ONE-DIMENSIONAL SYMMETRY FOR SEMILINEAR EQUATIONS WITH UNBOUNDED DRIFT

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Abstract. We consider semilinear equations with unbounded drift in the whole of $\mathbb{R}^n$ and we show that monotone solutions with finite energy are one-dimensional.

1. Introduction

In the paper [10] E. De Giorgi formulated the celebrated conjecture that bounded monotone solutions to the Allen-Cahn equation

$$(1) \quad \Delta u = u^3 - u$$

are necessarily one-dimensional (in the sense that the level sets are hyperplanes) at least if $n \leq 8$. This conjecture has been proved by Ghoussoub and Gui in [19](see also [3]) in dimension $n = 2$, and by Ambrosio and Cabré [2] in dimension $n = 3$ (see also [1]), and a counterexample has been given by del Pino, Kowalczyk and Wei in [11] for $n \geq 9$. Under the additional assumption that $u$ connects $-1$ to $1$, a proof has been presented by Savin [24] in dimension $n \leq 8$.

In this paper we consider the semilinear elliptic equation

$$(2) \quad \Delta u + c(z) u_z + \langle \nabla_y g(y), \nabla_y u \rangle + f(u) = 0,$$

where we write $x = (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}$. A solution $u$ of (2) of the form

$$u(x) = u_0(\langle \omega, x \rangle) \quad \forall x \in \mathbb{R}^n,$$  

where $u_0 : \mathbb{R} \rightarrow \mathbb{R}$ and $\omega \in \mathbb{R}^n$ with $|\omega| = 1$ will be called one-dimensional.

We are interested in symmetry results for solutions $u$ which are monotone in the $z$-variable, i.e. satisfy

$$u_z(x) > 0 \quad \forall x \in \mathbb{R}^n.$$

In particular, we will show that, under suitable assumptions, monotone solutions to (2) are necessarily one-dimensional (see Theorem 1.1).

Our methods rely on the geometric approach developed in [14] (see also [7, 8, 12, 13, 18, 25]), and our computations follow those in [15, 16], where the authors prove Liouville type results for stable solutions to elliptic equations in complete Riemannian manifolds with nonnegative Ricci curvature.

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1.1. Main result. Let us state the main result of this paper:

**Theorem 1.1.** Assume that \( f : \mathbb{R} \to \mathbb{R} \) is locally Lipschitz, \( g \in C^2(\mathbb{R}^{n-1}) \), \( c \in C^1(\mathbb{R}) \) and that

\[
c'(z)I_{n-1} \geq \nabla^2_y g(y) \quad \text{for every } (y,z) \in \mathbb{R}^{n-1} \times \mathbb{R},
\]

where \( I_{n-1} \) denotes the identity matrix on \( \mathbb{R}^{n-1} \). Let \( C \in C^2(\mathbb{R}) \) be a primitive of \( c \), and let \( u \) be a solution to (2) satisfying (4) and one of the following conditions:

a) \[
\int_{\mathbb{R}^n} |\nabla u|^2 e^{\varphi(y) + C(z)} \, dz \, dy < +\infty;
\]

b) For all \( z \in \mathbb{R} \)

\[
\int_{\mathbb{R}^{n-1}} |\nabla u|^2 e^{\varphi(y) + C(z)} \, dy \leq K \quad \text{for some } K > 0;
\]

c) \( n = 2 \) and for all \( (y,z) \in \mathbb{R}^n \)

\[
|\nabla u|^2 e^{\varphi(y) + C(z)} \leq K \quad \text{for some } K > 0;
\]

then \( u \) is one-dimensional.

In particular, if the strict inequality holds in (5) for some \( (y,z) \) then \( u \) depends only on \( z \).

From Theorem 1.1 we get the following corollaries which extend a result in [7], valid for the Ornstein-Uhlenbeck case \( C(z) = -z^2/2, g(y) = -|y|^2/2 \).

