Minimising convex combinations of low eigenvalues

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June 30, 2012

Abstract
We consider the variational problem

\[ \inf \{ \alpha \lambda_1(\Omega) + \beta \lambda_2(\Omega) + (1 - \alpha - \beta) \lambda_3(\Omega) \mid \Omega \text{ open in } \mathbb{R}^n, \ |\Omega| \leq 1 \} \]

for \( \alpha, \beta \in [0, 1], \ \alpha + \beta \leq 1 \), where \( \lambda_k(\Omega) \) is the \( k \)'th eigenvalue of the Dirichlet Laplacian acting in \( L^2(\Omega) \) and \( |\Omega| \) is the Lebesgue measure of \( \Omega \). We investigate for which values of \( \alpha, \beta \) a minimiser is connected.

Mathematics Subject Classification (2010): 49Q10; 49R50; 35P15.

Keywords: Variational problems, Dirichlet eigenvalues.
1 Introduction

If \( \Omega \subset \mathbb{R}^n \) is an open set with finite Lebesgue measure \( |\Omega| \), and \(-\Delta\) is the Dirichlet Laplacian acting in \( L^2(\Omega) \), then the spectrum of \(-\Delta\) consists of an increasing sequence of eigenvalues \( \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \) (see for example [9]). In this paper we are interested in minimising convex combinations of the first three eigenvalues among open sets of fixed Lebesgue measure. A min-max formula holds for eigenvalues (see [10, Section 1.3]),

\[
\lambda_k(\Omega) = \min_{E_k \subset H^1_0(\Omega)} \max_{u \in E_k} \frac{\|Du\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2}, \quad k \in \mathbb{N},
\]

and the optimal functions \( u_k \) are called eigenfunctions.

The following two inequalities are important to our analysis. The Faber-Krahn inequality (Theorem 3.2.1 in [10]) asserts that

\[
\lambda_1(\Omega) \geq \lambda_1(B) \left( \frac{|B|}{|\Omega|} \right)^{2/n},
\]

where \( B \) is the ball of unit measure in \( \mathbb{R}^n \), and with equality if and only if \( \Omega \) is any ball (up to sets of capacity zero). The latter follows as for an open set \( \Omega \subseteq \mathbb{R}^n \) we have \(|t\Omega| = t^n|\Omega|\), and

\[
\lambda_k(t\Omega) = t^{-2} \lambda_k(\Omega) \quad \text{for all } k \in \mathbb{N},
\]

under a homothety of ratio \( t > 0 \). The Krahn-Szegö inequality (Theorem 4.1.1 in [10]) asserts that

\[
\lambda_2(\Omega) \geq 2^{2/n} \lambda_1(B) \left( \frac{|B|}{|\Omega|} \right)^{2/n},
\]

with equality if and only if \( \Omega \) is any disjoint union of two balls of equal measure. We will denote the union of two disjoint balls each of half measure by \( \Theta \). Note that the minimisers for \( \lambda_1(\Omega) \) and \( \lambda_2(\Omega) \) subject to the constraint \( |\Omega| \leq 1 \) are given by (2) and (4) respectively. The existence of a minimiser for \( \lambda_3(\Omega) \) (in the class of quasi-open sets of fixed measure) was proved in [7], but its shape is still an open problem. Connectedness in dimension \( n = 2, 3 \) of any minimiser for \( \lambda_3(\Omega) \) was proved by Wolf and Keller in [13], where they showed that the ball has lower third eigenvalue than any disconnected set of the same measure, and is a local minimiser in \( \mathbb{R}^2 \). The ball is conjectured to be the minimiser for \( \lambda_3(\Omega) \) in \( \mathbb{R}^2 \) [10]. Connectedness of minimisers for individual eigenvalues is studied in [3].

In this paper we consider the variational problem

\[
\inf \{ \mathcal{F}(\Omega) := \alpha \lambda_1(\Omega) + \beta \lambda_2(\Omega) + (1 - \alpha - \beta) \lambda_3(\Omega) \mid \Omega \text{ open in } \mathbb{R}^n, \ |\Omega| \leq 1 \},
\]

for \( \alpha, \beta \in [0, 1] \) such that \( \alpha + \beta \leq 1 \). The existence of a quasi-open minimiser for (5) follows from Theorem A of the recent paper [11]. Note that by (3) the unit measure in the constraint \( |\Omega| \leq 1 \) is for convenience only; everything works in the same way for any other positive constant. The aim of this paper is to show that a minimiser for (5) is connected for a range of values of \( \alpha, \beta \), and to discuss the remaining cases. The result for \( \mathbb{R}^2 \) is the most complete. The values of \( \alpha_n, \beta_n, \gamma_n \) are given below.

**Theorem 1.**

Any minimiser of (5) is connected for each of the cases.
(i) $\alpha + \beta = 1$, $\alpha > 0$,
(ii) $\alpha_n \leq \alpha \leq 1$,
(iii) $0 < \beta \leq \beta_n(1 - \alpha)$,
(iv) $\beta = 0$, $\gamma_n \leq \alpha \leq 1$.

**Theorem 2.** Let $n = 2$.

(a) Any minimiser of (5) is connected for each of the cases

(i) $\alpha + \beta = 1$, $\alpha > 0$,
(ii) $0.350 \approx \alpha_2 \leq \alpha \leq 1$,
(iii) $0 \leq \beta \leq \beta_2(1 - \alpha) \approx 0.725(1 - \alpha)$,

(b) Any disconnected minimiser of (5) satisfies $\lambda_1(\Omega) = \lambda_2(\Omega)$ and has exactly two components.

(c) If any minimiser of (5) is connected for $\alpha = 0$ and each $\beta \in [0,1)$, then any minimiser is connected unless $\beta = 1$.

We conjecture the following.

