}\lambda_k(\Omega) \quad \text{for all } k \in \mathbb{N}, \tag{3}$$

under a homothety of ratio $t > 0$. The Krahn-Szegö inequality (Theorem 4.1.1 in [10]) asserts that

$$\lambda_2(\Omega) \geq 2^{2/n}\lambda_1(B) \left(\frac{|B|}{|\Omega|}\right)^{2/n}, \tag{4}$$

with equality if and only if Ω is any disjoint union of two balls of equal measure. We will denote the union of two disjoint balls each of half measure by Θ. Note that the minimisers for $\lambda_1(\Omega)$ and $\lambda_2(\Omega)$ subject to the constraint $|\Omega| \leq 1$ are given by (2) and (4) respectively. The existence of a minimiser for $\lambda_3(\Omega)$ (in the class of quasi-open sets of fixed measure) was proved in [7], but its shape is still an open problem. Connectedness in dimension $n = 2, 3$ of any minimiser for $\lambda_3(\Omega)$ was proved by Wolf and Keller in [13], where they showed that the ball has lower third eigenvalue than any disconnected set of the same measure, and is a local minimiser in $\mathbb{R}^2$. The ball is conjectured to be the minimiser for $\lambda_3(\Omega)$ in $\mathbb{R}^2$ [10]. Connectedness of minimisers for individual eigenvalues is studied in [3].

In this paper we consider the variational problem

$$\inf\{\mathcal{F}(\Omega) := \alpha\lambda_1(\Omega) + \beta\lambda_2(\Omega) + (1 - \alpha - \beta)\lambda_3(\Omega) \mid \Omega \text{ open in } \mathbb{R}^n,\ |\Omega| \leq 1\}, \tag{5}$$

for $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. The existence of a quasi-open minimiser for (5) follows from Theorem A of the recent paper [11]. Note that by (3) the unit measure in the constraint $|\Omega| \leq 1$ is for convenience only; everything works in the same way for any other positive constant. The aim of this paper is to show that a minimiser for (5) is connected for a range of values of α, β, and to discuss the remaining cases. The result for $\mathbb{R}^2$ is the most complete. The values of $\alpha_n, \beta_n, \gamma_n$ are given below.

Theorem 1.

Any minimiser of (5) *is connected for each of the cases* *(i)* $\alpha+\beta=1$, $\alpha>0$,

(ii) $\alpha_n \le \alpha \le 1$,

(iii) $0<\beta\le\beta_n(1-\alpha)$,

(iv) $\beta=0$, $\gamma_n\le\alpha\le 1$.

Theorem 2. *Let* $n=2$.

(a) Any minimiser of (5) *is connected for each of the cases*

(i) $\alpha+\beta=1$, $\alpha>0$,

(ii) $0.350\approx\alpha_2\le\alpha\le 1$,

(iii) $0\le\beta\le\beta_2(1-\alpha)\approx 0.725(1-\alpha)$,

(b) Any disconnected minimiser of (5) *satisfies* $\lambda_1(\Omega)=\lambda_2(\Omega)$ *and has exactly two components.*

(c) If any minimiser of (5) *is connected for* $\alpha=0$ *and each* $\beta\in[0,1)$, *then any minimiser is connected unless* $\beta=1$.

We conjecture the following.

Conjecture 3. *Let* $n=2$*; a minimiser for the problem* (5) *can not be disconnected unless* $\beta=1$.

Throughout this paper let α_n satisfy

$$\alpha_n=\frac{\lambda_2(B)-2^{2/n}\lambda_1(B)}{\lambda_2(B)-\lambda_1(B)}$$

for $n=2,3,4$, and

$$\alpha_n\left[\left(\left(\frac{1-\alpha_n}{\alpha_n}\right)^{n/(n+2)}+1\right)^{2/n}-1\right]+(1-\alpha_n)\left[\left(\left(\frac{\alpha_n}{1-\alpha_n}\right)^{n/(n+2)}+1\right)^{2/n}-\frac{\lambda_2(B)}{\lambda_1(B)}\right]>0,$$

for $n\ge 5$. Let β_n satisfy

$$\beta_n\left[2^{2/(n+2)}\left(\frac{1-\beta_n}{\beta_n}\right)^{n/(n+2)}+1\right]^{2/n}$$
$$+\,2^{2/n}(1-\beta_n)\left[2^{-2/(n+2)}\left(\frac{\beta_n}{1-\beta_n}\right)^{n/(n+2)}+1\right]^{2/n}-\frac{\lambda_2(B)}{\lambda_1(B)}<0,$$

for $n=2,3$, and let $\beta_n=0$ for $n\ge 4$. Finally let $\gamma_2=\gamma_3=0$ and let γ_n for $n\ge 4$ satisfy

$$\gamma_n\left[\left(1+\left(\frac{\lambda_1(B)}{\lambda_2(B)}\right)^{n/2}\right)^{2/n}-1\right]+(1-\gamma_n)\left[3^{2/n}-\frac{\lambda_2(B)}{\lambda_1(B)}\right]>0.$$

The appoximate values for $n=2,3,4$ are:

n	α_n	β_n	γ_n
2	0.350	0.725	0
3	0.439	0.476	0
4	0.479	0	0.311

An important component in the proof of Theorem 1 is the following lemma.

Lemma 4. *Let $n \geq 2$. The disjoint union of two balls can be optimal for (5) only if $\beta = 1$.*

This paper is organised as follows. In Section 2 we prove Lemma 4 and we rule out a minimiser with three connected components in $\mathbb{R}^2$. Section 3 contains the proof of Theorem 1 and Section 4 the proof of Theorem 2. Note that Theorem 2 (a) is an immediate consequence of Theorem 1. For calculations we use that $\lambda_1(B) = \omega_n^{2/n} j_{n/2-1}^2$, and $\lambda_2(B) = \cdots = \lambda_{n+1}(B) = \omega_n^{2/n} j_{n/2}^2$, where ω_n denotes the volume of the ball of unit radius in $\mathbb{R}^n$, and j_ν is the first positive zero of the Bessel function J_ν. In $\mathbb{R}^2$ we have $\lambda_1(B) \approx 18.168$ and $\lambda_2(B) = \lambda_3(B) \approx 46.125$. In the proofs below Ω is used to denote the optimal disconnected candidate for a minimiser. Throughout the paper we will for convenience define the value $m_0 = \frac{\lambda_1(B)}{\lambda_2(B)}$, while the values of m_1, m_2 will denote, respectively, the lower and the upper bound for the measure of a connected component.