**Corollary 1.2.** Let \( C, g \) bounded above and satisfying (5). Assume also that \( n = 2 \) or

\[
\int_{\mathbb{R}^{n-1}} e^{\varphi(y)} \, dy < +\infty.
\]

Let \( u \in W^{1,\infty}(\mathbb{R}^n) \) be a solution to (2) satisfying (4), then \( u \) is one-dimensional.

**Proof.** If \( C, g \) are bounded above and \( u \in W^{1,\infty}(\mathbb{R}^n) \), then (8) holds, and moreover condition (10) implies (7). The thesis then follows directly from Theorem 1.1. \( \square \)

**Remark 1.3.** From [21, Th. 2.4 and Rem. 2.5] it follows that, if \( \nabla^2 g(y) \) and \( c'(z) \) are uniformly bounded below, every bounded solution to (2) belongs to \( W^{1,\infty}(\mathbb{R}^n) \).

**Corollary 1.4.** Let \( C, g \) be concave, satisfying (5) and \(-C, -g\) coercive. Let \( u \in L^\infty(\mathbb{R}^n) \) be a solution to (2) satisfying (4), then \( u \) is one-dimensional.

**Proof.** In [9, Thm 2.5, Cor. 4.3] it is proved that if \(-C, -g\) are convex and coercive then any (weak) solution to (2) such that

\[
\int_{\mathbb{R}^n} u^2 e^{\varphi(y) + C(z)} \, dz \, dy < +\infty
\]

also satisfies (6) (see Remark 2.3). In particular, any bounded solution to (2) satisfies (6), and we can conclude by Theorem 1.1. \( \square \)
When \( c(z) \equiv c \in \mathbb{R} \), solutions to (2) correspond to traveling (or standing if \( c = 0 \)) wave solutions to the reaction-diffusion equation:

\[
(11) \quad v_t = \Delta v + \langle \nabla g, \nabla v \rangle + f(v) \quad \text{in} \ \mathbb{R}^n \times (0, +\infty).
\]

A traveling wave solution is a particular solution \( v \) to (11), uniformly translating in the \( z \)-direction at constant speed \( c \), of the form

\[
v(t, x) = u(y, z - ct).
\]

We refer to [26, 29] and references therein for classical results about existence and uniqueness of traveling waves in infinite cylinders.

**Corollary 1.5.** Let \( g \) be concave and let \( v(t, x) = u(y, z - ct) \) be a traveling or a standing wave solution to (11). If \( u \) satisfies one of the three conditions of Theorem 1.1, then \( u \) is one-dimensional. Moreover \( u \) depends only on \( z \) unless \( n = 2 \), \( g \) is constant and (8) holds.

Conditions (6), (7), (8) are quite restrictive. However, traveling wave solutions satisfying these conditions are relevant to propagation and are sometimes called *variational traveling waves*. We refer to [22, 23] for a general analysis of such solutions, including necessary and sufficient conditions for existence.

If these conditions are not satisfied, equation (11) admits in general traveling waves which are not one-dimensional even in the case \( n = 2 \) and \( g = 0 \) (see [4, 5, 20]).

**2. A Ornstein-Uhlenbeck type equations.**

More generally, we shall consider the following equation of Ornstein-Uhlenbeck type:

\[
(12) \quad \Delta u + \langle \nabla G(x), \nabla u \rangle + f(u) = 0 \quad x \in \mathbb{R}^n,
\]

where \( f : \mathbb{R} \to \mathbb{R} \) is a locally Lipschitz function and \( G \in C^2(\mathbb{R}^n) \).

Notice that solutions to (12) are critical point of the functional

\[
(13) \quad I(u) := \int_{\mathbb{R}^n} \left( \frac{\| \nabla u \|^2}{2} + F(u) \right) e^{G(x)} \, dx,
\]

where \( F'(t) = -f(t) \). We define the function \( \lambda_G \in C^0(\mathbb{R}^n) \) as

\[
(14) \quad \lambda_G(x) := \text{maximal eigenvalue of } \nabla^2 G(x).
\]

Observe that, if \( G(x) := g(y) + C(z) \), then (12) reduces to (2), and \( \lambda_G(x) \geq C''(z) \) for every \( x \in \mathbb{R}^n \).

