A SHARP QUANTITATIVE ISOPERIMETRIC INEQUALITY IN HIGHER CODIMENSION

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ABSTRACT. We establish a quantitative isoperimetric inequality in higher codimension. In particular, we prove that for any closed $(n-1)$-dimensional manifold Γ in $\mathbb{R}^{n+k}$ the following inequality

$$\mathbf{D}(\Gamma) \geq C\mathbf{d}^2(\Gamma)$$

holds true. Here, $\mathbf{D}(\Gamma)$ stands for the isoperimetric gap of Γ, i.e. the deviation in measure of Γ from being a round sphere and $\mathbf{d}(\Gamma)$ denotes a natural generalization of the Fraenkel asymmetry index of Γ to higher codimensions.

1. INTRODUCTION

In 1986 in his seminal paper "Optimal isoperimetric inequalities" [2] Almgren proved in the context of currents the higher codimension counterpart of the classical isoperimetric inequality established by De Giorgi in [7]. In the particular case of smooth $(n-1)$-dimensional manifolds $\Gamma \subset \mathbb{R}^{n+k}$ without boundary, spanning an area minimizing smooth surface M, his inequality states that

$$\mathcal{H}^{n-1}(\Gamma) \geq \mathcal{H}^{n-1}(\partial D), \tag{1.1}$$

where D is an n-dimensional flat disk with the same area as M. Here, $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional surface measure. Moreover equality occurs if and only if Γ is the boundary of a flat disk.

A natural question is the stability of inequality (1.1). More precisely, one would like to show that if Γ fails to realize equality in the isoperimetric inequality (1.1) by a small factor δ, i.e. $\mathcal{H}^{n-1}(\Gamma) = \mathcal{H}^{n-1}(\partial D) + \delta$, then Γ is close to the boundary ∂D in a suitable quantitative sense measured in terms of δ. For the classical isoperimetric inequality in codimension zero, this stability issue was raised in the beginning of the last century by Bernstein and Bonnesen in the particular case of planar convex sets [3, 5]. Later on the first results in higher dimensions were established in [15] by Fuglede in the case of convex or nearly spherical sets. His main result states that if $E \subset \mathbb{R}^n$ is a nearly spherical set in the sense that

$$\partial E = \big\{(1+u(x))x : x \in S^{n-1}\big\}$$

for some $u\colon S^{n-1} \to \mathbb{R}$ with small C^1-norm, whose volume is equal to the volume of the unit ball $B_1 \subset \mathbb{R}^n$ and whose barycenter is at the origin, then

$$\mathcal{H}^{n-1}(\partial E) - \mathcal{H}^{n-1}(\partial B_1) \geq c(n)\|u\|^2_{W^{1,2}(S^{n-1})}.$$

In particular, this inequality implies that the isoperimetric gap on the left-hand side controls the square of the measure of the symmetric difference $E\Delta B_1$. The extension of Fuglede's result to general sets of finite perimeter was first obtained in [17] (see also [19, 20] for a similar, but non optimal inequality). The result proved in [17] states that there exists a constant C depending only on the dimension n such that if E is a set of finite perimeter with $|E| = |B_r|$, then

$$D(E) \geq C(n)\,\alpha^2(E). \tag{1.2}$$

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Here, $D(E)$ stands for the (normalized) *isoperimetric gap*

$$D(E) := \frac{\mathcal{H}^{n-1}(\partial E) - n\omega_n r^{n-1}}{n\omega_n r^{n-1}}$$

and $\alpha(E)$ is the so-called *Fraenkel asymmetry*

$$\alpha(E) := \min_x \left\{ \frac{|E \Delta B_r(x)|}{r^n} \right\}.$$

While the original proof in [17] used mainly symmetrization arguments, in [14] a new proof based on arguments from the theory of optimal mass transport appeared. These arguments allowed an extension of (1.2) also to anisotropic perimeter functionals. Both proofs are quite involved due to their ad hoc character, especially, since they do not use any deep result or heavy machinery from other fields of Analysis and Geometry. In a recent paper Cicalese and Leonardi [6] observed that it is possible to give a proof of the quantitative isoperimetric inequality by a selection principle based on a suitable penalization of the functional $E \mapsto \frac{D(E)}{\alpha^2(E)}$ and the use of the regularity theory for minimal surfaces.

In order to describe the main result of our paper we restrict ourselves to the case of smooth $(n-1)$-dimensional closed surfaces Γ in $\mathbb{R}^{n+k}$. Denoting by $Q(\Gamma)$ an area minimizing n-dimensional surface with boundary Γ the *isoperimetric gap* is defined by

$$\mathbf{D}(\Gamma) := \frac{\mathcal{H}^{n-1}(\Gamma) - \mathcal{H}^{n-1}(\partial D_\varrho)}{\mathcal{H}^{n-1}(\partial D_\varrho)},$$

where D_ϱ is an n-dimensional flat disk in $\mathbb{R}^{n+k}$ with the same area as $Q(\Gamma)$, i.e. $\mathcal{H}^n(D_\varrho) = \mathcal{H}^n(Q(\Gamma))$. Note that the area minimizing surface $Q(\Gamma)$ may have singularities even if Γ is smooth. To overcome this, the use of currents is unavoidable. However, in order to keep the introduction as simple as possible we describe the objects in the context of manifolds. The precise definition of the *asymmetry index* $\mathbf{d}(\Gamma)$ is more technical and requires the use of a certain seminorm $\mathbf{m}$ (see Section 3). The underlying geometric idea can be described as follows. Given any flat disk D_ϱ with the same area as $Q(\Gamma)$, first one considers an area minimizing cylindric type surface $\Sigma(D_\varrho)$ spanned by the boundary components Γ and ∂D_ϱ, and afterwards one takes the infimum of the surface area $\mathcal{H}^n(\Sigma(D_\varrho))$ amongst all possible disks D_ϱ:

$$\mathbf{d}(\Gamma) := \varrho^{-n} \inf \big\{ \mathcal{H}^n(\Sigma(D_\varrho)) : \mathcal{H}^n(D_\varrho) = \mathcal{H}^n(Q(\Gamma)) \big\}.$$

The aim of this paper is to state and prove in the context of currents the following heuristic quantitative version of Almgren's optimal isoperimetric inequality:

Theorem. *Let $n \geq 2$ and $k \geq 0$. There exists a constant $C = C(n,k) > 0$ such that for any $(n-1)$-dimensional closed surface $\Gamma \subset \mathbb{R}^{n+k}$ the following inequality holds:*

$$\mathbf{D}(\Gamma) \geq C\, \mathbf{d}^2(\Gamma). \tag{1.3}$$

Note that if Γ is the boundary of a smooth open set E contained in an n-dimensional hyperplane, then the asymmetry index $\mathbf{d}(\Gamma)$ coincides with the classical Fraenkel asymmetry index $\alpha(E)$. Hence, (1.3) reduces to (1.2). In particular this shows that the exponent 2 on the right-hand side of the inequality cannot be improved, since it is known to be optimal already for (1.2).

A few words on the proof are in order. As in [6] the overall strategy is to show first a Fuglede type inequality and then to reduce the general case to it via a regularity argument. However, here the situation is more delicate and involved due to the higher codimension. First of all, the analogue of Fuglede's result deals with a *spherical graph over S^{n-1} in $\mathbb{R}^{n+k}$*, i.e. a manifold Γ which can be parametrized by a map $X\colon S^{n-1} \to \mathbb{R}^{n+k}$ of the form

$$X(x) := (1+u(x))(x,0) + (0, v(x)) \qquad x \in S^{n-1},$$

where $u \in C^1(S^{n-1})$ and $v \in C^1(S^{n-1}, \mathbb{R}^k)$ have both small C^1-norms. In our case a substantial difficulty arises from the fact that, beside imposing the volume constraint $\mathcal{H}^n(Q(\Gamma)) = \omega_n$ and the barycenter condition $\mathrm{bar}(\Gamma) = 0$, we have also to fix the mixed second order moments. This can be done for instance by assuming that they are all equal to zero, i.e.

$$\int_\Gamma z_i z_j \, d\mathcal{H}^{n-1} = 0 \tag{1.4}$$

for any choice of $i = 1, \dots, n$ and $j = n+1, \dots, n+k$. Differently from the case $k = 0$ considered by Fuglede, in which v does not appear, the conditions (1.4) play a crucial role in the estimation of the $n \cdot k$ first order Fourier coefficients of v. The bounds on the first order Fourier coefficients of u and the zero order Fourier coefficients of u and v follow from the barycenter condition and the constraint on $\mathcal{H}^n(Q(\Gamma))$. Under the above assumptions on u and v we prove the following inequality (see Theorem 4.1)

$$\mathcal{H}^{n-1}(\Gamma) - \mathcal{H}^{n-1}(S^{n-1}) \geq c_1 \big[\|u\|^2_{W^{1,2}(S^{n-1})} + \|v\|^2_{W^{1,2}(S^{n-1}, \mathbb{R}^k)} \big] \geq c_o \mathbf{d}^2(\Gamma), \tag{1.5}$$

where $c_1 \geq c_o$ are constants depending only on n.

The next step is to reduce the general case to the previous one by a contradiction argument using the regularity theory for ω-minimizing currents. However, following [1] where a similar kind of penalization term was introduced we use a much simpler penalization then the one used in [6] in the treatment of the codimension zero case (see also [8, 16, 18]) which is also reminiscent of the Ekeland variational principle [12]. Our argument goes as follows. We argue by contradiction assuming that there exists a sequence of $(n-1)$-dimensional surfaces Γ_j all contained in a large ball B_R such that $\mathcal{H}^n(Q(\Gamma_j)) = \omega_n$ and $\mathbf{D}(\Gamma_j)/\mathbf{d}^2(\Gamma_j) \to 0$. Then, we construct a new sequence by considering the minimizers Γ'_j of the penalized functionals

$$\mathcal{F}_j(\Gamma) := \mathcal{H}^{n-1}(\Gamma) + C_1 |\mathbf{d}(\Gamma) - \mathbf{d}(\Gamma_j)| + \Lambda |\mathcal{H}^n(Q(\Gamma)) - \omega_n|$$

with $\Lambda > 2n$ and $C_1 > 0$ a suitable constant depending on c_o. It is not difficult to show that the surfaces Γ'_j converge in a weak sense to S^{n-1} and that also the ratio $\mathbf{D}(\Gamma'_j)/\mathbf{d}^2(\Gamma'_j) \to 0$. Moreover, the weak convergence ensures that the barycenters and the second order moments of Γ'_j converge to zero while the corresponding area minimizers $Q(\Gamma'_j)$ converge in a weak sense to a flat disk with boundary S^{n-1}. To derive a contradiction to the Fuglede type estimate (1.5), one first has to show that the surfaces Γ'_j can be chosen to satisfy (1.4). This is done by proving that (see Lemma 4.2) if Γ is a manifold with sufficiently small second order moments one can find a rotation close to the identity such that the mixed second order moments of the rotated manifold are all equal to zero. Since the penalized functional above is invariant under rotations the tilted surfaces are still minimizers. Thus, the last step in deriving the contradiction to (1.5) is to establish that the surfaces Γ'_j are spherical graphs converging to S^{n-1} in $C^{1,\alpha}$. This is the point where the regularity theory for ω-minimizing currents enters. In fact, the existence theory yields only that the minimizers Γ'_j are rectifiable currents minimizing an appropriate generalization of the functional $\mathcal{F}_j$ in the context of Geometric Measure Theory. It can also be shown that the penalization terms in the functional are of lower order, so that the surfaces (in fact currents) Γ'_j are ω-minimizers of the area (mass) functional. However, to show that they are spherical $C^{1,\alpha}$ graphs over S^{n-1} one has to transform locally to a situation where the regularity theory for ω-minimizing currents is applicable. This is done by flattening locally S^{n-1} and transforming to a flat case in which the ω-mass minimizers become ω-minimizers of a suitable elliptic integrand, and in which they converge to a flat $(n-1)$-dimensional disk with multiplicity one. At this stage the regularity theory from [4, 10] applies and yields that the Γ'_j are spherical graphs converging in $C^{1,\alpha}$ to S^{n-1}. But this is a contradiction to the higher codimension version of Fuglede's theorem as stated in (1.5).