Note that for disconnected sets the eigenvalues are obtained by collecting and reordering the eigenvalues of the components (see [10, Remark 1.2.4]). We will write that the $k-$th eigenfunction is supported on a component G of Ω when $\lambda_k(\Omega) = \lambda_i(G)$, for some $i \leq k$. We say that G supports l eigenvalues if it has l eigenvalues less than or equal to the largest eigenvalue of Ω that we are minimising. Note that a minimiser for (5) must have at least one of the first three eigenvalues supported on each component.

2 Preliminaries

Proof of Lemma 4. The idea of the proof is that letting the two disjoint equal balls slightly overlap we obtain a better candidate for a minimiser of (5), because the increase in the second eigenvalue is less than the decrease in the first and the third. We divide the proof in two steps, treating first the case of two balls with equal measure, then the case of balls with different size.

Step I. Let $B(\varepsilon) = B(0,1) \cap \{x \mid x_1 < 1 - \varepsilon\}$ and $\Omega(\varepsilon) = B(0,1) \cup B(2(1-\varepsilon)e_1, 1)$, where $x = (x_1, x_2, \cdots, x_n)$ and e_1 is the unit vector in the x_1 direction. Moreover $\tilde{\Omega}(\varepsilon) = |\Omega(\varepsilon)|^{-\frac{1}{n}}\Omega(\varepsilon)$ is the set rescaled to unit measure. It follows from Theorem 1 in [2] that

$$\lambda_1(B(\varepsilon))|B(\varepsilon)|^{2/n} = \lambda_1(B)|B|^{2/n} + o\left(\varepsilon^{(n+1)/2}\right). \tag{6}$$

By monotonicity we have $\lambda_1(\Omega(\varepsilon)) < \lambda_1(B)$ and $\lambda_3(\Omega(\varepsilon)) < \lambda_2(B)$, and so taking scaling into account gives $\lambda_1(\tilde{\Omega}(\varepsilon)) < \lambda_1(\Theta) - c_1\varepsilon^{(n+1)/2}$ and $\lambda_3(\tilde{\Omega}(\varepsilon)) < \lambda_3(\Theta) - c_2\varepsilon^{(n+1)/2}$, for some positive constants c_1, c_2. By the min-max principle (1) we can obtain an upper bound by choosing the subspace E_2 spanned by the first eigenfunction of $B(\varepsilon)$ and the first eigenfunction of $B(0,1) \cap \{x \mid x_1 > 1-\varepsilon\}$. Hence $\lambda_2(\Omega(\varepsilon)) \leq \lambda_1(B(\varepsilon))$, so we can apply (6) and use the scaling (3) to obtain $\lambda_2(\tilde{\Omega}(\varepsilon)) \leq \lambda_2(\Theta) + o(\varepsilon^{(n+1)/2})$.

For $\beta < 1$ and for sufficiently small $\varepsilon > 0$, this gives

$$\begin{aligned}\alpha\lambda_1(\tilde{\Omega}(\varepsilon)) + \beta\lambda_2(\tilde{\Omega}(\varepsilon)) + (1-\alpha-\beta)\lambda_3(\tilde{\Omega}(\varepsilon))\\ < \alpha\lambda_1(\Theta) + \beta\lambda_2(\Theta) + (1-\alpha-\beta)\lambda_3(\Theta).\end{aligned}$$

Step II. Let $r_1 > r_2$ and Ω be the disjoint union of two balls with radii r_1, r_2 such that the first two eigenfunctions are supported on different components. We write $\tilde{\Omega} = |\Omega|^{-\frac{1}{n}}\Omega$ for the set rescaled to unit measure. Then, let $B_{r_1} = B(0, r_1)$, $B_{r_2} = B(r_1 + r_2 - \frac{\varepsilon}{2}(\frac{1}{r_1} + \frac{1}{r_2}), r_1)$, $\Omega(\varepsilon) = B_{r_1} \cup B_{r_2}$, $B_1(\varepsilon) = B_{r_1} \cap \{x \mid x_1 < r_1 - \frac{\varepsilon}{2r_1}\}$ and $B_2(\varepsilon) = B_{r_2} \cap \{x \mid x_1 > r_2 - \frac{\varepsilon}{2r_2}\}$. We write $\tilde{\Omega}(\varepsilon) = |\Omega(\varepsilon)|^{-\frac{1}{n}}\Omega(\varepsilon)$ for the rescaled set. By monotonicity we have $\lambda_1(\Omega(\varepsilon)) < \lambda_1(B_{r_1})$ and $\lambda_3(\Omega(\varepsilon)) < \lambda_2(B_{r_1})$, and again taking account of the scaling we have $\lambda_1(\tilde{\Omega}(\varepsilon)) < \lambda_1(\tilde{\Omega}) - c_1\varepsilon^{(n+1)/2}$ and $\lambda_3(\tilde{\Omega}(\varepsilon)) < \lambda_3(\tilde{\Omega}) - c_2\varepsilon^{(n+1)/2}$, for some positive constants c_1, c_2.

By the min-max principle (1), we can obtain an upper bound by choosing E_2 spanned by the first eigenfunction of B_{r_2} and the first eigenfunction of $\Omega(\varepsilon) \setminus B_{r_2}$, and for ε small enough we have $\lambda_1(B_{r_2}) \le \lambda_1(\Omega(\varepsilon) \setminus B_{r_2})$. Hence $\lambda_2(\Omega(\varepsilon)) \le \lambda_1(B_{r_2})$, and taking account of the scaling $\lambda_2(\tilde{\Omega}(\varepsilon)) \le \lambda_2(\tilde{\Omega}) - c_3\varepsilon^{(n+1)/2}$, for some positive c_3. In conclusion, for $\beta < 1$ and ε small enough, $\tilde{\Omega}(\varepsilon)$ is a better candidate than $\tilde{\Omega}$ for problem (5).

□

The following remark will be useful in the proof of Lemma 6 and in Section 4.

Remark 5. *Let $n = 2$. A disconnected set Ω can never be optimal for (5) if $\lambda_2(\Omega) \ge \lambda_2(B)$. Here the disk is better, since $\lambda_1(B) < \lambda_1(\Omega)$ by the Faber-Krahn inequality and $\lambda_3(B) < \lambda_3(\Omega)$ by [10, Corollary 5.2.2].*

Lemma 6. *Let $n = 2$. Any disconnected minimiser of* (5) *has exactly two components.*

Proof. For the case $\alpha+\beta = 1$ it is clear that a minimiser has at most two components. For $\alpha+\beta < 1$ the Faber-Krahn inequality implies that a disconnected minimiser with three components would be the union of three disjoint balls. If $\alpha > 0$, it is possible to apply Lemma 4 to the union of the balls supporting the second and the third eigenvalues. For $\alpha = 0$ this argument does not work, since neither λ_2 nor λ_3 are lowered. Hence we rule out the configuration with three connected components only for $n = 2$, by comparing it with B and Θ.