**2.1. \( h \)-stable solutions.** We denote by \( \mu \) the measure on \( \mathbb{R}^n \) with density \( e^{G(x)} \) w.r.t. the Lebesgue measure, and we let \( W^{k,p}_\mu(\mathbb{R}^n) \subset W^{k,p}_{\text{loc}}(\mathbb{R}^n) \), for \( k, p \in \mathbb{N} \), be the corresponding Sobolev spaces. Notice that, if \( G \) is concave, then \( \mu \) is a finite measure iff

\[
\lim_{|x| \to +\infty} G(x) = -\infty.
\]

We introduce now the notion of \( h \)-stability for solutions to (12).
Definition 2.1. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. A solution $u$ to (12) is $h$-stable if
\begin{equation}
\int_{\mathbb{R}^n} \left( |\nabla \varphi|^2 - f'(u)\varphi^2 \right) d\mu \geq \int_{\mathbb{R}^n} h(x)\varphi^2 d\mu \quad \forall \varphi \in C^1_c(\mathbb{R}^n).
\end{equation}
If $h \equiv 0$, then $u$ is said to be stable.

We recall that a function $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^n)$ is a weak solution to (12) if
\begin{equation}
\int_{\mathbb{R}^n} \left( \langle \nabla u, \nabla \varphi \rangle - f(u)\varphi \right) d\mu = 0 \quad \forall \varphi \in C^1_c(\mathbb{R}^n).
\end{equation}

Note that every critical point of the functional $I$ in (13) is a weak solution to (12). By classical elliptic regularity theory, if $u$ is a weak solution then $u \in C^{2,\alpha}(\mathbb{R}^n)$ for all $\alpha < 1$, in particular it is also a classical solution to (12).

Remark 2.2. The function $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^n)$ is a local minimizer of the functional $I$ in (13) if $I$ does not decrease under compactly supported perturbations, i.e.
\begin{equation}
I(u) \leq I(v) \quad \text{whenever } v \in W^{1,2}_{\text{loc}}(\mathbb{R}^n) \text{ and } \{u \neq v\} \subset K \subset \subset \mathbb{R}^n.
\end{equation}

Every local minimizer of $I$ is a stable weak solution to (12).

In [7] the authors show that, when $G(x) = -|x|^2/2$, monotone solutions to (12) are $-1$-stable (i.e. stable with respect to the constant function $h \equiv -1$).

In the following we will consider $h$-stable solutions to (12) which have finite energy, in the sense that
\begin{equation}
|\nabla u| \in L^2_\mu(\mathbb{R}^n).
\end{equation}
Note that if $G(y, z) = g(y) + C(z)$, this condition reduces to (6). When $n = 2$, we can substitute this condition with
\begin{equation}
|\nabla u|^2 e^{G} \in L^\infty(\mathbb{R}^n).
\end{equation}

Remark 2.3. If the measure $\mu$ is finite then $L^\infty(\mathbb{R}^n) \subset L^2_\mu(\mathbb{R}^n)$. If the function $G$ is concave, by [9, Thm 2.5, Cor. 4.3] this implies that every bounded solution to (12) belongs to $W^{2,2}_\mu(\mathbb{R}^n)$ and hence satisfies (17).

On the other hand, assumption (17) can be satisfied also when $\mu$ is not finite: for instance, if $G(x) = g(y)$ is such that (10) holds and $f(s) = s - s^3$, the function
\begin{equation}
u(z) = \tanh \left( \frac{z}{\sqrt{2}} \right)\end{equation}
is a monotone stable solution to (12) with finite energy.

3. $\lambda G$-stability and finite energy imply one-dimensional symmetry

We now show that $\lambda G$-stable solutions to (12), where $\lambda G$ is defined in (14), which satisfy (17) or (18) are one-dimensional. Similar results for stable solutions have been obtained in the setting of Riemannian manifolds with nonnegative Ricci curvature in [15, 16].

Given a differentiable function $v : \mathbb{R}^n \to \mathbb{R}$, we set $v_i := \partial_i v$ for all $i = 1, \ldots, n$. 

