A METRIC APPROACH TO ELASTIC REFORMATIONS

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Abstract. We study a variational framework to compare shapes, modeled as Radon measures on $\mathbb{R}^N$, in order to quantify how they differ from isometric copies. To this purpose we discuss some notions of *weak deformations* termed *reformations* as well as integral functionals having some kind of *isometries* as minimizers. The approach pursued is based on the notion of pointwise Lipschitz constant leading to a space metric framework. In particular, to compare general shapes, we study this *reformation* problem by using the notion of transport plan and Wasserstein distances as in optimal mass transportation theory.

Contents

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Introduction

One of the main goal in shape analysis relies in detecting and quantifying differences between shapes. The interest for such studies concerns a wide range of applications, especially those within the computer vision community, in particular in pattern recognition, image segmentation, and computation anatomy (see [50, 12, 20]). In recent years many authors have focused their attention on the notions of *shape space* and *shape metric* to the aim of establishing a general framework in which the analysis of shapes crucially depends on their invariance with respect to suitable geometric transformations (see [25, 12, 29, 58]). A natural suggestion in this direction comes from continuum mechanics since the variational theory of elasticity can be used to compare the initial and final shape of a deformable material body, i.e. to establish how the two shapes differ from an isometry of the euclidean space. Therefore some authors begin to study the possible links between elastic energies and distances in shape spaces (see [25, 65, 66]).

On the other side, by arguing from a mechanical perspective, we know that a large class of physical manifestations (fractures, fragmentations, material instabilities) require more general kinematical tools than those available in the context of Sobolev maps, hence it seems reasonable to exploit a more general mathematical framework to obtain more accurate descriptions of more complex physical problems.

In this paper we model (material) shapes as Radon measures on subsets $X, Y \subset \mathbb{R}^N$ and study a variational model to the aim of quantifying how a target shape ν on Y differs from an isometric copy of μ on X. To this purpose we scrutiny some notions of *weak* deformations, which we denote by the term *reformations*, as well as energy like (or cost) functionals having some kind of *isometries* between μ and ν as minimizers.

In the first part of the paper (Sections 1,2,3) we study the variational problem of *reformation* of two shapes μ and ν through functions called *reformation maps*, while in the second part (Sections 4,5,6) we relax the problem by considering a formulation in terms of *transport plans* which leads to a variational framework as in optimal transport theory.

For reader convenience we have added an appendix containing some basic tools from analysis in metric spaces.

Description of the variational model and main results

To *quantify* how two shapes $X, Y \subset \mathbb{R}^N$ are close to be *isometric*, an usual way relies in considering $Y = u(X)$ for maps u belonging to a suitable class of admissible maps. The two shapes are *isometric* if there exists $u : X \to Y$ such that $u(X) = Y$ and

$$|u(x) - u(y)| = |x - y|, \quad \forall x, y \in X.$$

Equivalently, the last condition means that the map u has bi-Lipschitz constant $L = 1$. Let us recall that a map $u : x \to Y$ is said to be bi-Lipschitz with constant L if

$$\frac{1}{L}|x - y| \leq |u(x) - u(y)| \leq L|x - y|, \quad \forall x, y \in X.$$

Therefore, the two shapes X, Y could be considered close to be *isometric* as the bi-Lipschitz constant L is close to one, so assuming the bi-Lipschitz constant as a quantifier of the closeness to the isometry. This approach has the disadvantage to involve a global condition. For instance, the shapes in Figure 0.1 looks very close to be isometric but the bi-Lipschitz constant is quite large and far from $L = 1$, whatever the size of the bending part. To avoid this difficulty some

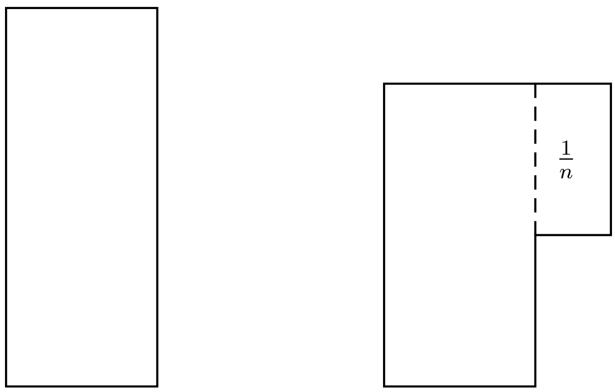

Figure 0.1. Bending a rectangle.

localization procedure is needed. This can be done by an analytical approach. An isometry u is of course an affine map $u(x) = Ax + b$ and $\nabla u = A$ is an orthogonal matrix. Actually, under some regularity assumptions, by Liouville

Rigidity Theorem the orthogonality of the Jacobian matrix characterizes the isometric maps (see also Theorem 3.8). Hence, a reasonable way to quantify how two shapes are isometric is that of measuring how ∇u is close to be an orthogonal matrix. This program can be carried on by selecting a function W reaching its minimal value at the orthogonal matrices. Then, by Liouville Rigidity Theorem, it follows that the isometries characterize the minimal possible value of the functional $I(u) = \int_\Omega W(\nabla u)\; dx$. This approach is pursued in [66] for smooth 2-dimensional domains where the admissible maps are incompressible diffeomorphisms, i.e. u such that $\det(\nabla u) = 1$.
In order to characterize the isometries, a polyconvex function W having minimal value at orthogonal matrices is selected. Therefore, to quantify how two domains $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ are close to be isometric one considers the variational problem

$$\min \left\{ \int_{\Omega_1} W(\nabla u)\; dx \mid u(\Omega_1) = \Omega_2,\ u \in \mathcal{D} \right\},$$

where $\mathcal{D}$ denotes the set of incompressible diffeomorphisms. This approach has of course many restrictions. For instance, to compare a connected domain to a disconnected one, or for non-smooth domains, many regularity questions arise. In continuum mechanics, one usually looks for minimizers $u : \overline{\Omega} \to \mathbb{R}^N$ of the stored energy $I(u) = \int_\Omega W(\nabla u)\; dx$ in an admissible class of deformations usually consisting of Sobolev functions which are locally orientation preserving, i.e. $\det \nabla u(x) > 0$ for a.e. $x \in \Omega$.

A main goal of our approach relies in exploiting possible extensions of the variational scheme of elasticity in order to compare more general shapes as those in Figure 4.1, also allowing *fragmentations*. However, a purely measurable setting does not work to compare shapes as shown in Example 3.6 and, on the other hand, to compare a more extended class of shapes we have to reduce regularity requirements. So, a useful compromise relies in working on a general metric framework.
We remark that an approach like the one followed in [66] cannot be pursued in a metric framework, indeed the mapping $A \mapsto \varphi(\|A\|)$ is polyconvex only if φ is a positive convex and strictly increasing function (see for instance [16]), therefore the minimal value cannot be reached at orthogonal matrices A, since they have $\|A\| = 1$.
We denote by $\mathcal{P}(X)$ the space of probability measures on the metric space X. Assume the material shapes are given by probability measures $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$, (to fix ideas consider $\mu = \mathcal{L}^N \llcorner X$, $\nu = \mathcal{L}^N \llcorner Y$).

In this paper we assume the *pointwise Lipschitz constant* $\mathrm{Lip}(u)(x)$ (see Definition 1.1) as a local descriptor to measure how an admissible map u is an *expansion* or a *contraction*. Note that the *pointwise Lipschitz constant* $\mathrm{Lip}(u)(x)$ coincides with the operator norm $\|\nabla u(x)\|$ whenever u is a differentiable map. Hence the local expansion and contraction due to the map u at any point x are respectively represented by the functions $e_u(x)$ and $c_u(x)$ (see Definition 2.1) depending on $\mathrm{Lip}(u)(x)$.
Let $H(x), K(x) > 0$ be given. We require the admissible maps $u : X \to Y$ satisfy the conditions

$$u_{\#}\mu = \nu, \tag{0.1}$$

$$e_u \leq K, \;\; c_u \leq H. \tag{0.2}$$

These maps will be called *reformation maps*, and the set of such functions will be denoted by $\mathrm{Ref}(\mu;\nu)^{H,K}$ (see Definition 2.2). We consider the local reformation energy $r_u = e_u + c_u$ and the total reformation energy $\mathscr{R}(u) = \int_X r_u(x)\, d\mu$, so in Theorem 3.10 we show that the variational problem

$$\min\{\mathscr{R}(u) \mid u \in \mathrm{Ref}(\mu;\nu)^{H,K}\} \tag{0.3}$$

admits solutions whenever $\mathrm{Ref}(\mu;\nu)^{H,K} \neq \emptyset$. Therefore, to quantify how the two measures μ, ν are isometric we look to the *elastic reformation energy* between μ and ν defined by

$$\mathcal{E}(\mu,\nu) := \inf\{\mathscr{R}(u) \mid u \in \mathrm{Ref}(\mu;\nu)^{H,K}\}. \tag{0.4}$$

In Theorem 3.15 we show that the value of (0.4) is attained and equal to 2 if and only if the two shapes are isometric.
In Section 4 we extend the scenario to deal with the case of non-existence of maps satisfying (2.4) and this happens, for instance, when fragmentation occurs. In such a case the notion of transport plan coming from the optimal mass transportation or Monge-Kantorovich theory is useful. A transport plan between μ and ν is a measure $\gamma \in \mathcal{P}(X \times Y)$, having μ, ν as marginals, namely $(\pi_1)_{\#}\gamma = \mu, (\pi_2)_{\#}\gamma = \nu$, where $\pi_{1,2}$ are the projections of $X \times Y$ on its factors. The notion of transport plan could be considered as a generalization of a transport map, i.e. $u : X \to Y$, such that $u_{\#}\mu = \nu$, and so as a *weak* notion of reformation of μ into ν. We shall refer to such measures as *reformation plans*. Actually, to every transport map corresponds the transport plan given by $(I \times u)_{\#}\mu$. The shapes μ, ν could be compared by considering the local mass transportation displacement in the target configuration.

More precisely, by Disintegration Theorem (see [3, Section 2.5]) every transport plan $\gamma \in \mathcal{P}(X \times Y)$ can be written as $\gamma = f(x) \otimes \mu$, where

$$f : X \to (\mathcal{P}(Y), W) \tag{0.5}$$

is called *disintegration map* and $\mathcal{P}(Y)$ is equipped with the Wasserstein distance W. This point of view leads to formulate the reformation problem in terms of disintegration map f and related metric expansion and contraction energies (see Definition 4.6). So, in this setting, reformation maps take value in the space of probabilities, endowed with the Wasserstein metric, over the target domain. The main advantage of this approach relies in its generality and in its connections with fertile topics as optimal mass transportation and geometric measure theory. However, many interesting open questions arise as the regularity needed on f to capture relevant geometrical and physical properties of the shapes.
In Section 5 we show several examples of shape reformations attainable by disintegration maps but not attainable by any regular transport map.
In Section 6 we study the main aspects of the variational problems of reformation in the enlarged context of generalized reformations, showing in Theorem 6.4 how isometric measures can be characterized by means of the reformation energy. In Theorem 6.8 we prove the existence of minimizing reformation plans for a constrained variational problem.

1. The pointwise Lipschitz constant

In this section we introduce the notion of *pointwise Lipschitz constant* and scrutiny some properties related to this tool since it will play a crucial role in this paper.

Definition 1.1. *Let (X, d_X), (Y, d_Y) be two metric spaces and let $f : (X, d_X) \to (Y, d_Y)$. The pointwise Lipschitz constant* $\mathrm{Lip}(f)(x_0)$ *of f at $x_0 \in X$ is defined by*

$$\mathrm{Lip}(f)(x_0) := \begin{cases} \limsup\limits_{x \to x_0} \dfrac{d_Y(f(x), f(x_0))}{d_X(x, x_0)} & \text{if } x_0 \text{ is a non-isolated point,} \\ 0 & \text{if } x_0 \text{ is an isolated point.} \end{cases} \tag{1.1}$$

It is readily seen that $\mathrm{Lip}(f)$ is a measurable function. The pointwise Lipschitz constant leads to a global Lipschitz's constant on convex sets.

Lemma 1.2 (Lemma 14.4 of [10])**.** *Let $L > 0$, $X \subset \mathbb{R}^N$ a convex set and let $f : X \to (Y, d)$ be a function such that* $\mathrm{Lip}(f)(x) \leq L \ \forall x \in X$*. Then f is L-Lipschitz.*

A result similar to the previous lemma holds true for *quasi-convex* metric spaces X (see [19]).
A metric space (X, d) is C-quasi-convex if there exists a constant $C > 0$ such that for each pair of points $x, y \in X$, there exists a curve γ connecting x and y with $l(\gamma) \leq Cd(x, y)$. As one can expect, a metric space is quasi-convex if, and only if, it is bi-Lipschitz homeomorphic to some length space. For X C-convex, the function f of Lemma 1.2 is just CL-Lipschitz.

The pointwise Lipschitz constant is also related to the notion of metric differential (see [7, 43, 44, 51]). A function $f : X \subset \mathbb{R}^N \to (Y, d)$ is said to be metrically differentiable at a point $x_0 \in X$ if there exists a (unique) on $\mathbb{R}^N$, denoted by $MD(f, x_0)$, such that for every $y, z \in X$ the following formula holds true

$$d(f(y), f(z)) - MD(f, x_0)(y - z) = o(\|y - x_0\| + \|z - x_0\|). \tag{1.2}$$

Let $U \subset \mathbb{R}^N$ be an open set and let $f : U \to (Y, d)$ be a Lipschitz function. Hence, for every fixed $p \in Y$ the function

$$x \mapsto d(f(x), p) : U \to \mathbb{R}_+ \tag{1.3}$$

is a Lipschitz function and by Rademacher Theorem it is a.e. differentiable in U. Moreover (see [43, 44]), it turns out that f is *metrically* differentiable at almost every point.

The following lemma establishes a link between the pointwise Lipschitz constant, the distance function (see [6] for dual Banach spaces) and the metric differential (see (1.7) below).