**Conjecture 3.** Let $n = 2$; a minimiser for the problem (5) can not be disconnected unless $\beta = 1$.

Throughout this paper let $\alpha_n$ satisfy

$$\alpha_n = \frac{\lambda_2(B) - 2^{2/n}\lambda_1(B)}{\lambda_2(B) - \lambda_1(B)}$$

for $n = 2, 3, 4$, and

$$\alpha_n \left[ \left( \frac{1 - \alpha_n}{\alpha_n} \right)^{n/(n+2)} + 1 \right]^{2/n} - 1 + (1 - \alpha_n) \left[ \left( \frac{\alpha_n}{1 - \alpha_n} \right)^{n/(n+2)} + 1 \right]^{2/n} - \frac{\lambda_2(B)}{\lambda_1(B)} > 0,$$

for $n \geq 5$. Let $\beta_n$ satisfy

$$\beta_n \left[ 2^{2/(n+2)} \left( \frac{1 - \beta_n}{\beta_n} \right)^{n/(n+2)} + 1 \right]^{2/n} + 2^{2/n}(1 - \beta_n) \left[ 2^{-2/(n+2)} \left( \frac{\beta_n}{1 - \beta_n} \right)^{n/(n+2)} + 1 \right]^{2/n} - \frac{\lambda_2(B)}{\lambda_1(B)} < 0,$$

for $n = 2, 3$, and let $\beta_n = 0$ for $n \geq 4$. Finally let $\gamma_2 = \gamma_3 = 0$ and let $\gamma_n$ for $n \geq 4$ satisfy

$$\gamma_n \left[ \left( 1 + \left( \frac{\lambda_1(B)}{\lambda_2(B)} \right)^{n/2} \right)^{2/n} - 1 \right] + (1 - \gamma_n) \left[ 2^{2/n} - \frac{\lambda_2(B)}{\lambda_1(B)} \right] > 0.$$

The approximate values for $n = 2, 3, 4$ are:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha_n$</th>
<th>$\beta_n$</th>
<th>$\gamma_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.350</td>
<td>0.725</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0.439</td>
<td>0.476</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0.479</td>
<td>0</td>
<td>0.311</td>
</tr>
</tbody>
</table>

An important component in the proof of Theorem 1 is the following lemma.
Lemma 4. Let $n \geq 2$. The disjoint union of two balls can be optimal for (5) only if $\beta = 1$.

This paper is organised as follows. In Section 2 we prove Lemma 4 and we rule out a minimiser with three connected components in $\mathbb{R}^2$. Section 3 contains the proof of Theorem 1 and Section 4 the proof of Theorem 2. Note that Theorem 2 (a) is an immediate consequence of Theorem 1. For calculations we use that $\lambda_1(B) = \omega_n^{2/n}/j_{n/2-1}$, and $\lambda_2(B) = \cdots = \lambda_{n+1}(B) = \omega_n^{2/n}/j_{n/2}$, where $\omega_n$ denotes the volume of the ball of unit radius in $\mathbb{R}^n$, and $j_n$ is the first positive zero of the Bessel function $J_n$. In $\mathbb{R}^2$ we have $\lambda_1(B) \approx 18.168$ and $\lambda_2(B) = \lambda_3(B) \approx 46.125$. In the proofs below $\Omega$ is used to denote the optimal disconnected candidate for a minimiser. Throughout the paper we will for convenience define the value $m_0 = \frac{\lambda_1(B)}{\lambda_2(B)}$, while the values of $m_1, m_2$ will denote, respectively, the lower and the upper bound for the measure of a connected component.

Note that for disconnected sets the eigenvalues are obtained by collecting and reordering the eigenvalues less than or equal to the largest eigenvalue of $\Omega$ that we are accounting for. Hence $\lambda_1(\tilde{\Omega}(\epsilon)) = \lambda_1(B(\epsilon)) < \lambda_2(B(\epsilon)) = \lambda_2(\tilde{\Omega}(\epsilon))$, and again taking account of the scaling we have $\lambda_1(\tilde{\Omega}(\epsilon)) < \lambda_1(\tilde{\Omega}) - c_1 \epsilon^{(n+1)/2}$ and $\lambda_3(\tilde{\Omega}(\epsilon)) < \lambda_3(\tilde{\Omega}) - c_2 \epsilon^{(n+1)/2}$, for some positive constants $c_1, c_2$.

2 Preliminaries

Proof of Lemma 4. The idea of the proof is that letting the two disjoint equal balls slightly overlap we obtain a better candidate for a minimiser of (5), because the increase in the second eigenvalue is less than the decrease in the first and the third. We divide the proof in two steps, treating first the case of two balls with equal measure, then the case of balls with different size.

Step I. Let $B(\epsilon) = B(0,1) \cap \{x \mid |x| < 1 - \epsilon\}$ and $\Omega(\epsilon) = B(0,1) \cup B(1 - \epsilon) e_1, 1)$, where $x = (x_1, x_2, \ldots, x_n)$ and $e_1$ is the unit vector in the $x_1$ direction. Moreover $\Omega(\epsilon) = |\Omega(\epsilon)|^{-\frac{1}{n}} \Omega(\epsilon)$ is the set rescaled to unit measure. It follows from Theorem 1 in [2] that

$$\lambda_1(B(\epsilon)) |B(\epsilon)|^{2/n} = \lambda_1(B)|B|^{2/n} + O\left(\epsilon^{(n+1)/2}\right).$$

(6)

By monotonicity we have $\lambda_1(\Omega(\epsilon)) < \lambda_1(B)$ and $\lambda_3(\Omega(\epsilon)) < \lambda_2(B)$, and so taking scaling into account gives $\lambda_1(\tilde{\Omega}(\epsilon)) < \lambda_1(\Theta) - \epsilon e^{(n+1)/2}$ and $\lambda_3(\tilde{\Omega}(\epsilon)) < \lambda_3(\Theta) - \epsilon e^{(n+1)/2}$, for some positive constants $c_1, c_2$. By the min-max principle (1) we can obtain an upper bound by choosing the subspace $E_2$ spanned by the first eigenfunction of $B(\epsilon)$ and the first eigenfunction of $B(0,1) \cap \{x \mid |x| > 1 - \epsilon\}$. Hence $\lambda_2(\tilde{\Omega}(\epsilon)) \leq \lambda_1(B(\epsilon))$, so we can apply (6) and use the scaling (3) to obtain $\lambda_2(\tilde{\Omega}(\epsilon)) \leq \lambda_2(\Theta) + O(\epsilon^{(n+1)/2})$.

For $\beta < 1$ and for sufficiently small $\epsilon > 0$, this gives

$$a \lambda_1(\tilde{\Omega}(\epsilon)) + \lambda_2(\tilde{\Omega}(\epsilon)) + (1 - a - \beta) \lambda_3(\tilde{\Omega}(\epsilon)) < a \lambda_1(\Theta) + \beta \lambda_2(\Theta) + (1 - a - \beta) \lambda_3(\Theta).$$

Step II. Let $r_1 > r_2$ and $\Omega$ be the disjoint union of two balls with radii $r_1, r_2$ such that the first two eigenfunctions are supported on different components. We write $\tilde{\Omega} = |\Omega|^{-\frac{1}{n}} \Omega$ for the set rescaled to unit measure. Then, let $B_{r_1} = B(0, r_1)$, $B_{r_2} = B(r_1 + r_2 - \frac{1}{2} \left(\frac{1}{n} + \frac{1}{2}\right), r_1)$, $\Omega(\epsilon) = B_{r_1} \cup B_{r_2}$, $B_1(\epsilon) = B_{r_1} \cap \{x \mid x_1 < r_1 - \frac{\epsilon}{\sqrt{n}}\}$ and $B_2(\epsilon) = B_{r_2} \cap \{x \mid x_1 > r_2 - \frac{\epsilon}{\sqrt{n}}\}$. We write $\tilde{\Omega}(\epsilon) = |\Omega(\epsilon)|^{-\frac{1}{n}} \Omega(\epsilon)$ for the rescaled set. By monotonicity we have $\lambda_1(\Omega(\epsilon)) < \lambda_1(B_{r_1})$ and $\lambda_3(\Omega(\epsilon)) < \lambda_2(B_{r_1})$, and again taking account of the scaling we have $\lambda_1(\tilde{\Omega}(\epsilon)) < \lambda_1(\tilde{\Omega}) - c_1 \epsilon^{(n+1)/2}$ and $\lambda_3(\tilde{\Omega}(\epsilon)) < \lambda_3(\tilde{\Omega}) - c_2 \epsilon^{(n+1)/2}$, for some positive constants $c_1, c_2$. 