2. Notation and statement of the result

Let $n \in \mathbb{N}$, $k \in \mathbb{N}_o$ and $0 \le m \le n$. Then m-dimensional surfaces in $\mathbb{R}^{n+k}$ will be modelled by *locally rectifiable integer multiplicity currents with finite mass* in $\mathbb{R}^{n+k}$. Such currents T, of dimension m, can be represented by an $(\mathcal{H}^m, m)$ rectifiable mesurable supporting set $M_T \subset \mathbb{R}^{n+k}$, an $\mathcal{H}^m$ summable multiplicity function $\vartheta_T \colon M_T \to \mathbb{N}$, and an $\mathcal{H}^m$ measurable orientation $\vec{T} \colon M_T \to \bigwedge_m \mathbb{R}^{n+k}$, i.e. $\vec{T}$ is the exterior product of an orthonormal basis in the m-dimensional measure theoretic tangent space $\mathrm{Tan}(\mathcal{H}^m \llcorner M_T, x)$ of M_T which exists at $\mathcal{H}^m$ almost all points $x \in M_T$. We set $\vartheta_T \equiv 0$, $\vec{T} \equiv 0$ outside M_T and denote by $\|T\| = \vartheta_T \mathcal{H}^m \llcorner M_T$ the (Radon) measure associated with T and by $\mathbf{M}(T) = \|T\|(\mathbb{R}^{n+k}) = \int \vartheta_T \, d\mathcal{H}^m$ the mass (or m area) of T. Note that the summability of ϑ_T is equivalent to the finiteness of the mass $\mathbf{M}(T)$. Here we follow the terminology of [13]. By definition, an m current is a continuous linear functional on the space of compactly supported smooth m forms on $\mathbb{R}^{n+k}$ which we denote by $\alpha \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^{n+k}, \bigwedge^m \mathbb{R}^{n+k})$. In terms of the quantities $\|T\|$ and $\vec{T}$ the pairing of currents and differential forms is given by

$$T(\alpha) = \int_{M_T} \langle \alpha, \vec{T} \rangle \vartheta_T \, d\mathcal{H}^m = \int_{\mathbb{R}^{n+k}} \langle \alpha, \vec{T} \rangle \, d\|T\|,$$

and it is defined whenever α is a bounded Baire form of degree m. The set of all locally rectifiable integer multiplicity m currents is denoted by $\mathcal{R}_m(\mathbb{R}^{n+k})$.

The boundary current ∂T is then defined by taking formally the dual of the exterior derivative, i.e. $\partial T(\beta) = T(d\beta)$ for compactly supported smooth $m-1$ forms β on $\mathbb{R}^{n+k}$. For an open (and more generally a Baire) set $U \subset \mathbb{R}^{n+k}$ we define the mass of T in U by

$$\mathbf{M}_U(T) := (\|T\| \llcorner U)(\mathbb{R}^{n+k}) = \int_{M_T \cap U} \vartheta_T \, d\mathcal{H}^m.$$

On the set of closed m-dimensional surfaces, i.e. for $T \in \mathcal{R}_m(\mathbb{R}^{n+k})$ with $\partial T = 0$ and $1 \le m < n+k$, we now define a seminorm measuring the mass of a minimal surface spanned by T. More precisely, given T as above there exists a mass minimizing current $Q(T) \in \mathcal{R}_{m+1}(\mathbb{R}^{n+k})$ with boundary $\partial Q(T) = T$. The mass of $Q(T)$ is denoted by $\mathbf{m}(T)$, i.e

$$\mathbf{m}(T) := \mathbf{M}(Q(T)) \equiv \inf_{\substack{P \in \mathcal{R}_{m+1}(\mathbb{R}^{n+k}), \\ \partial P = T}} \mathbf{M}(P).$$

When writing $Q(T)$ we always understand that we have specified one particular mass minimizer with boundary T. We note that there might be several mass minimizers. Our arguments however will not depend on a particular choice. Moreover, in case that $\mathrm{spt}\, T$ is compact we know from [21, Remark 34.2(2)] that $\mathrm{spt}\, Q(T) \subset$ convex hull of $\mathrm{spt}\, T$ for any mass minimizer $Q(T)$.

To give the precise formulation of our main result we have to introduce the notion of a *flat n-dimensional disk* in $\mathbb{R}^{n+k}$. The *Euclidian current* $\mathbf{E}^n$ on $\mathbb{R}^n$ is defined by

$$\mathbf{E}^n(\alpha) := \int_{\mathbb{R}^n} \langle \alpha, e_1 \wedge \cdots \wedge e_n \rangle \, d\mathcal{L}^n \quad \text{for any } \alpha \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^n, \textstyle\bigwedge^n \mathbb{R}^n).$$

Here $\mathcal{L}^n$ denotes the Lebesgue measure on $\mathbb{R}^n$. For an $\mathcal{L}^n$ measurable set $A \subset \mathbb{R}^n$ the current $\mathbf{E}^n \llcorner A$ is defined as usual via

$$(\mathbf{E}^n \llcorner A)(\alpha) = \int_A \langle \alpha, e_1 \wedge \cdots \wedge e_n \rangle \, d\mathcal{L}^n.$$

Then, by an *n-dimensional flat disk* in $\mathbb{R}^{n+k}$ we mean a current $T \in \mathcal{R}_n(\mathbb{R}^{n+k})$ of the form $T := \Phi_\#(\mathbf{E}^n \llcorner D)$ where D is any open ball in $\mathbb{R}^n$ and $\Phi \colon \mathbb{R}^n \to \mathbb{R}^{n+k}$ an isometric embedding. In order not to overburden our presentation with notation we will use the short hand notation $[\![D]\!]$ instead of $\Phi_\#(\mathbf{E}^n \llcorner D)$. By $[\![D_r]\!]$ we denote a flat disk of radius $r > 0$. We use a similar notation for currents associated to oriented, compact, m-dimensional submanifolds $M \subset \mathbb{R}^{n+k}$. Indeed, if ξ denotes an orientation m-vector field on M, then the corresponding m-current $[\![M]\!] \in \mathcal{R}_m(\mathbb{R}^{n+k})$ is defined by $[\![M]\!](\alpha) := \int_M \langle \alpha, \xi \rangle \, d\mathcal{H}^m$ for all $\alpha \in \mathcal{D}^m(\mathbb{R}^{n+k})$.

Now, let $n \geq 2$. As introduced above, we use for a current $T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with $\partial T = 0$ the abbreviation $\mathbf{m}(T)$ to denote the minimal mass spanned by T. Moreover, by $\varrho(T)$ we denote the radius of any flat n-dimensional disk $[\![D]\!]$ whose mass is equal to the minimal mass spanned by T, that is $\mathbf{m}(T) = \mathbf{M}([\![D]\!]) = \omega_n \varrho(T)^n$, so that

$$\varrho(T) := \sqrt[n]{\frac{\mathbf{m}(T)}{\omega_n}}.$$

Then, the *isoperimetric gap* is given by

$$\mathbf{D}(T) := \frac{\mathbf{M}(T) - n\omega_n \varrho(T)^{n-1}}{n\omega_n \varrho(T)^{n-1}}.$$

Note that the isoperimetric gap is invariant with respect to translations, rotations and dilations. Next, we observe that $\mathbf{m}(T - \partial[\![D_{\varrho(T)}]\!])$ measures how close T and $\partial[\![D_{\varrho(T)}]\!]$ are. Of course, when taking an arbitrary disk of radius $\varrho(T)$ this distance can be very large. Therefore, in order to measure the deviation of the surface from round spheres of radius $\varrho(T)$ we shall take the infimum over all such spheres. This quantity we call the *asymmetry index* of T, and it is a measure for the deviation of T from being a round sphere. Hence, for $T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with $\partial T = 0$ we define

$$\mathbf{d}(T) := \inf_{[\![D_{\varrho(T)}]\!]} \frac{\mathbf{m}(T - \partial[\![D_{\varrho(T)}]\!])}{\varrho(T)^n},$$

where now the infimum is taken over all flat n-dimensional disks $[\![D_{\varrho(T)}]\!]$ of radius $\varrho(T)$, i.e. about those disks with mass equal to the minimal mass $\mathbf{m}(T)$ spanned by T. Note that also $\mathbf{d}(T)$ is invariant under translations, rotations and dilations. Now we are in the position to state our result.

Theorem 2.1. *Let $n \geq 2$ and $k \geq 0$. Then, there exists a constant $C > 0$ depending only on n and k such that for any $T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with $\partial T = 0$ the sharp quantitative isoperimetric inequality holds*

$$\mathbf{D}(T) \geq C\, \mathbf{d}^2(T). \tag{2.1}$$

3. Facts from geometric measure theory

For later use we recall some facts from Geometric Measure Theory which can be retrieved either from [13] or [21]. We start with the definition of the flat seminorm. For a given open set U and an m-dimensional current T with locally finite boundary mass, i.e. $\mathbf{M}_W(\partial T) < \infty$ for any $W \Subset \mathbb{R}^{n+k}$, the *flat semi norm* is defined by

$$\mathbf{F}_U(T) := \inf_{T = S + \partial P} \big(\mathbf{M}_U(S) + \mathbf{M}_U(P)\big),$$

where the infimum is taken over all $S \in \mathcal{R}_m(\mathbb{R}^{n+k})$ and $P \in \mathcal{R}_{m+1}(\mathbb{R}^{n+k})$. In the case $U = \mathbb{R}^{n+k}$ we write $\mathbf{F} := \mathbf{F}_{\mathbb{R}^{n+k}}$. The topology induced by the semi norms $\mathbf{F}_U$ for $U \subset \mathbb{R}^{n+k}$ open and bounded is called the $\mathbf{F}_{\mathrm{loc}}$-topology on $\mathcal{R}_m(\mathbb{R}^{n+k})$. The following theorem ensures that for sequences the $\mathbf{F}_{\mathrm{loc}}$-topology and the weak topology on $\mathcal{R}_m(\mathbb{R}^{n+k})$ are identical, cf. [21, Theorem 31.2]. Note that we state the following two theorems only for locally rectifiable integer multiplicity m-currents with finite mass. The original versions certainly include m-currents with only locally finite mass.

Theorem 3.1. *Let $T, \{T_j\} \subset \mathcal{R}_m(\mathbb{R}^{n+k})$ be m-currents with*

$$\sup_{j \in \mathbb{N}} \big(\mathbf{M}_U(T_j) + \mathbf{M}_U(\partial T_j)\big) < \infty \quad \textit{for all } U \Subset \mathbb{R}^{n+k}.$$

Then $T_j \to T$ with respect to the $\mathbf{F}_{\rm loc}$-topology if and only if $T_j \to T$ with respect to the weak topology.

For later purposes we recall the *compactness theorem* of Federer and Fleming, see [13, Theorem 4.2.17] or [21, Theorem 27.3].

Theorem 3.2. *If $\{T_j\} \subset \mathcal{R}_m(\mathbb{R}^{n+k})$ is a sequence of m-currents in $\mathbb{R}^{n+k}$ with*

$$\sup_{j\in\mathbb{N}} \big(\mathbf{M}_U(T_j) + \mathbf{M}_U(\partial T_j)\big) < \infty \quad \textit{for all } U \Subset \mathbb{R}^{n+k},$$

then there is an m-current $T \in \mathcal{R}_m(\mathbb{R}^{n+k})$ and a subsequence $\{T_j\}$ such that $T_j \to T$ with respect to the $\mathbf{F}_{\rm loc}$-topology.

By Theorem 3.1 the compactness in Theorem 3.2 also holds with respect to the weak topology. This allows to extract a weakly convergent subsequence from any sequence of currents $T_j \in \mathcal{R}_m(\mathbb{R}^{n+k})$ satisfying a suitable mass bound. Together with a lower semicontinuity property of certain functionals this yields the existence of a minimizer, as for example in the case of the mass $\mathbf{M}$ (which is easily seen to be lower semicontinuous with respect to weak convergence of currents).

We note that the flat norm $\mathbf{F}$ and the seminorm $\mathbf{m}$ are almost equivalent. First one observes that $\mathbf{F}_U \le \mathbf{m}$ holds for any open set $U \subset \mathbb{R}^{n+k}$. On the other hand, the following lemma, whose proof is an easy consequence of the isoperimetric inequality, ensures that also a reverse type inequality holds true for currents with compact support.