Let $\mathcal{G}(\cdot) = \beta\lambda_2(\cdot) + (1-\beta)\lambda_3(\cdot)$, and write Ω_i, $i = 1, 2, 3$, for the three components of Ω. Assuming $\lambda_i(\Omega) = \lambda_1(\Omega_i)$ for $i = 1, 2, 3$ gives $|\Omega_1| \ge |\Omega_2| \ge |\Omega_3|$. We write $m = |\Omega_1|$ and note that $|\Omega_2| = m$, as for $|\Omega_1| > |\Omega_2|$ we could enlarge Ω_2 and shrink Ω_1, lowering the functional. Thus $|\Omega_3| = 1 - 2m$, and the following constraints on m hold:

1) Remark 5 implies $\lambda_2(B) > \lambda_2(\Omega) = \lambda_1(\Omega_2) = \frac{\lambda_1(B)}{m}$, so $m > \frac{\lambda_1(B)}{\lambda_2(B)} = m_1 \approx 0.394$.

2) We must have $\frac{\lambda_2(B)}{m} = \lambda_2(\Omega_1) \ge \lambda_1(\Omega_3) = \frac{\lambda_1(B)}{1-2m}$, as otherwise we can reduce to only two components. This inequality implies

$$m \le \frac{\lambda_2(B)}{\lambda_1(B) + 2\lambda_2(B)} = m_2 \approx 0.418.$$

Coming back to the study of $\mathcal{G}$, we can use the scaling properties of eigenvalues and the bounds above to obtain

$$\mathcal{G}(\Omega) = \beta\lambda_2(\Omega) + (1-\beta)\lambda_3(\Omega) = \left\{\frac{\beta}{m} + \frac{(1-\beta)}{(1-2m)}\right\}\lambda_1(B) \ge \left\{\frac{\beta}{m_2} + \frac{(1-\beta)}{(1-2m_1)}\right\}\lambda_1(B). \tag{7}$$

Now we look for those β for which the unit ball B gives a lower value of $\mathcal{G}$ than this lower bound. In particular we are looking for those β that satisfy

$$\mathcal{G}(B) - \mathcal{G}(\Omega) \le \lambda_2(B) - \left\{\frac{\beta}{m_2} + \frac{(1-\beta)}{(1-2m_1)}\right\}\lambda_1(B) < 0,$$

i.e.

$$\beta < \frac{\frac{1}{(1-2m_1)} - \frac{1}{m_1}}{\left\{\frac{1}{(1-2m_1)} - \frac{1}{m_2}\right\}} \approx 0.936.$$

For this range of β three balls can not be optimal when minimising $\mathcal{G}$.
The remaining β are ruled out by comparing Ω with Θ. Using (7), three connected components can not be optimal when

$$\mathcal{G}(\Theta) - \mathcal{G}(\Omega) \leq 2\beta\lambda_1(B) + 2(1-\beta)\lambda_2(B) - \left\{ \frac{\beta}{m_2} + \frac{(1-\beta)}{(1-2m_1)} \right\} \lambda_1(B) < 0,$$

i.e. when

$$\beta > \frac{2\lambda_2(B) - \frac{\lambda_1(B)}{1-2m_1}}{2\lambda_2(B) - \left(2 + \frac{1}{1-2m_1} - \frac{1}{m_2}\right)\lambda_1(B)} \approx 0.479.$$

Since the two ranges we obtained on β cover all cases, a minimiser for (5) can never have three components in $\mathbb{R}^2$. □

3 Proof of Theorem 1.

3.1 Proof of Theorem 1 (i) and (ii).

Proof of Theorem 1 (i) and (ii). We first consider the situation $\alpha + \beta = 1$. Note that this result for $\mathbb{R}^2$ is also discussed in [12, Chapter 2], but the details of a proof are not given. A disconnected minimiser Ω must by the Faber-Krahn inequality be the union of two disjoint balls with measures m and $1-m$ respectively. Hence an immediate application of Lemma 4 rules out this configuration in any dimension when $\beta < 1$, and (i) is proved.

To prove (ii) we need a different argument, but start again from the case $\alpha + \beta = 1$, and note that

$$\alpha\lambda_1(\Omega) + (1-\alpha)\lambda_2(\Omega) = \lambda_1(B)\left(\frac{\alpha}{m^{2/n}} + \frac{1-\alpha}{(1-m)^{2/n}}\right).$$

For $\alpha \in (\frac{1}{2}, 1]$, differentiating this with respect to m to obtain a lower bound and comparing with the value for the unit ball rules out this configuration if

$$\begin{aligned} \alpha\lambda_1(B)\left[\left(\frac{1-\alpha}{\alpha}\right)^{n/(n+2)} + 1\right]^{2/n} &+ (1-\alpha)\lambda_1(B)\left[\left(\frac{\alpha}{1-\alpha}\right)^{n/(n+2)} + 1\right]^{2/n} \\ &> \alpha\lambda_1(B) + (1-\alpha)\lambda_2(B). \end{aligned} \tag{8}$$

For $\alpha \in (0, \frac{1}{2}]$ the constraint $m \geq 1/2$ implies that the optimal disconnected configuration is two disjoint balls of equal measure. This is ruled out by comparison with the unit ball when

$$\lambda_1(B)2^{2/n} > \alpha\lambda_1(B) + (1-\alpha)\lambda_2(B). \tag{9}$$

Combining (9) with (8) we have that the ball is better than any disconnected set for $1 \geq \alpha \geq \alpha_n$.

Finally to extend beyond the situation $\alpha + \beta = 1$, just note that for $1 \geq \alpha \geq \alpha_n$

$$\begin{aligned} \alpha\lambda_1(\Omega) + \beta\lambda_2(\Omega) + (1-\alpha-\beta)\lambda_3(\Omega) &\geq \alpha\lambda_1(\Omega) + (1-\alpha)\lambda_2(\Omega) \\ &> \alpha\lambda_1(B) + (1-\alpha)\lambda_2(B) = \alpha\lambda_1(B) + \beta\lambda_2(B) + (1-\alpha-\beta)\lambda_3(B), \end{aligned}$$

using the fact that $\lambda_2(\Omega) \leq \lambda_3(\Omega)$ while $\lambda_2(B) = \lambda_3(B)$. This concludes (ii). □

3.2 Proof of Theorem 1 (iii) and (iv).

Proof of Theorem 1 (iii) and (iv). The case $\alpha = 0$: We first consider the case $\alpha = 0$. Note that it is known that the disk is a critical point for $\alpha = 0, \beta \leq 1/2$. (See [10] p. 93.) Note also that a disconnected minimiser for $\alpha = 0$ would consist of a ball supporting the second eigenvalue, and another set supporting the first and third eigenvalues. This is because a configuration with a ball supporting the third eigenvalue and a set supporting the others would be three balls by the Krahn-Szegö inequality. An optimal configuration with a ball supporting the first eigenvalue would satisfy $\lambda_1(\Omega) = \lambda_2(\Omega)$, because the quantity we are minimising for $\alpha = 0$ does not involve $\lambda_1(\Omega)$. Both are of the form of a ball supporting $\lambda_2(\Omega)$ and a set supporting $\lambda_1(\Omega)$ and $\lambda_3(\Omega)$, and so it remains only to rule out this configuration.