Lemma 3.1. Let \( u \in W^{1,2}_{\text{loc}}(\mathbb{R}^n) \) be a weak solution to (12). Then for any \( i = 1, \ldots, n \) and \( \varphi \in C^1_c(\mathbb{R}^n) \) we have
\[
(19) \quad \int_{\mathbb{R}^n} \left( \langle \nabla u_i, \nabla \varphi \rangle - \langle \nabla u, \nabla G_i \rangle \varphi - f'(u)u_i\varphi \right) d\mu(x) = 0.
\]

Proof. It suffices to prove (19) for \( \varphi \in C_c^\infty(\mathbb{R}^n) \). From (16), applied with \( \varphi \) replaced by \( \varphi_i \), we get
\[
0 = \int_{\mathbb{R}^n} \langle \nabla u_i, \nabla \varphi_i \rangle - f(u)\varphi_i \ d\mu(x)
\]
\[
= \int_{\mathbb{R}^n} -\langle \nabla u_i, \nabla \varphi \rangle - \langle \nabla u, \nabla \varphi \rangle G_i + f'(u)u_i\varphi + f(u)\varphi G_i \ d\mu(x)
\]
\[
= \int_{\mathbb{R}^n} -\langle \nabla u_i, \nabla \varphi \rangle - \langle \nabla u, \nabla (\varphi G_i) \rangle + \langle \nabla u, \nabla G_i \rangle \varphi + f'(u)u_i\varphi + f(u)\varphi G_i \ d\mu(x).
\]
Recalling (16), applied with \( \varphi \) replaced by \( \varphi G_i \), we obtain the thesis. \( \square \)

Proposition 3.2. Let \( h \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( u \) be a \( h \)-stable solution to (12). Then for every \( \varphi \in C^1_c(\mathbb{R}^n) \) we have
\[
(21) \quad \int_{\mathbb{R}^n} \left( \|\nabla u_i\|^2 - \|\nabla u\|^2 + \langle (h(x)I_n - \nabla^2 G(x)) \nabla u, \nabla u \rangle \right) \varphi^2 d\mu(x) \leq \int_{\mathbb{R}^n} |\nabla u_i|^2 |\nabla \varphi|^2 d\mu(x).
\]

Proof. Let \( \varphi \in C^1_c(\mathbb{R}^n) \). Using (19) with test function \( u_i \varphi^2 \) we obtain
\[
(22) \quad \int_{\mathbb{R}^n} \langle \nabla u_i, \nabla (u_i \varphi^2) \rangle - f'(u)u_i^2 \varphi^2 d\mu(x) = \int_{\mathbb{R}^n} \langle \nabla u, \nabla G_i \rangle u_i \varphi^2 d\mu(x).
\]

Summing over \( i \), (22) gives
\[
(23) \quad \int_{\mathbb{R}^n} \|\nabla^2 u\|^2 \varphi^2 + \frac{1}{2} \langle \nabla \|\nabla u\|^2, \nabla \varphi^2 \rangle - f'(u)\|\nabla u\|^2 \varphi^2 d\mu(x) = \int_{\mathbb{R}^n} \langle \nabla^2 G(x) \nabla u, \nabla u \rangle \varphi^2 d\mu(x).
\]

Using (15) with test function \( |\nabla u|\varphi \) we then get
\[
(24) \quad \int_{\mathbb{R}^n} -h(x)|\nabla u|^2 \varphi^2 d\mu(x) \leq \int_{\mathbb{R}^n} |\nabla (|\nabla u|\varphi)|^2 - f'(u)|\nabla u|^2 \varphi^2 d\mu(x)
\]
\[
\quad = \int_{\mathbb{R}^n} \varphi^2 - |\nabla u|^2 |\nabla \varphi|^2 + \frac{1}{2} \langle \nabla |\nabla u|^2, \nabla \varphi^2 \rangle - f'(u)|\nabla u|^2 \varphi^2 d\mu(x).
\]

Substituting (23) in (24) we get the result. \( \square \)