Lemma 1.3. *Let $f : U \subset \mathbb{R}^N \to (Y, d)$ be a Lipschitz function over a separable metric space Y. Then, for a.e. $x_0 \in U$ it results*

$$\mathrm{Lip}(f)(x_0) = \sup_{p \in Y} \|\nabla d(f(x_0), p)\|. \tag{1.4}$$

Proof. We assume $\|\nabla d(f(x_0), p)\| = 0$ if the function $x \mapsto d(f(x), p)$ is not differentiable at x_0. For $p \in Y$ we compute

$$\begin{aligned} \langle \nabla d(f(x_0), p), v \rangle &= \lim_{t \to 0^+} \frac{d(f(x_0 + tv), p) - d(f(x_0), p)}{t} \\ &\leq \lim_{t \to 0^+} \frac{d(f(x_0 + tv), f(x_0))}{t} \leq \mathrm{Lip}(f)(x_0). \end{aligned}$$

Taking the supremum with respect to v and then respect to p, we have

$$\sup_{p\in Y} \|\nabla d(f(x_0), p)\| \leq \mathrm{Lip}(f)(x_0).$$

To get the opposite inequality, we use a slight modification of the proof of [7, Theorem 4.1.6]. Since Y is separable, we fix a countable dense set $\{p_n\} \subset Y$, then for every $x_1, x_2 \in U$ we have

$$d(f(x_1), f(x_2)) = \sup_n |d(f(x_1), p_n) - d(f(x_2), p_n)|. \tag{1.5}$$

Consider the Lipschitz function $\varphi_n(t) = d(f(x_0 + tv), p_n)$ and set $m(t) = \sup_n |\dot{\varphi}_n(t)|$. Observe that $|m(t)| \leq \mathrm{Lip}(f)$. By the Lipschitz condition, we may suppose that, for every $n \in \mathbb{N}$, $t = 0$ is a differentiability point for φ_n and also $t = 0$ is a Lebesgue point for $m \in L^\infty$. By (1.5) we obtain

$$\frac{d(f(x_0 + tv), f(x_0))}{t} \leq \sup_n \frac{1}{t} \int_0^t |\dot{\varphi}_n(s)| ds \leq \frac{1}{t} \int_0^t m(s) ds.$$

Letting $t \to 0^+$ we get (see Prop. 1 and Th. 2 of [44])

$$MD(f, x_0)(v) \leq m(0) \leq \sup_n \|\nabla d(f(x_0), p_n)\|. \tag{1.6}$$

On the other hand, by (1.2) we get

$$\frac{d(f(x), f(x_0))}{\|x - x_0\|} = MD(f, x_0)\left(\frac{x - x_0}{\|x - x_0\|}\right) + \frac{o(\|x - x_0\|)}{\|x - x_0\|}$$

Letting $x \to x_0$, by (1.6) we get

$$\mathrm{Lip}(f)(x_0) \leq \sup_n \|\nabla d(f(x_0), p_n)\|.$$

□

Observe that by the proof of the previous lemma, the following equality also holds true

$$\mathrm{Lip}(f)(x_0) = \sup_{v\in\mathbb{R}^N,\ |v|=1} |MD(f, x_0)(v)|. \tag{1.7}$$

Lemma 1.4. *Let (Y, d) be a separable metric space. Assume $U \subset \mathbb{R}^N$ is an open set, $(f_n)_{n\in\mathbb{N}}$ be a sequence of (locally) equi-Lipschitz functions $f_n : U \to (Y, d)$ and let $f : U \to (Y, d)$. If $f_n \to f$ (locally) uniformly on U then*

$$\int_U \mathrm{Lip}(f)(x)\ dx \leq \liminf_{n\to+\infty} \int_U \mathrm{Lip}(f_n)(x)\ dx. \tag{1.8}$$

Proof. By uniform convergence f is a (locally) Lipschitz function, moreover we have that $d(f_n(\cdot), p) \rightharpoonup d(f(\cdot), p)$ weakly$*$ in $W^{1,\infty}_{loc}(U)$. Therefore, for every p, by weak l.s.c. of the gradient norm (see also Ch. III Th. 3.3 of [60]) and using (1.4) we get

$$\int_U \|\nabla d(f(x), p)\| dx \le \liminf_{n\to+\infty} \int_U \|\nabla d(f_n(x), p)\| dx \le \liminf_{n\to+\infty} \int_U \mathrm{Lip}(f_n)(x)\ dx.$$

Since Y is separable, as in the proof of Lemma 1.3, denoting by $g_n = \|\nabla d(f(x), p_n)\|$, by (1.4) we may assume that $\mathrm{Lip}(f)(x) = \lim_n g_n(x)$. Moreover, observe that $|g_n(x)| \le \mathrm{Lip}(f)$. Hence, passing to the limit under the integral sign we finally obtain

$$\int_U \mathrm{Lip}(f)(x)\ dx = \lim_{n\to+\infty} \int_U g_n(x) dx \le \liminf_{n\to+\infty} \int_U \mathrm{Lip}(f_n)(x)\ dx.$$

□

Remark 1.5. The above Lemma holds true of course for the function $\mathrm{Lip}^p(f)$, for any $p \ge 1$. If $Y \subset \mathbb{R}^N$, the uniform convergence can be replaced by the weak convergence on the Sobolev space $W^{1,p}(U)$. In such a case, Lemma 1.4 just states the lower semicontinuity of the p-Dirichlet energy in Sobolev spaces, since if $u : \mathbb{R}^N \to \mathbb{R}^N$ is differentiable at x then $\mathrm{Lip}(u)(x) = \|\nabla u(x)\|$ (see also [60, Ch. 3 Theorem 3.3] and [61] for a related semicontinuity result).

2. Reformation maps

In this section we introduce the class of *reformation maps* and establish some properties of these functions. Though the definition of reformation map holds for general metric measure spaces, as a first step we restrict our analysis to the euclidean framework of subsets of $\mathbb{R}^N$.
Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set, $X = \overline{\Omega}$, $Y \subset \mathbb{R}^N$, $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$.

Definition 2.1 (Expansion and contraction energy)**.** *Let $x_0 \in X$ and $u : X \to Y$. The pointwise expansion energy of u at x_0 is defined by*

$$e_u(x_0) := \mathrm{Lip}(u)(x_0) = \limsup_{x\to x_0} \frac{|u(x) - u(x_0)|}{|x - x_0|}. \tag{2.1}$$

The pointwise contraction energy of u at x_0 is defined by

$$c_u(x_0) := \limsup_{x\to x_0} \frac{|x - x_0|}{|u(x) - u(x_0)|}. \tag{2.2}$$

The pointwise reformation energy of u *at* x_0 *is defined by*

$$r_u(x_0) = e_u(x_0) + c_u(x_0). \tag{2.3}$$

Definition 2.2 (Reformation maps)**.** *Given* $H, K : X \to]0, +\infty[$*,* $H, K \in L^1(X, \mu)$ *and a fixed covering* $\mathcal{A}$ *of* X *made by balls, we define the set of reformation maps between* μ *and* ν*, which we shall denote by* $\mathrm{Ref}(\mu;\nu)^{H,K}$*, as the set of maps* $u : X \to Y$ *such that the following conditions hold true:*

$$u_{\#}\mu = \nu, \tag{2.4}$$

$$\forall x \in X\ \exists B(x,r) \in \mathcal{A}\ s.t.\ c_u(y) \leq H(x),\ e_u(y) \leq K(x)\ \forall y \in \overline{B}(x,r) \cap \Omega, \tag{2.5}$$

where $u_{\#}\mu$ *is the probability measure on* Y *defined by* $u_{\#}\mu(A) = \mu(u^{-1}(A))$ *for every Borel set* A *of* Y*.*

The point in the above definition is that the functions c_u, e_u are locally bounded from the above by $H(x), K(x)$ which may depend just on the point x an not by the map $u \in \mathrm{Ref}(\mu;\nu)^{H,K}$.Therefore, the reformation maps are characterized by locally uniformly bounded expansion and contraction. Notice that, by the bounds (2.5), any $u \in \mathrm{Ref}(\mu;\nu)^{H,K}$ is continuous and, by Stepanov Theorem (see [36]), is a.e. differentiable in Ω. In particular, by Lemma 1.2 reformation maps are locally Lipschitz on Ω.

Remark 2.3. In a mechanical perspective, the constraints stated in (2.5) could be considered as a bound on the maximum expansion or contraction experienced by the material Ω. In this setting, the assumption that the bounds $H(x), K(x)$ do not depend on the map u in (2.5) corresponds to a constitutive property of the material under consideration. We point out that bounds like (2.5) are in some sense necessary to control the geometry of the reformations. For instance, in the case of $\nu = \delta_{y_0}$ we have $e_u = 0$, $c_u = +\infty$ for any map u satisfying (2.4). On the other hand, mapping a bar into a bended one (see Fig. 2.1) by two piecewise isometries u_1, u_2 such that $u_1(x_0) \neq u_2(x_0)$, we necessarily create a *fracture* at the point x_0. It results $e_u(x_0) = +\infty$ at the discontinuity point x_0. See also Example 3.6. Therefore, roughly speaking, the bound $c_u \leq H$ means no implosion , while $e_u \leq K$ means no fractures.

Remark 2.4. The constraint $c_u \leq H$ in (2.5) is related to inversion properties, both local or global, of reformation maps, see [26, 27, 41]. Observe that for differentiable maps with non-vanishing Jacobian we always have

$$c_u = \|(\nabla u)^{-1}\|, \quad e_u = \|\nabla u\|.$$

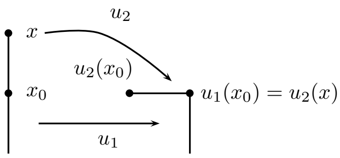

FIGURE 2.1. Mapping a bar into a bended one.

Therefore, it would be an interesting question to consider just pointwise bounds $H(x), K(x)$. This is similar in spirit to the passage from functions with bounded distortion to functions with finite distortion (see the monograph [40]). However, in such a case, inversion properties becomes more subtle and further assumptions are needed, see for instance [47, 48, 59] for inversion results of Sobolev maps. Anyway, the main interest of reformation maps relies in this perspective in considering just metric objects (see Section 4 below). However, in a purely metric framework, such pointwise conditions are not enough to guarantee inversion properties. Consider for instance the map $u : \mathbb{R} \to \mathbb{R}, u(x) = |x|$ having $e_u = c_u = 1$ at every point. Therefore, also uniform bounds alone are not enough to get satisfactory inversion properties. Under differentiability assumptions in $\mathbb{R}^N$, pointwise bounds are in fact enough, see Lemma 2.10. However, in general this is not true. For instance (see [55]), it is possible to find everywhere differentiable maps with everywhere invertible differential on Hilbert spaces which are not neither open or locally one-to-one. In the metric setting different restrictions arise (see [27] for a detailed discussion).

Remark 2.5. The local uniform bounds in Def. 2.2 are also related to quasi-isometries, see for instance [10, 41]. In such case uniform bounds

$$m \leq D^- f(x) \leq D^+ f(x) \leq M,$$

where

$$D^- f(x_0) = \liminf_{x \to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|}, \qquad D^+ f(x_0) = \limsup_{x \to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|}$$

denote the local distortion of distances, are required. Observe that $e_u(x) = D^+ f(x)$, while $c_u(x) = \frac{1}{D^- f(x_0)}$. Indeed,

$$\frac{|x - x_0|}{|f(x) - f(x_0)|} = \frac{1}{\frac{|f(x)-f(x_0)|}{|x-x_0|}} \leq \frac{1}{\inf_B \frac{|f(x)-f(x_0)|}{|x-x_0|}}.$$

Taking the supremum over $B = B(x_0, r)$ and then letting $r \to 0^+$ we get $c_u(x) \leq \frac{1}{D^- f(x_0)}$. In a similar way the opposite inequality follows.

Remark 2.6. The mass conservation property (2.4) is a generalized version of incompressibility and it can be always satisfied (provided μ has no atom, see [54]) by some measurable map u. Actually, condition (2.4) is equivalent to the following change of variable formula

$$\int_X f(u(x))\, d\mu = \int_Y f(y)\, d\nu, \tag{2.6}$$

for every continuous function $f : Y \to \mathbb{R}$. In the setting of Mass Transportation, maps u satisfying (2.4) are called *transport maps*.

Remark 2.7. Since $\mathrm{Ref}(\mu;\nu)^{H,K}$ is made by *nice* functions, to compare the present approach with other ones, as for instance that of [66], one should also require the condition $u(X) = Y$ in the case of $\mu = \mathcal{L}^N \llcorner X$, $\nu = \mathcal{L}^N \llcorner Y$. However, we point out that this surjection requirement is actually a severe constraint. Indeed, for instance, to find Lipschitz functions $u : X \to Y$, $u(X) = Y$ for general compact sets in dimension $N \geq 3$ (see [1]), as far as we know, is still an open question. Moreover, also bi-Lipschitz functions between *nice* sets are not easy to find (see [22, 33]). Anyway, in such a case many restrictions on the target space Y may be needed (connectedness, for instance).

In the following we prove some properties enjoyed by reformation maps. A first estimate easy to verify is an immediate consequence of (2.5) and is given by the following proposition.

Proposition 2.8. *Let $u \in \mathrm{Ref}(\mu;\nu)^{H,K}$. Then, for every $x_0 \in X$ there exists $r > 0$ such that*

$$\frac{1}{H(x_0)}|x - x_0| \leq |u(x) - u(x_0)| \leq K(x_0)|x - x_0| \quad \forall\, x \in X \cap \overline{B}(x_0, r). \tag{2.7}$$

Lemma 2.9. *Let $u \in \mathrm{Ref}(\mu;\nu)^{H,K}$. Then u is a discrete map, i.e. for every $y \in Y$, $u^{-1}(y)$ is a finite set.*

Proof. Let $y \in Y$. By (2.7), if $x_0 \in u^{-1}(y)$ we have that $u(x) \neq u(x_0)$ for every $x \in \overline{B}(x_0, r)$. Hence, x_0 is an isolated point of $u^{-1}(y)$. We claim that $u^{-1}(y)$ is a finite set. Indeed, otherwise, since X is compact, we find a sequence $x_n \to x_0 \in X$ such that $x_n \in u^{-1}(y)$. By continuity of u we have also $x_0 \in u^{-1}(y)$. This is a contradiction since x_0 is an isolated point of $u^{-1}(y)$. $\square$

For discrete continuous maps the *local degree* or *local index* $i(x_0, u)$ of $u : X \to Y$ at $x_0 \in X$ is defined as follows

$$i(x_0, u) = \deg(u(x_0), u, \overline{B}(x_0, r)), \tag{2.8}$$

where $\deg(y, u, B)$ denotes the topological degree (see [21, 60]).
Let $u^{-1}(y) = \{x_1, \cdots, x_h\}$, we have the following relation

$$\deg(y, u, Y) = \sum_{j=1}^{h} i(x_j, u). \tag{2.9}$$

Observe that if u is locally injective in a neighborhood of $x \in X$ then $|i(x, u)| = 1$.
We say that u is a *sense-preserving (reversing) continuous map* if the local index $i(x, u)$ has constant sign in X. Notice that each homeomorphism on a domain is either sense-preserving or sense-reversing (see [21, Theorem 3.35]). Moreover, sense-preserving or sense-reversing differentiable maps have constant Jacobian sign, (see [21, Lemma 5.9]), since

$$i(x_0, u) = sign(Ju(x_0)), \tag{2.10}$$

where $Ju := \det\nabla u$, providing $Ju(x_0) \neq 0$.

Lemma 2.10. *Let* $u \in \mathrm{Ref}(\mu; \nu)^{H,K}$. *If* u *is differentiable at* $x_0 \in \Omega$ *then* $Ju(x_0) \neq 0$.

Proof. Suppose by contradiction that $Ju(x_0) = 0$. Then we find a vector $|v| = 1$ such that $\nabla u(x_0) \cdot v = 0$. Fixed $\varepsilon > 0$, since u is differentiable, there exists $\delta > 0$ such that $|u(x_0 + tv) - u(x_0)| < \varepsilon t$, whenever $|t| < \delta$. On the other hand, by (2.7), there exists $0 < t < \delta$ such that $\frac{t}{H} < |u(x_0 + tv) - u(x_0)| < \varepsilon t$. Hence $\frac{1}{H} < \varepsilon$. Letting $\varepsilon \to 0^+$ we get a contradiction. □

By the previous lemma, any $u \in C^1(X; Y) \cap \mathrm{Ref}(\mu; \nu)^{H,K}$ (or even an everywhere differentiable reformation map) is locally invertible on Ω (see [34] for a related inversion result and [55] for an elementary analytical proof).
If u is only a.e. differentiable, by Lemma 2.10 we have $Ju(x) \neq 0$ for a.e. x. However, it is well known that in general this condition does not ensure the local invertibility of Sobolev maps (see for instance [47]). By the way, the condition $Ju > 0$ on an open set Ω is a standard requirement (see for instance [59]), ensuring that u is locally invertible for a.e. $x \in \Omega$. The restriction to sense-preserving maps is also made in [27] to derive local inversion results. To this

aim, also assumptions on the boundedness of $HK \leq M$, for sufficiently small M are necessary. In our context, since we are interested in comparing domains, also in a metric framework, restrictions to open maps seem more natural.