Lemma 3.3. *Let $R > 0$. Then, for any $T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with $\partial T = 0$ and $\operatorname{spt} T \subset B_R$ there holds*

$$\mathbf{m}(T) \le \big[C(n)\,\mathbf{M}(T) + 1\big]\mathbf{F}_{B_{2R}}(T).$$

Proof. We first choose $S \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ and $P \in \mathcal{R}_n(\mathbb{R}^{n+k})$ realizing $\mathbf{F}_{B_{2R}}(T)$ up to an error $\varepsilon > 0$, i.e. $S + \partial P = T$ and $\mathbf{M}_{B_{2R}}(S) + \mathbf{M}_{B_{2R}}(P) < \mathbf{F}_{B_{2R}}(T) + \varepsilon$. Since $\operatorname{spt} T \subset B_R$, we may assume without loss of generality that $\operatorname{spt} S, \operatorname{spt} P \subset \overline{B}_R$. Indeed, otherwise we replace S and P with the corresponding projections $p_\#(S)$ and $p_\#(P)$ onto $\overline{B}_R$, which still satisfy the equality $T = p_\#(T) = p_\#(S + \partial P) = p_\#(S) + \partial p_\#(P)$ but have smaller mass on B_{2R}. Then, from Theorem 3.4 below we observe that

$$\begin{aligned}
\mathbf{m}(T) &\le (\mathbf{m}(T))^{\frac1n}[\mathbf{m}(S) + \mathbf{m}(\partial P)]^{\frac{n-1}{n}}\\
&\le \mathbf{m}(T)^{\frac1n}\Big[\gamma_n \mathbf{M}(S)^{\frac{n}{n-1}} + \mathbf{M}(P)\Big]^{\frac{n-1}{n}}\\
&\le \mathbf{m}(T)^{\frac1n}\gamma_n^{\frac{n-1}{n}}\mathbf{M}(S) + \mathbf{m}(T)^{\frac1n}\mathbf{M}(P)^{\frac{n-1}{n}}\\
&\le \big(\gamma_n \mathbf{M}(T)^{\frac{n}{n-1}}\big)^{\frac1n}\gamma_n^{\frac{n-1}{n}}\mathbf{M}(S) + \tfrac1n \mathbf{m}(T) + \tfrac{n-1}{n}\mathbf{M}(P)\\
&= \gamma_n \mathbf{M}(T)^{\frac{1}{n-1}}\mathbf{M}(S) + \tfrac1n \mathbf{m}(T) + \tfrac{n-1}{n}\mathbf{M}(P).
\end{aligned}$$

Taking into account that $\mathbf{M}(S) = \mathbf{M}_{B_{2R}}(S)$, $\mathbf{M}(P) = \mathbf{M}_{B_{2R}}(P)$, we get

$$\begin{aligned}
\mathbf{m}(T) &\le \tfrac{n}{n-1}\gamma_n \mathbf{M}(T)^{\frac{1}{n-1}}\mathbf{M}_{B_{2R}}(S) + \mathbf{M}_{B_{2R}}(P)\\
&\le \Big[\big(\tfrac{n\gamma_n}{n-1}\big)^{n-1}\mathbf{M}(T) + 1\Big]\mathbf{M}_{B_{2R}}(S) + \mathbf{M}_{B_{2R}}(P)\\
&\le \Big[\big(\tfrac{n\gamma_n}{n-1}\big)^{n-1}\mathbf{M}(T) + 1\Big]\big(\mathbf{M}_{B_{2R}}(S) + \mathbf{M}_{B_{2R}}(P)\big)\\
&\le \Big[\big(\tfrac{n\gamma_n}{n-1}\big)^{n-1}\mathbf{M}(T) + 1\Big]\big(\mathbf{F}_{B_{2R}}(T) + \varepsilon\big).
\end{aligned}$$

Letting $\varepsilon \downarrow 0$ the assertion of the lemma follows. □

In the proof of the quantitative isoperimetric inequality it will be convenient to work with a non re-scaled version of the asymmetry index $\mathbf{d}$. Hence, for $T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$

with $\partial T = 0$ we define

$$\mathbf{d}_1(T) := \inf_{[\![D_1]\!]} \mathbf{m}(T - \partial[\![D_1]\!]),$$

where the infimum is taken over all flat n-dimensional disks $[\![D_1]\!]$ of radius 1. Note that $\mathbf{d}_1(T)$ is invariant under translations and rotations and that $\mathbf{d}(T) = \mathbf{d}_1(T)$ if $\mathbf{m}(T) = \omega_n$.

Finally, the following optimal isoperimetric inequality can be retrieved from [2, Theorem 9].

Theorem 3.4. *Suppose that $T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with $\partial T = 0$ and that $Q(T)$ is a mass minimizing current with boundary T. Then, there holds*

$$\mathbf{M}(Q(T)) \le \gamma_n \mathbf{M}(T)^{\frac{n}{n-1}}$$

where $\gamma_n := n^{-\frac{n}{n-1}} \omega_n^{-\frac{1}{n-1}}$ denotes the optimal isoperimetric constant. Equality holds if and only if $Q(T)$ is a flat n-dimensional disk $[\![D]\!]$ in $\mathbb{R}^{n+k}$.

4. A version of Fuglede's Theorem in higher codimension

We start with some notation. Coordinates $z \in \mathbb{R}^{n+k}$ are written as $z = (x, y)$. Here $n \ge 2$ and $k \ge 0$. The case $k = 0$ corresponds to the classical case treated by Fuglede. For this reason we restrict ourselves to the case $k \ge 1$. Throughout this section we consider an $(n-1)$-dimensional surface $\Gamma \subset \mathbb{R}^{n+k}$ which can be parametrized globally by a map $X\colon S^{n-1} \to \mathbb{R}^{n+k}$ from the sphere S^{n-1} into $\mathbb{R}^{n+k}$ as follows:

$$X(x) := (1 + u(x))(x, 0) + (0, v(x)) \qquad x \in S^{n-1}. \tag{4.1}$$

Here $u\colon S^{n-1} \to \mathbb{R}$ is a scalar valued function and $v\colon S^{n-1} \to \mathbb{R}^k$ is a vector valued function. We call such a surface Γ a *spherical graph* over S^{n-1}; actually such a surface is a global section in the normal bundle over S^{n-1}. For spherical graphs we have

Theorem 4.1 (Fuglede's theorem for spherical graphs in higher codimension). *There exist $\varepsilon_o \in (0, 1]$ and $C_1 > C_o > 0$ depending only on n such that there holds: Whenever Γ is a spherical C^1-graph over S^{n-1} in $\mathbb{R}^{n+k}$ such that the defining functions $u\colon S^{n-1} \to \mathbb{R}$ and $v\colon S^{n-1} \to \mathbb{R}^k$ satisfy*

$$\|u\|_{C^1(S^{n-1})} + \|v\|_{C^1(S^{n-1}, \mathbb{R}^k)} \le \varepsilon_o, \tag{4.2}$$

and such that the area of a mass minimizing current Q spanned by Γ is equal to the area of a flat n-dimensional disk of radius 1, that is

$$\mathbf{m}([\![\Gamma]\!]) \equiv \inf\big\{\mathbf{M}(Q) : Q \in \mathcal{R}_n(\mathbb{R}^{n+k}), \partial Q = [\![\Gamma]\!]\big\} = \omega_n, \tag{4.3}$$

whose barycenter and mixed second order moments are zero, that is

$$\operatorname{bar}(\Gamma) := \int_\Gamma z \, d\mathcal{H}^{n-1} = 0, \tag{4.4}$$

and

$$\int_\Gamma x_i y_\alpha \, d\mathcal{H}^{n-1} = 0 \tag{4.5}$$

for every choice of $i = 1, \dots, n$ and $\alpha = 1, \dots, k$, then the following quantitative isoperimetric inequality holds:

$$\frac{\mathcal{H}^{n-1}(\Gamma) - n\omega_n}{n\omega_n} \ge C_1 \big[\|u\|^2_{W^{1,2}} + \|v\|^2_{W^{1,2}}\big] \ge C_o \, \mathbf{d}^2([\![\Gamma]\!]).$$

Proof. The proof will be divided into several steps.

Step 1. Lower bound for the isoperimetric gap. We first compute the area element of the surface Γ with the help of the parametrization X from (4.1). For this we evaluate the $(n-1)$-Jacobian JX of X from the matrix representation of $\nabla_\tau X$ with respect to an orthonormal basis $\tau_1, \dots, \tau_{n-1}$ in the tangent space to S^{n-1} and the associated orthonormal basis $(\tau_1, 0), \dots, (\tau_{n-1}, 0), (x, 0)$, $(0, e_1), \dots, (0, e_k)$ in $\mathbb{R}^{n+k}$. In this representation we have

$$DX = \begin{pmatrix} (1+u) & 0 & \dots & 0 \\ 0 & (1+u) & & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & & (1+u) \\ \nabla_{\tau_1} u & \nabla_{\tau_2} u & \dots & \nabla_{\tau_{n-1}} u \\ \nabla_{\tau_1} v_1 & \nabla_{\tau_2} v_1 & \dots & \nabla_{\tau_{n-1}} v_1 \\ \vdots & & \ddots & \vdots \\ \nabla_{\tau_1} v_k & \nabla_{\tau_2} v_k & \dots & \nabla_{\tau_{n-1}} v_k \end{pmatrix}$$

and the Jacobian can easily be computed as follows

$$\begin{aligned} [JX]^2 &= (1+u)^{2(n-1)} + (1+u)^{2(n-2)} \big(|\nabla_\tau u|^2 + |\nabla_\tau v|^2\big) \\ &\quad + \sum_{\alpha=2}^{\min\{k+1, n-1\}} (1+u)^{2(n-1-\alpha)} M_\alpha(\nabla_\tau u, \nabla_\tau v)^2, \end{aligned}$$

where $M_\alpha(\nabla_\tau u, \nabla_\tau v)^2$ is the sum of the squares of the $\alpha \times \alpha$ minors of the matrix $(\nabla_\tau u, \nabla_\tau v)$. This leads us to

$$[JX]^2 = (1+u)^{2(n-1)} + (1+u)^{2(n-2)} \big(|\nabla_\tau u|^2 + |\nabla_\tau v|^2\big) + R_1,$$

where assumption (4.2) ensures that the remainder R_1 is pointwise bounded on S^{n-1} by

$$|R_1| \le c(n)\,(1+|u|)^{2(n-3)} \big(|\nabla_\tau u|^2 + |\nabla_\tau v|^2\big)^2 \le c(n)\,\varepsilon_o \big(|\nabla_\tau u|^2 + |\nabla_\tau v|^2\big).$$

¿From the last identity we obtain

$$[JX]^2 = 1 + 2(n-1)u + (n-1)(2n-3)u^2 + |\nabla_\tau u|^2 + |\nabla_\tau v|^2 + R_2, \tag{4.6}$$

where the remainder R_2 satisfies

$$|R_2| \le c(n)\,\varepsilon_o \big(|u|^2 + |\nabla_\tau u|^2 + |\nabla_\tau v|^2\big). \tag{4.7}$$

At this point we use the inequality $\sqrt{1+a} \ge 1 + \frac{1}{2}a - \frac{1}{8}a^2 - |a|^3$ which is valid for $|a| \le 1/2$. We apply this inequality with the obvious choice

$$a = 2(n-1)u + (n-1)(2n-3)u^2 + |\nabla_\tau u|^2 + |\nabla_\tau v|^2 + R_2.$$

Note that $|a| \le 1/2$ if we choose $\varepsilon_o > 0$ small enough. In this way we obtain

$$JX \ge 1 + (n-1)u + \tfrac{(n-1)(n-2)}{2} u^2 + \tfrac{1}{2}\big(|\nabla_\tau u|^2 + |\nabla_\tau v|^2\big) + R_2,$$

with a possibly different remainder R_2 which still satisfies (4.7). This allows us to estimate

$$\begin{aligned} \mathcal{H}^{n-1}(\Gamma) - n\omega_n &= \int_{S^{n-1}} (JX - 1)\, d\mathcal{H}^{n-1} \\ &\ge (n-1) \int_{S^{n-1}} u\, d\mathcal{H}^{n-1} + \tfrac{(n-1)(n-2)}{2} \int_{S^{n-1}} u^2\, d\mathcal{H}^{n-1} \\ &\quad + \tfrac{1}{2} \int_{S^{n-1}} \big(|\nabla_\tau u|^2 + |\nabla_\tau v|^2\big)\, d\mathcal{H}^{n-1} - c(n)\,\varepsilon_o \big[\|u\|^2_{W^{1,2}} + \|v\|^2_{W^{1,2}}\big]. \end{aligned} \tag{4.8}$$