By the min-max principle (1), we can obtain an upper bound by choosing $E_2$ spanned by the first eigenfunction of $B_{e_2}$ and the first eigenfunction of $\Omega(\varepsilon) \setminus B_{e_2}$, and for $\varepsilon$ small enough we have $\lambda_1(B_{e_2}) < \lambda_1(\Omega(\varepsilon) \setminus B_{e_2})$. Hence $\lambda_2(\Omega(\varepsilon)) < \lambda_1(B_{e_2})$, and taking account of the scaling $\lambda_2(\Omega(\varepsilon)) \leq \lambda_2(\Omega) - c_3\varepsilon^{(n+1)/2}$, for some positive $c_3$. In conclusion, for $\beta < 1$ and $\varepsilon$ small enough, $\Omega(\varepsilon)$ is a better candidate than $\Omega$ for problem (5).

The following remark will be useful in the proof of Lemma 6 and in Section 4.

**Remark 5.** Let $n = 2$. A disconnected set $\Omega$ can never be optimal for (5) if $\lambda_2(\Omega) > \lambda_2(B)$. Here the disk is better, since $\lambda_1(\Omega) < \lambda_1(\Omega)$ by the Faber-Krahn inequality and $\lambda_3(B) < \lambda_3(\Omega)$ by [10, Corollary 5.2.2].

**Lemma 6.** Let $n = 2$. Any disconnected minimiser of (5) has exactly two components.

**Proof.** For the case $\alpha + \beta = 1$ it is clear that a minimiser has at most two components. For $\alpha + \beta < 1$ the Faber-Krahn inequality implies that a disconnected minimiser with three components would be the union of three disjoint balls. If $\alpha > 0$, it is possible to apply Lemma 4 to the union of the balls supporting the second and the third eigenvalues. For $\alpha = 0$ this argument does not work, since neither $\lambda_2$ nor $\lambda_3$ are lowered. Hence we rule out the configuration with three connected components only for $n = 2$, by comparing it with $B$ and $\Theta$.

Let $\mathcal{G}(\cdot) = \beta \lambda_2(\cdot) + (1 - \beta)\lambda_3(\cdot)$, and write $\Omega_i$, $i = 1, 2, 3$, for the three components of $\Omega$. Assuming $\lambda_i(\Omega) = \lambda_i(\Omega_i)$ for $i = 1, 2, 3$ gives $|\Omega_1| \geq |\Omega_2| \geq |\Omega_3|$. We write $m = |\Omega_1|$ and note that $|\Omega_2| = m$, as for $|\Omega_1| > |\Omega_2|$ we could enlarge $\Omega_2$ and shrink $\Omega_1$, lowering the functional. Thus $|\Omega_3| = 1 - 2m$, and the following constraints on $m$ hold:

1) Remark 5 implies $\lambda_2(B) > \lambda_2(\Omega) = \lambda_1(\Omega_2) = \frac{\lambda_1(\Omega_2)}{m}$, so $m > \frac{\lambda_1(\Omega_2)}{\lambda_2(\Omega)} = m_1 \approx 0.394$.

2) We must have $\frac{\lambda_2(B)}{m} = \lambda_2(\Omega_1) \geq \lambda_1(\Omega_3) = \frac{\lambda_1(\Omega_3)}{m},$ as otherwise we can reduce to only two components. This inequality implies

$$m \leq \frac{\lambda_2(B)}{\lambda_1(B) + 2\lambda_2(B)} = m_2 \approx 0.418.$$ 

Coming back to the study of $\mathcal{G}$, we can use the scaling properties of eigenvalues and the bounds above to obtain

$$\mathcal{G}(\Omega) = \beta \lambda_2(\Omega) + (1 - \beta)\lambda_3(\Omega) = \left\{ \frac{\beta}{m} + \frac{(1 - \beta)}{(1 - 2m)} \right\} \lambda_1(B) \geq \left\{ \frac{\beta}{m_2} + \frac{(1 - \beta)}{(1 - 2m_1)} \right\} \lambda_1(B).$$

(7)

Now we look for those $\beta$ for which the unit ball $B$ gives a lower value of $\mathcal{G}$ than this lower bound. In particular we are looking for those $\beta$ that satisfy

$$\mathcal{G}(B) - \mathcal{G}(\Omega) \leq \lambda_2(B) - \left\{ \frac{\beta}{m_2} + \frac{(1 - \beta)}{(1 - 2m_1)} \right\} \lambda_1(B) < 0,$$

i.e.

$$\beta < \frac{1}{(1 - 2m_1)} - \frac{1}{m_1} \approx 0.936.$$
For this range of $\beta$ three balls can not be optimal when minimising $G$.

The remaining $\beta$ are ruled out by comparing $\Omega$ with $\Theta$. Using (7), three connected components can not be optimal when

\[ G(\Theta) - G(\Omega) \leq 2\beta \lambda_1(B) + 2(1 - \beta) \lambda_2(B) - \left\{ \frac{\beta}{m_2} + \frac{(1 - \beta)}{(1 - m_1)} \right\} \lambda_1(B) < 0, \]

i.e. when

\[ \beta > \frac{2\lambda_2(B) - \frac{\lambda_1(B)}{1 - \frac{m_1}{m_2}}}{2\lambda_2(B) - \left( 2 + \frac{1}{1 - \frac{m_1}{m_2}} \right) \lambda_1(B)} \approx 0.479. \]

Since the two ranges we obtained on $\beta$ cover all cases, a minimiser for (5) can never have three components in $\mathbb{R}^2$.