Corollary 3.3. Recalling that \( \|\nabla^2 u\|^2 - \|\nabla u\|^2 \geq 0 \) (see Remark 3.4), from (21) it follows
\[
(25) \quad \int_{\mathbb{R}^n} \langle (h(x)I_n - \nabla^2 G(x)) \nabla u, \nabla u \rangle \varphi^2 d\mu(x) \leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \varphi|^2 d\mu(x).
\]

If \( h \geq \lambda_G \), from (21) and the definition of \( \lambda_G \) in (14) it follows
\[
(26) \quad \int_{\mathbb{R}^n} \left( \|\nabla^2 u\|^2 - \|\nabla u\|^2 \right) \varphi^2 d\mu(x) \leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \varphi|^2 d\mu(x).
\]
Remark 3.4. The Poincaré type formula (26) was first obtained by Sternberb and Zumbrun [28]. Notice that the quantity $|\nabla^2 u|^2 - |\nabla |\nabla u||^2$ has a geometric interpretation, in the sense that it can be expressed in terms of the principal curvatures of level sets of $u$. More precisely, letting

$$L_{u,x} := \{ y \in \mathbb{R}^n \mid u(y) = u(x) \},$$

we denote by $\nabla_T u$ the tangential gradient of $u$ along $L_{u,x} \cap \{ \nabla u \neq 0 \}$, and by $k_1, \ldots, k_{n-1}$ the principal curvatures of $L_{u,x} \cap \{ \nabla u \neq 0 \}$. Then the following formula holds (as proved in Lemma 2.1 in [27])

$$|\nabla^2 u|^2 - |\nabla |\nabla u||^2 = |\nabla_T |\nabla u||^2 + |\nabla u|^2 \sum_{j=1}^{n-1} k_j^2 \quad \text{on } L_{u,x} \cap \{ \nabla u \neq 0 \},$$

so that (26) becomes

$$\int_{\{ \nabla u \neq 0 \}} (|\nabla u|^2 K^2 + |\nabla_T |\nabla u||^2) \phi^2 \, d\mu(x) + \int_{\{ \nabla u = 0 \}} (|\nabla^2 u|^2 - |\nabla |\nabla u||^2) \phi^2 \, d\mu(x) \leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \phi|^2 \, d\mu(x).$$

where $K := \sum_{j=1}^{n-1} k_j^2$. By Stampacchia's Theorem, since $\mu << \mathcal{L}^n$, we get

$$\nabla |\nabla u|(x) = 0 \quad \mu\text{-a.e } x \in \{ |\nabla u| = 0 \}$$

$$\nabla u_j(x) = 0 \quad \mu\text{-a.e } x \in \{ |\nabla u| = 0 \} \subseteq \{ u_j = 0 \}$$

for any $j = 1, \ldots, n$. Hence (28) gives

$$\int_{\{ \nabla u \neq 0 \}} (|\nabla u|^2 K^2 + |\nabla_T |\nabla u||^2) \phi^2 \, d\mu(x) \leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \phi|^2 \, d\mu(x).$$

We refer to [28] and [14] for more details.

We now state the main result of this section.

**Theorem 3.5.** Assume that $G \in C^2(\mathbb{R}^n)$ and $h \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $h \geq \lambda_G$. Let $u$ be a $h$-stable solution to (12) such that one of the following conditions hold:

i) $u$ satisfies (17);

ii) $n = 2$ and $u$ satisfies (18).

Then $u$ is one-dimensional, i.e. there exists $\omega \in \mathbb{S}^{n-1}$ and $u_0 : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$u(x) = u_0(\langle \omega, x \rangle) \quad \forall x \in \mathbb{R}^n.$$

Moreover,

$$\langle (h(x)I_n - \nabla^2 G(x)) \nabla u, \nabla u \rangle = 0 \quad \forall x \in \mathbb{R}^n.$$

In particular, if $u_0$ is not constant, there are $C$ and $g$ of class $C^2$ such that

$$G(x) = C(\langle x, \omega \rangle) + g(x'),$$

where $x' := x - (x, \omega) \omega$, and $\lambda_G(x) = h(x) = C''(\langle x, \omega \rangle)$ for all $x \in \mathbb{R}^n$. 