We refer [9] for a proof of the following result.

Theorem 2.11. *Let $u : \Omega \to \mathbb{R}^N$ be a continuous open and discrete map. Then u is sense-preserving or sense-reversing.*

Lemma 2.12. *Let $u : \Omega \to \mathbb{R}^N$ be a discrete sense-preserving (reversing) continuous map such that $|i(x_0, u)| = 1$. Then u is injective in a neighborhood of x_0.*

Proof. We use here the same arguments of [60, Ch. II Theorem 6.6]. Suppose $i(x_0, u) = 1$. The other case is analogous. By contradiction, if u is not injective we have two distinct sequences $(x_n^1)_{n\in\mathbb{N}}$, $(x_n^2)_{n\in\mathbb{N}}$ converging to x_0 such that for every $n \in \mathbb{N}$: $u(x_n^1) = u(x_n^2) = y_n$. By continuity we also have $u(x_n^1) \to y_0 = u(x_0)$. Since the degree is constant in a neighborhood of y_0, for $n \in \mathbb{N}$ large enough and suitable small radius $r > 0$ we have

$$\deg(y_n, u, \overline{B}(x_0, r)) = \deg(y_0, u, \overline{B}(x_0, r)) = i(x_0, u) = 1. \tag{2.11}$$

On the other hand, since u is sense-preserving and by (2.9) we have

$$\deg(y_n, u, \overline{B}(x_0, r)) \geq i(x_n^1, u) + i(x_n^2, u) = 2,$$

contradicting (2.11). □

Theorem 2.13 (Invertibility of incompressible maps)**.** *Assume $u \in \mathrm{Ref}(\mu; \nu)^{H,K}$ is an incompressible map (see for instance* [66]*), i.e.*

$$|Ju(x)| = 1 \text{ for a.e. } x \in \Omega. \tag{2.12}$$

Then $u_{|\Omega}$ is globally invertible.

Proof. Of course, condition (2.4) holds true for injective such maps u. Anyway, let us begin by showing that u is an open map. Let U be an open subset of Ω. We have to prove that $V = u(U)$ is open. Let $y_0 \in V$, $y_0 = f(x_0)$ for a $x_0 \in U$. By (2.7) we find $C = \overline{B}(x_0, r)$ such that $y_0 \notin u(\partial C)$. Therefore, it is well defined $\deg(y_0, u, C)$ which is constant in a neighborhood of y_0. If u is differentiable at x_0, by Lemma 2.9 and Lemma 2.10 we have

$$|\deg(y, u, C)| = |\deg(y_0, u, C)| = 1,$$

in a neighborhood V_{y_0} of y_0. Since $\deg(y, u, C) \neq 0$, it follows that $V_{y_0} \subset u(C) \subset V$, since otherwise the degree would be zero.

On the other hand, denoting by $N(y,u,K) := \mathrm{card}\{u^{-1}(y)\cap K\}$ the multiplicity function, by the Area Formula and the push-forward condition (2.6) we compute

$$\mathcal{L}^N(u(B(x_0,r))) \le \int_{u(B(x_0,r))} N(y,u,B(x_0,r))\;dy = \int_{u^{-1}(u(B(x_0,r)))} |Ju|\;dx =$$

$$\mathcal{L}^N(u^{-1}(u(B(x_0,r)))) = \mathcal{L}^N(u(B(x_0,r))).$$

Therefore, for a.e. $y \in u(B(x_0,r))$, it results $N(y,u,B(x_0,r)) = 1$. Hence, if x_0 is a non-differentiability point, by the Lusin (N)-property, there exists a differentiability point $x \in B(x_0,\delta)$ of u and an open neighborhood V_{y_0} of y_0 such that $u(x) \in V_{y_0}$ and $\deg(u(x),u,C) \neq 0$. As above it follows that $V_{y_0} \subset u(C) \subset V$. Hence $u_{|\Omega}$ is open. By Theorem 2.11 we have that u is sense-preserving, or sense-reversing. Since actually it results $|i(x_0,u)| = 1$, the statement follows by Lemma 2.12. □

Remark 2.14. If a reformation map satisfies (2.12), then $|i(x,u)| = 1$ for a.e. $x \in \Omega$. Therefore, by Lemma 2.12 such map u is a.e. locally invertible on Ω. As previously discussed, actually to obtain this invertibility property it is enough to require $|Ju| \le 1$ a.e. in Ω. Since $|Ju| \le \|\nabla u\|^N$, this happens for instance for reformation maps u satisfying the condition $e_u \le 1$.

Theorem 2.15 (Invertibility of small reformations)**.** *Let $u \in \mathrm{Ref}(\mu;\nu)^{H,K}$ be such that $e_u(x) < \sqrt[N]{2}$ for a.e. x. Then u is globally invertible.*

Proof. Recalling the assumption that both μ and ν have density given by characteristics functions of an open set, by the constraint $c_u \le H$ we find (see [52, Prop. 1.1, Sec. 3]) an open dense subset $U \subset X$ on which u is locally bi-Lipschitz. It follows that the multiplicity function $N(y,u,U)$ is locally constant (see [2]). We first prove that u is globally invertible on U. For $y = u(x) \in u(U)$, we prove that $N(y,u,U) = 1$. Observe that by the Domain Invariance Theorem (see for instance [21]), $u_{|U}$ is open. Let $B = B(x,r)$ be a ball on which u is bi-Lipschitz. We may suppose that $D = N(y,u,B)$ is constant on $u(B)$. Since $|Ju| \le \|\nabla u\|^N$, by the Area Formula we compute

$$\begin{aligned} D\mathcal{L}^N(u(B)) &= \int_{u(B)} N(y,u,B)dy = \int_{u^{-1}(u(B))} |Ju(x)|dx \\ &\le \int_{u^{-1}(u(B))} \|\nabla u\|^N dx = \int_{u^{-1}(u(B))} e_u(x)^N dx. \end{aligned} \tag{2.13}$$

Therefore, if the map u satisfies the small expansion condition $e_u < \sqrt[N]{2}$, by (2.13) and the push-forward condition we get

$$D\mathcal{L}^N(u(B)) < 2\mathcal{L}^N(u^{-1}(u(B))) = 2\mathcal{L}^N(u(B)),$$

hence the map u is globally invertible on U. By uniform continuity, u uniquely extends to the whole X and therefore letting to global invertibility on X. □

Theorem 2.16. *Let $u \in \mathrm{Ref}^{H,K}(\mu;\nu)$ be an open map and suppose that $\forall x \in X : H(x)K(x) < 2$. Then $u_{|\Omega}$ is locally invertible.*

Proof. By Lemma 2.9 and Theorem 2.11 it follows that u is sense-preserving or sense-reversing. Hence, by Theorem II of [27] the thesis follows. □

The condition $e_u c_u \leq \alpha$ may be required to hold just a.e. by reducing the upper bound $\alpha < \sqrt[N]{2}$ (see [27]). Observe that the map $u(x) = |x|$ in one dimension is not a counterexample to Theorem 2.15 since u is not mass preserving around $x_0 = 0$. If Ω is a ball, or in some classes of convex sets, for sufficiently small α the map u is actually globally invertible (see [28]).

3. The variational problem of elastic reformation

Definition 3.1. *Let $u : X \to Y$, $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$, such that $u_{\#}\mu = \nu$. We define the total reformation energy $\mathscr{R}(u)$ of a reformation map u of μ into ν as follows*

$$\mathscr{R}(u) := \int_X r_u(x)\ d\mu. \tag{3.1}$$

We recall that $\mathscr{R}(u) < +\infty$ for every $u \in \mathrm{Ref}(\mu;\nu)^{H,K}$, since we will always assume

$$H(x),\ K(x) \in L^1(X,\mu), \tag{3.2}$$

where H, K are given in Definition 2.2. We have the following

Lemma 3.2. *Let $u : X \to Y$. Then*

$$c_u(x) \geq \frac{1}{e_u(x)} \quad \forall x \in X. \tag{3.3}$$

Proof. It suffices to recall that $c_u(x) = \frac{1}{D^- f(x)}$, see Remark 2.5. □

Observe that for every $u : X \to Y$ such that $u_{\#}\mu = \nu$, by Lemma 3.2 it results $\mathscr{R}(u) \geq 2$.

Actually, Definition 3.1 is motivated by the trivial fact that the real function

$f(x) = x + 1/x$ reaches its minimum value at $f(1) = 2$. Moreover, observing that at any $x_0 \in X$

$$r_u(x_0) \geq e_u(x_0) + \frac{1}{e_u(x_0)} \geq 2, \tag{3.4}$$

we have that $r_u(x_0)$ reaches its minimum value if $u : X \to Y$ is an isometric mapping, i.e. $|u(x) - u(y)| = |x - y|$, $\forall x, y \in X$. Therefore $\mathscr{R}(u)$ can be viewed as a measure detecting how u is far from being an isometric map.

Lemma 3.3. *Let $u : X \to Y$ be a local homeomorphism. Then*

$$c_u(x) = e_{u^{-1}}(u(x)) \quad \forall x \in X. \tag{3.5}$$

Proof. Fix $x \in X$, $\delta_1 > 0$ and let $B_1 = B(u(x), \delta_1)$. By the local homeomorphism condition, there exists a $\delta > 0$ such that $u(B_\delta) \subset B_1$ and u is invertible on $B_\delta = B(x, \delta)$. For every $y \in B_\delta$ we have

$$\frac{|y - x|}{|u(y) - u(x)|} = \frac{|u^{-1}(u(y)) - u^{-1}(u(x))|}{|u(y) - u(x)|} \leq \sup_{z \in B_1} \frac{|u^{-1}(z) - u^{-1}(u(x))|}{|z - u(x)|}.$$

Taking the supremum with respect to $y \in B_\delta$ and letting $\delta_1 \to 0^+$, we get $c_u(x) \leq e_{u^{-1}}(u(x))$. Analogously we deduce the opposite inequality. □

Definition 3.4. *We define the elastic reformation energy between μ and ν as*

$$\mathcal{E}(\mu, \nu) := \inf\{\mathscr{R}(u) \mid u \in \mathrm{Ref}(\mu; \nu)^{H,K}\}. \tag{3.6}$$

In general, the above elastic reformation energy is not symmetric. For instance, if $\mu = \mathcal{L}^N \llcorner B$, for a ball B, and ν a Dirac delta, we have $\mathcal{E}(\mu, \nu) = +\infty$. Reversing the shapes, we see that $\mathcal{E}(\nu, \mu)$ has no meaning simply because $\mathrm{Ref}(\nu; \mu)^{H,K} = \emptyset$. Moreover, also in nice cases, the matter is that reformation maps could be not invertible. Assuming invertibility for $u \in \mathrm{Ref}(\mu; \nu)^{H,K}$, setting $v := u^{-1}$, by using Lemma 3.3 we have

$$\mathscr{R}(u) = \int_X e_u \, d\mu + \int_X c_u \, d\mu = \int_X e_u(u^{-1}(u(x))) \, d\mu + \int_X e_{u^{-1}}(u(x)) \, d\mu =$$
$$\int_Y e_{v^{-1}}(v(y)) \, d\nu + \int_Y e_v(y) \, d\nu = \int_Y c_v(y) \, d\nu + \int_Y e_v(y) \, d\nu = \mathscr{R}(v).$$

Since $v \in \mathrm{Ref}(\nu; \mu)^{H,K}$, we get $\mathcal{E}(\mu, \nu) = \mathcal{E}(\nu, \mu)$. Therefore, symmetry issues essentially correspond to invertibility of maps.

The question is now to establish conditions to ensure the infimum in (3.6) is attained. It is easily seen that

$$\mathscr{R}(u) = 2 \text{ if and only if } r_u(x) = 2 \quad \text{for } \mu - a.e.\ x \in X. \tag{3.7}$$

Lemma 3.5. *Let* $x_0 \in X$, $u : X \to Y$. *Then* $r_u(x_0) = 2$ *if and only if*

$$\forall \varepsilon > 0 : \frac{1}{1+\varepsilon}|x - x_0| \leq |u(x) - u(x_0)| \leq (1+\varepsilon)|x - x_0|, \quad \forall x \in X \cap B(x_0, r_\varepsilon).$$

Proof. Assume $r_u(x_0) = 2$, then

$$2 = e_u(x_0) + c_u(x_0) \geq e_u(x_0) + \frac{1}{e_u(x_0)} \geq 2,$$

so

$$e_u(x_0) + \frac{1}{e_u(x_0)} = 2 \Rightarrow (e_u(x_0) - 1)^2 = 0 \Rightarrow e_u(x_0) = c_u(x_0) = 1.$$

Fix $\varepsilon > 0$, then $e_u(x_0) < 1 + \varepsilon$ implies that u satisfies

$$|u(x) - u(x_0)| \leq (1+\varepsilon)|x - x_0|, \quad \forall x \in X \cap B(x_0, r_\varepsilon). \tag{3.8}$$

By using the condition $c_u(x_0) < 1 + \varepsilon$, eventually by decreasing the radius r_ε, we get the opposite inequality. Vice versa, if both the inequalities locally hold, then it results $2 \leq r_u(x_0) = e_u(x_0) + c_u(x_0) \leq 1 + 1 = 2$. □

Therefore, the maps $u : X \to Y$ such that $r_u = 2$ are in some sense *pointwise* locally quasi-isometric, (see [53] for the relation with quasi-conformal maps).

In the following we shall try to characterize in a more precise way the reformation maps $u \in \mathrm{Ref}(\mu; \nu)^{H,K}$, if any, realizing the minimum energy level $\mathscr{R}(u) = 2$. We also want to prevent pathological situations as the one described in Example 3.6 below in which the map $u : X \to Y$ is merely a.e. continuous (it is actually a.e. invertible and differentiable).

In particular, by (3.4) (see also the proof of Lemma 3.5) it results $e_u(x) = 1$ for μ-a.e. $x \in X$. Moreover, $e_u(x) < +\infty$ implies u continuous at x. Then these reformation maps u are at least a.e. continuous functions. However, this mild regularity is too poor to preserve geometric (or physical) properties as we show in the next example.