3 Proof of Theorem 1.

3.1 Proof of Theorem 1 (i) and (ii).

Proof of Theorem 1 (i) and (ii). We first consider the situation $\alpha + \beta = 1$. Note that this result for $\mathbb{R}^2$ is also discussed in [12, Chapter 2], but the details of a proof are not given. A disconnected minimiser $\Omega$ must by the Faber-Krahn inequality be the union of two disjoint balls with measures $m$ and $1 - m$ respectively. Hence an immediate application of Lemma 4 rules out this configuration in any dimension when $\beta < 1$, and (i) is proved.

To prove (ii) we need a different argument, but start again from the case $\alpha + \beta = 1$, and note that

\[ \alpha \lambda_1(\Omega) + (1 - \alpha) \lambda_2(\Omega) = \lambda_1(B) \left( \frac{\alpha}{m^{2/n}} + \frac{1 - \alpha}{(1 - m)^{2/n}} \right). \]

For $\alpha \in \left( \frac{1}{2}, 1 \right]$, differentiating this with respect to $m$ to obtain a lower bound and comparing with the value for the unit ball rules out this configuration if

\[ \alpha \lambda_1(B) \left( \frac{1 - \alpha}{\alpha} \right)^{n/(n+2)} + 1 \right) + (1 - \alpha) \lambda_1(B) \left( \frac{\alpha}{1 - \alpha} \right)^{n/(n+2)} + 1 \right) \geq \lambda_1(B) + (1 - \alpha) \lambda_2(B). \]

For $\alpha \in (0, \frac{1}{2}]$ the constraint $m \geq 1/2$ implies that the optimal disconnected configuration is two disjoint balls of equal measure. This is ruled out by comparison with the unit ball when

\[ \lambda_1(B) 2^{2/n} > \alpha \lambda_1(B) + (1 - \alpha) \lambda_2(B). \]

Combining (9) with (8) we have that the ball is better than any disconnected set for $1 \geq \alpha \geq \alpha_n$.

Finally to extend beyond the situation $\alpha + \beta = 1$, just note that for $1 \geq \alpha \geq \alpha_n$

\[ \alpha \lambda_1(\Omega) + \beta \lambda_2(\Omega) + (1 - \alpha - \beta) \lambda_3(\Omega) \geq \alpha \lambda_1(\Omega) + (1 - \alpha) \lambda_2(\Omega) \]

\[ > \alpha \lambda_1(B) + (1 - \alpha) \lambda_2(B) = \alpha \lambda_1(B) + \beta \lambda_2(B) + (1 - \alpha - \beta) \lambda_3(B), \]

using the fact that $\lambda_2(\Omega) \leq \lambda_3(\Omega)$ while $\lambda_2(B) = \lambda_3(B)$. This concludes (ii).
3.2 Proof of Theorem 1 (iii) and (iv).

*Proof of Theorem 1 (iii) and (iv). The case $\alpha = 0$: We first consider the case $\alpha = 0$. Note that it is known that the disk is a critical point for $\alpha = 0, \beta \leq 1/2$. (See [10] p. 93.) Note also that a disconnected minimiser for $\alpha = 0$ would consist of a ball supporting the second eigenvalue, and another set supporting the first and third eigenvalues. This is because a configuration with a ball supporting the third eigenvalue and a set supporting the others would be three balls by the Krahn-Szegö inequality. An optimal configuration with a ball supporting the first eigenvalue would satisfy $\lambda_1(\Omega) = \lambda_2(\Omega)$, because the quantity we are minimising for $\alpha = 0$ does not involve $\lambda_1(\Omega)$. Both are of the form of a ball supporting $\lambda_2(\Omega)$ and a set supporting $\lambda_1(\Omega)$ and $\lambda_3(\Omega)$, and so it remains only to rule out this configuration.

For $\beta \in [0, 1)$, the Krahn-Szegő inequality gives the lower bound
\[
\beta \lambda_2(\Omega) + (1 - \beta) \lambda_3(\Omega) \geq \lambda_1(B) \left( \frac{\beta}{m^{2/n}} + \frac{2^{2/n}(1 - \beta)}{(1 - m)^{2/n}} \right),
\]
(10)

For $\beta \in [0, 1/3]$ we set $m = 1/3$, as we hit the constraint $m \geq 1/3$. Substituting into (10) and comparing with $\beta \lambda_2(B) + (1 - \beta) \lambda_3(B) = \lambda_2(B)$ gives that any minimiser in this range is connected for $n = 2, 3$. For $\beta \in (1/3, 1)$ optimising (10) with respect to $m$ gives
\[
\frac{1}{m} = 2^{2/(n+2)} \left( \frac{1 - \beta}{\beta} \right)^{n/(n+2)} + 1, \quad \frac{1}{1 - m} = 2^{-2/(n+2)} \left( \frac{\beta}{1 - \beta} \right)^{n/(n+2)} + 1.
\]

Again comparing with $\lambda_2(B)$ gives that any minimiser is connected for $1/3 < \beta \leq \beta_n$. This only gives information for $n = 2, 3$, and we conclude connectedness for:

- $n = 2$: $\beta \in [0, 0.725)$,
- $n = 3$: $\beta \in [0, 0.476)$.

The case $\beta = 0$: We now consider the case $\beta = 0$. To prove connectedness we first see that a disconnected minimiser would consist of a ball supporting the third eigenvalue, and another set supporting the first and second eigenvalues. This is because a configuration with a disk supporting the first eigenvalue and a set supporting the others would be three balls using that the minimiser for the second eigenvalue is the union of two disjoint balls. An optimal configuration with a ball supporting the second eigenvalue would have $\lambda_2(\Omega) = \lambda_3(\Omega)$, as scaling down the ball to obtain this does not affect $\lambda_1(\Omega), \lambda_3(\Omega)$. Both these possible configurations are just special cases of a ball supporting the $\lambda_3(\Omega)$ and a set supporting $\lambda_1(\Omega), \lambda_2(\Omega)$, and so it only remains to rule out this possibility. This is done by obtaining lower bounds for the first and third eigenvalues and using comparison with a ball. Letting the third eigenvalue be supported on a ball of measure $m$, we have $\lambda_3(\Omega) = \frac{\lambda_3(B)}{m^{2/n}}$, while the Faber-Krahn and the Krahn-Szegö inequalities respectively give
\[
\lambda_1(\Omega) \geq \frac{\lambda_1(B)}{(1 - m)^{2/n}} \quad \text{and} \quad \frac{\lambda_1(B)}{m^{2/n}} = \lambda_3(\Omega) \geq \lambda_2(\Omega) \geq 2^{2/n} \frac{\lambda_1(B)}{(1 - m)^{2/n}},
\]
which implies $\frac{1}{m} \geq 3$, and so $m \leq \frac{1}{3}$. With explicit computations it is then easy to see that for $n = 2, 3$
\[
\lambda_3(B) = \lambda_2(B) \leq \lambda_3(\Omega).
\]

By the Faber-Krahn inequality, the ball strictly lowers the first eigenvalue, so we rule out this configuration for all $\alpha \in [0, 1]$ when $n = 2, 3$. This proves Theorem 1 (iv) for $n = 2, 3$. 