Proof. Let us fix $R > 1$ and let us define $\varphi(x) := \Phi(|x|)$ where $\Phi \in C^\infty(\mathbb{R})$, $|\Phi'(t)| \leq 3$ for any $t \in [R, R + 1]$

\begin{equation}
\Phi(t) := \begin{cases} 
1 & \text{if } t \leq R \\
0 & \text{if } t \geq R + 1.
\end{cases}
\end{equation}

Obviously $\varphi \in C^\infty_c(\mathbb{R}^n)$ and $|\nabla \varphi(x)| \leq |\Phi'(|x|)| \leq 3$. Hence for every $R > 1$ (29) yields

\begin{equation}
\int_{\{\nabla u \neq 0\} \cap B_R} \left(|\nabla u|^2 \kappa^2 + |\nabla T| \nabla u|^2\right) \, d\mu(x) \leq 9 \int_{B_{R+1} \setminus B_R} |\nabla u|^2 \, d\mu(x)
\end{equation}

where $B_R := \{y \in \mathbb{R}^n \mid |y| < R\}$.

If $\nabla u \in L^2(\mathbb{R}^n)$, then

\begin{equation}
\lim_{R \to \infty} \int_{B_{R+1} \setminus B_R} |\nabla u|^2 \, d\mu(x) = 0.
\end{equation}

Hence (34) and (35) yield

\begin{equation}
k_j(x) = 0 \quad \text{and} \quad |\nabla T| |\nabla u||| (x) = 0
\end{equation}

for every $j = 1, \ldots, n - 1$ and every $x \in \{\nabla u \neq 0\}$. From this and Lemma 2.11 in [14] we get the one-dimensional symmetry of $u$.

If $n = 2$ and $|\nabla u|^2 e^G \in L^\infty(\mathbb{R}^n)$, we take in (29) the following test function

\begin{equation}
\varphi(x) = \max \left[0, \min \left(1, \frac{\ln R^2 - \ln |x|}{\ln R}\right)\right],
\end{equation}

Reasoning as in [14, Cor. 2.6], we then obtain

\begin{equation*}
\int_{\{\nabla u \neq 0\} \cap B_R} \left(|\nabla u|^2 \kappa^2 + |\nabla T| |\nabla u|\right) \, d\mu(x) \leq \int_{B_{R^2} \setminus B_R} \frac{1}{|x|^2 (\ln R)^2} |\nabla u|^2 e^G(x) \, dx.
\end{equation*}

When $R \to +\infty$, since $|\nabla u|^2 e^G(x)$ is bounded, the r.h.s. term of the previous inequality vanishes, and we conclude again that $u$ is one-dimensional.

Assume now that $u$ is not constant. If we take in (25) the same test functions as above, we get

\begin{equation*}
\int_{\mathbb{R}^n} \langle (h(x)I_n - \nabla^2 G(x)) \nabla u, \nabla u \rangle \, d\mu(x) = 0.
\end{equation*}

Using the fact that $u(x) = u_0(\omega, x)$, we obtain that $\langle (h(x)I_n - \nabla^2 G(x)) \omega, \omega \rangle = 0$ for all $x$ such that $u_0'(\omega, x) \neq 0$. Since $u$ is not constant and is a solution to the elliptic equation (12), the set of points such that $u_0'(\omega, x) = 0$ has zero measure, so, by the regularity of $G$ we conclude that

\begin{equation*}
\langle (h(x)I_n - \nabla^2 G(x)) \omega, \omega \rangle = 0 \quad \forall \ x \in \mathbb{R}^n,
\end{equation*}

which gives (31) and (32). \hfill \Box

Theorem 3.5 directly implies the following Liouville type result (cfr. [15]).

**Corollary 3.6.** Let $h \in C^0(\mathbb{R}^n)$ with $h \geq \lambda_G$, and $u$ be a $h$-stable solution solution to (12) with finite energy. If $\lambda_G(x) < h(x)$ for some $x \in \mathbb{R}^n$, then $u$ is constant. In particular, if $u$ is a stable solution and $\lambda_G(x) < 0$ for some $x \in \mathbb{R}^n$, then $u$ is constant.
Remark 3.7. Recalling Remark 2.3, when the measure $\mu$ is finite and $G$ is concave, Theorem 3.5 implies that bounded solutions to (12) which are $\lambda G$-stable are one-dimensional.