Example 3.6. Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded open set and $Q \subset \mathbb{R}^N$ be a cube such that $\mathcal{L}^N(\Omega) = \mathcal{L}^N(Q)$ (see figure 3.1). For $n \geq 1$ large enough, Ω contains a certain number of disjoint squares Q_n of length $\frac{1}{n}$. Then consider the map u_n which isometrically moves every square Q_n inside Q in a disjoint way. On the remainder of Ω, consider the contained squares Q_m, $m > n$, and then the map u_m which coincides with u_n on the squares Q_n and moves by an isometry the squares Q_m inside Q in a disjoint way. By this procedure it is then defined a sequence $(u_n)_{n\in\mathbb{N}}$. Taking the limit $u = \lim_{n\to+\infty} u_n$ we obtain a

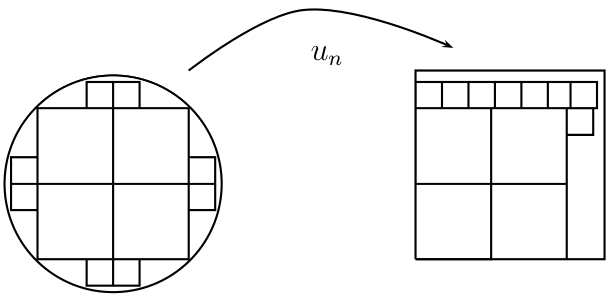

FIGURE 3.1. A piece-wise isometric map from the circle into a square.

measurable map $u : \Omega \to Q$ such that $u_{\#}\mu = \nu$, where $\mu = \mathcal{L}^N \llcorner \Omega$, $\nu = \mathcal{L}^N \llcorner Q$, and $r_u(x_0) = 2$ for a.e. $x_0 \in \Omega$. Therefore, every bounded smooth open set can be reformed into a square at minimal energy.

In order to preserve some geometric and physical properties of the shapes under consideration, we then need more regularity on the admissible reformations. We have the following

Lemma 3.7. *Let $x_0 \in \Omega$, $u : X \to Y$. If u is differentiable at x_0 then*

$$r_u(x_0) = 2 \Rightarrow \nabla u(x_0) \in O(N),$$

where $O(N)$ denotes the set of orthogonal matrices.

Proof. By (3.4) we have $c_u(x_0) = e_u(x_0) = 1$. Hence, for every $v \in \mathbb{R}^N$, taking $x = x_0 + \delta v$ we get

$$c_u(x_0) = e_u(x_0) = 1 \Rightarrow \frac{|\nabla u(x_0) \cdot v|}{|v|} = 1 \Rightarrow \nabla u(x_0) \in O(N).$$

□

By Liouville Theorem (see for instance [14]) it follows that every $u \in C^1(X;Y)$ such that $\mathscr{R}(u) = 2$ is actually an isometry. There are several generalizations of Liouville Rigidity Theorem, however (see [14, 18, 24]) these results are not directly applicable in our context since they generally require a constant sign for the Jacobian, as the condition $\nabla u(x) \in SO(N)$ for a.e. $x \in X$. For instance, the map $u(x) = x$ if $x_1 \geq 0$ and $u(x) = (-x_1, x_2, \dots, x_N)$ if $x_1 \leq 0$ belongs to the Sobolev space $W^{1,2}(\Omega, \mathbb{R}^N)$, $\nabla u(x) \in O(N)$ for a.e. x, but u is

not an isometry.
Since reformations have to preserve the volume, we have the following result.

Theorem 3.8 (Rigidity)**.** *Let $U \subset \mathbb{R}^N$ be an open connected bounded set. Let $u : \overline{U} \to \mathbb{R}^N$ be a continuous, locally Lipschitz, open map such that $\mathcal{L}^N(U) = \mathcal{L}^N(u(U))$ and satisfying the following conditions*

(i) $u(\partial U) \subset \partial(u(U))$

(ii) *u is a.e. differentiable and $\nabla u \in O(N)$ a.e. on U.*

Then u is an affine function.

Proof. By (ii) it follows that u is locally a 1-Lipschitz function (see [15, Proposition 3.4]). By the Area Formula and (ii) we infer

$$\mathcal{L}^N(u(U)) = \mathcal{L}^N(U) = \int_U |Ju| \, dx = \int_{\mathbb{R}^N} N(y, u, U) \, dy.$$

Therefore, $N(y, u, U) = 1$ for a.e. $y \in u(U)$. Observe that

$$u(U) \subset \mathbb{R}^N \setminus \partial u(U) \subset \mathbb{R}^N \setminus u(\partial U). \tag{3.9}$$

Then, $\forall x \in U$ $\deg(u(x), u, U)$ is well defined. Since u is a.e. differentiable, for a.e. $x \in U$ it results (see Lemma 5.9 of [21])

$$|\deg(u(x), u, U)| = |\text{sign } (Ju(x))| = 1.$$

On the other hand, since $u(U)$ is connected, by (3.9), $u(U)$ is contained in a connected component of $\mathbb{R}^N \setminus u(\partial U)$. Therefore, the degree is constant on $u(U)$and so the sign of the Jacobian Ju is a.e. fixed. The conclusion follows by Liouville Theorem for Sobolev maps (see for instance [14]). □

For a related rigidity result involving local homeomorphisms see [60]. For quasi-isometries over Banach spaces see [10, Cor. 14.8].

Remark 3.9. Condition (i) holds of course for invertible maps u. If we are dealing with locally invertible maps, since continuous and locally invertible maps are open maps, actually by (i) the equality $u(\partial U) = \partial(u(U))$ holds true. Moreover, if the map $u : \partial U \to \partial(u(U))$ is injective, then u is globally invertible (see also [49]). Another classical condition for global invertibility holds for simply connected, or simply connectedly exhausted, target $\overline{u(U)}$ (see [2, 60]). Moreover, suppose to have a continuous, locally invertible, surjective function $u : X \to Y$ such that $\nabla u(x) \in O(N)$ for a.e. x. Then, $D = N(y, u, U)$ is constant (see [2]) and by Area Formula we have

$$D\mathcal{L}^N(Y) = \int_Y N(y,u,U)dy = \int_X |Ju(x)|dx = \mathcal{L}^N(X).$$

Hence, if $\mathcal{L}^N(X) = \mathcal{L}^N(Y)$, it follows that $N(y,u,U) = 1$ and hence u is globally invertible.

Theorem 3.10. *Let $\mu \in \mathcal{P}(\Omega)$ and $\nu \in P(Y)$ so that $\mu = \mathcal{L}^N \llcorner \Omega$, $\nu = \mathcal{L}^N \llcorner Y$. Suppose that for $H, K \in L^1(X,\mu)$ provided by Definition 2.2 the inequality $H(x)K(x) < 2$ is satisfied. Then the variational problem*

$$\min\{\mathscr{R}(u) \mid u \in \mathrm{Ref}(\mu;\nu)^{H,K}, u \text{ open } \} \tag{3.10}$$

admits solutions whenever $\{u \in \mathrm{Ref}(\mu;\nu)^{H,K}, u \text{ open } \} \neq \emptyset$.

Proof. Since $\mu = \mathcal{L}^N \llcorner \Omega$, we may assume that $X = \Omega$. Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence. Given $x_0 \in \Omega$, by Definition 2.2 and Lemma 1.2 it follows that the sequence $(u_n)_{n\in\mathbb{N}}$ is locally equi-Lipschitz on $\overline{B}(x_0,r)$. By the Ascoli-Arzelá Theorem we extract a subsequence converging, uniformly on compact subsets of Ω, to a continuous map u. For this continuous limit map $u : \Omega \to \mathbb{R}^N$ it is easily seen that $u_\# \mu = \nu$. It remains to prove $u \in \mathrm{Ref}(\mu;\nu)^{H,K}$, namely $e_u(x) \leq K(x_0), c_u(x) \leq H(x_0)$ for every $x \in X \cap \overline{B}(x_0,r)$.

Since $X = \Omega$, by Lemma 1.2 we get the Lipschitz condition

$$|u_n(x_1) - u_n(x_2)| \leq K(x_0)|x_1 - x_2|$$

for every $x_1, x_2 \in \overline{B}(x_0,r) \subset \Omega$. Passing to the limit as $n \to +\infty$ and then as $r \to 0^+$, we obtain $e_u(x) \leq K(x_0)$.

Observe that by Theorem 2.16 the maps u_n are locally invertible. Therefore, using Lemma 1.2 and Lemma 3.3 the inverse maps u_n^{-1} are also locally equi-Lipschitz. Moreover, by the theory of quasi-isometric mappings (see [41, Theorem III], [10]) the maps u_n are equi-Lipschitz on the balls $B(x_0, \frac{r}{HK})$. In this way we find a common neighborhood $U_{x_0} := B(x_0, \frac{r}{HK}) \subset B(x_0,r)$ in which the functions u_n are all simultaneously invertible (see also [48, Proposition 7], [32]). It follows that u is also locally invertible. Indeed, suppose by contradiction to get two distinct sequences $(x_h^1)_{h\in\mathbb{N}}$, $(x_h^2)_{h\in\mathbb{N}}$ converging to x_0 such that $u(x_h^1) = u(x_h^2)$ $\forall h \in \mathbb{N}$. Let $\varepsilon > 0$ be fixed. By uniform convergence, we find a large integer n such that $|u_n(x) - u(x)| < \varepsilon$ for every $x \in U_{x_0}$. Now, for a large h we may assume $x_h^1, x_h^2 \in U_{x_0}$ and we compute

$$|x_1^h - x_2^h| = |u_n^{-1}(u_n(x_h^1)) - u_n^{-1}(u_n(x_h^2))| \leq H|u_n(x_h^1) - u_n(x_h^2)| \leq$$

$$H\left(|u_n(x_h^1)-u(x_h^1)|+|u(x_h^1)-u(x_h^2)|+|u(x_h^2)-u_n(x_h^2)|\right)\le 2H\varepsilon,$$

where H is a common Lipschitz constant for u_n^{-1}. By the arbitrariness of ε we get the contradiction $x_h^1=x_h^2$. Observe that u is open by the Domain Invariance Theorem. Hence $u(\Omega)$ is actually an open set. Let $x_1\in\overline{B}(x_0,r)$ and $y_1=u(x_1)$ be fixed. Adding a constant, we may also suppose $u_n(x_1)=y_1$. By using [41, Th. II] it results $B_1:=B(y_1,\frac{r}{H})\subset u_n(B(x_1,r))$, where the u_n^{-1} are simultaneously defined.

By Lemma 3.3 we get $e_{u_n^{-1}}(y)\le H(x_0)$ for every $y\in B_1$. By Lemma 1.2 it follows

$$|u_n^{-1}(y)-u_n^{-1}(y_1)|\le H(x_0)|y-y_1|,\quad \forall y\in B_1.$$

On the other hand, the maps u_n are bi-Lipschitz on $U_1=B(x_1,\frac{r_1}{HK})$. For the common neighborhood U_1 of x_1 we have

$$|x-x_1|\le H(x_0)|u_n(x)-u_n(x_1)|,\quad \forall x\in U_1.$$

Passing to the limit as $n\to+\infty$ and then as $x\to x_1$ we get $c_u(x_1)\le H(x_0)$. Hence $u\in\mathrm{Ref}(\mu;\nu)^{H,K}$.

Fixed $\varepsilon>0$, we find $\delta>0$ such that $\int_E(H(x)+K(x))\,d\mu<\varepsilon$ whenever $\mathcal{L}^N(E)<\delta$. By using a Vitali covering, we cover Ω, up to a measurable set E such that $\mathcal{L}^N(E)=\delta>0$, by a finite number of disjoint neighborhoods U_i on which $u_n\to u$ uniformly and invertibility holds. Since $u_{\#}\mu=\nu$ we compute

$$\mathscr{R}(u)\le\sum_{i=1}^{l}\left(\int_{U_i}\mathrm{Lip}(u)(x)d\mu+\int_{U_i}\mathrm{Lip}(u^{-1})(u(x))d\mu\right)+$$
$$\int_E(H(x)+K(x))d\mu\le\sum_{i=1}^{l}\left(\int_{U_i}\mathrm{Lip}(u)(x)d\mu+\int_{u(U_i)}\mathrm{Lip}(u^{-1})(y)\ d\nu\right)+\varepsilon.$$

By Lemma 1.4 we get

$$\begin{aligned}\mathscr{R}(u)&\le\sum_{i=1}^{l}\liminf_{n\to+\infty}\left(\int_{U_i}\mathrm{Lip}(u_n)(x)d\mu+\int_{u(U_i)}\mathrm{Lip}(u_n^{-1})(y)\ d\nu\right)+\varepsilon\\&\le\liminf_{n\to+\infty}\sum_{i=1}^{l}\left(\int_{U_i}\mathrm{Lip}(u_n)(x)d\mu+\int_{u(U_i)}\mathrm{Lip}(u_n^{-1})(y)\ d\nu\right)+\varepsilon\\&\le\liminf_{n\to+\infty}\left(\int_\Omega e_{u_n}(x)\ d\mu+\int_\Omega c_{u_n}(x)\ d\mu\right)+\varepsilon=\liminf_{n\to+\infty}\mathscr{R}(u_n)+\varepsilon.\end{aligned}$$

Letting $\varepsilon\to0^+$ we get the thesis. □

Remark 3.11. By using essentially the same tools employed in the proof of Theorem 3.10 and according to Theorem 2.15 and Theorem 2.13 one can prove existence results for the variational problems $\min\{\mathscr{R}(u) \mid u \in A_i\}$, where

$A_1 = \{u \in \mathrm{Ref}(\mu;\nu)^{H,K}, e_u < \sqrt[N]{2}\}, A_2 = \{u \in \mathrm{Ref}(\mu;\nu)^{H,K}, u \text{ incompressible}\},$
$A_3 = \{u \in \mathrm{Ref}(\mu;\nu)^{H,K}, u \text{ (locally) invertible}\}.$

Remark 3.12. If for reformation maps the surjection property $u(X) = Y$ is required, we may argue as follows. To check that u is onto, let us fix $y_0 \in Y$. Observe that u_n^{-1} are locally equi-Lipschitz. Arguing as in the proof of Theorem in 3.10 for the sequence u_n^{-1} we find a common neighborhood $B(y_0, r)$ such that u_n^{-1} are simultaneously homeomorphisms. Therefore, since $u_n(X) = Y$, we find a sequence $x_n \to x_0$ such that $u_n(x_n) = y_0$. Then we have

$$\begin{aligned} |u(x_0) - y_0| &\leq |u(x_0) - u(x_n)| + |u(x_n) - u_n(x_n)| + |u_n(x_n) - y_0| \\ &\leq |u(x_0) - u(x_n)| + \|u - u_n\|_\infty \to 0 \end{aligned}$$

as $n \to +\infty$.

Remark 3.13. Observe that thanks to the compactness of $\mathrm{Ref}(\mu;\nu)^{H,K}$, no coercitivity conditions on the energy functional $\mathscr{R}$ are needed (See [60, Ch. II Section 9] for related results in the setting of mappings with bounded distortion). In the case of $H, K \in L^p(X, \mu)$ the above minimization result could be obtained by using Rellich-Kondrakov compactness in Sobolev spaces and the l.s.c of the p-Dirichlet energy.

Remark 3.14. For X compact, considering finite coverings, it turns out that $H, K \in L^\infty$. Therefore, in such a case we may consider H, K as two universal constants. However the proof of Theorem 3.10 works as well for the non-compact case. It would be interesting to develop an analogous theory under weaker requirement on the functions H, K. For instance, supposing $H, K \in L^p$ with $p > N$, by Morrey's inequality

$$|u(x) - u(y)| \leq C(N,p)|x - y|^{1-\frac{N}{p}} \|\nabla u\|_p$$

it follows that the sequences u_n, u_n^{-1} of the proof of Theorem 3.10 are locally equi-Holder. Hence we get existence of minimizers for example in the set A_3 as in Remark 3.11. To check that the set A_3 is closed one can also use the results of [8, 62, 32].

Isometric measures are characterized by the following statement.