Step 2. Consequences of the mass assumption (4.3)**.** In the case that

$$\int_{S^{n-1}} u\, d\mathcal{H}^{n-1} \ge 0$$

the estimate (4.8) will be sufficient to complete the proof. However, in the negative case, which can be viewed as the more difficult case since the leading first order term in (4.8) is negative, we shall need an improvement of (4.8). This improvement can be achieved by utilizing assumption (4.3), i.e. the fact that the minimal mass $\mathbf{m}(\Gamma)$ spanned by Γ is equal to ω_n. The precise argument is as follows. We consider the cone

$$C \equiv C(x,\varrho) := \varrho\big[(1+u(x))(x,0) + (0,v(x))\big] \qquad x \in S^{n-1}, \varrho \in (0,1]$$

over Γ. Using the minimality of $\mathbf{m}(\Gamma)$ we see that

$$\omega_n = \mathbf{m}(\Gamma) \le \mathcal{H}^n(C) = \int_0^1 \int_{S^{n-1}} JC \, d\mathcal{H}^{n-1}\, d\varrho, \tag{4.9}$$

where JC is the n-Jacobian of C. In order to utilize the properties of the right-hand side we need to compute the area element of the cone C. For the partial derivatives we have

$$\nabla_{\tau_i} C(x,\varrho) = \varrho \nabla_{\tau_i} X(x) = \varrho\big[(1+u(x))(\tau_i,0) + (x,0)\nabla_{\tau_i} u(x) + (0,\nabla_{\tau_i} v(x))\big]$$

for $i = 1,\dots,n-1$ and

$$\nabla_\varrho C(x,\varrho) = (1+u(x))(x,0) + (0,v(x)).$$

The area element $I := \nabla_\varrho C \wedge \bigwedge_{i=1}^{n-1} \nabla_{\tau_i} C$ can now be rewritten in the form

$$I = \varrho^{n-1}\Bigg[(1+u)(x,0) \wedge \bigwedge_{i=1}^{n-1} \nabla_{\tau_i} X + (0,v) \wedge \bigwedge_{i=1}^{n-1} \nabla_{\tau_i} X\Bigg] =: \varrho^{n-1}(I_1 + I_2).$$

For I_1 we have

$$\begin{aligned}
I_1 &:= (1+u)(x,0) \wedge \bigwedge_{i=1}^{n-1} \big[(1+u)(\tau_i,0) + (x,0)\nabla_{\tau_i} u + (0,\nabla_{\tau_i} v)\big] \\
&= (1+u)(x,0) \wedge \bigwedge_{i=1}^{n-1} \big[(1+u)(\tau_i,0) + (0,\nabla_{\tau_i} v)\big] \\
&= (1+u)^n (x,0) \wedge (\tau_1,0) \wedge \dots \wedge (\tau_{n-1},0) \\
&\quad + (1+u)^{n-1} \sum_{i=1}^{n-1} (x,0) \wedge (\tau_1,0) \wedge \dots \wedge (0,\nabla_{\tau_i} v) \wedge \dots (\tau_{n-1},0) + R_{31},
\end{aligned}$$

where the remainder can be estimated as follows:

$$|R_{31}| \le c(n)\, |\nabla_\tau v|^2.$$

For I_2 we similarly compute

$$\begin{aligned}
I_2 &:= (0,v) \wedge \bigwedge_{i=1}^{n-1} \big[(1+u)(\tau_i,0) + (x,0)\nabla_{\tau_i} u + (0,\nabla_{\tau_i} v)\big] \\
&= (1+u)^{n-1} (0,v) \wedge (\tau_1,0) \wedge \dots \wedge (\tau_{n-1},0) + R_{32},
\end{aligned}$$

where now the remainder R_{32} can be bounded by

$$|R_{32}| \le c(n)\, |v| \big(|\nabla_\tau u| + |\nabla_\tau v|\big) \le c(n)\, \big(|\nabla_\tau u|^2 + |v|^2 + |\nabla_\tau v|^2\big).$$

Combining the preceding estimates we arrive at

$$\begin{aligned}
[JC]^2 &= \varrho^{2(n-1)}\Big[(1+u)^{2n} + (1+u)^{2(n-1)}\big(|v|^2 + |\nabla_\tau v|^2\big) + R_3\Big] \\
&= \varrho^{2(n-1)} (1+u)^{2n} \bigg[1 + \frac{|v|^2 + |\nabla_\tau v|^2}{(1+u)^2} + R_3\bigg],
\end{aligned}$$

where R_3 has changed in the last line. Nevertheless, with the help of (4.2) we find that

$$|R_3| \le c(n)\, \varepsilon_o \big(|\nabla_\tau u|^2 + |v|^2 + |\nabla_\tau v|^2\big).$$

Using (4.9), the expansion of JC from above and (4.2) we see that

$$\omega_n \le \int_0^1 \int_{S^{n-1}} JC \, d\mathcal{H}^{n-1}\, d\varrho$$

$$
\begin{aligned}
&\le \tfrac{1}{n}\int_{S^{n-1}}(1+u)^n\sqrt{1+\frac{|v|^2+|\nabla_\tau v|^2}{(1+u)^2}+R_3}\,d\mathcal{H}^{n-1}\\
&= \tfrac{1}{n}\int_{S^{n-1}}(1+u)^n\,d\mathcal{H}^{n-1}+II\\
&\le \omega_n+\int_{S^{n-1}}u\,d\mathcal{H}^{n-1}+\tfrac{n-1}{2}\int_{S^{n-1}}u^2\,d\mathcal{H}^{n-1}+c(n)\,\varepsilon_o\|u\|^2_{L^2}+II
\end{aligned}
$$

where we have abbreviated

$$
II:=\tfrac{1}{n}\int_{S^{n-1}}(1+u)^n\left[\sqrt{1+\frac{|v|^2+|\nabla_\tau v|^2}{(1+u)^2}+R_3}-1\right]d\mathcal{H}^{n-1}.
$$

To estimate II we use $\sqrt{1+a}\le 1+\frac{1}{2}a$ for $a\ge -1$ and obtain

$$
\begin{aligned}
II &\le \tfrac{1}{2n}\int_{S^{n-1}}(1+u)^{n-2}\big(|v|^2+|\nabla_\tau v|^2\big)\,d\mathcal{H}^{n-1}+c(n)\,\varepsilon_o\big[\|u\|^2_{W^{1,2}}+\|v\|^2_{W^{1,2}}\big]\\
&\le \tfrac{1}{2n}\int_{S^{n-1}}\big(|v|^2+|\nabla_\tau v|^2\big)\,d\mathcal{H}^{n-1}+c(n)\,\varepsilon_o\big[\|u\|^2_{W^{1,2}}+\|v\|^2_{W^{1,2}}\big].
\end{aligned}
$$

Here we also used that $(1+u)^{n-2}\le 1+c(n)\varepsilon_o$ by (4.2). Joining this with the preceding estimate we obtain

$$
\begin{aligned}
\int_{S^{n-1}}u\,d\mathcal{H}^{n-1}\ge -\tfrac{n-1}{2}\int_{S^{n-1}}u^2\,d\mathcal{H}^{n-1}-\tfrac{1}{2n}\int_{S^{n-1}}\big(|v|^2+|\nabla_\tau v|^2\big)\,d\mathcal{H}^{n-1}\\
-c(n)\,\varepsilon_o\big[\|u\|^2_{W^{1,2}}+\|v\|^2_{W^{1,2}}\big]. \qquad (4.10)
\end{aligned}
$$

Plugging the last inequality into (4.8) we obtain the desired improvement of (4.8), that is

$$
\begin{aligned}
\mathcal{H}^{n-1}(\Gamma)-n\omega_n \ge{}& \tfrac{1}{2}\left[\int_{S^{n-1}}|\nabla_\tau u|^2\,d\mathcal{H}^{n-1}-(n-1)\int_{S^{n-1}}u^2\,d\mathcal{H}^{n-1}\right]\\
&+\tfrac{1}{2n}\left[\int_{S^{n-1}}|\nabla_\tau v|^2\,d\mathcal{H}^{n-1}-(n-1)\int_{S^{n-1}}|v|^2\,d\mathcal{H}^{n-1}\right]\\
&-c(n)\,\varepsilon_o\big[\|u\|^2_{W^{1,2}}+\|v\|^2_{W^{1,2}}\big]. \qquad (4.11)
\end{aligned}
$$

Step 3. Consequences of the barycenter assumption (4.4)**.** The next prerequisites for the final proof are estimates which can be derived from the *barycenter condition* (4.4). Using the first n entries in (4.4) we infer with the help of the area formula for $i=1,\dots,n$ that

$$
0=\int_\Gamma x_i\,d\mathcal{H}^{n-1}=\int_{S^{n-1}}(1+u)x_iJX\,d\mathcal{H}^{n-1}.
$$

Using also the fact that $\int_{S^{n-1}}x_i\,d\mathcal{H}^{n-1}=0$ for $i=1,\dots,n$ we compute

$$
\begin{aligned}
\int_{S^{n-1}}ux_i\,d\mathcal{H}^{n-1}&=\int_{S^{n-1}}ux_i(1-JX)\,d\mathcal{H}^{n-1}+\int_{S^{n-1}}(1+u)x_iJX\,d\mathcal{H}^{n-1}\\
&\quad+\int_{S^{n-1}}x_i(1-JX)\,d\mathcal{H}^{n-1}\\
&=\int_{S^{n-1}}ux_i(1-JX)\,d\mathcal{H}^{n-1}+\int_{S^{n-1}}x_i(1-JX)\,d\mathcal{H}^{n-1}\\
&=\int_{S^{n-1}}ux_i\,\frac{1-[JX]^2}{1+JX}\,d\mathcal{H}^{n-1}+\int_{S^{n-1}}x_i\,\frac{1-[JX]^2}{1+JX}\,d\mathcal{H}^{n-1}.
\end{aligned}
$$

Using (4.6) in both integrals on the right-hand side we find that

$$
\int_{S^{n-1}}ux_i\,d\mathcal{H}^{n-1}=\int_{S^{n-1}}R_4\,d\mathcal{H}^{n-1}-2(n-1)\int_{S^{n-1}}\frac{ux_i}{1+JX}\,d\mathcal{H}^{n-1},
$$

where

$$
|R_4|\le c(n)\,\big(|u|^2+|\nabla_\tau u|^2+|\nabla_\tau v|^2\big).
$$

Adding $(n-1)\int_{S^{n-1}} ux_i\, d\mathcal{H}^{n-1}$ on both sides of the preceding inequality we obtain

$$\begin{aligned} n\int_{S^{n-1}} ux_i\, d\mathcal{H}^{n-1} &= (n-1)\int_{S^{n-1}} ux_i\Big[1-\frac{2}{1+JX}\Big]\, d\mathcal{H}^{n-1} + \int_{S^{n-1}} R_4\, d\mathcal{H}^{n-1} \\ &= -(n-1)\int_{S^{n-1}} ux_i\,\frac{1-JX}{1+JX}\, d\mathcal{H}^{n-1} + \int_{S^{n-1}} R_4\, d\mathcal{H}^{n-1}. \end{aligned}$$

The first integral on the right-hand side can now be estimated with the help of (4.6) by

$$\begin{aligned} \bigg|\int_{S^{n-1}} ux_i\,\frac{1-JX}{1+JX}\, d\mathcal{H}^{n-1}\bigg| &= \bigg|\int_{S^{n-1}} ux_i\frac{1-[JX]^2}{(1+JX)^2}\, d\mathcal{H}^{n-1}\bigg| \\ &\le c(n)\big[\|u\|^2_{W^{1,2}} + \|v\|^2_{W^{1,2}}\big]. \end{aligned}$$

Joining the preceding estimates we finally arrive at

$$\bigg|\int_{S^{n-1}} ux_i\, d\mathcal{H}^{n-1}\bigg| \le c(n)\big[\|u\|^2_{W^{1,2}} + \|v\|^2_{W^{1,2}}\big]. \tag{4.12}$$

For components y_α with $\alpha = 1,\dots,k$, i.e. those ones corresponding to the functions v_α, we argue as before, in the case of the components x_i. Using again the area formula and the barycenter condition (4.4) for the y_α-components we have

$$\begin{aligned} \int_{S^{n-1}} v\, d\mathcal{H}^{n-1} &= \int_{S^{n-1}} vJX\, d\mathcal{H}^{n-1} + \int_{S^{n-1}} v(1-JX)\, d\mathcal{H}^{n-1} \\ &= \int_\Gamma y\, d\mathcal{H}^{n-1} + \int_{S^{n-1}} v(1-JX)\, d\mathcal{H}^{n-1} \\ &= \int_{S^{n-1}} v(1-JX)\, d\mathcal{H}^{n-1} = \int_{S^{n-1}} v\,\frac{1-[JX]^2}{1+JX}\, d\mathcal{H}^{n-1}. \end{aligned}$$

Together with (4.6) this leads us to

$$\bigg|\int_{S^{n-1}} v\, d\mathcal{H}^{n-1}\bigg| \le c(n)\big[\|u\|^2_{W^{1,2}} + \|v\|^2_{W^{1,2}}\big]. \tag{4.13}$$

Step 4. Consequences of assumption (4.5) on the mixed second order moments. ¿From (4.5) we get for $i = 1,\dots,n$ and $\alpha = 1,\dots,k$ that

$$\begin{aligned} 0 &= \int_\Gamma x_i y_\alpha\, d\mathcal{H}^{n-1} \\ &= \int_{S^{n-1}} (1+u)x_i v_\alpha JX\, d\mathcal{H}^{n-1} \\ &= \int_{S^{n-1}} x_i v_\alpha\, d\mathcal{H}^{n-1} + \int_{S^{n-1}} \Big[x_i v_\alpha uJX - x_i v_\alpha\,\frac{1-[JX]^2}{1+JX}\Big]\, d\mathcal{H}^{n-1}. \end{aligned}$$

¿From (4.2) and (4.6) we therefore conclude that

$$\bigg|\int_{S^{n-1}} x_i v\, d\mathcal{H}^{n-1}\bigg| \le c(n)\,\big[\|u\|^2_{W^{1,2}} + \|v\|^2_{W^{1,2}}\big] \tag{4.14}$$

holds for $i = 1,\dots,n$.