For $\beta \in [0, 1)$, the Krahn-Szegö inequality gives the lower bound

$$\beta\lambda_2(\Omega) + (1-\beta)\lambda_3(\Omega) \geq \lambda_1(B)\left(\frac{\beta}{m^{2/n}} + \frac{2^{2/n}(1-\beta)}{(1-m)^{2/n}}\right). \tag{10}$$

For $\beta \in [0, 1/3]$ we set $m = 1/3$, as we hit the constraint $m \geq 1/3$. Substituting into (10) and comparing with $\beta\lambda_2(B) + (1-\beta)\lambda_3(B) = \lambda_2(B)$ gives that any minimiser in this range is connected for $n = 2, 3$. For $\beta \in (1/3, 1)$ optimising (10) with respect to m gives

$$\frac{1}{m} = 2^{2/(n+2)}\left(\frac{1-\beta}{\beta}\right)^{n/(n+2)} + 1, \qquad \frac{1}{1-m} = 2^{-2/(n+2)}\left(\frac{\beta}{1-\beta}\right)^{n/(n+2)} + 1.$$

Again comparing with $\lambda_2(B)$ gives that any minimiser is connected for $1/3 < \beta \leq \beta_n$. This only gives information for $n = 2, 3$, and we conclude connectedness for:

$n = 2: \quad \beta \in [0, 0.725),$

$n = 3: \quad \beta \in [0, 0.476).$

The case $\beta = 0$: We now consider the case $\beta = 0$. To prove connectedness we first see that a disconnected minimiser would consist of a ball supporting the third eigenvalue, and another set supporting the first and second eigenvalues. This is because a configuration with a disk supporting the first eigenvalue and a set supporting the others would be three balls using that the minimiser for the second eigenvalue is the union of two disjoint balls. An optimal configuration with a ball supporting the second eigenvalue would have $\lambda_2(\Omega) = \lambda_3(\Omega)$, as scaling down the ball to obtain this does not effect $\lambda_1(\Omega), \lambda_3(\Omega)$. Both these possible configurations are just special cases of a ball supporting the $\lambda_3(\Omega)$ and a set supporting $\lambda_1(\Omega), \lambda_2(\Omega)$, and so it only remains to rule out this possibility. This is done by obtaining lower bounds for the first and third eigenvalues and using comparison with a ball. Letting the third eigenvalue be supported on a ball of measure m, we have $\lambda_3(\Omega) = \frac{\lambda_1(B)}{m^{2/n}}$, while the Faber-Krahn and the Krahn-Szegö inequalities respectively give

$$\lambda_1(\Omega) \geq \frac{\lambda_1(B)}{(1-m)^{2/n}} \quad \text{and} \quad \frac{\lambda_1(B)}{m^{2/n}} = \lambda_3(\Omega) \geq \lambda_2(\Omega) \geq 2^{2/n}\frac{\lambda_1(B)}{(1-m)^{2/n}},$$

which implies $\frac{1}{m} \geq 3$, and so $m \leq \frac{1}{3}$. With explicit computations it is then easy to see that for $n = 2, 3$

$$\lambda_3(B) = \lambda_2(B) \leq \lambda_3(\Omega).$$

By the Faber-Krahn inequality, the ball strictly lowers the first eigenvalue, so we rule out this configuration for all $\alpha \in [0, 1]$ when $n = 2, 3$. This proves Theorem 1 (iv) for $n = 2, 3$.

For $n \geq 4$ we must be more precise and obtain only partial estimates. If $\lambda_3(\Omega) = \frac{\lambda_1(B)}{m^{2/n}} \geq \frac{\lambda_2(B)}{(1-m)^{2/n}}$, then as we assume $\beta = 0$, the set supporting the first two eigenvalues should be a ball B_1. This would contradict the optimality of Ω, as we would have $\lambda_3(B_1) = \lambda_2(B_1) \leq \lambda_3(\Omega)$ and $|B_1| < 1$. So we conclude $\frac{\lambda_1(B)}{m^{2/n}} < \frac{\lambda_2(B)}{(1-m)^{2/n}}$, and so $m > \frac{m_0^{n/2}}{1+m_0^{n/2}}$, which gives the bound

$$\frac{1}{1-m} > 1 + m_0^{n/2}.$$

This gives the lower bound

$$\alpha\lambda_1(\Omega) + (1-\alpha)\lambda_3(\Omega) \geq \alpha\lambda_1(B)\left(1 + m_0^{n/2}\right)^{2/n} + (1-\alpha)\lambda_1(B)3^{2/n}. \tag{11}$$

We then have connectedness for example in the following cases:

$n = 2: \quad \alpha \in [0, 1],$

$n = 3: \quad \alpha \in [0, 1],$

$n = 4: \quad \alpha \in (0, 1],$

$n = 5: \quad \alpha \in [0.467, 1],$

$n = 6: \quad \alpha \in [0.547, 1].$

The case $0 < \beta \leq \beta_n(1-\alpha)$*:* Now finally consider the case $0 < \beta \leq \beta_n(1-\alpha)$. Recall from the above that

$$\alpha\lambda_2(\Omega) + (1-\alpha)\lambda_3(\Omega) > \alpha\lambda_2(B) + (1-\alpha)\lambda_3(B),$$

for $\alpha \in [0, \beta_n)$. This implies

$$\frac{\beta}{1-\alpha}\lambda_2(\Omega) + \frac{(1-\alpha-\beta)}{1-\alpha}\lambda_3(\Omega) > \frac{\beta}{1-\alpha}\lambda_2(B) + \frac{(1-\alpha-\beta)}{1-\alpha}\lambda_3(B),$$

for $\frac{\beta}{1-\alpha} \in [0, \beta_n)$, and so

$$\beta\lambda_2(\Omega) + (1-\alpha-\beta)\lambda_3(\Omega) > \beta\lambda_2(B) + (1-\alpha-\beta)\lambda_3(B),$$

for $\beta \in [0, \beta_n(1-\alpha))$. Together with $\lambda_1(\Omega) \geq \lambda_1(B)$ this gives the result. $\square$

4 Proof of Theorem 2 (b) and (c)

We begin with a lemma, which asserts that a disconnected minimiser must have multiple eigenvalues. The idea of the proof is that if every eigenvalue is simple, then small variations of the connected components (in the sense of shrinking one and enlarging the other) contradict the optimality of such a disconnected set. For simplicity we will often write $\lambda_i = \lambda_i(\Omega)$ and $\gamma = 1-\alpha-\beta$, and as before define $m_0 = \frac{\lambda_1(B)}{\lambda_2(B)} \approx 0.394$.