Theorem 3.15. *Let $\mu \in \mathcal{P}(\Omega)$ and $\nu \in \mathcal{P}(Y)$, so that $\mu = \mathcal{L}^N \llcorner \Omega$, $\nu = \mathcal{L}^N \llcorner Y$, for a given bounded set Y. Then, $\mathcal{E}(\mu, \nu) = 2$ if and only if there exists an isometry u such that $u_\# \mu = \nu$.*

Proof. By Theorem 3.10 we get a minimizer $u : \Omega \to \mathbb{R}^N$ which belongs to $\mathrm{Ref}(\mu; \nu)^{H,K}$. By Theorem 2.15 and Remark 2.15 it follows that u is globally invertible. By Lemma 3.7 and Theorem 3.8 it follows that u is a local isometry, then (see for instance [10, Th. 14.1]), u is an isometric map.

□

By Theorem 3.15 we have that by reforming a flat configuration μ in a corrugated one ν it results $\mathcal{E}(\mu, \nu) > 2$. This last fact gives an alternative proof of the so-called Grinfeld instability (see [23]), indeed, by the changing of the geometry, any possible reformation must expand or contract some portion of the body.

4. Generalized reformations

The notion of reformation introduced in the previous section has some restrictions, indeed it is easy to exhibit examples, like the one in Figure 4.1, in which every reformation map has a large cost while allowing fractures of the body leads to map the initial measure by using local isometries.

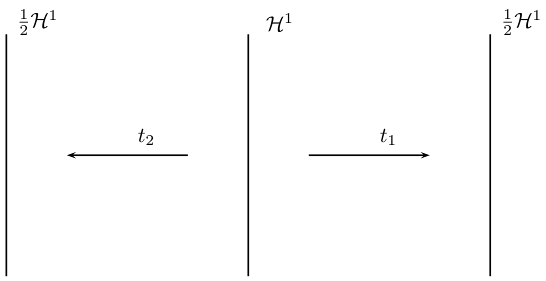

Figure 4.1. An isometric fractured reformation.

Here we introduce a notion of *generalized reformation.* Our approach relies on measure theoretic tools mostly developed in the field of optimal mass transportation (see [4, 64]) where maps satisfying $u_\# \mu = \nu$ are called transport maps. A natural generalization of the transport map (reformation map) is given by the notion of transport plan. A transport plan between two probability measures $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$ is a measure $\gamma \in \mathcal{P}(X \times Y)$ such that $\pi^1_\# \gamma = \mu$, $\pi^2_\# \gamma = \nu$, where π^i, $i = 1, 2$ denote the projections of $X \times Y$ on its factors. A

transport map u induces the transport plan $\gamma_u := (I \times u)_{\#}\mu$, where I is the identity map of X. Observe that the set of transport plans with marginals μ and ν, denoted by $\Pi(\mu, \nu)$, is never empty since it always contains the transport plan $\mu \otimes \nu$.
We shall call *generalized reformation*, or *reformation plan*, of μ into ν any transport plan γ with marginals μ and ν.
Let us recall some known results which will play a crucial role in the following (we refer to [3, 4]).

Definition 4.1. *Let $\mathcal{M}(Y)$ be the space of Radon measures on Y. A map $\lambda : X \to \mathcal{M}(Y)$ is said to be Borel if for any open set $B \subset Y$ the function $x \in X \mapsto \lambda_x(B)$ is a real valued Borel map. Equivalently, $x \mapsto \lambda_x$ is a Borel map if for any Borel and bounded map $\varphi : X \times Y \to \mathbb{R}$ it results that the map*

$$x \in X \mapsto \int_Y \varphi(x, y) d\lambda_x$$

is Borel.

Theorem 4.2 (Disintegration theorem)**.** *Let $\gamma \in \mathcal{P}(X \times Y)$ be given and let $\pi^1 : X \times Y \to X$ be the first projection map of $X \times Y$, we set $\mu = (\pi^1)_{\#}\gamma$. Then for $\mu - a.e.$ $x \in X$ there exists $\nu_x \in \mathcal{P}(Y)$ such that*

(i) *the map $x \mapsto \nu_x$ is Borel,*

(ii) $\forall \varphi \in \mathcal{C}_b(X \times Y) : \int_{X \times Y} \varphi(x, y) d\gamma = \int_X \left(\int_Y \varphi(x, y) d\nu_x(y) \right) d\mu(x).$

Moreover the measures ν_x are uniquely determined up to a negligible set with respect to μ.

Let $\gamma \in \Pi(\mu, \nu)$, as usual we will write $\gamma = \nu_x \otimes \mu$, assuming that ν_x satisfy the condition (i) and (ii) of Theorem 4.2. Obviously the transport plan $\mu \otimes \nu$ corresponds to the constant map $x \mapsto \nu_x = \nu$. Let $u : X \to Y$, observe that for the transport plan $\gamma_u := (I \times u)_{\#}\mu$, the Disintegration Theorem yields $\gamma_u = \delta_{u(x)} \otimes \mu$.

Remark 4.3. Let $X \subset \mathbb{R}^N$, we recall that the barycenter of a measure $\mu \in \mathcal{P}(X)$ is given by

$$\beta(\mu) = \int_X x \, d\mu.$$

If $\gamma = \nu_x \otimes \mu$, then, by Theorem 4.2 the map $x \mapsto \beta(\nu_x)$ is measurable. It is possible to define a generalized pointwise expansion and compression energy through the pointwise Lipschitz constant of the map $x \mapsto \varphi(x) := \beta(\nu_x)$. Observe that for a transport map u, since $\beta(\delta_x) = x$, we have

$$r_\varphi(x_0) = r_u(x_0).$$

However, it may happen that the map φ is an isometry although the target are quite far from being *isometric* as described in Figure 4.2 .

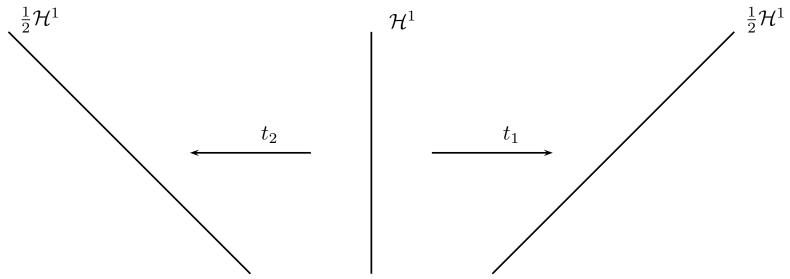

FIGURE 4.2. A barycenter isometric reformation.

In the sequel we will introduce the notion of generalized pointwise compression and expansion energy through the notion of 1-Wasserstein distance of measures.

Definition 4.4. *Let $\mu, \nu \in \mathcal{P}(X)$, the 1-Wasserstein distance between μ and ν is defined by*

$$W(\mu, \nu) = \inf_{\gamma \in \Pi(\mu,\nu)} \int_X d(x, y) \; d\gamma(x, y). \tag{4.1}$$

Let us recall that by Kantorovich duality (see [4, 30, 64]) the 1-Wasserstein distance between μ and ν can be expressed as follows

$$W(\mu, \nu) = \sup \left\{ \int_X \varphi \; d(\mu - \nu) \mid \varphi \in \mathrm{Lip}_1(X) \right\}. \tag{4.2}$$

Lemma 4.5. *The balls of $(\mathcal{P}(Y), W)$ are 1-convex.*

Proof. Let $\mu \in \mathcal{P}(Y)$, $r > 0$ be fixed, we consider $\nu_1, \nu_2 \in B := B(\mu, r) \subset \mathcal{P}(Y)$. For every $t \in [0, 1]$, let $\nu_t := t\nu_1 + (1 - t)\nu_2$. Then, by considering (4.2), for any fixed $\varphi \in \mathrm{Lip}_1(Y)$, we compute

$$\begin{aligned} \int_Y \varphi \; d(\nu_t - \mu) &= t \int_Y \varphi \; d(\nu_1 - \mu) + (1 - t) \int_Y \varphi \; d(\nu_2 - \mu) \\ &\leq tW(\nu_1, \mu) + (1 - t)W(\nu_2, \mu) \leq r \end{aligned}$$

Passing to the supremum with respect to $\varphi \in \mathrm{Lip}_1(Y)$ we get $W(\nu_t, \mu) \le r$, hence $\nu_t \in B$ $\forall t \in [0,1]$. Observing that $W(\nu_{t+h}, \nu_t) = hW(\nu_1, \nu_2)$ it follows that the length of the curve ν_t (see Appendix A) amounts to $l(\nu_t) = \int_0^1 |\dot{\nu}_t| dt = W(\nu_1, \nu_2)$. □

As stated in Section 1, over the metric space $(\mathcal{P}(Y), W)$ the above Lemma allows to derive local Lipschitz conditions from just pointwise Lipschitz bounds (see also [19]). Let $\gamma = \nu_x \otimes \mu$, the function

$$f : X \to (\mathcal{P}(Y), W), \quad f(x) = \nu_x. \tag{4.3}$$

will be called *disintegration map*. Let us introduce the notion of generalized compression and expansion energy in terms of the disintegration map f.

Definition 4.6 (Generalized expansion and compression energy). *For any reformation plan $\gamma = \nu_x \otimes \mu$ of μ into ν we define the pointwise expansion energy*

$$e_\gamma(x_0) := \limsup_{x \to x_0} \frac{W(\nu_x, \nu_{x_0})}{|x - x_0|}, \tag{4.4}$$

and the pointwise compression energy

$$c_\gamma(x_0) = \limsup_{x \to x_0} \frac{|x - x_0|}{W(\nu_x, \nu_{x_0})}. \tag{4.5}$$

By using (4.3) we can state

$$e_\gamma(x) = e_f(x), \;\; c_\gamma(x) = c_f(x). \tag{4.6}$$

The pointwise reformation energy is then defined by

$$r_\gamma(x_0) = e_\gamma(x_0) + c_\gamma(x_0).$$

Remark 4.7. Notice that, since $W(\delta_x, \delta_y) = |x - y|$, if γ is a reformation plan induced by a map $u : X \to Y$, say $\gamma_u = (I \times u)_{\#}\mu$ and f_u is the disintegration map of γ, then it results

$$r_\gamma(x_0) = r_{f_u}(x_0) = r_u(x_0).$$

Definition 4.8. *Given $H, K : X \to]0, +\infty[$, $H, K \in L^1(X, \mu)$ and a fixed covering $\mathcal{A}$ of X made by balls, we define the set $\mathrm{GRef}(\mu;\nu)^{H,K} \subset \Pi(\mu,\nu)$ as the subset of reformation plans γ of μ into ν such that*

$$\forall x_0 \in X : \; \exists \, B(x_0, r) \in \mathcal{A} \; s.t. \; e_\gamma(x) \le K(x_0), \; c_\gamma(x) \le H(x_0) \tag{4.7}$$

$\forall x \in \Omega \cap \overline{B}(x_0, r)$.

Remark 4.9. By (4.4)-(4.6) the role played by the disintegration map is clear, hence one is led to argue as in the previous section trying to establish the analogous of Theorem 2.15 in the case of disintegration maps. Unfortunately in the general case of metric spaces some tools as degree theory are not available. Therefore, it is not clear if local invertibility follows by (4.7).

Nevertheless, by restricting the analysis to the case of *small reformations*, i.e. satisfying $HK \leq \mu_0$, for enough small constant μ_0, it is possible to prove some global invertibility results suitable to the present case. For instance, assuming that Ω is a ball and that f is a local homeomorphism, then there exists a constant μ_0 such that $HK < \mu_0$ implies f globally invertible (see [41, 10] and [28] for other classes of domains Ω).

Definition 4.10. *Let us define*

$$\mathrm{GRef}_0(\mu,\nu)^{H,K} = \{\gamma \in \mathrm{GRef}(\mu,\nu)^{H,K} | \gamma = f(x)\otimes\mu, f : \Omega \to \mathcal{P}(Y) \text{ invertible } \}. \tag{4.8}$$

5. Finding reformation plans

In the following examples we show that it is possible to compare shapes with regular disintegration maps despite no regular transport map does exist.

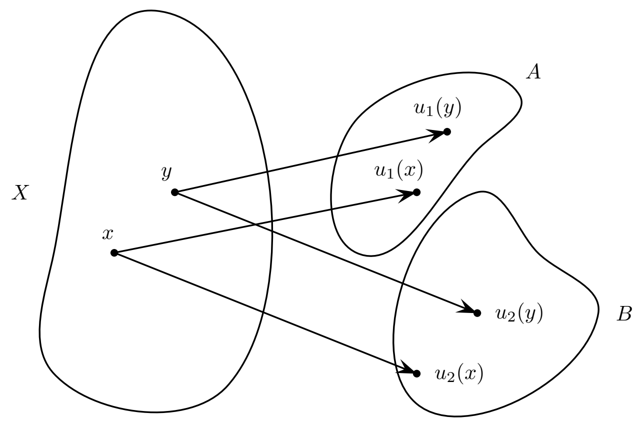

Figure 5.1. A disconnected target reformation

Example 5.1. Consider a regular domain $X \subset \mathbb{R}^N$ splitted into $Y = A \cup B$ for two disjoint regular domains $A, B \subset \mathbb{R}^N$ in such a way $1 = \mathcal{L}^N(X) = \mathcal{L}^N(A) + \mathcal{L}^N(B)$. We find (see [33, 66]) two diffeomorphisms $u_1 : X \to A$,

$u_2 : X \to B$ so that $|Ju_1| = \mathcal{L}^N(A)$, $|Ju_2| = \mathcal{L}^N(B)$.
Diffeomorphisms with constant Jacobian can be constructed by using the results of [17]. Indeed, let $\varphi : \Omega \to \Omega_1$ be a diffeomorphism. Assume for instance $J\varphi(x) > 0\ \forall x \in \Omega$ and let $f(x) = \frac{\mathcal{L}^N(\Omega)}{\mathcal{L}^N(\Omega_1)} J\varphi(x)$. Then

$$\int_\Omega f(x)\ dx = \frac{\mathcal{L}^N(\Omega)}{\mathcal{L}^N(\Omega_1)} \int_\Omega J\varphi(x)\ dx = \mathcal{L}^N(\Omega).$$

By the results of [17], there exists a diffeomorphism $u : \Omega \to \Omega$ such that $Ju = f$. Setting $\psi = \varphi \circ u^{-1} : \Omega \to \Omega_1$ it follows that $J\psi = \frac{\mathcal{L}^N(\Omega)}{\mathcal{L}^N(\Omega_1)}$.

Let $\nu_x = \mathcal{L}^N(A)\delta_{u_1(x)} + \mathcal{L}^N(B)\delta_{u_2(x)}$, then the reformation plan $\gamma := \nu_x \otimes \mu$ has $\mu = \mathcal{L}^N \llcorner X$ and $\nu = \mathcal{L}^N \llcorner Y$ as marginals. We claim that the function $f(x) = \nu_x$ is, at least locally, bi-Lipschitz. Indeed it results

$$W(\nu_x, \nu_{x_0}) = \mathcal{L}^N(A)|u_1(x) - u_1(x_0)| + \mathcal{L}^N(B)|u_2(x) - u_2(x_0)|.$$

Since u_1, u_2 are diffeomorphisms, we find constants $K_{1,2}, H_{1,2}, H, K$ such that

$$\begin{aligned}
\frac{1}{H}|x - x_0| &\leq \frac{\mathcal{L}^N(A)}{H_1}|x - x_0| + \frac{\mathcal{L}^N(B)}{H_2}|x - x_0| \\
&\leq \mathcal{L}^N(A)|u_1(x) - u_1(x_0)| + \mathcal{L}^N(B)|u_2(x) - u_2(x_0)| \\
&= W(\nu_x, \nu_{x_0}),
\end{aligned}$$

$$\begin{aligned}
W(\nu_x, \nu_{x_0}) &= \mathcal{L}^N(A)|u_1(x) - u_1(x_0)| + |\mathcal{L}^N(B)|u_2(x) - u_2(x_0)| \\
&\leq \mathcal{L}^N(A)K_1|x - x_0| + \mathcal{L}^N(B)K_2|x - x_0| \\
&\leq K|x - x_0|.
\end{aligned}$$

Remark 5.2. The above construction is possible also for a class of star-shaped domains as in [22, Theorem 5.4] by considering bi-Lipschitz maps in place of diffeomorphisms.