Step 5. The final conclusion. We consider the expansion of u and v into the corresponding Fourier series

$$u = \sum_{j=0}^{\infty}\sum_{\ell=1}^{m_j} a_{j,\ell}Y_{j,\ell} \quad \text{and} \quad v = \sum_{j=0}^{\infty}\sum_{\ell=1}^{m_j} b_{j,\ell}Y_{j,\ell}$$

where $\{Y_{j,\ell}\colon j\in\mathbb{N}_o,\ \ell = 1,\dots,m_j\}$ stands for the orthonormal basis of spherical harmonics in $L^2(S^{n-1})$, i.e we have

$$-\Delta_{S^{n-1}}Y_{j,\ell} = j(j+n-2)Y_{j,\ell} \quad \text{for } j\in\mathbb{N}_o,\ \ell = 1,\dots,m_j.$$

Here, m_j denotes the dimension of the eigenspace associated to the eigenvalue $j(j+n-2)$. Note that $m_1 = n$ and the precise value of m_j is given for $j \geq 2$ by

$$m_j := \binom{n+j-1}{n-1} - \binom{n+j-3}{n-1}.$$

Moreover, we have

$$\int_{S^{n-1}} Y_{j_1,\ell_1} Y_{j_2,\ell_2}\, d\mathcal{H}^{n-1} = \delta_{j_1,j_2}\delta_{\ell_1,\ell_2}.$$

The coefficients of the Fourier expansions of u and v are obtained by

$$a_{j,\ell} := \int_{S^{n-1}} u Y_{j,\ell}\, d\mathcal{H}^{n-1} \in \mathbb{R} \quad \text{and} \quad b_{j,\ell} := \int_{S^{n-1}} v Y_{j,\ell}\, d\mathcal{H}^{n-1} \in \mathbb{R}^k .$$

In terms of the Fourier coefficients the L^2-norms of u and v can be expressed as follows

$$\int_{S^{n-1}} u^2\, d\mathcal{H}^{n-1} = \sum_{j=0}^{\infty}\sum_{\ell=1}^{m_j} a_{j,\ell}^2, \qquad \int_{S^{n-1}} |v|^2\, d\mathcal{H}^{n-1} = \sum_{j=0}^{\infty}\sum_{\ell=1}^{m_j} |b_{j,\ell}|^2.$$

Further, the L^2-norms of the gradients of u and v are given by

$$\int_{S^{n-1}} |\nabla u|^2\, d\mathcal{H}^{n-1} = \sum_{j=1}^{\infty}\sum_{\ell=1}^{m_j} j(j+n-2)a_{j,\ell}^2$$

and

$$\int_{S^{n-1}} |\nabla v|^2\, d\mathcal{H}^{n-1} = \sum_{j=1}^{\infty}\sum_{\ell=1}^{m_j} j(j+n-2)|b_{j,\ell}|^2.$$

We note that $Y_o \equiv 1/\sqrt{n\omega_n}$ and $Y_{1,\ell}(x) = x_\ell/\sqrt{\omega_n}$ for $\ell = 1,\dots,n$ so that the zero order coefficients a_o and b_o are given by $a_o = (1/\sqrt{n\omega_n})\int_{S^{n-1}} u\, d\mathcal{H}^{n-1}$ respectively $(b_o)_\alpha = (1/\sqrt{n\omega_n})\int_{S^{n-1}} v_\alpha\, d\mathcal{H}^{n-1}$ for $\alpha = 1,\dots,k$, and the first order coefficients are given by $a_{1,\ell} = (1/\sqrt{\omega_n})\int_{S^{n-1}} x_\ell u\, d\mathcal{H}^{n-1}$ respectively $(b_{1,\ell})_\alpha = (1/\sqrt{\omega_n})\int_{S^{n-1}} x_\ell v_\alpha\, d\mathcal{H}^{n-1}$ for $\ell = 1,\dots,n$ and $\alpha = 1,\dots,k$. These integrals have been estimated before and we recall the bounds here. For convenience in notation we abbreviate

$$I(\mu) := \sum_{j=\mu}^{\infty}\sum_{\ell=1}^{m_j} \big[j(j+n-2)+1\big]\big(a_{j,\ell}^2 + |b_{j,\ell}|^2\big) \qquad \text{for } \mu \in \mathbb{N}_0$$

and note that $I(0) = \|u\|_{W^{1,2}}^2 + \|v\|_{W^{1,2}}^2$. ¿From (4.12) and (4.2) we infer the following bound for $a_1 := (a_{1,1},\dots,a_{1,n})$:

$$\begin{aligned} |a_1|^2 &= \sum_{\ell=1}^{n} (a_{1,\ell})^2 \leq c\big[\|u\|_{W^{1,2}}^2 + \|v\|_{W^{1,2}}^2\big]^2 \\ &\leq c\,\varepsilon_o\big[\|u\|_{W^{1,2}}^2 + \|v\|_{W^{1,2}}^2\big] = c(n)\,\varepsilon_o I(0). \end{aligned} \tag{4.15}$$

Similarly, from (4.13) and (4.2) we obtain for $b_o \in \mathbb{R}^k$ that

$$|b_o|^2 = \sum_{\alpha=1}^{k} \big((b_o)_\alpha\big)^2 \leq c\left[\|u\|_{W^{1,2}}^2 + \|v\|_{W^{1,2}}^2\right]^2 \leq c(n)\,\varepsilon_o I(0). \tag{4.16}$$

¿From (4.14) and (4.2) we obtain for $b_1 := (b_{1,1},\dots,b_{1,n})$ that

$$|b_1|^2 = \sum_{\ell=1}^{n}\sum_{\alpha=1}^{k} \big((b_{1,\ell})_\alpha\big)^2 \leq c\left[\|u\|_{W^{1,2}}^2 + \|v\|_{W^{1,2}}^2\right]^2 \leq c(n)\,\varepsilon_o I(0). \tag{4.17}$$

With respect to a_o we recall that we have to distinguish between the cases that either $a_o \geq 0$ or that $a_o < 0$. In the first case, i.e. when $a_o \geq 0$ we start from (4.8) omitting the positive term $\int_{S^{n-1}} u^2\, d\mathcal{H}^{n-1}$ on the right-hand side. Rewriting the resulting inequality in terms of the Fourier-coefficients we obtain

$$\begin{aligned}\mathcal{H}^{n-1}(\Gamma) - n\omega_n &\geq (n-1)\sqrt{n\omega_n}\, a_o + \tfrac{1}{2}\sum_{j=1}^{\infty}\sum_{\ell=1}^{m_j} j(j+n-2)\big(a_{j,\ell}^2 + |b_{j,\ell}|^2\big) - c\,\varepsilon_o I(0)\\ &\geq (n-1)\sqrt{n\omega_n}\, a_o + \tfrac{1}{4}I(1) - c\,\varepsilon_o I(0)\\ &= (n-1)\sqrt{n\omega_n}\, a_o + (\tfrac{1}{4} - c\,\varepsilon)I(0) - \tfrac{1}{4}\big(a_o^2 + |b_o|^2\big).\end{aligned}$$

Here, we have used that $j(j+n-2) \geq \frac{1}{2}[j(j+n-2)+1]$ for $j \geq 1$ in the second last line. Using the bound (4.16) for $|b_o|^2$ we find that

$$\mathcal{H}^{n-1}(\Gamma) - n\omega_n \geq (n-1)\sqrt{n\omega_n}\, a_o + (\tfrac{1}{4} - c\,\varepsilon_o)I(0) - \tfrac{1}{4}a_o^2,$$

with a modified constant c still depending only on n. ¿From (4.2) we deduce that $a_o \leq \sqrt{n\omega_n}\varepsilon_o$ and hence $\sqrt{n\omega_n}a_o \geq a_o^2/\varepsilon_o$. Therefore, choosing ε_o small enough we have $(n-1)\sqrt{n\omega_n}\, a_o \geq \frac{1}{4}a_o^2$ which yields that

$$\mathcal{H}^{n-1}(\Gamma) - n\omega_n \geq (\tfrac{1}{4} - c\,\varepsilon_o)I(0).$$

Therefore, choosing ε_o sufficiently small we get

$$\mathcal{H}^{n-1}(\Gamma) - n\omega_n \geq \tfrac{1}{8}I(0) = \tfrac{1}{8}\big[\|u\|_{W^{1,2}}^2 + \|v\|_{W^{1,2}}^2\big]. \tag{4.18}$$

We now turn our attention to the case $a_o < 0$. Here, we have the improvement from (4.11) at hand, which can be rewritten in terms of the Fourier-coefficients as

$$\mathcal{H}^{n-1}(\Gamma) - n\omega_n \geq \tfrac{1}{2}II(0) - c\,\varepsilon_o I(0),$$

where

$$II(\mu) := \sum_{j=\mu}^{\infty}\sum_{\ell=1}^{m_j}\big[j(j+n-2) - (n-1)\big]\big(a_{j,\ell}^2 + \tfrac{1}{n}|b_{j,\ell}|^2\big) \qquad \text{for } \mu \in \mathbb{N}_0.$$

The term $II(0)$ we rewrite as follows

$$II(0) = -(n-1)\big(a_o^2 + \tfrac{1}{n}|b_o|^2\big) + II(2).$$

Since $j(j+n-2) - (n-1) \geq \frac{1}{2}[j(j+n-2)+1]$ for $j \geq 2$ we have $II(2) \geq \frac{1}{2n}I(2)$ and therefore

$$\begin{aligned}II(0) &\geq -(n-1)\big(a_o^2 + \tfrac{1}{n}|b_o|^2\big) + \tfrac{1}{2n}I(2)\\ &= -(n-1)\big(a_o^2 + \tfrac{1}{n}|b_o|^2\big) + \tfrac{1}{2n}I(0) - \tfrac{1}{2n}\big(a_o^2 + |b_o|^2\big) - \tfrac{1}{2}\big(|a_1|^2 + |b_1|^2\big)\\ &\geq \tfrac{1}{2n}I(0) - n\big(a_o^2 + |b_o|^2\big) - \tfrac{1}{2}\big(|a_1|^2 + |b_1|^2\big).\end{aligned}$$

Inserting this above we have

$$\mathcal{H}^{n-1}(\Gamma) - n\omega_n \geq \big(\tfrac{1}{4n} - c\,\varepsilon_o\big)I(0) - \tfrac{n}{2}\big(a_o^2 + |b_o|^2\big) - \tfrac{1}{4}\big(|a_1|^2 + |b_1|^2\big),$$

Since $a_o < 0$ we infer from (4.10) and (4.2) the following estimate for a_o:

$$a_o^2 \leq c\big[\|u\|_{W^{1,2}}^2 + \|v\|_{W^{1,2}}^2\big]^2 \leq c\,\varepsilon_o\big[\|u\|_{W^{1,2}}^2 + \|v\|_{W^{1,2}}^2\big] \leq c\,\varepsilon_o I(0).$$

Using also the inequalities (4.15), (4.16) and (4.17) we obtain from the second last inequality

$$\mathcal{H}^{n-1}(\Gamma) - n\omega_n \geq \big(\tfrac{1}{4n} - c\,\varepsilon_o\big)I(0) \geq \tfrac{1}{8n}I(0) = \tfrac{1}{8n}\big[\|u\|_{W^{1,2}}^2 + \|v\|_{W^{1,2}}^2\big], \tag{4.19}$$

provided $\varepsilon_o > 0$ is chosen small enough in dependence of n. This finishes the proof in the case $a_o < 0$. In any case we have the bound from below for the quantity $\mathcal{H}^{n-1}(\Gamma) - n\omega_n$ in terms of the $W^{1,2}$-norms of u and v with the constant $\frac{1}{8n}$.