Lemma 7. *A disconnected minimiser* Ω *for (5) in* $\mathbb{R}^2$ *can not have both* $\lambda_1(\Omega) \neq \lambda_2(\Omega)$ *and* $\lambda_2(\Omega) \neq \lambda_3(\Omega)$*.*

Proof. Note that we only need to consider the cases for problem (5) that are not treated in Theorem 2 (a). Additionally, the case of three components is ruled out by Lemma 6. The proof of the remaining cases is divided into three steps.

Step I. We consider the case of a set $\Omega = \Omega_1 \cup \Omega_2$, with Ω_1 supporting u_1, while Ω_2 supports u_2 and u_3. From the hypoteses of the Step, $\lambda_1 = \lambda_1(\Omega_1)$, $\lambda_2 = \lambda_1(\Omega_2)$ and $\lambda_3 = \lambda_2(\Omega_2)$, and by Faber-Krahn Ω_1 is a ball. We define $m = |\Omega_1|$, so $1 - m = |\Omega_2|$. The following constraints on m hold:

1) $m > \frac{\lambda_1(B)}{\lambda_2(B)} = m_1 \approx 0.394$, since $\frac{\lambda_1(B)}{m} = \lambda_1(\Omega_1) \leq \lambda_1(\Omega_2) = \lambda_2 < \lambda_2(B)$ (see Remark 5).

2) $\frac{\lambda_2(B)}{m} = \lambda_2(\Omega_1) \geq \lambda_2(\Omega_2) > \frac{\lambda_2(\Theta)}{(1-m)} = \frac{2\lambda_1(B)}{(1-m)}$, so $m < \frac{\lambda_2(B)}{2\lambda_1(B)+\lambda_2(B)} = m_2 \approx 0.559$.

Now we can shrink Ω_1 and enlarge Ω_2, in order to obtain two new sets of the same shape $\widetilde{\Omega}_1$, $\widetilde{\Omega}_2$, such that $|\widetilde{\Omega}_1| = m - \varepsilon$, while $|\widetilde{\Omega}_2| = 1 - m + \varepsilon$. Writing $\widetilde{\lambda}_i$ for the eigenvalues of $\widetilde{\Omega}_1 \cup \widetilde{\Omega}_2$ we obtain the following ratios (for $\varepsilon \ll 1$):

$$\frac{\widetilde{\lambda}_1}{\lambda_1} = \frac{m}{m-\varepsilon} \approx 1 + \frac{\varepsilon}{m}; \qquad \frac{\widetilde{\lambda}_2}{\lambda_2} = \frac{\widetilde{\lambda}_3}{\lambda_3} = \frac{(1-m)}{1-m+\varepsilon} \approx 1 - \frac{\varepsilon}{1-m}.$$

The optimality of Ω implies $\mathcal{F}(\Omega) \leq \mathcal{F}(\widetilde{\Omega}_1 \cup \widetilde{\Omega}_2)$, that means

$$\left(\frac{\alpha\lambda_1}{m} - \frac{\beta\lambda_2 + \gamma\lambda_3}{1-m}\right)\varepsilon + o(\varepsilon) \geq 0.$$

Taking either $\varepsilon > 0$ or $\varepsilon < 0$ (this is possible since we are supposing that the eigenvalues are simple) gives that the expression in the brackets must be zero, hence $\frac{\alpha\lambda_1}{m} = \frac{\beta\lambda_2+\gamma\lambda_3}{1-m}$.

In order to conclude this first step it suffices to consider $\varepsilon > 0$. Since $\widetilde{\lambda}_3 \leq \lambda_3$, we have a contradiction if $\mathcal{F}(\widetilde{\Omega}_1 \cup \widetilde{\Omega}_2) < \mathcal{F}(\Omega)$, i.e. when

$$\alpha\widetilde{\lambda}_1 + \beta\widetilde{\lambda}_2 < \alpha\lambda_1 + \beta\lambda_2. \tag{12}$$

Equation (12) holds if and only if

$$\beta\lambda_2\left(1 - \frac{\widetilde{\lambda}_2}{\lambda_2}\right) > \alpha\lambda_1\left(\frac{\widetilde{\lambda}_1}{\lambda_1} - 1\right) \qquad \Longleftrightarrow \qquad \beta > \alpha\left(\frac{\lambda_1}{\lambda_2}\right)\left(\frac{1-m}{m}\right).$$

Using $\frac{\lambda_1}{\lambda_2} \leq \frac{\lambda_1(B)}{m}\frac{1-m}{\lambda_2(\Theta)} = \frac{1-m}{2m}$ and the above constraints on m gives $\frac{1-m}{m} \leq \frac{1-m_1}{m_1}$. So if $\beta > \frac{\alpha}{2}\left(\frac{1-m_1}{m_1}\right)^2 \approx 1.18\alpha$, the set $\Omega = \Omega_1 \cup \Omega_2$ can not be optimal. The case $\beta \leq 1.18\alpha$ was treated in Theorem 1 (a), and so this concludes Step I.

Step II. We now consider the case of a set $\Omega = \Omega_1 \cup \Omega_2$, with Ω_1 supporting u_1 and u_3, while Ω_2 supports u_2. Clearly $\lambda_1 = \lambda_1(\Omega_1)$, $\lambda_2 = \lambda_1(\Omega_2)$ and $\lambda_3 = \lambda_2(\Omega_1)$, and again it is better to take Ω_2 to be a ball. Write $m = |\Omega_1|$ and $1 - m = |\Omega_2|$. The following constraints on m hold:

1) $\frac{\lambda_1(B)}{m} < \lambda_1(\Omega_1) \leq \lambda_1(\Omega_2) = \frac{\lambda_1(B)}{(1-m)}$, so $m > 1/2 = m_1$.