Moreover, generalized reformation maps are useful to compare *near-isometric* shapes.

Example 5.3. Consider a rectangle R and a bended one with the bended size of $\frac{1}{n}$ (see Figure 0.1). Consider the maps

$$u_1(x) = \left(1 - \frac{1}{n}\right)(Ax + a), \quad u_2(x) = \frac{1}{n}(Bx + b)$$

for orthogonal matrices A, B and then the reformation plan

$$\gamma = \left(\left(1 - \frac{1}{n}\right) \delta_{u_1(x)} + \frac{1}{n}\delta_{u_2(x)} \right) \otimes \mu,$$

where $\mu = \mathcal{L}^N \llcorner R$. We compute

$$W(\nu_x, \nu_{x_0}) = \left(1 - \frac{1}{n}\right) |u_1(x) - u_1(x_0)| + \frac{1}{n}|u_2(x) - u_2(x_0)| = \left(\left((1 - \frac{1}{n}\right)^2 + \frac{1}{n^2} \right) |x - x_0|.$$

Therefore the function $f(x) = \nu_x$ is, at least locally, bi-Lipschitz and

$$e_\gamma(x_0) = \left(1 - \frac{1}{n}\right)^2 + \frac{1}{n^2} \to 1$$

as $n \to +\infty$, while $c_\gamma(x_0) = \frac{1}{e_\gamma(x_0)}$.

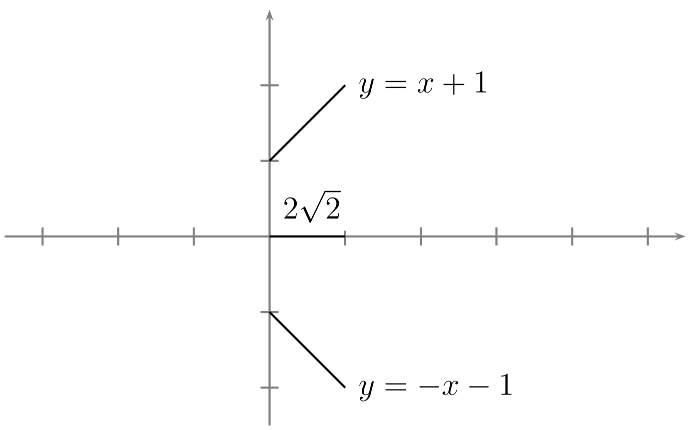

FIGURE 5.2. An horizontal segment, with mass $2\sqrt{2}$, splitted into two different ones.

Example 5.4. Consider the situation displayed in Figure 5.3. Defining $\nu_x = \frac{1}{2}\left(\delta_{x+1} + \delta_{-x-1}\right)$, we find $W(\nu_x, \nu_{x_0}) = \sqrt{2}|x - x_0|$. Hence $r_\gamma = e_\gamma + c_\gamma = \sqrt{2} + \frac{1}{\sqrt{2}}$.

Example 5.5. Let $X \subset \mathbb{R}^N$ be a measurable set with $\mathcal{L}^N(\partial X) = 0$. We find an increasing sequence of polyhedral sets X_n such that $X = \bigcup_{n \geq 1} X_n$ up to

a negligible set. Let $Y \subset \mathbb{R}^N$ be the unitary cube, $\mathcal{L}^N(X) = \mathcal{L}^N(Y)$ and let $Y_n \subset Y$ be a rectangle such that $\mathcal{L}^N(Y_n) = \mathcal{L}^N(X_n)$ $\forall n \in \mathbb{N}$. Let $\mu = \mathcal{L}^N \llcorner X$, $\nu = \mathcal{L}^N \llcorner Y$.
We find a sequence $(u_n)_{n\in\mathbb{N}}$ so that $\forall n \in \mathbb{N}$ $u_n : X_n \to Y_n$ is a bi-Lipschitz map with $Ju_n = 1$. The volume constraint implies that $K_n := \mathrm{Lip}(u_n) \leq K$, $H_n := \mathrm{Lip}(u_n^{-1}) \leq H$. In particular, for every $x, y \in X_n$ we have

$$\frac{1}{H}|x-y| \leq |u_n(x) - u_n(y)| \leq K|x-y|.$$

By Lipschitz extension, we may consider u_n as defined on the whole X. By Ascoli-Arzelá Theorem we find $u_n \to u$ uniformly. It follows that

$$\frac{1}{H}|x-y| \leq |u(x) - u(y)| \leq K|x-y|,$$

up to a zero measure set. Moreover

$$\begin{aligned}
\int_X f(u(x))\ dx &= \lim_{n\to+\infty} \int_X f(u_n(x))\ dx \\
&= \lim_{n\to+\infty} \left(\int_{X_n} f(u_n(x))\ dx + \int_{X\setminus X_n} f(u_n(x))\ dx \right) \\
&= \lim_{n\to+\infty} \int_{Y_n} f(y)\ dy = \int_Y f(y)\ dy.
\end{aligned}$$

Hence, $u_{\#}\mu = \nu$.

6. Variational problems on generalized reformations

The notion of generalized reformation involves the Lipschitz pointwise constant of maps in a metric space framework. For the associated integral energies it is natural to consider some notion of Sobolev spaces in a metric setting. There exist different notions of metric Sobolev spaces which coincide provided some mild assumptions such as a *doubling condition*, a Poincarè inequality and a power of integrability $1 < p < +\infty$ are satisfied. We refer the reader to the Appendix B and the references therein for further informations. In particular, the requirement on the power $p > 1$ will be important to state a general existence result (see Theorem 6.8 below) for the variational problem related to generalized reformations. Actually, these kind of assumptions seem to form a natural context to work with in the setting of metric analysis. Therefore, along all this section we will assume

$$\begin{aligned} &X = \overline{\Omega} \subset \mathbb{R}^N \text{ compact and satisfying (B.3) and (B.4)}, \\ &Y \subset \mathbb{R}^N \text{ compact.} \end{aligned} \tag{6.1}$$

Definition 6.1. *Let $\gamma \in \Pi(\mu, \nu)$. We define the reformation energy of γ as follows*

$$\mathscr{R}(\gamma) = \int_X (c_\gamma + e_\gamma)\ d\mu. \tag{6.2}$$

Remark 6.2. With abuse of notation we are using the same symbol $\mathscr{R}$ to denote the reformation energy functional defined on the space of reformation maps and the analogous defined on the space of reformation plans. Since in the paper it always appear with its argument specified, there is no risk of confusion.

Theorem 6.3. *Let $\gamma = f(x) \otimes \mu \in \mathrm{GRef}(\mu;\nu)^{H,K}$ be such that $\mathscr{R}(\gamma) = 2$, μ absolutely continuous with respect to the Lebesgue measure. Then there exists an open dense subset of Ω on which the disintegration map f is a local isometry (with respect to the Wasserstein distance).*

Proof. First observe that since Ω is quasiconvex (see for instance [46, Lemma 6.1]), then f is a Lipschitz function. We have $e_\gamma = c_\gamma = 1$ a.e. By [52, Prop. 1.1, Sec. 3], there exists an open dense subset $U \subset \Omega$ on which f is locally bi-Lipschitz. Therefore, consider a bi-Lipschitz map $f : B \to \mathcal{P}(Y)$ for an open ball $B \subset U$. For $x_1, x_2 \in B$, by using Fubini Theorem, we find a curve η connecting x_1, x_2 as in [15, Prop. 3.4] in such a way for a.e. t it results $e_\gamma(\eta(t)) = 1$ and $l(\eta) \le |x_1 - x_2| + \varepsilon$. Since f is Lipschitz, the curve $\rho : [0,1] \to (\mathcal{P}(Y), W)$, defined by $\rho_t = f(\eta(t))$ is Lipschitz too. Hence, it admits a tangent vector v (see Theorem A.2). Fixed $u \in \mathrm{Lip}_1(Y)$, by standard approximation argument we may suppose that $u \in \mathcal{C}^1$. Therefore, by using the continuity equation (A.4) we compute

$$\int_Y u\ d\left(f(x_1) - f(x_2)\right) = \int_Y u\ d(\rho_1 - \rho_0) = \int_0^1 \frac{d}{dt}\left(\int_Y u\ d\rho_t\right) dt =$$

$$= \int_0^1 \int_Y \langle du, v\rangle d\rho_t\ dt \le \int_0^1 \int_Y |v| d\rho_t\ dt = \int_0^1 |\dot\rho|(t)\ dt \le \int_0^1 e_\gamma(\eta_t)|\dot\eta| dt \le$$
$$\le l(\eta) \le |x_1 - x_2| + \varepsilon.$$

Taking the supremum with respect to u and letting $\varepsilon \to 0^+$ we get the 1-Lipschitz property

$$W(f(x_1), f(x_2)) \le |x_1 - x_2|.$$

To get the opposite inequality, we argue as follows. Set $\rho_0 = f(x_1), \rho_1 = f(x_2)$, let us consider a geodesic $\rho_t : [0,1] \to \mathcal{P}(Y)$ between ρ_0 and ρ_1, i.e. $l(\rho) = W(f(x_1), f(x_2))$. Since f is bi-Lipschitz, there exists an injective Lipschitz curve $\gamma : [0,1] \to B$ connecting x_1, x_2 such that $\rho_t = f(\gamma(t))$. Again by using a Fubini type argument, we find a sequence of Lipschitz injective curves $(\gamma_n)_{n\in\mathbb{N}}$ so that $\gamma_n \to \gamma$ uniformly and $\mathrm{Lip}(f^{-1})(f(\gamma_n(t))) = 1$ for a.e. $t \in [0,1]$. Therefore, we get $\sigma_n = f(\gamma_n) \to \rho$ uniformly in $(\mathcal{P}(Y), W)$. By the upper semicontinuity of the Hausdorff measure along the sequence σ_n (see for instance [11, Lemma 4.1]), recalling that for injective curves it results $l(\sigma) = \mathcal{H}^1(\sigma([0,1]))$ (see [7]), fixed $\varepsilon > 0$, we find a Lipschitz curve σ connecting ρ_0 and ρ_1 such that $\mathrm{Lip}(f^{-1})(\sigma(t)) = 1$ for a.e. $t \in [0,1]$ and $l(\sigma) \le W(f(x_1), f(x_2)) + \varepsilon$. Finally, we compute

$$|x_1 - x_2| = |f^{-1}(\sigma(0)) - f^{-1}(\sigma(1))| = \left| \int_0^1 \frac{d}{dt} f^{-1}(\sigma(t))\, dt \right| \le$$

$$\le \int_0^1 |\dot\sigma|_W(t)\, dt = l(\sigma) \le W(f(x_1), f(x_2)) + \varepsilon.$$

Letting $\varepsilon \to 0^+$ we get the thesis. □

Theorem 6.3 should be compared with Theorem 3.17. The main restriction is on invertibility which is just on an open dense subset. We may say that this open set is of full measure, actually coinciding with the whole space, just for the case of small reformations as done in Theorem 6.4 below. There are different restrictions in doing so for the general case. A first matter relies in characterizing the set where a map is locally invertible on a metric setting. A second one relies on the fact that the integral functional $\mathscr{R}$ gives a.e. informations, while invertibility requires global conditions. Therefore the matter is on passing from a.e. conditions to everywhere ones. In the results concerning reformation maps, this difficulty was overcome by using degree theory in $\mathbb{R}^N$. Therefore, something similar to degree theory over metric spaces should be needed in order to handle with this kind of questions.

Let us introduce the notation

$$\mathcal{E}_G(\mu,\nu) = \inf\{\mathscr{R}(\gamma) \mid \gamma \in \mathrm{GRef}_0(\mu;\nu)^{H,K}\}. \tag{6.3}$$

Concerning symmetry properties of the above generalized reformation energy, the same reasonings made for transport maps, compare with Definition 3.4, hold as well. We remark here that this time the question of symmetry is not just a question on invertibility. For instance, the transport plan $\gamma = f \otimes \mu$ between

μ and ν, considered in Figure 4.1 is isometric, i.e. $W(f(x), f(x_0)) = |x - x_0|$. However, reversing the target measures we see that the transport plan between ν and μ is just locally isometric and no transport plan $g \otimes \nu$ between ν and μ is isometric. The fact is that the corresponding disintegration maps are of the form

$$g : Y \to \mathcal{P}(X).$$

Therefore, symmetry questions are quite involved and here we do not further consider them.

We state the following characterization of the lowest possible value of the generalized reformation energy.

Theorem 6.4. *If $\mathcal{E}_G(\mu, \nu) = 2$, with $\mu = \mathcal{L}^N \llcorner \Omega$, then the infimum is attained at a local isometric reformation plan.*

Proof. Since $\mu = \mathcal{L}^N \llcorner \Omega$, we may assume $X = \Omega$. Let γ_n be a minimizing sequence. By compactness of $\mathcal{P}(X \times Y)$, by passing to a subsequence, we may assume that $\gamma_n \overset{*}{\rightharpoonup} \gamma$. It follows that γ is also a transport plan between μ and ν. By disintegration, we also assume that $\gamma_n = f_n(x) \otimes \mu$, $\gamma = \nu_x \otimes \mu$. For any fixed $\varphi \in \mathcal{C}(X)$, $\psi \in \mathcal{C}(Y)$, we get

$$\int_X \varphi(x) \left(\int_Y \psi(y) d\nu_x \right) d\mu = \int_{X \times Y} \varphi(x)\psi(y)\ d\gamma = \lim_{n \to +\infty} \int_{X \times Y} \varphi(x)\psi(y)\ d\gamma_n =$$

$$= \lim_{n \to +\infty} \int_X \varphi(x) \left(\int_Y \psi(y) df_n(x) \right) d\mu. \tag{6.4}$$

By density of continuous functions, it follows that $\int_Y \psi(y) df_n(x) \rightharpoonup \int_Y \psi(y) d\nu_x$ in Lebesgue spaces of integrable functions.