At this stage it remains to derive a bound from above for the asymmetry index in terms of the L^2 norms of u and v. For this we use the homotopy formula. We connect S^{n-1} and Γ by the affine homotopy $h(t,x) := tX(x) + (1-t)(x,0)$, $t \in [0,1]$, $x \in S^{n-1}$. Then $h(1,S^{n-1}) = \Gamma$ and $h(0,S^{n-1}) = S^{n-1}$. The area of the affine connection can be computed by the area formula. To be precise we have (with $\vec{e}(x) = \tau_1 \wedge \cdots \wedge \tau_{n-1}$ denoting the orienting vector field of S^{n-1})

$$\begin{aligned}
\mathbf{d}(\llbracket\Gamma\rrbracket) &\le \mathbf{m}(\llbracket\Gamma\rrbracket - \partial\llbracket D_1\rrbracket)\\
&\le \mathcal{H}^n\big(h_\#(\llbracket 0,1\rrbracket \times S^{n-1})\big)\\
&= \int_0^1 \int_{S^{n-1}} \Big|\big(X-(x,0)\big) \wedge \textstyle\bigwedge_{n-1} D_\tau h(t,x)\vec{e}(x)\Big|\, d\mathcal{H}^{n-1}dt\\
&\le \sup_{t\in[0,1]} \int_{S^{n-1}} |X-(x,0)| \prod_{i=1}^{n-1} \Big[t\sqrt{(1+u)^2 + |\nabla_{\tau_i} u|^2 + |\nabla_{\tau_i} v|^2} + (1-t)\Big] d\mathcal{H}^{n-1}\\
&\le 2^{n-1}\int_{S^{n-1}} |X-(x,0)|\, d\mathcal{H}^{n-1} = 2^{n-1}\int_{S^{n-1}} |(xu,0)+(0,v)|\, d\mathcal{H}^{n-1}\\
&= 2^{n-1}\int_{S^{n-1}} \sqrt{|u|^2+|v|^2}\, d\mathcal{H}^{n-1},
\end{aligned}$$

where in the second last we used (4.2). With the help of Hölder's inequality and (4.18) if $a_o \ge 0$, respectively (4.19) if $a_o < 0$ we further estimate

$$\begin{aligned}
\mathbf{d}(\llbracket\Gamma\rrbracket) &\le 2^{n-1}\sqrt{n\omega_n}\bigg(\int_{S^{n-1}} |u|^2 + |v|^2\, d\mathcal{H}^{n-1}\bigg)^{\frac12}\\
&\le 2^{n-1}\sqrt{n\omega_n}\,\big[\|u\|^2_{W^{1,2}} + \|v\|^2_{W^{1,2}}\big]^{\frac12}\\
&\le 2^n n\sqrt{2\omega_n}\sqrt{\mathcal{H}^{n-1}(\Gamma) - n\omega_n}.
\end{aligned}$$

This proves the quantitative isoperimetric inequality for spherical graphs in higher codimension with a constant $C_o = [2^{2n+1}n^3\omega_n^2]^{-1}$. □

The next Lemma provides the possibility to tilt (rotate) $(n-1)$-currents with second order moments close to those ones of the flat $(n-1)$-dimensional unit sphere in such a way that the mixed second order moments of the tilted current vanish. Later on this will enable us to guarantee that certain penalized currents arising from a sequence of currents contradicting the quantitative isoperimetric inequality can be adjusted in such a way that the mixed second order moments vanish. This adjustment will be important for the application of the higher codimension version of Fuglede's theorem, i.e. Theorem 4.1.

Lemma 4.2. *There exists a constant $\varepsilon_o = \varepsilon_o(n,k) \in (0,1]$ such that there holds: Whenever $T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ has compact support and second order moments defined by*

$$M := \omega_n^{-1}\int z\otimes z\, d\|T\| \in \mathbb{R}^{(n+k)\times(n+k)}$$

satisfying

$$\|M - \mathbb{I}_n\| \le \varepsilon \quad \textit{for some } \varepsilon \in (0,\varepsilon_o], \tag{4.20}$$

where $\mathbb{I}_n \colon \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ is defined by $\mathbb{I}_n(x,y) := (x,0)$, there exists $R \in S\mathbb{O}(n+k)$ with

$$\|R - \mathbb{I}\| \le c(n,k)\,\varepsilon$$

such that for the second order moments of $(R^{-1})_\# T$, defined by

$$M_{(R^{-1})_\# T} := \omega_n^{-1}\int z\otimes z\, d\|(R^{-1})_\# T\|$$

holds

$$\|M_{(R^{-1})_\# T} - \mathbb{I}_n\| \le 2\varepsilon$$

and

$$\big(M_{(R^{-1})_\# T}\big)_{i,n+\alpha} \equiv \int x_i y_\alpha\, d\|(R^{-1})_\# T\| = 0$$

for $i = 1,\dots,n$ and $\alpha = 1,\dots,k$.

Proof. We note that

$$\mathbb{I}_n \equiv \omega_n^{-1} \int_{S^{n-1}\times\{0\}} z \otimes z \, d\mathcal{H}^{n-1}(z).$$

Therefore, the smallness assumption (4.20) ensures that the second order moments of T are close to the second order moments of $S^{n-1} \times \{0\} \subset \mathbb{R}^{n+k}$. In particular, the mixed second order moments of T are small. The idea is to consider the map $\Phi\colon S\mathbb{O}(n+k) \to \mathbb{R}^{(n+k)\times(n+k)}$ defined by

$$\Phi(R) := \int z \otimes z \, d\|(R^{-1})_{\#} T\| = \int Rz \otimes Rz \, d\|T\|.$$

Evaluating Φ at the identity we find that $\Phi(\mathbb{I}) = M$. Next we compute the differential of Φ at the identity. For a skew-symmetric matrix $S \in \mathfrak{so}(n+k)$ we consider its exponential $\exp(tS) \in S\mathbb{O}(n+k)$ and compute

$$\begin{aligned}
\langle D\Phi(\mathbb{I}); S\rangle &= \frac{d}{dt}\Big|_{t=0} \Phi(\exp(tS)) \\
&= \frac{d}{dt}\Big|_{t=0} \int \exp(tS)z \otimes \exp(tS)z \, d\|T\| \\
&= \int \big[Sz \otimes z + z \otimes Sz\big] d\|T\|.
\end{aligned}$$

Now, we fix a matrix $A \in L(\mathbb{R}^k, \mathbb{R}^n)$, which is at our disposal, and define a skew-symmetric matrix $S \in \mathfrak{so}(n+k)$ by

$$S := \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} \tag{4.21}$$

and compute $Sz = (Ay, -A^t x)$. For the following computations we denote by e_i, $i = 1, \ldots, n$ the standard basis in $\mathbb{R}^n$ and by e_α, $\alpha = 1, \ldots, k$ the standard basis in $\mathbb{R}^k$. The standard basis in $\mathbb{R}^{n+k}$ we denote by $\tau_1, \ldots, \tau_{n+k}$ and note that $\tau_i = (e_i, 0)$ for $i = 1, \ldots, n$ and $\tau_{n+\alpha} = (0, e_\alpha)$ for $\alpha = 1, \ldots, k$. Then, for $i = 1, \ldots, n$ and $\alpha = 1, \ldots, k$ we have $(Sz \otimes z)_{i,n+\alpha} = y_\alpha(\tau_i \cdot Sz) = y_\alpha(e_i \cdot Ay)$ and $(z \otimes Sz)_{i,n+\alpha} = x_i(\tau_{n+\alpha} \cdot Sz) = -x_i(e_\alpha \cdot A^t x)$ and hence

$$\langle D\Phi_{i,n+\alpha}(\mathbb{I}); S\rangle = \int \big[y_\alpha(e_i \cdot Ay) - x_i(e_\alpha \cdot A^t x)\big] d\|T\|.$$

Next, we compute

$$e_i \cdot Ay = \sum_{\beta=1}^{k} y_\beta(e_i \cdot Ae_\beta) \quad \text{and} \quad e_\alpha \cdot A^t x = \sum_{\ell=1}^{n} x_\ell(e_\ell \cdot Ae_\alpha).$$

Recalling the definition of the second moments and writing $M_{i,j} := \tau_i \cdot M\tau_j$ for $i, j = 1, \ldots, n+k$ we therefore have

$$\langle D\Phi_{i,n+\alpha}(\mathbb{I}); S\rangle = \sum_{\beta=1}^{k} (e_i \cdot Ae_\beta) M_{n+\alpha,n+\beta} - \sum_{\ell=1}^{n} (e_\ell \cdot Ae_\alpha) M_{i,\ell}.$$

We now choose A according to

$$e_i \cdot Ae_\alpha = M_{i,n+\alpha} \qquad \text{for } i = 1, \ldots, n \text{ and } \alpha = 1, \ldots, k \tag{4.22}$$

and find that

$$\begin{aligned}
\langle D\Phi_{i,n+\alpha}(\mathbb{I}); S\rangle &= \sum_{\beta=1}^{k} M_{i,n+\beta} M_{n+\alpha,n+\beta} - \sum_{\ell=1}^{n} M_{\ell,n+\alpha} M_{i,\ell} \\
&= -M_{i,i} M_{i,n+\alpha} + \sum_{\beta=1}^{k} M_{i,n+\beta} M_{n+\alpha,n+\beta} - \sum_{\ell\neq i,\ell=1}^{n} M_{\ell,n+\alpha} M_{i,\ell}.
\end{aligned}$$

Therefore, by Taylor's formula and the fact that $\Phi(\mathbb{I}) = M$ we obtain for the mixed moments of $(R^{-1})_{\#}T$ with $R = \exp(S)$, i.e. for the components with $i = 1, \dots, n$ and $n+\alpha$ with $\alpha = 1, \dots, k$, that there holds

$$\begin{aligned}
&|\Phi_{i,n+\alpha}(\exp(S))| \\
&\quad\le |\Phi_{i,n+\alpha}(\mathbb{I}) + \langle D\Phi_{i,n+\alpha}(\mathbb{I}), S\rangle| \\
&\qquad + \tfrac{1}{2} \sup_{t\in[0,1]} \Big| D^2\Phi_{i,n+\alpha}(\exp(tS))\big(\exp(tS)S, \exp(tS)S\big) \\
&\qquad\qquad\qquad\qquad + \big\langle D\Phi_{i,n+\alpha}(\exp(tS)), \exp(tS)S^2 \big\rangle \Big| \\
&\quad\le |\Phi_{i,n+\alpha}(\mathbb{I}) + \langle D\Phi_{i,n+\alpha}(\mathbb{I}), S\rangle| \\
&\qquad + \tfrac{1}{2} \sup_{O\in S\mathbb{O}(n+k)} \|D^2\Phi_{i,n+\alpha}(O)\| \|S\|^2 + \tfrac{1}{2} \sup_{O\in S\mathbb{O}(n+k)} \|D\Phi_{i,n+\alpha}(O)\| \|S\|^2 \\
&\quad\le \Big| (1 - M_{i,i})\, M_{i,n+\alpha} + \sum_{\beta=1}^{k} M_{i,n+\beta} M_{n+\alpha,n+\beta} - \sum_{\ell\ne i,\ell=1}^{n} M_{\ell,n+\alpha} M_{i,\ell} \Big| \\
&\qquad + c(n,k) \sum_{\ell=1}^{n} \sum_{\beta=1}^{k} M_{\ell,n+\beta}^2 .
\end{aligned} \tag{4.23}$$