For $n \geq 4$ we must be more precise and obtain only partial estimates. If $\lambda_3(\Omega) = \frac{\lambda_3(B)}{(1-m)^2/n}$, then as we assume $\beta = 0$, the set supporting the first two eigenvalues should be a ball $B_1$. This would contradict the optimality of $\Omega$, as we would have $\lambda_3(B_1) = \lambda_2(B_1) \leq \lambda_3(\Omega)$ and $|B_1| < 1$. So we conclude $\frac{\lambda_3(B)}{m_2/n} < \frac{\lambda_2(B)}{(1-m)^2/n}$, and so $m > \frac{m_0^{n/2}}{1+n_0}$, which gives the bound

$$\frac{1}{1-m} > 1 + m_0^{n/2}.$$ 

This gives the lower bound

$$\alpha \lambda_1(\Omega) + (1-\alpha)\lambda_3(\Omega) \geq \alpha \lambda_1(B) \left(1 + m_0^{n/2}\right)^{2/n} + (1-\alpha)\lambda_1(B)3^{2/n}. \quad (11)$$

We then have connectedness for example in the following cases:

- $n = 2$: $\alpha \in [0, 1]$,
- $n = 3$: $\alpha \in [0, 1]$,
- $n = 4$: $\alpha \in (0, 1]$,
- $n = 5$: $\alpha \in [0.467, 1]$,
- $n = 6$: $\alpha \in [0.547, 1]$.

The case $0 < \beta \leq \beta_n(1-\alpha)$: Now finally consider the case $0 < \beta \leq \beta_n(1-\alpha)$. Recall from the above that

$$\alpha \lambda_2(\Omega) + (1-\alpha)\lambda_3(\Omega) > \alpha \lambda_2(B) + (1-\alpha)\lambda_3(B),$$

for $\alpha \in [0, \beta_n)$. This implies

$$\frac{\beta}{1-\alpha} \lambda_2(\Omega) + \frac{1}{1-\alpha} \lambda_3(\Omega) > \frac{\beta}{1-\alpha} \lambda_2(B) + \frac{1}{1-\alpha} \lambda_3(B),$$

for $\frac{\beta}{1-\alpha} \in [0, \beta_n)$, and so

$$\beta \lambda_2(\Omega) + (1-\alpha-\beta)\lambda_3(\Omega) > \beta \lambda_2(B) + (1-\alpha-\beta)\lambda_3(B),$$

for $\beta \in [0, \beta_n(1-\alpha))$. Together with $\lambda_1(\Omega) \geq \lambda_1(B)$ this gives the result. \qed

### 4 Proof of Theorem 2 (b) and (c)

We begin with a lemma, which asserts that a disconnected minimiser must have multiple eigenvalues. The idea of the proof is that if every eigenvalue is simple, then small variations of the connected components (in the sense of shrinking one and enlarging the other) contradict the optimality of such a disconnected set. For simplicity we will often write $\lambda_i = \lambda_i(\Omega)$ and $\gamma = 1-\alpha-\beta$, and as before define $m_0 = \frac{\lambda_1(B)}{\lambda_2(B)} \approx 0.394$.

**Lemma 7.** A disconnected minimiser $\Omega$ for (5) in $\mathbb{R}^2$ can not have both $\lambda_1(\Omega) \neq \lambda_2(\Omega)$ and $\lambda_2(\Omega) \neq \lambda_3(\Omega)$. 


Proof. Note that we only need to consider the cases for problem (5) that are not treated in Theorem 2 (a). Additionally, the case of three components is ruled out by Lemma 6. The proof of the remaining cases is divided into three steps.

Step I. We consider the case of a set $\Omega = \Omega_1 \cup \Omega_2$, with $\Omega_1$ supporting $u_1$, while $\Omega_2$ supports $u_2$ and $u_3$. From the hypotheses of the Step, $\lambda_1 = \lambda_1(\Omega_1)$, $\lambda_2 = \lambda_1(\Omega_2)$ and $\lambda_3 = \lambda_2(\Omega_2)$, and by Faber-Krahn $\Omega_1$ is a ball. We define $m = |\Omega_1|$, so $1 - m = |\Omega_2|$. The following constraints on $m$ hold:

1) $m > \frac{\lambda_1(B)}{\lambda_1(\Omega_1)} = m_1 \approx 0.394$, since $\frac{\lambda_1(B)}{m} = \lambda_1(\Omega_1)$, $\lambda_1(\Omega_2) < \lambda_2(\Omega_2)$ (see Remark 5).

2) $\frac{\lambda_2(B)}{m} = \lambda_2(\Omega_1) \geq \lambda_2(\Omega_2) > \frac{\lambda_2(B)}{(1-m)} = 2 \lambda_2(B) \frac{(1-B)}{(1-m)}$, so $m < \frac{\lambda_2(B)}{(1-B)} \approx 0.559$.

Now we can shrink $\Omega_1$ and enlarge $\Omega_2$, in order to obtain two new sets of the same shape $\tilde{\Omega}_1$, $\tilde{\Omega}_2$, such that $|\tilde{\Omega}_1| = m - \varepsilon$, while $|\tilde{\Omega}_2| = 1 - m + \varepsilon$. Writing $\tilde{\lambda}_i$ for the eigenvalues of $\tilde{\Omega}_1 \cup \tilde{\Omega}_2$ we obtain the following ratios for $m << 1$:

$$\frac{\tilde{\lambda}_1}{\lambda_1} = m - \varepsilon \approx 1 + \frac{\varepsilon}{m}; \quad \frac{\tilde{\lambda}_2}{\lambda_2} = \frac{(1-m)}{1-m+\varepsilon} \approx 1 - \frac{\varepsilon}{1-m}.$$  

The optimality of $\Omega$ implies $F(\Omega) \leq F(\tilde{\Omega}_1 \cup \tilde{\Omega}_2)$, that means

$$\alpha \lambda_1 + \beta \lambda_2 + \gamma \lambda_3 < \alpha \tilde{\lambda}_1 + \beta \tilde{\lambda}_2 + \gamma \tilde{\lambda}_3 + \varepsilon o(\varepsilon) \geq 0.$$  

Taking either $\varepsilon > 0$ or $\varepsilon < 0$ (this is possible since we are supposing that the eigenvalues are simple) gives that the expression in the brackets must be zero, hence $\frac{\alpha \lambda_1}{m} = \frac{\beta \lambda_2 + \gamma \lambda_3}{1-m}$.