Since X is quasiconvex, by definition of generalized reformations, it follows that the sequence f_n is equi-Lipschitz on X. By Ascoli-Arzelá Theorem, by passing to a subsequence we have that $f_n \to f$ uniformly on compact subsets. Since the disintegration is uniquely determined, it follows that $f(x) = \nu_x$ for μ-a.e. x. Indeed, since the Wasserstein distance metrizes the weak* convergence of measures (Y is compact), for every $\psi \in \mathcal{C}(Y)$ we have

$$\int_Y \psi df_n(x) \to \int_Y \psi df(x).$$

Hence, for every $\varphi \in \mathcal{C}(X)$, passing to the limit under the integral sign and by (6.4) we get

$$\int_X \varphi(x)\left(\int_Y \psi df(x)\right) d\mu =$$
$$\lim_{n\to+\infty}\int_X \varphi(x)\left(\int_Y \psi df_n(x)\right) d\mu = \int_X \varphi(x)\left(\int_Y \psi d\nu_x\right) d\mu.$$

By lemma 4.5, f_n, f_n^{-1} are both locally equi-Lipschitz. It follows that also f is invertible. Indeed, if $y_0 = f(x_1) = f(x_2)$, as in the proof of Theorem 3.10, the inverse maps f_n^{-1} are well defined on a small ball $B(y_0,\delta)$. For a common Lipschitz constant H we compute

$$|x_1 - x_2| = |f_n^{-1}(f_n(x_1)) - f_n^{-1}(f_n(x_2))| \le HW(f_n(x_1), f_n(x_2)) \le$$
$$H\left(W(f_n(x_1), f(x_1)) + W(f(x_1), f(x_2)) + W(f(x_2), f_n(x_2))\right).$$

Letting $n \to +\infty$ we get $x_1 = x_2$. Therefore, $f \in \mathrm{GRef}_0(\mu;\nu)^{H,K}$. Since

$$2 \le \int_X \left(e_{\gamma_n} + \frac{1}{e_{\gamma_n}}\right) d\mu \le \mathscr{R}(\gamma_n) \quad \forall n \in \mathbb{N},$$

passing to the limit we get

$$\lim_{n\to+\infty}\int_X g_n(x) d\mu = 2,$$

where $g_n(x) = e_{\gamma_n} + \frac{1}{e_{\gamma_n}}$. Passing to a subsequence we have $g_n \to 2$ a.e. Since $g_n(x) = \varphi(e_{\gamma_n})$ for $\varphi(t) = t + \frac{1}{t}$, by continuity of φ it follows that $e_{\gamma_n} \to 1$ a.e. On the other hand, $c_{\gamma_n} \ge \frac{1}{e_{\gamma_n}}$ yielding $\liminf_{n\to+\infty} c_{\gamma_n} \ge 1$ a.e. Since γ_n is a minimizing sequence for $\mathscr{R}$, we get

$$2 = \lim_{n\to+\infty}\mathscr{R}(\gamma_n) = 1 + \lim_{n\to+\infty}\int_X c_{\gamma_n} d\mu.$$

and by Fatou Lemma we infer

$$1 \le \int_X \liminf_{n\to+\infty} c_{\gamma_n} d\mu \le \lim_{n\to+\infty}\int_X c_{\gamma_n} = 1.$$

Therefore, by passing to a subsequence, we also have that $c_{\gamma_n} \to 1$ a.e. Arguing as in the proof of Theorem 6.3, we locally find in Ω a curve $\eta : [0,1] \to \mathcal{P}(Y)$ such that $e_{\gamma_n}(\eta(t)) \to 1$ a.e. and $l(\eta) \le |x_1 - x_2| + \varepsilon$. Therefore we get

$$W(f_n(x_1), f_n(x_2)) \le \int_0^1 e_{\gamma_n}(\eta(t))|\dot\eta|(t)dt.$$

Passing to the limit we obtain

$$W(f(x_1), f(x_2)) \leq l(\eta) \leq |x_1 - x_2| + \varepsilon.$$

Letting $\varepsilon \to 0^+$ we obtain the 1-Lipschitz condition

$$W(f(x_1), f(x_2)) \leq |x_1 - x_2|.$$

Arguing again as in the proof of Theorem 6.3, we (locally) obtain

$$W(f(x_1), f(x_2)) = |x_1 - x_2|,$$

hence $\mathscr{R}(\gamma) = 2$. □

Remark 6.5. To recover a global isometry in the above results as in Theorem 3.15 one should establishes some metric version of Liouville Rigidity Theorems as in Theorem 3.8.

A natural question concerns the validity of an existence result as in Theorem 3.10. However, we observe that the approach pursued in the proof of such result involve the push-forward of the transport map. Therefore, for generalized reformations, the push-forward of the disintegrations maps is involved. This point of view leads to consider a variational problem over transport classes as introduced in [31]. The definition of transport classes is the following

Definition 6.6. *Let $\gamma, \eta \in \Pi(\mu, \nu)$ with $\gamma = f(x) \otimes \mu$, $\eta = g(x) \otimes \mu$ be given. We shall say that γ and η are equivalent (by disintegration), in symbols $\gamma \approx \eta$, if $f_\# \mu = g_\# \mu$.*
For any $\eta \in \Pi(\mu, \nu)$ with $\eta = g(x) \otimes \mu$, we shall call transport class any equivalence class of a transport plan η and it will be denoted by $[\eta]$, i.e.

$$[\eta] = \{\gamma = f(x) \otimes \mu \mid f_\# \mu = g_\# \mu\}. \tag{6.5}$$

For a transport map u the disintegration map is given by $x \mapsto \delta_{u(x)}$. In [31] it is shown that every such disintegration map leads to the same push-forwarded measure. In other words, all the reformation plans of the form $(I \times u)_\# \mu$ belong to the same transport class. Moreover, the following result holds true

Proposition 6.7. *Let $u : X \to Y$ be such that $u_\# \mu = \nu$ and let $\eta = (I \times u)_\# \mu = \delta_{u(x)} \otimes \mu$. If $\gamma \in [\eta]$ then there exists $v : X \to Y$ such that $\gamma = \delta_{v(x)} \otimes \mu$, i.e. the transport plan γ is concentrated on the graph of v.*

In this perspective, fixed $v : X \to Y$ such that $v_\# \mu = \nu$, the variational problem (3.10) studied in Section 3 could be rephrased as follows

$$\min\{\mathscr{R}(u) \mid u \in \mathrm{Ref}(\mu; \nu)^{H,K}\} = \min_{\mathrm{GRef}(\mu;\nu)^{H,K}} \{\mathscr{R}(\gamma) \mid \gamma \in [(I \times v)_\# \mu]\}. \tag{6.6}$$

However, by passing to transport plans, different transport classes arise. By the above discussion it seems natural to fix a transport plan $\eta \in \Pi(\mu,\nu)$, $\eta = g(x) \otimes \mu$ and to consider the variational problem

$$\min_{\mathrm{GRef}(\mu;\nu)^{H,K}} \left\{ \mathscr{R}(\gamma) \mid \gamma \in [\eta] \right\}. \tag{6.7}$$

Theorem 6.8. *(Existence of optimal reformation plans) Assume* (6.1) *and* $\mu = \mathcal{L}^N \llcorner \Omega$. *Let* $\eta \in \mathrm{GRef}_0(\mu;\nu)^{H,K}$ *be given. Then, for every* $p > 1$ *the variational problem*

$$\min_{\mathrm{GRef}_0(\mu;\nu)^{H,K}} \left\{ \mathscr{R}^p(\gamma) := \int_X (c_\gamma^p + e_\gamma^p) d\mu \mid \gamma \in [\eta] \right\} \tag{6.8}$$

admits solutions.

Proof. Let $\gamma_n = f_n(x) \otimes \mu$ be a minimizing sequence. Let $f_n \to f$ uniformly with respect to the Wasserstein distance as in the proof of Theorem 6.4. By Lemma 1.4 we get the lower semicontinuity of the term $\int_X e_\gamma^p(x) d\mu$. Moreover, by Lemma 3.3 we get

$$\int_X c_\gamma^p(x) d\mu = \int_X \mathrm{Lip}^p(f^{-1})(f(x))\ d\mu \tag{6.9}$$

Since (6.1) X satisfies the doubling condition given in Definition B.4 and the Poincaré inequality given in Definition B.5, we can apply the theory of Sobolev spaces over the subset $f(X)$ of the metric space $(\mathcal{P}(Y), W, f_{\#}\mu)$ (see Appendix B). Moreover (see [57]), since for $p > 1$ the pointwise Lipschitz constant $\mathrm{Lip}(g)$ is the minimal generalized upper gradient of the locally Lipschitz map g ([57, Theorem 5.9]) and the Cheeger p-energy (B.1) is lower semicontinuous with respect to L^p convergence ([57, Theorem 2.8]), by using (6.9) we have

$$\int_X c_\gamma^p(x) d\mu = \int_{\mathcal{P}(Y)} \mathrm{Lip}^p(f^{-1})(y)\ d(f_{\#}\mu) \le \liminf_{n\to+\infty} \int_{\mathcal{P}(Y)} \mathrm{Lip}^p(f_n^{-1})(y)\ d(f_{\#}\mu). \tag{6.10}$$

By taking into account the condition $(f_n)_{\#}\mu = f_{\#}\mu\ \forall n \in \mathbb{N}$, we get

$$\int_X c_\gamma^p(x) d\mu \le \liminf_{n\to+\infty} \int_{\mathcal{P}(Y)} \mathrm{Lip}^p(f_n^{-1})(y)\ d((f_n)_{\#}\mu) = \liminf_{n\to+\infty} \int_X c_{\gamma_n}^p(x) d\mu.$$

□

6.1. **Small reformation plans.** Let $\gamma \in \mathrm{GRef}(\mu;\nu)^{H,K}$ and $f : X \to \mathcal{P}(Y)$ be the correspondent disintegration map. Following the proof of Theorem 2.15, f is locally invertible on an open dense subset U and $N(y, f, U) = D$ is locally constant. In order to prove that actually $N(y, f, U) = 1$, fix a small ball B on which f is bi-Lipschitz. By using the Metric Area Formula (see [6, 43, 44]) we have

$$D\mathcal{H}^N(f(B)) = \int_{f(B)} N(y, f, B)\ d\mathcal{H}^N(y) = \int_{f^{-1}(f(B))} J(MD(f, x))\ dx \leq$$
$$\leq \int_{f^{-1}(f(B))} e_f(x)^N\ dx \leq K^N \mathcal{L}^N(f^{-1}(f(B))),$$

where $MD(f, x)$ denotes the metric differential introduced in Section 1, while for any seminorm P the metric Jacobian is defined by

$$J(P) = N\omega_N \left(\int_{S^{N-1}} P(v)^{-N} d\mathcal{H}^{N-1}(v) \right)^{-1}.$$

For $V = f(B)$ and $\mu = \mathcal{L}^N \llcorner X$ we are led to

$$D\mathcal{H}^N(V) \leq K^N \mathcal{L}^N(f^{-1}(V)) = K^N f_{\#}\mu(V).$$

Therefore, invertibility for *small* K as in Theorem 2.15 depends on the transport class correspondent to $\Lambda = f_{\#}\mu$. Such invertibility property could be obtained for $\Lambda(V) \leq \mathcal{H}^N(V)$. For instance, consider the isometric embedding $y \mapsto \delta_y$. Let $i(Y) = \Delta \subset \mathcal{P}(Y)$ be the set of Dirac deltas. It follows that $\mathcal{H}^N(\Delta) = \mathcal{H}^N(i(Y)) = \mathcal{L}^N(Y) = 1$. Consider Λ as the probability measure over $(\mathcal{P}(Y), W)$ defined by $\Lambda(F) = \int_F \chi_\Delta(\lambda)\ d\mathcal{H}^N(\lambda)$. In such case we have that if $K < \sqrt[N]{2}$ then f is globally invertible. Therefore, fixed a transport plan $\eta = f(x) \otimes \mu$ such that $f_{\#}\mu = \Lambda$, we get existence of the variational problem of minimizing $\mathscr{R}^p(\gamma)$ over the set

$$\{\gamma \in \mathrm{GRef}(\mu, \nu)^{H,K}\ :\ \gamma \in [\eta]\},$$

provided of course that such set of reformation plans is not empty.

Appendix A. Curves in metric spaces

For reader convenience here we just summarize some basic results. For analysis in metric spaces we refer to [5, 7, 36, 37]. For Lipschitz function on a metric space (X, d) we introduce the metric derivative according to the following definition.

Definition A.1. *Given a curve $\rho : [a, b] \to (X, d)$, the metric derivative at the point $t \in]a, b[$ is given by*

$$\lim_{h \to 0} \frac{d(\rho(t+h), \rho(t))}{h} \tag{A.1}$$

whenever it exists and in this case we denote it by $|\dot{\rho}|(t)$.

Of course, the above notion of metric derivative coincides with the metric differential (1.2). If $\rho : [a, b] \to (X, d)$ is a Lipschitz curve, by metric Rademacher Theorem the metric derivative of ρ exists at $\mathcal{L}^1$-a.e. point in $[a, b]$. Furthermore, the length of the Lipschitz curve ρ is given by

$$l(\rho) = \int_a^b |\dot{\rho}|(t) dt. \tag{A.2}$$

We restrict to the case of $\mathcal{P}(X) := (\mathcal{P}(\Omega), W)$. The following theorem relates absolutely continuous curves in $\mathcal{P}(X)$ to the continuity equation.

Theorem A.2. *Let $t \mapsto \rho_t \in \mathcal{P}(X), t \in [0, 1]$, be a curve. If ρ_t is absolutely continuous and $|\dot{\rho}| \in L^1(0, 1)$ is its metric derivative, then there exists a Borel vector field $v : (t, x) \mapsto v_t(x)$ such that*

$$v_t \in L^p(X, \rho_t) \quad \textit{and} \quad \|v_t\|_{L^p(X,\rho_t)} \leq |\dot{\rho}|(t) \quad \textit{for} \quad \mathcal{L}^1 - a.e.\ t \in [0, 1] \tag{A.3}$$

and the continuity equation

$$\dot{\rho}_t + \operatorname{div}(v\rho_t) = 0 \quad \textit{in} \quad (0, 1) \times X, \tag{A.4}$$

where the divergence operator with respect to the spatial variables is satisfied in the sense of distributions.
Conversely, if ρ_t satisfies the continuity equation (A.4) *for some vector fields v_t such that $\|v_t\|_{L^p(\rho_t)} \in L^1(0, 1)$, then $t \mapsto \rho_t$ is absolutely continuous and*

$$|\dot{\rho}|(t) \leq \|v_t\|_{L^p(X,\rho_t)} \quad \textit{for} \quad \mathcal{L}^1 - a.e.\ t \in [0, 1].$$

Remark A.3. The minimality property (A.3) uniquely determines a tangent field v_t. We will refer to v_t as the tangent vector associated to the curve $t \mapsto \rho_t$. The continuity equation (A.4) has been used in the Monge-Kantorovich theory since its beginning for many applications. The fact that it characterizes the absolutely continuous curves on the space of probability measures equipped with the Wasserstein metric was only recently pointed out and the full proof is contained in [5].

An immediate consequence of the continuity equation is the following

Lemma A.4. *For every solution (ρ_t, v_t) of the continuity equation (A.4) and for every $f \in \mathcal{C}^1(X)$ it results*

$$\frac{d}{dt}\left(\int_X f(x)d\rho_t\right) = \int_X \langle \nabla f(x), v_t(x)\rangle d\rho_t \tag{A.5}$$

in the sense of distributions.

Actually, it turns out that the map $f \mapsto \int_X f d\rho_t$ belongs to $W^{1,1}_{loc}(0,1)$. Therefore, formula (A.5) holds for a.e. $t \in (0,1)$. We refer the reader to [4, 5, 30] for proofs and more details.

Appendix B. Sobolev spaces on metric spaces

There are several ways to generalize the notion of Sobolev spaces into a metric framework, see for instance [13, 19, 35, 38, 46, 57, 63]. The approach based on the notion of *upper gradient* (see [13, 38, 57, 63]) seems to be more appropriate to the context of this paper.