2) $\frac{\lambda_1(B)}{(1-m)} = \lambda_2(\Omega) < \lambda_2(B)$ by Remark 5, so $m < \frac{\lambda_2(B)-\lambda_1(B)}{\lambda_2(B)} = 1 - m_0 = m_2 \approx 0.606$.

As in the previous case we shrink Ω_1 to $\widetilde{\Omega}_1$ and we enlarge Ω_2 to $\widetilde{\Omega}_2$, so that $|\widetilde{\Omega}_1| = m - \varepsilon$, while $|\widetilde{\Omega}_2| = 1 - m + \varepsilon$. With the same arguments of the previous Step, if Ω is optimal then $\mathcal{F}(\widetilde{\Omega}_1 \cup \widetilde{\Omega}_2) \geq \mathcal{F}(\Omega)$ and so

$$\left(\frac{\alpha\lambda_1 + \gamma\lambda_3}{m} - \frac{\beta\lambda_2}{1-m}\right)\varepsilon + o(\varepsilon) \geq 0.$$

Taking again either $\varepsilon > 0$ or $\varepsilon < 0$ gives $\frac{\beta\lambda_2}{1-m} = \frac{\alpha\lambda_1+\gamma\lambda_3}{m}$. Now, since Ω_2 is a ball and thanks to the bounds on m, we can rewrite the complete functional in a more interesting way

$$\mathcal{F}(\Omega) = \alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3 = \frac{m}{1-m}\beta\lambda_2 + \beta\lambda_2 = \frac{\beta\lambda_2}{1-m} = \frac{\beta\lambda_1(B)}{(1-m)^2} \geq \frac{\beta\lambda_1(B)}{(1-m_1)^2} \geq 4\beta\lambda_1(B).$$

Comparing this lower bound with the case of the ball gives a contradiction for β such that

$$\mathcal{F}(B) - \mathcal{F}(\Omega) \leq \alpha\lambda_1(B) + (1-\alpha)\lambda_2(B) - 4\beta\lambda_1(B) < 0,$$

i.e. for

$$\beta > \frac{1}{4m_0} - \frac{\alpha}{4}\left(\frac{1}{m_0} - 1\right) \approx 0.635 - 0.385\alpha. \tag{13}$$

In order to consider the cases that are not treated in Theorem 2 (a), we look at the equations $\alpha = \alpha_2$, $\beta = \frac{1}{4m_0} - \frac{\alpha}{4}\left(\frac{1}{m_0} - 1\right)$, and $\beta = \beta_2(1-\alpha)$. The remaining cases can be viewed as inside a small triangle in the α-β plane, with vertices approximately given by $A = (0.275; 0.529)$, $B = (0.350; 0.500)$ and $C = (0.350; 0.474)$ (See Figure 1).

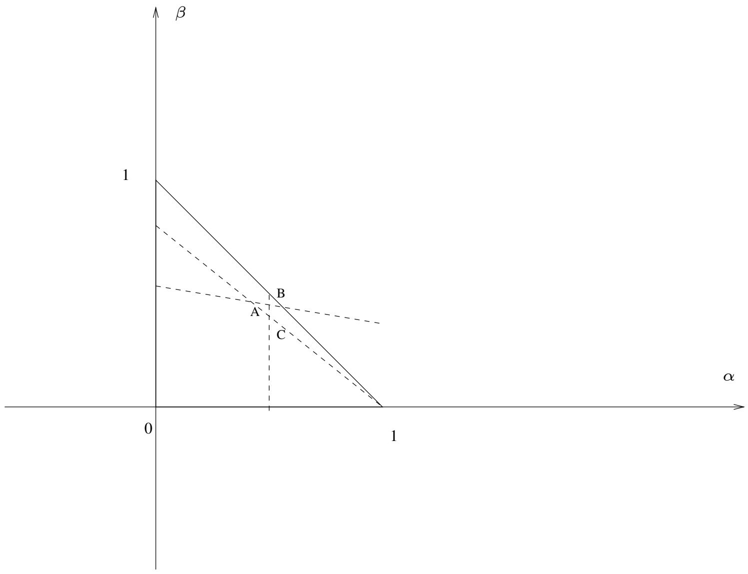

Figure 1: The small triangle in the α-β plane

For these remaining points, it is possible to show that a ball is better than Ω, i.e. that $\mathcal{F}(B) - \mathcal{F}(\Omega) < 0$. In fact, the following relations hold between the eigenvalues of the ball and those of Ω (using $m \in (m_1, m_2)$, the Faber-Krahn and the Krahn-Szegö inequalities):

$$\begin{aligned}
\lambda_1(B) - \lambda_1 &\leq \left(1 - \frac{1}{m}\right)\lambda_1(B) \leq \left(1 - \frac{1}{m_2}\right)\lambda_1(B), \\
\lambda_2(B) - \lambda_2 &= \lambda_2(B) - \frac{\lambda_1(B)}{1-m} \leq \lambda_2(B) - 2\lambda_1(B), \\
\lambda_3(B) - \lambda_3 &\leq \lambda_2(B) - \frac{2\lambda_1(B)}{m} \leq \lambda_2(B) - \frac{2\lambda_1(B)}{m_2}.
\end{aligned} \tag{14}$$

From (14) we get

$$\mathcal{F}(B) - \mathcal{F}(\Omega) \leq \alpha\lambda_1(B)\left(1 - \frac{1}{m_2}\right) + \beta\left(\lambda_2(B) - 2\lambda_1(B)\right) + (1-\alpha-\beta)\left(\lambda_2(B) - \frac{2\lambda_1(B)}{m_2}\right).$$

Hence the ball is better than Ω if

$$\beta < \frac{2\lambda_1(B) - m_2\lambda_2(B)}{2\lambda_1(B) - 2m_2\lambda_1(B)} + \alpha\frac{(-\lambda_1(B) + m_2(\lambda_2(B) - \lambda_1(B)))}{2\lambda_1(B) - 2m_2\lambda_1(B)} \approx 0.58 - 0.08\alpha.$$

This inequality together with (13) concludes Step II.