Here, we used the fact that there exists a constant $c(n,k) < \infty$ such that if

$$\|M - \mathbb{I}_n\| \le 1 \tag{4.24}$$

then

$$\|D\Phi\| + \|D^2\Phi\| \le 4 \int |z|^2 \, d\|T\| \le c(n,k),$$

and thanks to (4.20), condition (4.24) is trivially satisfied. Similarly, we compute for $i, j = 1, \dots, n$ that

$$\langle D\Phi_{i,j}(\mathbb{I}); S\rangle = 2 \sum_{\beta=1}^{k} M_{i,n+\beta} M_{j,n+\beta}$$

which, in view of (4.24), leads us to

$$\begin{aligned}
|\Phi_{i,j}(\exp(S)) - \delta_{i,j}| &\le |\Phi_{i,j}(\mathbb{I}) - \delta_{i,j} + \langle D\Phi_{i,j}(\mathbb{I}); S\rangle| + c(n,k)\, \|S\|^2 \\
&\le \Big| M_{i,j} - \delta_{i,j} + 2 \sum_{\beta=1}^{k} M_{i,n+\beta} M_{j,n+\beta} \Big| + c(n,k) \sum_{\ell=1}^{n} \sum_{\beta=1}^{k} M_{\ell,n+\beta}^2 \\
&\le |M_{i,j} - \delta_{i,j}| + c(n,k) \sum_{\ell=1}^{n} \sum_{\beta=1}^{k} M_{\ell,n+\beta}^2 .
\end{aligned} \tag{4.25}$$

Furthermore, for $\alpha, \beta = 1, \dots, k$ we find that

$$\langle D\Phi_{n+\alpha,n+\beta}(\mathbb{I}); S\rangle = -2 \sum_{\ell=1}^{n} M_{\ell,n+\alpha} M_{\ell,n+\beta}$$

and hence, still using the fact that (4.24) holds,

$$\begin{aligned}
|\Phi_{n+\alpha,n+\beta}(\exp(S))| &\le |\Phi_{n+\alpha,n+\beta}(\mathbb{I}) + \langle D\Phi_{n+\alpha,n+\beta}(\mathbb{I}); S\rangle| + c(n,k)\, \|S\|^2 \\
&\le \Big| M_{n+\alpha,n+\beta} - 2 \sum_{\ell=1}^{n} M_{\ell,n+\alpha} M_{\ell,n+\beta} \Big| + c(n,k) \sum_{\ell=1}^{n} \sum_{\beta=1}^{k} M_{\ell,n+\beta}^2 \\
&\le |M_{n+\alpha,n+\beta}| + c(n,k) \sum_{\ell=1}^{n} \sum_{\beta=1}^{k} M_{\ell,n+\beta}^2 .
\end{aligned} \tag{4.26}$$

Finally, we also have

$$\|\exp(S)-\mathbb{I}\|^2 \le c(n,k)\,\|S\|^2 \le c(n,k)\sum_{\ell=1}^{n}\sum_{\beta=1}^{k} M_{\ell,n+\beta}^2. \tag{4.27}$$

Here, we used the definitions of S and A from (4.21) and (4.22) and the fact that by (4.20) the mixed second order moments satisfy (4.24), and therefore we have $\|S\| \le c(n,k)$.

Now, we want to iterate this procedure. We set $M^{(0)} := M$ and $R^{(0)} := \mathbb{I}$ and define iteratively for $h \in \mathbb{N}_0$

$$S^{(h+1)} := \begin{pmatrix} 0 & A^{(h+1)} \\ -(A^{(h+1)})^t & 0 \end{pmatrix} \quad \text{where } e_i \cdot A^{(h+1)} e_\alpha := M^{(h)}_{i,n+\alpha}$$

and

$$M^{(h+1)} := \Phi(R^{(h+1)}) \quad \text{where } R^{(h+1)} := \exp(S^{(h+1)})R^{(h)}.$$

Moreover, for $h \in \mathbb{N}_0$ we define

$$a^{(h)} := \Big(\sum_{i=1}^{n}\sum_{\alpha=1}^{k} |M^{(h)}_{i,n+\alpha}|^2\Big)^{\frac12}$$

and

$$b^{(h)} := \Big(\sum_{i=1}^{n}\sum_{j=1}^{n} |M^{(h)}_{i,j}-\delta_{i,j}|^2 + \sum_{\alpha=1}^{k}\sum_{\beta=1}^{k} |M^{(h)}_{n+\alpha,n+\beta}|^2\Big)^{\frac12}.$$

Then, from (4.20) we know that

$$a^{(0)} \le \varepsilon \quad \text{and} \quad b^{(0)} \le \varepsilon. \tag{4.28}$$

Moreover, from the preceding computations, i.e. from (4.23), (4.25), (4.26) and (4.27) we infer that for $h \in \mathbb{N}_0$, provided (4.24) holds true for $M^{(h)}$, then

$$a^{(h+1)} \le \tilde c\, a^{(h)}\big(a^{(h)}+b^{(h)}\big), \quad b^{(h+1)} \le b^{(h)} + \tilde c\,\big(a^{(h)}\big)^2 \tag{4.29}$$

and

$$\|\exp(S^{(h)})-\mathbb{I}\| \le \tilde c\, a^{(h)} \tag{4.30}$$

for some constant $\tilde c = \tilde c(n,k) \ge 1$. In the following we assume that $3\tilde c\varepsilon \le \frac12$. We will prove by induction that

$$a^{(h)} \le (3\tilde c)^h \varepsilon^{h+1} \quad \text{and} \quad b^{(h)} \le \varepsilon \sum_{\ell=0}^{h} (3\tilde c\varepsilon)^\ell \tag{4.31}$$

holds for any $h \in \mathbb{N}_0$. For $h=0$ the assertion (4.31) is obviously satisfied by (4.28). Now, assume that (4.31) holds for some $h \ge 0$. ¿From (4.29), (4.31) and the fact that $3\tilde c\varepsilon \le \frac12$ we find that

$$\begin{aligned} a^{(h+1)} &\le \tilde c\, a^{(h)}\big(a^{(h)}+b^{(h)}\big) \le \tilde c\,(3\tilde c)^h\varepsilon^{h+1}\Big[(3\tilde c)^h\varepsilon^{h+1} + \varepsilon\sum_{\ell=0}^{h}(3\tilde c\varepsilon)^\ell\Big] \\ &= (3\tilde c)^{h+1}\varepsilon^{h+2}\Big[\tfrac13(3\tilde c\varepsilon)^h + \tfrac13\sum_{\ell=0}^{h}(3\tilde c\varepsilon)^\ell\Big] \le (3\tilde c)^{h+1}\varepsilon^{h+2}. \end{aligned}$$

Further, from (4.29), (4.31) and the fact that $3\tilde c\varepsilon \le \frac12$ we infer

$$b^{(h+1)} \le b^{(h)} + \tilde c\,\big(a^{(h)}\big)^2 \le \varepsilon\sum_{\ell=0}^{h}(3\tilde c\varepsilon)^\ell + \tilde c\,(3\tilde c)^{2h}\varepsilon^{2h+2} \le \varepsilon\sum_{\ell=0}^{h+1}(3\tilde c\varepsilon)^\ell.$$

The last two inequalities establish the assertion (4.31). We note that $b^{(h)} \le 2\varepsilon$ and $a^{(h)} \le \varepsilon$. Then $\|M^{(h)}-\mathbb{I}_n\| \le 3\varepsilon \le 1$ and therefore the condition (4.24) is fulfilled for any $h \in \mathbb{N}_0$.

Next, we prove that $R^{(h)}$ is a Cauchy sequence. This follows from (4.30), (4.31) and $3\tilde c\varepsilon \le \frac{1}{2}$ since

$$\begin{aligned}\|R^{(h+\ell)} - R^{(h)}\| &\le \sum_{i=0}^{\ell-1} \|R^{(h+i+1)} - R^{(h+i)}\| \le \sum_{i=0}^{\ell-1} \big\|\exp(S^{(h+i+1)}) - \mathbb{I}\big\|\|R^{(h+i)}\| \\ &\le \tilde c \sum_{i=0}^{\ell-1} a^{(h+i+1)} \le \varepsilon\tilde c(3\tilde c\varepsilon)^{h+1}\sum_{i=0}^{\ell-1}(3\tilde c\varepsilon)^i \le \varepsilon\tilde c(3\tilde c\varepsilon)^h \le 2^{-h}\tilde c\,\varepsilon.\end{aligned}$$

Therefore, there exists $R_\infty \in S\mathbb{O}(n+k)$ such that $R^{(h)} \to R_\infty$ as $h \to \infty$ and from the preceding inequality with $h = 0$ and $\ell \to \infty$ we obtain

$$\|R_\infty - \mathbb{I}\| \le \tilde c\,\varepsilon.$$

Next, we observe that $a^{(h)} \to 0$ as $h \to \infty$. But this means that

$$\int x_i y_\alpha \, d\|(R_\infty^{-1})_\# T\| = 0$$

for any $i = 1, \ldots, n$ and $\alpha = 1, \ldots, k$. We remark that by (4.31) we also have

$$\|M_\infty - \mathbb{I}_n\| \le 2\varepsilon, \quad \text{where } M_\infty := \omega_n^{-1} \int z \otimes z \, d\|(R_\infty^{-1})_\# T\|.$$

This completes the proof of the lemma. □

5. A penalization procedure

We start this section with the definition of an auxiliary functional which will play a crucial role in the final proof of the quantitative isoperimetric inequality. For given constants $C_1, \delta \ge 0$ and $\Lambda \ge 1$ we define the variational functional $\mathcal{F}\colon \mathcal{R}_{n-1}(\mathbb{R}^{n+k}) \to [0,\infty)$ by

$$\mathcal{F}(T) := \mathbf{M}(T) + C_1|\mathbf{d}_1(T) - \delta| + \Lambda|\mathbf{m}(T) - \omega_n|.$$

The presence of the two penalization terms forces a minimizer on one hand to have an asymmetry index close to δ (and since δ will be small in the application, close to zero), and on the other hand to make $\mathbf{m}(T)$ close to ω_n. Heuristically, this means that minimizers will be close to a flat n-dimensional disk. However, a subtle interplay between the area term and the two penalization terms will take place. The following lemma ensures the existence of $\mathcal{F}$-minimizers.

Lemma 5.1. *Let $R \ge 1$. Then, there exists a minimizer $S \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ of the variational problem*

$$\min\big\{\mathcal{F}(T) : T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k}) \text{ with } \partial T = 0 \text{ and } \operatorname{spt} T \subset \overline{B}_R\big\}. \tag{5.1}$$

Proof. We use the direct method of the calculus of variations. Let $\{S_j\}_{j=1}^\infty$ be a minimizing sequence, i.e. $S_j \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with $\partial S_j = 0$ and $\operatorname{spt} S_j \subset \overline{B}_R$ and

$$\lim_{j\to\infty} \mathcal{F}(S_j) = \inf\big\{\mathcal{F}(T)\colon T \in R_{n-1}(\mathbb{R}^{n+k}),\ \partial T = 0,\ \operatorname{spt} T \subset \overline{B}_R\big\}.$$

¿From the definition of $\mathcal{F}$ we infer that

$$\sup_{j\ge1}\big[\mathbf{M}(S_j) + \mathbf{m}(S_j)\big] \le C < \infty.$$

For each S_j we choose a mass minimizer $Q(S_j)$ with boundary $\partial Q(S_j) = S_j$ and $\operatorname{spt} Q(S_j) \subset \overline{B}_R$. Since $\mathbf{M}(Q(S_j)) = \mathbf{m}(S_j)$ we have

$$\sup_{j\ge1}\big[\mathbf{M}(S_j) + \mathbf{M}(Q(S_j))\big] \le C < \infty.$$

In this situation the compactness Theorem 3.2 ensures the existence of a current $\widetilde Q \in \mathcal{R}_n(\mathbb{R}^{n+k})$ and a (not relabeled) subsequence such that $Q(S_j) \to \widetilde Q$ with respect to the

$\mathbf{F}_{\rm loc}$-topology. By Theorem 3.1 we also have $Q(S_j) \rightharpoonup \widetilde{Q}$ with respect to weak convergence in $\mathcal{R}_n(\mathbb{R}^{n+k})$. This implies $S_j \rightharpoonup \widetilde{S} := \partial\widetilde{Q}$ weakly in $\mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ and moreover $\operatorname{spt}\widetilde{S} \subset \overline{B}_R$. Now, the compactness theorem for mass minimizing currents [21, Theorem 34.5] ensures that $\widetilde{Q}$ is mass minimizing with respect to its boundary $\widetilde{S}$, i.e. we know that $\widetilde{Q} = Q(\widetilde{S})$. Moreover, from [21, Theorem 34.5] we also conclude that

$$\lim_{j\to\infty} \mathbf{M}(Q(S_j)) = \mathbf{M}(Q(\widetilde{S})). \tag{5.2}$$