Definition B.1. *Let (X, d_X), (Y, d_Y) be metric spaces, let $U \subset X$ be an open subset and let $u : U \to Y$ be a given map. A Borel function $g : U \to [0, +\infty]$ is said to be an upper gradient of u if for any unit speed curve $\gamma : [0, l] \to X$ it results*

$$d_Y(u(\gamma(0)), u(\gamma(l))) \leq \int_0^l g(\gamma(s))\, ds.$$

If $u : U \to Y$ is Lipschitz, then the pointwise Lipschitz constant $\mathrm{Lip}(u)$ is an upper gradient for u, see [13, 19, 63]. For $u \in L^p(U, Y)$, the *Cheeger type p-energy* is defined as follows

$$E_p(u) = \inf_{(u_n, g_n)} \liminf_{n\to+\infty} |g_n|^p_{L^p}, \tag{B.1}$$

where the infimum is taken over the sequences (u_n, g_n) such that g_n is an upper gradient of u_n and $u_n \to u, g_n \to g$ in L^p. By definition (B.1) it immediately follows

$$E_p(u) \leq \liminf_{n\to+\infty} E_p(u_n) \tag{B.2}$$

whenever $u_n \to u$ in L^p. The Cheeger metric $(1,p)$-Sobolev space is defined as

$$H^{1,p}(U, Y) = \{u \in L^p(U, Y) \ : \ E_p(u) < +\infty\}.$$

We need two more definitions.

Definition B.2. *A function $g \in L^p$ is called a generalized upper gradient for $u \in H^{1,p}(U,Y)$ if there exists a sequence (u_n, g_n) such that g_n is an upper gradient for u_n and $u_n \to u, g_n \to g$ in L^p.*

From Definition B.1 it follows that $|g|^p_{L^p} \geq E_p(u)$ whenever g is a generalized upper gradient for u.

Definition B.3. *A generalized upper gradient g for a map $u \in H^{1,p}(U,Y)$ is said to be minimal if it satisfies $|g|^p_{L^p} = E_p(u)$*

Under some regularity requirement on the target metric space Y, it may be proved (see [57]) that every $u \in H^{1,p}(U,Y)$, with $1 < p < +\infty$ admits a unique minimal generalized upper gradient g_u. This minimal generalized upper gradient coincides with the pointwise Lipschitz constant $\mathrm{Lip}(u)$ under some geometrical property of the spaces $(X,\mu), Y$ (see [57, Theorem 5.9]). In particular, a crucial role is played by the *doubling condition* and a *weak Poincaré* $(1,p)$*-inequality* for the space (X,μ).

Definition B.4. *A measure μ over a metric space X is said to be "doubling" if μ is finite on bounded sets and there exists a constant C such that for every $x \in X$ and every $r > 0$ the following inequality holds*

$$\mu(B(x,2r)) \leq C\mu(B(x,r)). \tag{B.3}$$

Definition B.5. *Let $1 \leq p < +\infty$. A metric measure space (X,d,μ) is said to satisfy the weak Poincaré $(1,p)$-inequality if, for any $s > 0$, there exist constants $C, \Lambda \geq 1$ such that, for any open ball $B(x,r)$ with $0 < r \leq s$, function $f \in L^1(B(x,\Lambda r)))$ and upper gradient $g : B(x,\Lambda r)) \to [0,+\infty]$ for f, the following inequality holds*

$$⨍_{B(x,r)} \left| f - ⨍_{B(x,r)} f \, d\mu \right| \, d\mu \leq C \left(⨍_{B(x,\Lambda r)} g^p \, d\mu \right)^{\frac{1}{p}} \tag{B.4}$$

Observe that under some geometrical requirement on X, the Poincaré inequality (B.4) may be required to hold just for Lipschitz functions f (see [37, 38]). The euclidean space $\mathbb{R}^N$ equipped with the Lebesgue measure $\mathcal{L}^N$ is doubling and satisfies the above Poincaré inequality with $\Lambda = 1$. Given a square Q and $\mu = \mathcal{L}^N \llcorner Q$, by the inequality

$$\frac{1}{2^N}\mathcal{L}^N(B(x,r)) \leq \mu(B(x,r)) \leq \mathcal{L}^N(B(x,r)),$$

holding for every ball $B(x,r)$ of Q and the usual Poincaré inequality on convex sets, it follows that (Q,μ) is doubling and supports the Poincaré inequality

(B.4). Since the doubling condition and the Poincaré inequality are stable under bi-Lipschitz maps, every diffeomorphic (or bi-Lipschitz), with volume preserving maps, domain Ω (as balls, see for instance [22, 33]) with the same volume of the square Q, equipped with the measure $\nu = \mathcal{L}^N \llcorner \Omega$ is doubling and supports the Poincaré inequality (B.4). For more details on the doubling and Poincaré inequality we refer the reader for instance to [7, 13, 38, 46].

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References

[1] G. Alberti, M. Csornyei, D. Preiss, Structure of null sets in the plane and applications, European Congress of Mathematics, Eur. Math. Soc., Zurich (2005), 3–22.

[2] A. Ambrosetti, G. Prodi, *A Primer in Non-Linear Analysis*, Cambridge University Press, 1993.

[3] L. Ambrosio, N. Fusco & D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Clarendon Press, Oxford, 2000.

[4] L. Ambrosio, Lecture Notes on Transport Problems, in: *Mathematical Aspects of Evolving Interfaces*, Lecture Notes in Mathematics **1812**, Springer, Berlin, 2003, 1–52.

[5] L. Ambrosio, N. Gigli, G. Savaré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, Lecture Notes in Mathematics, ETH Zürich, Birkhäuser, 2005.

[6] L. Ambrosio, B. Kircheim, Rectifiable sets in metric and Banach spaces, *Math. Ann.* **318** (3) (2000) 527–555.

[7] L. Ambrosio, P. Tilli, *Topics on Analysis in Metric Spaces*, Oxford University Press, 2004.

[8] E. Barvínek, I. Daler, J. Franku, Convergence of sequences of inverse functions, Archivium Mathematicum (1991), Vol. 27 **3-4**, 201-204.

[9] C. Biasi, C. Gutierrez. E. L. Dos Santos, The Implicit Function Theorem for Continuous Functions, *Topol. Methods Nonlinear Anal.* **32** (2008), no. 1, 177–185.

[10] Y. Benyamini, J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, Vol.1, AMS, 2000.

[11] G. Buttazzo, A. Pratelli, S. Solimini, E. Stepanov, *Optimal Urban Networks via Mass Transportation*, Springer, 2009. and Optimal Transport, *Appl. Math. Optim.* **63** (2011), 1–9.

[12] N. Charon, A. Trouvé, Functional Currents: A New Mathematical Tools to Model and Analyse Functional Shapes, J. Math. Imaging Vis., published online 17 January 2013.

[13] J. Cheeger, Differentiability of Lipschitz Functions on Metric Measure Spaces, *GAFA, Geom. funct. anal.* **9** (1999) 428–517.

[14] P. G. Ciarlet, *An Introduction to Differential Geometry with Applications to Elasticity*, 2005, Springer.

[15] B. Dacorogna, P. Marcellini, E. Paolini, Lipschitz-continuous local isometric immersions: rigid maps and origami, *Journal de Mathematiques Pures et Appliquees*, **90** (2008), 66–81.

[16] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Springer.

[17] B. Dacorogna, J. Moser, On a partial differential equation involving the Jacobian determinant, *Ann. Inst. H. Poincarè* **7**, n. 1 (1990), 1–26.

[18] C. De Lellis, L. Szekelyhidi, Simple proof or two-well rigidity, *C. R. Acad. Sci. Paris*, Ser. 1 343 (2006), 367–370.

[19] E. Durand-Cartagena, J.A. Jaramillo, Pointwise Lipschitz functions on metric spaces, *J. Math. Anal. Appl.* **363** (2010), 525–548.

[20] S. Durrleman, A. Trouvè, N. Ayache, X. Pennec, Statistical models of sets of curves and surfaces based on currents, Medical Imaging Analysis (2009).

[21] I. Fonseca, W. Gangbo, *Degree Theory in Analysis and Applications*, Clarendom Press Oxford, 1995.

[22] I. Fonseca, G. Parry, Equilibrium Configurations of Defective Crystals, *Archive for Rational Mechanics and Analysis* **97** (1987), 189–223.

[23] R. Fosdick, L. Granieri, F. Maddalena, Reformation instability in elastic solids, *J. of Elasticity*, published online 28 june 2011.

[24] G. Friesecke, R. D. James, S. Muller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity, *Communications in Pure and Applied Mathematics*, Vol. LV (2002), 1461–1506.

[25] M. Fuchs, B. Jutter, O. Scherzer, H. Yang, Shape metrics based on elastic deformations, *Journal of Mathematical Imaging and Vision* **35**, no, 1 (2009), 86–102.

[26] I. Garrido, O. Gutú, J.A. Jaramillo, Global inversion and covering maps on length spaces, *Nonlinear Analysis* **73** (2010), 1364–1374.

[27] J. Gevirtz, Metric conditions that imply local invertibility, *Communications in Pure and Applied Mathematics* **23** (1969), 243–264.

[28] J. Gevirtz, Injectvity in Banach spaces and the Mazur-Ulam Theorem on Isometries, *Transactions AMS* **274**, no. 1 (1982), 307–318.

[29] J. Glaunes, M. Vaillant, Surface matching via currents, Proceedings of Information Processing in Medical Imaging (IPMI), Lecture Notes in Computer Science, **3565**(2006), 381-392.

[30] L. Granieri, *Optimal Transport and Minimizing Measures*, LAP Lambert Academic Publishing, 2010.

[31] L.Granieri, F. Maddalena, *Monge-Kantorovich Transport Problems and Disintegration Maps*, forthcoming on *Esaim COCV.*

[32] L. Granieri, G. Mola, Sequences of inverse and implicit functions, preprint.
[33] J. A. Griepentrog, W. Höppner, H. C. Kaiser, J. Rehberg, A bi-Lipschitz continuous, volume preserving map from the unit ball onto a cube, *Note Mat.* **1** (2008), 177–193.
[34] C. Gutierrez, C. Biasi, Finite Branched Coverings in a Generalized Inverse Mapping Theorem, *Int. Journal of Math. Analysis* **2** no. 4 (2008), 169–179.
[35] P. Hajlasz, Sobolev Spaces on an Arbitrary Metric Space, *Potential Analysis* **5** (1996), 403–415.
[36] J. Heinonen, *Lectures on Lipschitz Analysis*, Report University of Jyväskylä Department of Mathematics and Statistics, **100**, 2005.
[37] J. Heinonen, *Lectures on analysis on metric spaces*, Springer, 2001.
[38] J. Heinonen, P. Koskela, A note on Lipschitz functions, upper gradients, and the Poincarè inequality, *N.Z. Math. J.* **28** (1999), 37–42.
[39] S. Hencl, J. Mali, Jacobians of Sobolev homeomorphisms, *Calc. Var.* **38** (2010), 233–242.
[40] T. Iwaniec, G. Martin, *Geometric Function Theory and Non-Linear Analysis*, Clarendon Press, Oxford, 2001.
[41] F. John, On Quasi-Isometric Mappings I, *Communications in Pure and Applied Mathematics* **21** (1968), 77–110.
[42] A. Kalamajska, On Compactness of embedding for Sobolev spaces defined on metric spaces, *Annales Academiae Scientiarum Fennicae Mathematica* **24** (1999), 123–132.
[43] M. B. Karmanova, Area and Coarea Formulas for the mappings of Sobolev classes with values in a metric space, *Siberian Mathematical Journal*, Vol. 48, No. 4 (2007), 621–628.
[44] B. Kirchheim, Rectifiable Metric Spaces: Local Structure and Regularity of the Hausdorff Measure, *Proceedings of the American Mathematical Society*, **121**, No. 1 (1994), 113–123.
[45] P. Koskela, *Sobolev Spaces and Quasiconformal Mappings on Metric Spaces*, European Congress of Mathematics, Vol. I (Barcelona, 2000), 457–467, Progr. Math., 201, Birkhäuser, Basel, 2001.
[46] P. Koskela, Metric Sobolec spaces, *Nonlinear analysis, function spaces and applications* **7**, (2003) 132–147.
[47] L. V. Kovalev, Jani Onninen, Kai Rajala, Invertibility of Sobolev mappings under minimal hypotheses, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27** (2010), no. 2, 517–528.
[48] L. V. Kovalev, Jani Onninen, On invertibility of Sobolev mappings, *Journal für die reine und angewandte Mathematik*, **656**, (2011), 1–16.
[49] Holger Lang, *A condition that a continuously deformed, simply connected body does not penetrate itself*, AGTM report, University of Kaiserslautern, Germany, No. 271, 2007.
[50] Y. Laurent, *Shapes and Diffeomorphisms*, Springer, 2010.
[51] A. Lytchak, Differentiation in Metric Spaces, *St. Petersburg Math. J.* **16** (2005), No. 6, 1017–1041.
[52] A. Lytchak, Open map Theorem for Metric Spaces, *St. Petersburg Math. J.* **17** (2006), No. 3, 477–491.
[53] V.M. Miklyukov, On maps almost quasi-conformally close to quasi-isometries, *J. Anal. Math.* **100** (2006), 375–396.
[54] A. Pratelli, *Existence of optimal transport maps and regularity of the transport density in mass transportation problems*, Ph.D. Thesis, Scuola Normale Superiore, Pisa, 2003.
[55] J. S. Raymond, *Local inversion for differentiable functions and Darboux property*

[56] S. Rickman, *Quasiregular mappings*, Results in Mathematics and Related Areas (3) **26**, Springer-Verlag, Berlin, 1993.
[57] Shin-Ichi Ohta, Cheeger Type Sobolev Spaces for Metric Space Targets, *Potential Analysis* **20** (2004), 149–175.
[58] A. Srivastava, C. samir, S.H. Joshi, M. Daoudi, Elastic shapes models for face analysis using curvilinear coordinates, *J. Math. Imaging Vision* **33** (2009), no. 2, 253–265.
[59] R. Van Der Putten, A note on the local invertibility of Sobolev functions, *Math. Scand.* **83** (1998), 255–264.
[60] Yu. G. Reshetnyak, *Space Mappings with Bounded Distortion*, AMS, 1989.
[61] Yu. G. Reshetnyak, Sobolev-type classes of functions with values in a metric space, *Siberian Mathematical Journal*, **38** (1997) No. 3, 567–583.
[62] M. Seeotharama Gowda, R. Sznajder, Weak univalence and connectedness of inverse images of continuus functions, Mathematics of Operations Reaserch **24 (1999)**, No. 1, 255-261.
[63] N. Shanmugalingam, Newtonian spaces. An extension of Sobolev spaces to metric measure spaces, *Revista Matematica Iberoamericana* **16** (2000) no. 2.
[64] C. Villani, *Optimal Transport, Old and New*, Springer, 2009.
[65] B. Wirth, L. Bar, M. Rumpf, G. Sapiro, A Continuum mechanical approach to geodesics in shape space, *Int. J. Comput. Vis.* **93** (2011), no. 3, 293–318.
[66] G. Wolansky, Incompressible, Quasi-Isometric Deformations of 2-Dimensional Domains, *SIAM J. Imaging Sciences*, **2** (2009), No. 4, . 1031–1048.

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