In order to conclude this first step it suffices to consider $\varepsilon > 0$. Since $\tilde{\lambda}_3 \leq \lambda_3$, we have a contradiction if $F(\tilde{\Omega}_1 \cup \tilde{\Omega}_2) < F(\Omega)$, i.e. when

$$\alpha \tilde{\lambda}_1 + \beta \tilde{\lambda}_2 < \alpha \lambda_1 + \beta \lambda_2.$$  

Equation (12) holds if and only if

$$\beta \lambda_2 \left(1 - \frac{\tilde{\lambda}_2}{\lambda_2}\right) > \alpha \lambda_1 \left(\frac{\tilde{\lambda}_1}{\lambda_1} - 1\right) \iff \beta > \alpha \left(\frac{\lambda_1}{\lambda_2}\right) \left(1 - \frac{m}{m_1}\right).$$  

Using $\frac{\lambda_1}{\lambda_2} \leq \frac{\lambda_1(B)}{\lambda_2(\Theta)} = \frac{1-m}{2m}$ and the above constraints on $m$ gives $\frac{1-m}{m_1} \leq \frac{1-m}{m_1}$. So if $\beta > \frac{\alpha}{\frac{1-m}{m_1}} \approx 1.18 \alpha$, the set $\Omega = \tilde{\Omega}_1 \cup \tilde{\Omega}_2$ cannot be optimal. The case $\beta \leq 1.18 \alpha$ was treated in Theorem 1 (a), and so this concludes Step I.

Step II. We now consider the case of a set $\Omega = \Omega_1 \cup \Omega_2$, with $\Omega_1$ supporting $u_1$ and $u_3$, while $\Omega_2$ supports $u_2$. Clearly $\lambda_1 = \lambda_1(\Omega_1)$, $\lambda_2 = \lambda_1(\Omega_2)$ and $\lambda_3 = \lambda_2(\Omega_1)$, and again it is better to take $\Omega_2$ to be a ball. Write $m = |\Omega_1|$ and $1 - m = |\Omega_2|$. The following constraints on $m$ hold:

1) $\frac{\lambda_1(B)}{m} < \lambda_1(\Omega_1) \leq \lambda_1(\Omega_2) = \frac{\lambda_1(B)}{(1-m)}$, so $m > 1/2 = m_1$.

2) $\frac{\lambda_1(B)}{(1-m)} = \lambda_2(\Omega) < \lambda_2(B)$ by Remark 5, so $m < \frac{\lambda_2(B) - \lambda_1(B)}{\lambda_2(B)} = 1 - m_0 = m_2 \approx 0.606$.

As in the previous case we shrink $\Omega_1$ to $\tilde{\Omega}_1$ and we enlarge $\Omega_2$ to $\tilde{\Omega}_2$, so that $|\tilde{\Omega}_1| = m - \varepsilon$, while $|\tilde{\Omega}_2| = 1 - m + \varepsilon$. With the same arguments of the previous Step, if $\Omega$ is optimal then $F(\tilde{\Omega}_1 \cup \tilde{\Omega}_2) \geq F(\Omega)$ and so

$$\left(\frac{\alpha \lambda_1 + \gamma \lambda_3}{m} - \frac{\beta \lambda_2}{1-m}\right) \varepsilon + o(\varepsilon) \geq 0.$$  

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Taking again either $\varepsilon > 0$ or $\varepsilon < 0$ gives $\frac{\beta \lambda_2}{1 - m} = \frac{\alpha \lambda_1 + \gamma \lambda_3}{m}$. Now, since $\Omega_2$ is a ball and thanks to the bounds on $m$, we can rewrite the complete functional in a more interesting way

$$F(\Omega) = \alpha \lambda_1 + \beta \lambda_2 + \gamma \lambda_3 = \frac{m}{1 - m} \beta \lambda_2 + \beta \lambda_2 = \frac{\beta \lambda_1(B)}{1 - m} = \frac{\beta \lambda_1(B)}{(1 - m)^2} \geq \frac{\beta \lambda_1(B)}{(1 - m_1)^2} \geq 4 \beta \lambda_1(B).$$

Comparing this lower bound with the case of the ball gives a contradiction for $\beta$ such that

$$F(B) - F(\Omega) \leq \alpha \lambda_1(B) + (1 - \alpha) \lambda_2(B) - 4 \beta \lambda_1(B) < 0,$$

i.e. for

$$\beta > \frac{1}{4m_0} - \frac{\alpha}{4} \left( \frac{1}{m_0} - 1 \right) \approx 0.635 - 0.385 \alpha. \quad (13)$$

In order to consider the cases that are not treated in Theorem 2 (a), we look at the equations $\alpha = \alpha_2$, $\beta = \frac{1}{4m_0} - \frac{\alpha}{4} \left( \frac{1}{m_0} - 1 \right)$, and $\beta = \beta_2(1 - \alpha)$. The remaining cases can be viewed as inside a small triangle in the $\alpha$-$\beta$ plane, with vertices approximately given by $A = (0.275; 0.529)$, $B = (0.350; 0.500)$ and $C = (0.350; 0.474)$ (See Figure 1).

![Figure 1: The small triangle in the $\alpha$-$\beta$ plane](image)

For these remaining points, it is possible to show that a ball is better than $\Omega$, i.e. that $F(B) - F(\Omega) < 0$. In fact, the following relations hold between the eigenvalues of the ball and those of $\Omega$ (using $m \in (m_1, m_2)$, the Faber-Krahn and the Krahn-Szegö inequalities):

1. $\lambda_1(B) - \lambda_1 \leq \left( 1 - \frac{1}{m} \right) \lambda_1(B) \leq \left( 1 - \frac{1}{m_2} \right) \lambda_1(B)$,
2. $\lambda_2(B) - \lambda_2 = \lambda_2(B) - \frac{\lambda_1(B)}{1 - m} \leq \lambda_2(B) - 2 \lambda_1(B)$,
3. $\lambda_3(B) - \lambda_3 \leq \lambda_2(B) - \frac{2 \lambda_1(B)}{m} \leq \lambda_2(B) - \frac{2 \lambda_1(B)}{m_2}$. \quad (14)
From (14) we get
\[
\mathcal{F}(B) - \mathcal{F}(\Omega) \leq \alpha \lambda_1(B) \left(1 - \frac{1}{m_2}\right) + \beta \left(\lambda_2(B) - 2\lambda_1(B)\right) + (1 - \alpha - \beta) \left(\lambda_2(B) - \frac{2\lambda_1(B)}{m_2}\right).
\]

Hence the ball is better than \( \Omega \) if
\[
\beta < \frac{2\lambda_1(B) - m_2\lambda_2(B)}{2\lambda_1(B) - 2m_2\lambda_1(B)} + \alpha \frac{-\lambda_1(B) + m_2(\lambda_2(B) - \lambda_1(B))}{2\lambda_1(B) - 2m_2\lambda_1(B)} \approx 0.58 - 0.08\alpha.
\]

This inequality together with (13) concludes Step II.