4. Monotonicity implies $\lambda G$-stability

In this section we assume that, for every $x \in \mathbb{R}^n$, $e_n$ is the eigenvector associated to the maximal eigenvalue $\lambda_G(x)$ of $\nabla^2 G(x)$. This implies that there exist two functions $g$ and $C$ such that

\[ G(x) = g(y) + C(z) \text{ and } \lambda_G(x) = C''(z). \]

We prove that solutions to (12) which are monotone along the $z$-axis are stable.

Theorem 4.1. Assume that $G$ satisfies (38) and $u$ is a solution to (12) satisfying (4). Then $u$ is $\lambda G$-stable.

Proof. Equation (19) with $i = n$ reads

\[ \int_{\mathbb{R}^n} \langle \nabla u_{z}, \nabla \varphi \rangle - C''(z)u_z \varphi - f'(u)u_z \varphi \, d\mu(x) = 0. \]

Let $\varphi \in C^1_c(\mathbb{R}^n)$. Taking as test function $\frac{\varphi^2}{u_z}$ in (39), we get

\begin{align*}
0 &= \int_{\mathbb{R}^n} \langle \nabla u_{z}, \nabla \left( \frac{\varphi^2}{u_z} \right) \rangle - C''(z) \varphi^2 - f'(u) \varphi^2 \, d\mu(x) \\
&= \int_{\mathbb{R}^n} |\nabla \varphi|^2 - \left| \frac{\varphi}{u_z} \nabla u - \nabla \varphi \right|^2 - C''(z) \varphi^2 - f'(u) \varphi^2 \, d\mu(x) \\
&\leq \int_{\mathbb{R}^n} |\nabla \varphi|^2 - C''(z) \varphi^2 - f'(u) \varphi^2 \, d\mu(x),
\end{align*}

which is the stability condition (15). \qed

5. Proof of Theorem 1.1

Observe that in (2), $G(x) = g(y) + C(z)$, and by assumption $C''(z) \geq \nabla^2 g(y)$. So (38) holds, and by Theorem 4.1 every solution to (2) satisfying (4) is $\lambda G$-stable.

If either a) or c) holds, the thesis follows from Theorem 3.5.

Let us assume that $u$ satisfies b). We define $\psi_R(y) := \Phi(|y|)$ where $\Phi$ is as in (33) and $\varphi_S(z)$ as follows. We fix $S > 1$ and let

\[ \varphi_S(z) := \begin{cases} 
3 & \text{if } |z| \leq S \\
4 - \frac{z^2}{S^2} & \text{if } S \leq |z| \leq 2S \\
0 & \text{if } |z| \geq 2S.
\end{cases} \]

We compute (29) with test function $\psi_R(y) \varphi_S(z)$ and obtain, recalling (7),

\[ \int_{\{ \nabla u \neq 0 \}} \left( |\nabla u|^2 K^2 + |\nabla_T |\nabla u||^2 \right) \psi_R^2 \varphi_S^2 \, d\mu(x) \leq \int_{\mathbb{R}^n} |\nabla u|^2 \varphi_S^2(z) \nabla^2 \psi_R(y) d\mu(x) \]

\[ \leq \frac{4}{S^2} \int_{\mathbb{R}^n} |\nabla u|^2 \nabla^2 \psi_R(y) d\mu(x) \leq \frac{36K}{S^2}. \]
If we let $R \to +\infty$ we obtain
\[
\int_{\{\nabla u \neq 0\} \cap \{|z| \leq S\}} \left( |\nabla u|^2 K^2 + |\nabla T| \nabla u|^2 \right) \, d\mu(x) \leq \frac{4K^2}{S^2}.
\]
Letting $S \to +\infty$ we then obtain (36) and we conclude as in the proof of Theorem 3.5. □

REFERENCES


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