Step III. We now consider the case of a set $\Omega = \Omega_1 \cup \Omega_2$, with Ω_1 supporting u_1 and u_2, while Ω_2 supports u_3. Clearly $\lambda_1 = \lambda_1(\Omega_1)$, $\lambda_2 = \lambda_2(\Omega_1)$ and $\lambda_3 = \lambda_1(\Omega_2)$, and it is better to take Ω_2 to be a ball. Let $m = |\Omega_1|$ and $1-m = |\Omega_2|$. Note that if $\lambda_3(\Omega) = \lambda_1(\Omega_2) \geq m\lambda_3(\Omega_1)$, then Ω can not be optimal. In fact in this case it is better to take the connected set obtained by enlarging Ω_1 till unit measure, since this lowers both λ_1 and λ_2 (by monotonicity), while also the third eigenvalue is lower, by hypothesis. The following constraints on m hold:

1) $\lambda_2(B) > \lambda_2(\Omega) = \lambda_2(\Omega_1) \geq \frac{2\lambda_1(B)}{m}$ (see Remark 5), so $m > 2m_0 = m_1 \approx 0.788$.
2) In order to have Ω optimal, an upper bound on m follows from inequality $m\lambda_3(\Omega_1) > \lambda_1(\Omega_2)$ explained above and from the fact that $\lambda_2(\Omega) < \lambda_2(B)$ (see Remark 5). Using also the Ashbaugh-Benguria Theorem (see [1]) gives

$$\frac{\lambda_1(B)}{(1-m)} = \lambda_1(\Omega_2) < m\lambda_3(\Omega_1) \leq m\frac{\lambda_2(B)}{\lambda_1(B)}\lambda_2(\Omega_1) < m\frac{\lambda_2(B)^2}{\lambda_1(B)}.$$

This means $m^2 - m + m_0^2 < 0$, which gives the upper bound

$$m < \frac{\lambda_2(B) + \sqrt{\lambda_2(B)^2 - 4\lambda_1(B)^2}}{2\lambda_2(B)} = m_2 \approx 0.808.$$

As in the previous steps we can enlarge Ω_1 to $\widetilde{\Omega}_1$ and we can shrink Ω_2 to $\widetilde{\Omega}_2$, in order that $|\widetilde{\Omega}_1| = m + \varepsilon$, while $|\widetilde{\Omega}_2| = 1 - m - \varepsilon$. The following ratios between the eigenvalues of $\widetilde{\Omega}_1 \cup \widetilde{\Omega}_2$ and those of Ω hold (for $\varepsilon \ll 1$) :

$$\frac{\widetilde{\lambda}_1}{\lambda_1} = \frac{\widetilde{\lambda}_2}{\lambda_2} = \frac{m}{m+\varepsilon} \approx 1 - \frac{\varepsilon}{m}; \qquad \frac{\widetilde{\lambda}_3}{\lambda_3} = \frac{1-m}{1-m-\varepsilon} \approx 1 + \frac{\varepsilon}{1-m}.$$

In order to be optimal, Ω must satisfy

$$\alpha\widetilde{\lambda}_1 + \beta\widetilde{\lambda}_2 + (1-\alpha-\beta)\widetilde{\lambda}_3 \geq \alpha\lambda_1 + \beta\lambda_2 + (1-\alpha-\beta)\lambda_3.$$

An analogous argument to that in Step I and Step II gives a contradiction for $\beta \gtrsim 0.914 - 0.948\alpha$. Actually we can obtain a better result observing that Ω is worse than Ω_1 enlarged to unit measure (which we will call $\overline{\Omega}$ in the following) if β is suitably large. We denote by $\{\overline{\lambda}_i\}$ the eigenvalues of $\overline{\Omega}$ and we again write $\gamma = 1 - \alpha - \beta$ for the sake of simplicity. The following relations between the eigenvalues hold: $\overline{\lambda}_1 = m\lambda_1$, $\overline{\lambda}_2 = m\lambda_2$ and $\overline{\lambda}_3 = m\lambda_3(\Omega_1) \leq \frac{m}{m_0}\lambda_2$, using the Ashbaugh-Benguria Theorem. This gives

$$\mathcal{F}(\overline{\Omega}) = \alpha\overline{\lambda}_1 + \beta\overline{\lambda}_2 + \gamma\overline{\lambda}_3 \leq \mathcal{F}(\Omega) + \alpha(m-1)\lambda_1 + \beta(m-1)\lambda_2 + \gamma\left(\frac{m}{m_1} - 1\right)\lambda_2.$$

Clearly Ω can not be optimal when $\mathcal{F}(\overline{\Omega}) - \mathcal{F}(\Omega) < 0$, which holds if

$$\alpha(m-1)\lambda_1 + \left[\beta(m-1) + \gamma\left(\frac{m}{m_0} - 1\right)\right]\lambda_2 < 0.$$

Again using the Ashbaugh-Benguria Theorem gives that the result follows if

$$\left[\alpha(m-1)m_0 + \beta(m-1) + \gamma\left(\frac{m}{m_0} - 1\right)\right]\lambda_2 < 0.$$

Since $m \in (m_1, m_2)$ and the function in brackets is clearly increasing in m, Ω can not be optimal when

$$\frac{m_2}{m_0} - 1 + \alpha\left((m_2-1)m_0 + 1 - \frac{m_2}{m_0}\right) + \beta\left(m_2 - \frac{m_2}{m_0}\right) < 0,$$

i.e. for

$$\beta > \frac{\left(\frac{m_2}{m_0} - 1\right) + \alpha\left((m_2-1)m_0 + 1 - \frac{m_2}{m_0}\right)}{\frac{m_2}{m_0} - m_2} \approx 0.845 - 0.906\alpha. \tag{15}$$

In conclusion we have an estimate that tells us that when β is suitably big, then $\Omega = \Omega_1 \cup \Omega_2$ can not be optimal. Writing $\gamma = 1 - \alpha - \beta$, we now finally show that Ω can not be optimal also when γ is not very small. We use a technique very similar to the case β big. For this suppose Ω is optimal for the problem (5) and let $|\varepsilon| \ll 1$. Then if we enlarge Ω_1 to $\widetilde{\Omega}_1$ with measure $m+\varepsilon$ and we shrink Ω_2 to $\widetilde{\Omega}_2$ with measure $1-m-\varepsilon$, calling λ_i the eigenvalues of $\Omega = \Omega_1 \cup \Omega_2$, while $\widetilde{\lambda}_i$ are the eigenvalues of $\widetilde{\Omega} = \widetilde{\Omega}_1 \cup \widetilde{\Omega}_2$, we must have $\mathcal{F}(\Omega) - \mathcal{F}(\widetilde{\Omega}) \leq 0$. On the other hand, with analogous computations to those in Step II,

$$\mathcal{F}(\Omega) - \mathcal{F}(\widetilde{\Omega}) = \left(\frac{\gamma\lambda_3}{1-m} - \frac{\alpha\lambda_1 + \beta\lambda_2}{m}\right)\varepsilon + o(\varepsilon), \tag{16}$$

and hence the expression in brackets must be zero, as otherwise taking $\varepsilon > 0$ or $\varepsilon < 0$ (this is possible since we are treating only the case of simple eigenvalues) contradicts the optimality of Ω. So if Ω is optimal then $\alpha\lambda_1 + \beta\lambda_2 = \gamma\lambda_3\frac{m}{1-m}$. Since $m \mapsto \frac{1}{(1-m)^2}$ is increasing we have the lower bound

$$\mathcal{F}(\Omega, \alpha, \beta) = \gamma\lambda_3\frac{m}{1-m} + \gamma\lambda_3 \geq \gamma\lambda_1(B)\frac{1}{(1-2m_1)^2}.$$