**Step III.** We now consider the case of a set \( \Omega = \Omega_1 \cup \Omega_2 \), with \( \Omega_1 \) supporting \( u_1 \) and \( u_2 \), while \( \Omega_2 \) supports \( u_3 \). Clearly \( \lambda_1 = \lambda_1(\Omega_1) \), \( \lambda_2 = \lambda_2(\Omega_1) \) and \( \lambda_3 = \lambda_1(\Omega_2) \), and it is better to take \( \Omega_2 \) to be a ball. Let \( m = |\Omega_1| \) and \( 1 - m = |\Omega_2| \). Note that if \( \lambda_3(\Omega) = \lambda_1(\Omega_2) \geq m\lambda_3(\Omega_1) \), then \( \Omega \) can not be optimal. In fact in this case it is better to take the connected set obtained by enlarging \( \Omega_1 \) till unit measure, since this lowers both \( \lambda_1 \) and \( \lambda_2 \) (by monotonicity), while also the third eigenvalue is lower, by hypothesis. The following constraints on \( m \) hold:

1) \( \lambda_2(B) > \lambda_2(\Omega) = \lambda_2(\Omega_1) \geq \frac{2\lambda_1(B)}{m} \) (see Remark 5), so \( m > 2m_0 = m_1 \approx 0.788 \).

2) In order to have \( \Omega \) optimal, an upper bound on \( m \) follows from inequality \( m\lambda_3(\Omega_1) > \lambda_1(\Omega_2) \) explained above and from the fact that \( \lambda_2(\Omega) < \lambda_2(B) \) (see Remark 5). Using also the Ashbaugh-Benguria Theorem (see [1]) gives

\[
\frac{\lambda_1(B)}{1 - m} = \lambda_1(\Omega_2) < m\lambda_3(\Omega_1) \leq m \frac{\lambda_2(B)}{\lambda_1(B)} \lambda_2(\Omega_1) < m^2 \frac{\lambda_2(B)}{\lambda_1(B)}.
\]

This means \( m^2 - m + m_0^2 < 0 \), which gives the upper bound

\[
m < \frac{\lambda_2(B) + \sqrt{\lambda_2(B)^2 - 4\lambda_1(B)^2}}{2\lambda_2(B)} = m_2 \approx 0.808.
\]

As in the previous steps we can enlarge \( \Omega_1 \) to \( \Omega_1 \) and we can shrink \( \Omega_2 \) to \( \Omega_2 \), in order that \( |\Omega_1| = m + \varepsilon \), while \( |\Omega_2| = 1 - m - \varepsilon \). The following ratios between the eigenvalues of \( \Omega_1 \cup \Omega_2 \) and those of \( \Omega \) hold (for \( \varepsilon \ll 1 \)):

\[
\frac{\lambda_1}{\lambda_1} = \frac{\lambda_2}{\lambda_2} = \frac{m}{m + \varepsilon} \approx 1 - \frac{\varepsilon}{m}; \quad \frac{\lambda_3}{\lambda_3} = \frac{1 - m}{1 - m - \varepsilon} \approx 1 + \frac{\varepsilon}{1 - m}.
\]

In order to be optimal, \( \Omega \) must satisfy
\[
\alpha \tilde{\lambda}_1 + \beta \tilde{\lambda}_2 + (1 - \alpha - \beta) \tilde{\lambda}_3 \geq \alpha \lambda_1 + \beta \lambda_2 + (1 - \alpha - \beta) \lambda_3.
\]

An analogous argument to that in Step I and Step II gives a contradiction for \( \beta \geq 0.914 - 0.948\alpha \). Actually we can obtain a better result observing that \( \Omega \) is worse than \( \Omega_1 \) enlarged to unit measure (which we will call \( \Pi \) in the following) if \( \beta \) is suitably large. We denote by \( \{\tilde{\lambda}_i\} \) the eigenvalues of \( \Pi \) and we again write \( \gamma = 1 - \alpha - \beta \) for the sake of simplicity. The following relations between the eigenvalues hold: \( \tilde{\lambda}_1 = m\lambda_1 \), \( \tilde{\lambda}_2 = m\lambda_2 \) and \( \tilde{\lambda}_3 = m\lambda_3(\Omega_1) \leq \frac{m}{m_1}\lambda_2 \), using the Ashbaugh-Benguria Theorem. This gives

\[
\mathcal{F}(\Pi) = \alpha \tilde{\lambda}_1 + \beta \tilde{\lambda}_2 + \gamma \tilde{\lambda}_3 \leq \mathcal{F}(\Omega) + \alpha(m - 1)\lambda_1 + \beta(m - 1)\lambda_2 + \gamma \left(\frac{m}{m_1} - 1\right)\lambda_2.
\]
Clearly Ω can not be optimal when $F(Ω) - F(Ω^*) < 0$, which holds if
\[ α(m - 1)λ_1 + \left[ β(m - 1) + γ \left( \frac{m}{m_0} - 1 \right) \right] λ_2 < 0. \]

Again using the Ashbaugh-Benguria Theorem gives that the result follows if
\[ \left[ α(m - 1) + β(m - 1) + γ \left( \frac{m}{m_0} - 1 \right) \right] λ_2 < 0. \]

Since $m \in (m_1, m_2)$ and the function in brackets is clearly increasing in $m$, Ω can not be optimal when
\[ \frac{m_2}{m_0} - 1 + α \left( m_2 - 1 \right) m_0 + 1 - \frac{m_2}{m_0} + β \left( m_2 - \frac{m_2}{m_0} \right) < 0, \]
i.e. for
\[ β > \left( \frac{m_2}{m_0} - 1 \right) + α \left( m_2 - 1 \right) m_0 + 1 - \frac{m_2}{m_0} \approx 0.845 - 0.906α. \]

In conclusion we have an estimate that tells us that when β is suitably big, then Ω = Ω_1 ∪ Ω_2 can not be optimal. Writing $γ = 1 - α - β$, we now finally show that Ω can not be optimal also when γ is not very small. We use a technique very similar to the case β big. For this suppose Ω is optimal for the problem (5) and let $|ε| \ll 1$. Then if we enlarge Ω_1 to Ω_1 with measure $m + ε$ and we shrink Ω_2 to Ω_2 with measure $1 - m - ε$, calling $λ_1$ the eigenvalues of Ω = Ω_1 ∪ Ω_2, while $λ_1$ are the eigenvalues of $Ω = Ω_1 ∪ Ω_2$, we must have $F(Ω) - F(Ω^*) ≤ 0$. On the other hand, with analogous computations to those in Step II,
\[ F(Ω) - F(Ω^*) = \left( \frac{γλ_3}{1 - m} - \frac{αλ_1 + βλ_2}{m} \right) ε + o(ε), \]
and hence the expression in brackets must be zero, as otherwise taking $ε > 0$ or $ε < 0$ (this is possible since we are treating only the case of simple eigenvalues) contradicts the optimality of Ω. So if Ω is optimal then $αλ_1 + βλ_2 = γλ_3 \frac{m}{1 - m}$. Since $m \rightarrow \frac{1}{1 - m}$ is increasing we have the lower bound
\[ F(Ω, α, β) = γλ_3 \frac{m}{1 - m} + γλ_3 ≥ γλ_1(B) \frac{1}{1 - 2m_1^2}. \]

We can show that, for $γ$ suitably big, comparing the functional for Θ with the lower bound above gives an absurd. In fact, the functional for the two balls is given by
\[ F(Θ, α, β) = (α + β)2λ_1(B) + γ2λ_2(B) = 2λ_1(B) + γ(2λ_2(B) - 2λ_1(B)). \]
Hence $F(Θ, α, β) < F(Ω, α, β)$ for $γ > \overline{γ} ≈ 0.104$, in which case two balls are better than our set Ω. Combining the cases in which either $γ > \overline{γ}$ or (15) holds concludes Step III and hence the proof of the lemma.

4.1 Proof of Theorem 2 (b).

Proof of Theorem 2 (b). It is proved in Lemma 7 that any disconnected minimiser has multiple eigenvalues. Let Ω be a connected minimiser for $\inf \{ αλ_1(Ω) + (1 - α)λ_3(Ω) \} \text{ open in } \mathbb{R}^2, |Ω| ≤ 1$. The case $λ_2(Ω) = λ_3(Ω)$ is then ruled out, as it would give
\[ αλ_1(Ω) + βλ_2(Ω) + (1 - α - β)λ_3(Ω) = αλ_1(Ω) + (1 - α)λ_3(Ω) > αλ_1(Ω) + (1 - α)λ_3(Ω), \]
\[ ≥ αλ_1(Ω) + βλ_2(Ω) + (1 - α - β)λ_3(Ω). \]
Therefore any disconnected minimiser must satisfy $\lambda_1(\Omega) = \lambda_2(\Omega)$ and can be viewed as the union of a disk supporting the first eigenvalue with a set supporting the second and third.

With $\lambda_1(\Omega) = \lambda_2(\Omega)$, a minimiser with three components would be the union of three disks with at least two being of equal measure. Lemma 4 rules out this configuration.

4.2 Proof of Theorem 2 (c).

Proof of Theorem 2 (c). Let $\alpha + \beta < 1$, and let $\tilde{\Omega}$ be a connected minimiser for $\inf \{(\alpha + \beta)\lambda_2(\Omega) + (1 - \alpha - \beta)\lambda_3(\Omega) \mid \Omega \text{ open in } \mathbb{R}^2, |\Omega| \leq 1\}$. Theorem 2 (b) then gives $\lambda_1(\Omega) = \lambda_2(\Omega)$, whereby $\alpha \lambda_1(\Omega) + \beta \lambda_2(\Omega) + (1 - \alpha - \beta)\lambda_3(\Omega) = (\alpha + \beta)\lambda_2(\Omega) + (1 - \alpha - \beta)\lambda_3(\Omega)$

$> (\alpha + \beta)\lambda_2(\tilde{\Omega}) + (1 - \alpha - \beta)\lambda_3(\tilde{\Omega})$

$\geq \alpha \lambda_1(\tilde{\Omega}) + \beta \lambda_2(\tilde{\Omega}) + (1 - \alpha - \beta)\lambda_3(\tilde{\Omega})$.

5 Appendix

This appendix is devoted to a different proof of the connectedness of the minimiser for the problem

$$\inf \{\alpha \lambda_1(\Omega) + (1 - \alpha)\lambda_2(\Omega) \mid \Omega \subseteq \mathbb{R}^2, \text{ open, with } |\Omega| = 1\},$$

(17)

for $\alpha > 0$. This corresponds to Theorem 2 (a) (ii). It was proved in Paragraph 3.1 that for problem (17) the unit ball is better than every disconnected set if $\alpha \in (\alpha_2, 1]$, while when $\alpha \in (0, 1/2)$ the best disconnected set is the disjoint union of two equal balls, say $\Theta$. We aim to give a proof of the fact that $\Theta$ can not be optimal for (17) unless $\alpha = 0$, that does not rely on Lemma 6. We focus on the case $\alpha \in (0, 1/2)$.

We need to introduce the set

$$\mathcal{E} = \{(\lambda_1(\Omega), \lambda_2(\Omega)) \mid \Omega \subseteq \mathbb{R}^2 \text{ open, with } |\Omega| = 1\}.$$

(18)

For a description of many properties of this set and a numerical approximation of it we refer to [6] or to [10, Chapter 6.4]. The property which interests us deals with the lower part of the boundary of $\mathcal{E}$, the curve $\mathcal{C}$ that joins the point $A = (\lambda_1(\Theta), \lambda_2(\Theta))$ and $B = (\lambda_1(B), \lambda_2(B))$. Wolf and Keller [13] proved that the curve $\mathcal{C}$ must be vertical at the point $B$ by a perturbation argument with nearly circular domains. They also suggested that $\mathcal{C}$ should be horizontal at $A$, and this was proved recently by Brasco, Nitsch and Pratelli [4]. This is the crucial point of our proof, as a minimiser for the convex combination $\alpha \lambda_1(\Omega) + (1 - \alpha)\lambda_2(\Omega)$ is given by the set corresponding to the first point in which the straight line $\alpha x + (1 - \alpha) y = a$ touches $\mathcal{E}$, by increasing $a$. In particular, for $\alpha = 0$ this line is $y = \lambda_2(\Theta) = 2\lambda_1(B)$ by the Krahn-Szegö inequality. On the other hand, for all $\alpha \in (0, 1/2)$, it is possible to find a set $\Omega$ that is linked to a line of the form $\alpha x + (1 - \alpha)y = a_\alpha$, with $a_\alpha < \lambda_2(\Theta) = 2\lambda_1(B)$, since the curve $\mathcal{C}$ has horizontal tangent. Hence $\Theta$ can not be the minimiser for (17) unless $\alpha = 0$.

Acknowledgments. We warmly thank Michiel van den Berg and Aldo Pratelli for their suggestions on the paper. This work has been supported by the ERC Starting Grant n. 258685 “AnOptSetCon”. 

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References