We can show that, for γ suitably big, comparing the functional for Θ with the lower bound above gives an absurd. In fact, the functional for the two balls is given by

$$\mathcal{F}(\Theta, \alpha, \beta) = (\alpha+\beta)2\lambda_1(B) + \gamma 2\lambda_2(B) = 2\lambda_1(B) + \gamma(2\lambda_2(B) - 2\lambda_1(B)).$$

Hence $\mathcal{F}(\Theta, \alpha, \beta) < \mathcal{F}(\Omega, \alpha, \beta)$ for $\gamma > \overline{\gamma} \approx 0.104$, in which case two balls are better than our set Ω. Combining the cases in which either $\gamma > \overline{\gamma}$ or (15) holds concludes Step III and hence the proof of the lemma. □

4.1 Proof of Theorem 2 (b).

Proof of Theorem 2 (b). It is proved in Lemma 7 that any disconnected minimiser has multiple eigenvalues. Let $\tilde{\Omega}$ be a connected minimiser for $\inf\{\alpha\lambda_1(\Omega) + (1-\alpha)\lambda_3(\Omega) \mid \Omega \text{ open in } \mathbb{R}^2,\ |\Omega| \leq 1\}$. The case $\lambda_2(\Omega) = \lambda_3(\Omega)$ is then ruled out, as it would give

$$\begin{aligned}\alpha\lambda_1(\Omega) + \beta\lambda_2(\Omega) + (1-\alpha-\beta)\lambda_3(\Omega) &= \alpha\lambda_1(\Omega) + (1-\alpha)\lambda_3(\Omega) > \alpha\lambda_1(\tilde{\Omega}) + (1-\alpha)\lambda_3(\tilde{\Omega}),\\ &\geq \alpha\lambda_1(\tilde{\Omega}) + \beta\lambda_2(\tilde{\Omega}) + (1-\alpha-\beta)\lambda_3(\tilde{\Omega}).\end{aligned}$$

Therefore any disconnected minimiser must satisfy $\lambda_1(\Omega) = \lambda_2(\Omega)$ and can be viewed as the union of a disk supporting the first eigenvalue with a set supporting the second and third.

With $\lambda_1(\Omega) = \lambda_2(\Omega)$, a minimiser with three components would be the union of three disks with at least two being of equal measure. Lemma 4 rules out this configuration. □

4.2 Proof of Theorem 2 (c).

Proof of Theorem 2 (c). Let $\alpha+\beta < 1$, and let $\tilde{\Omega}$ be a connected minimiser for $\inf\{(\alpha+\beta)\lambda_2(\Omega) + (1-\alpha-\beta)\lambda_3(\Omega) \mid \Omega \text{ open in } \mathbb{R}^2,\ |\Omega| \leq 1\}$. Theorem 2 (b) then gives $\lambda_1(\Omega) = \lambda_2(\Omega)$, whereby

$$\begin{aligned}\alpha\lambda_1(\Omega) + \beta\lambda_2(\Omega) + (1-\alpha-\beta)\lambda_3(\Omega) &= (\alpha+\beta)\lambda_2(\Omega) + (1-\alpha-\beta)\lambda_3(\Omega)\\ &> (\alpha+\beta)\lambda_2(\tilde{\Omega}) + (1-\alpha-\beta)\lambda_3(\tilde{\Omega}),\\ &\geq \alpha\lambda_1(\tilde{\Omega}) + \beta\lambda_2(\tilde{\Omega}) + (1-\alpha-\beta)\lambda_3(\tilde{\Omega}).\end{aligned}$$

□

5 Appendix

This appendix is devoted to a different proof of the connectedness of the minimiser for the problem

$$\inf\left\{\alpha\lambda_1(\Omega) + (1-\alpha)\lambda_2(\Omega) \mid \Omega \subseteq \mathbb{R}^2,\ \text{open, with } |\Omega| = 1\right\}, \tag{17}$$

for $\alpha > 0$. This corresponds to Theorem 2 (a) (ii). It was proved in Paragraph 3.1 that for problem (17) the unit ball is better than every disconnected set if $\alpha \in (\alpha_2, 1]$, while when $\alpha \in (0, 1/2)$ the best disconnected set is the disjoint union of two equal balls, say Θ. We aim to give a proof of the fact that Θ can not be optimal for (17) unless $\alpha = 0$, that does not rely on Lemma 6. We focus on the case $\alpha \in (0, 1/2)$.

We need to introduce the set

$$\mathcal{E} = \left\{(\lambda_1(\Omega), \lambda_2(\Omega)) \mid \Omega \subseteq \mathbb{R}^2 \text{ open, with } |\Omega| = 1\right\}. \tag{18}$$

For a description of many properties of this set and a numerical approximation of it we refer to [6] or to [10, Chapter 6.4]. The property which interests us deals with the lower part of the boundary of $\mathcal{E}$, the curve $\mathcal{C}$ that joins the point $A = (\lambda_1(\Theta), \lambda_2(\Theta))$ and $B = (\lambda_1(B), \lambda_2(B))$. Wolf and Keller [13] proved that the curve $\mathcal{C}$ must be vertical at the point B by a perturbation argument with nearly circular domains. They also suggested that $\mathcal{C}$ should be horizontal at A, and this was proved recently by Brasco, Nitsch and Pratelli [4]. This is the crucial point of our proof, as a minimiser for the convex combination $\alpha\lambda_1(\Omega) + (1-\alpha)\lambda_2(\Omega)$ is given by the set corresponding to the first point in which the straight line $\alpha x + (1-\alpha)y = a$ touches $\mathcal{E}$, by increasing a. In particular, for $\alpha = 0$ this line is $y = \lambda_2(\Theta) = 2\lambda_1(B)$ by the Krahn-Szegö inequality. On the other hand, for all $\alpha \in (0, 1/2)$, it is possible to find a set $\widetilde{\Omega}$ that is linked to a line of the form $\alpha x + (1-\alpha)y = a_\alpha$, with $a_\alpha < \lambda_2(\Theta) = 2\lambda_1(B)$, since the curve $\mathcal{C}$ has horizontal tangent. Hence Θ can not be the minimiser for (17) unless $\alpha = 0$.

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