Finally, we note that $\operatorname{spt} Q(\widetilde{S}) \subset \overline{B}_R$ by the convex hull property, since $\operatorname{spt}\widetilde{S}$ is contained in $\overline{B}_R$. At this point it remains to prove that there holds

$$\mathcal{F}(\widetilde{S}) \le \lim_{j\to\infty} \mathcal{F}(S_j). \tag{5.3}$$

We first note that the lower semi continuity of the mass with respect to weak convergence of currents implies

$$\mathbf{M}(\widetilde{S}) \le \liminf_{j\to\infty} \mathbf{M}(S_j). \tag{5.4}$$

Next, we let $[\![D]\!]$ be a flat n-dimensional disk with radius 1 realizing $\mathbf{d}_1(\widetilde{S})$ up to an error $\varepsilon > 0$, that is we choose $[\![D]\!]$ such that there holds $\mathbf{m}(\widetilde{S} - \partial[\![D]\!]) < \mathbf{d}_1(\widetilde{S}) + \varepsilon$. Since $S_j \rightharpoonup \widetilde{S}$ with respect to the $\mathbf{F}_{\rm loc}$-topology and since both currents are supported in $\overline{B}_R$ we conclude from Lemma 3.3 that $\mathbf{m}(S_j - \widetilde{S}) \le \varepsilon$ for $j \gg 1$. We therefore find that

$$\mathbf{d}_1(S_j) \le \mathbf{m}(S_j - \partial[\![D]\!]) \le \mathbf{m}(S_j - \widetilde{S}) + \mathbf{m}(\widetilde{S} - \partial[\![D]\!]) \le \mathbf{d}_1(\widetilde{S}) + 2\varepsilon.$$

Similarly, we can also obtain a reverse type estimate. For $j \in \mathbb{N}$ we denote by $[\![D_j]\!]$ flat n-dimensional disks of radius 1 realizing up to an error $\varepsilon > 0$ the quantities $\mathbf{d}_1(S_j)$, that is $\mathbf{m}(S_j - \partial[\![D_j]\!]) < \mathbf{d}_1(S_j) + \varepsilon$. We therefore find that

$$\mathbf{d}_1(\widetilde{S}) \le \mathbf{m}(\widetilde{S} - \partial[\![D_j]\!]) \le \mathbf{m}(\widetilde{S} - S_j) + \mathbf{m}(S_j - \partial[\![D_j]\!]) \le \mathbf{d}_1(S_j) + 2\varepsilon.$$

Combining the two preceding inequalities yields

$$\lim_{j\to\infty} \mathbf{d}_1(S_j) = \mathbf{d}_1(\widetilde{S}). \tag{5.5}$$

Joining (5.2), (5.4) and (5.5) and recalling the definition of the functional $\mathcal{F}$ yields the claim (5.3) and therefore finishes the proof of the lemma. □

Next, let us recall the notions of λ-minimizing and almost minimizing currents.

Definition 5.2. For $\lambda > 0$ we say that $S \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ is λ-minimizing in $\mathbb{R}^{n+k}$ if for any $P \in \mathcal{R}_n(\mathbb{R}^{n+k})$ there holds

$$\mathbf{M}(S) \le \mathbf{M}(S + \partial P) + \lambda\mathbf{M}(P).$$

For a given radius $\varrho_o > 0$ and a given modulus $\omega\colon (0, \varrho_o] \to [0, \infty)$ one says that a current $S \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ is $(\mathbf{M}, \omega)$-minimizing in $\mathbb{R}^{n+k}$ if

$$\mathbf{M}(S) \le \mathbf{M}(S + X) + \omega(\varrho)\,\mathbf{M}(S \llcorner K + X)$$

holds for any $X \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with $\partial X = 0$ and support contained in a compact set K which is contained in a ball of radius $\varrho \le \varrho_o$.

In the next lemma we establish that minimizers of the variational problem (5.1) are λ-minimizing and almost minimizing.

Lemma 5.3. *Let $C_1, \delta \ge 0$, $\Lambda \ge 1$ and $R \ge 1$. Suppose that $S \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ is a minimizer of the problem* (5.1). *Then, S is λ-minimizing in $\mathbb{R}^{n+k}$ with $\lambda := C_1 + \Lambda$. Moreover, S is $(\mathbf{M}, \omega)$-minimizing in $\mathbb{R}^{n+k}$ with $\omega(\varrho) := 4\lambda\varrho$ and $\varrho_o := 1/(2\lambda)$.*

Proof. By $p\colon \mathbb{R}^{n+k} \to \overline{B}_R$ we denote the spherical projection of $\mathbb{R}^{n+k}$ onto $\overline{B}_R$. Now, let $P \in \mathcal{R}_n(\mathbb{R}^{n+k})$. Since $\mathrm{spt}(S + \partial p_\# P) \subset \overline{B}_R$ and $\partial(S + \partial p_\# P) = 0$ we have that $S + \partial p_\# P$ is an admissible comparison current for the minimizer S. Therefore, by the minimality of S we have

$$\begin{aligned} &\mathbf{M}(S) + C_1|\mathbf{d}_1(S) - \delta| + \Lambda|\mathbf{m}(S) - \omega_n| \\ &\quad \le \mathbf{M}(S + p_\#\partial P) + C_1|\mathbf{d}_1(S + p_\#\partial P) - \delta| + \Lambda|\mathbf{m}(S + p_\#\partial P) - \omega_n|. \end{aligned}$$

Since $p_\# S = S$ (note that $\mathrm{spt}\, S \subset \overline{B}_R$) we have

$$\mathbf{M}(S + p_\#\partial P) = \mathbf{M}(p_\#(S + \partial P)) \le \mathbf{M}(S + \partial P),$$

and the last two inequalities therefore imply

$$\begin{aligned} \mathbf{M}(S) \le \mathbf{M}(S + \partial P) &+ C_1|\mathbf{d}_1(S + p_\#\partial P) - \mathbf{d}_1(S)| \\ &+ \Lambda|\mathbf{m}(S + p_\#\partial P) - \mathbf{m}(S)|. \end{aligned} \tag{5.6}$$

In the following we shall bound the last two terms by a constant times $\mathbf{M}(P)$. In order to proceed in this direction we choose a mass minimizer $Q(S)$ subject to the boundary S, and moreover a mass minimizer $Q(S + p_\#\partial P)$ with respect to the boundary $S + p_\#\partial P$; this means that $\mathbf{m}(S + p_\#\partial P) = \mathbf{M}(Q(S + p_\#\partial P))$ and $\mathbf{m}(S) = \mathbf{M}(Q(S))$. Moreover, both currents have support in $\overline{B}_R$ since $\mathrm{spt}\, S$ and $\mathrm{spt}(S + p_\#\partial P)$ are contained in $\overline{B}_R$. Using the fact that $\partial(p_\# P) = p_\#(\partial P)$ we find

$$\partial(Q(S) + p_\# P) = S + \partial(p_\# P) = S + p_\#\partial P,$$

and together with $\mathbf{M}(p_\# P) \le \mathbf{M}(P)$ and the minimizing property of $Q(S + p_\#\partial P)$ we obtain

$$\mathbf{m}(S + p_\#\partial P) \le \mathbf{M}(Q(S) + p_\# P) \le \mathbf{m}(S) + \mathbf{M}(P).$$

On the other hand, we also have

$$\partial(Q(S + p_\#\partial P) - p_\# P) = S + p_\#\partial P - \partial(p_\# P) = S.$$

This allows us to utilize the minimality of $Q(S)$ to deduce

$$\begin{aligned} \mathbf{m}(S) &\le \mathbf{M}(Q(S + p_\#\partial P) - p_\# P) \\ &\le \mathbf{M}(Q(S + p_\#\partial P)) + \mathbf{M}(p_\# P) \le \mathbf{m}(S + p_\#\partial P) + \mathbf{M}(P). \end{aligned}$$

Together, we have shown that

$$|\mathbf{m}(S + p_\#\partial R) - \mathbf{m}(S)| \le \mathbf{M}(P). \tag{5.7}$$

In order to estimate the second term on the right hand side of (5.6) we first recall that $\partial(p_\# P) = p_\#\partial P$ which allows us to compute

$$\mathbf{m}(p_\#\partial P) = \mathbf{m}(\partial(p_\# P)) \le \mathbf{M}(p_\# P) \le \mathbf{M}(P).$$

Denoting by $[\![D]\!]$ a flat n-dimensional unit disk in $\mathbb{R}^{n+k}$ realizing $\mathbf{d}_1(S)$ up to an error of $\varepsilon > 0$, that is $\mathbf{m}(S - \partial[\![D]\!]) < \mathbf{d}_1(S) + \varepsilon$ we find that

$$\begin{aligned} \mathbf{d}_1(S + p_\#\partial P) &\le \mathbf{m}(S + p_\#\partial P - \partial[\![D]\!]) \\ &\le \mathbf{m}(p_\#\partial P) + \mathbf{m}(S - \partial[\![D]\!]) \\ &\le \mathbf{M}(P) + \mathbf{d}_1(S) + \varepsilon. \end{aligned}$$

Similarly, denoting now with $[\![D]\!]$ a flat n-dimensional unit disk in $\mathbb{R}^{n+k}$ which realizes $\mathbf{d}_1(S + p_\#\partial P)$ up to an error of $\varepsilon > 0$, that is $\mathbf{m}(S + p_\#\partial P - \partial[\![D]\!]) < \mathbf{d}_1(S + p_\#\partial P) + \varepsilon$ we obtain

$$\begin{aligned} \mathbf{d}_1(S) \le \mathbf{m}(S - \partial[\![D]\!]) &\le \mathbf{m}(p_\#\partial P) + \mathbf{m}(S + p_\#\partial P - \partial[\![D]\!]) \\ &\le \mathbf{M}(P) + \mathbf{d}_1(S + p_\#\partial P) + \varepsilon. \end{aligned}$$

Joining the last two estimates and letting $\varepsilon \downarrow 0$ we infer that

$$|\mathbf{d}_1(S + p_\# \partial P) - \mathbf{d}_1(S)| \le \mathbf{M}(P). \tag{5.8}$$

Inserting (5.7) and (5.8) into (5.6) we arrive at

$$\mathbf{M}(S) \le \mathbf{M}(S + \partial P) + (C_1 + \Lambda)\mathbf{M}(P),$$

i.e. S is a λ-minimizing current in $\mathbb{R}^{n+k}$ in the sense of Definition 5.2 with $\lambda = C_1 + \Lambda$, and this proves the first assertion of the Lemma.

The second assertion, i.e. the $(\mathbf{M}, \omega)$ minimality is now an easy consequence of the λ-minimality. For this it is sufficient to consider the case when $x_o \in \mathbb{R}^{n+k}$ and $\varrho \in (0, 1/(2\lambda)]$ are such that $B_\varrho(x_o) \cap \overline{B}_R \ne \emptyset$, since in the case $B_\varrho(x_o) \cap \overline{B}_R = \emptyset$ the almost minimality holds trivially, because $\mathbf{M}(S \llcorner K) = 0$. Note that K is a compact subset of $B_\varrho(x_o)$ and the support of X is contained in K. Now, let $X \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with $\partial X = 0$ and $\operatorname{spt} X \subset K \subset B_\varrho(x_o)$ and choose $P := x_o \times X$. Then, $\operatorname{spt} P \subset B_\varrho(x_o)$ and $\partial P = X$. ¿From the λ-minimality and $\mathbf{M}(P) \le \frac{\varrho}{n}\mathbf{M}(X)$ we obtain

$$\begin{aligned}\mathbf{M}(S) &\le \mathbf{M}(S + X) + \lambda\varrho\mathbf{M}(X) \\ &\le \mathbf{M}(S + X) + \lambda\varrho\big(\mathbf{M}(S \llcorner K + X) + \mathbf{M}(S \llcorner K)\big).\end{aligned}$$

Since $(S + X) \llcorner (\mathbb{R}^{n+k} \setminus K) = S \llcorner (\mathbb{R}^{n+k} \setminus K)$ the preceding estimate is equivalent to

$$\mathbf{M}(S \llcorner K) \le \mathbf{M}(S \llcorner K + X) + \lambda\varrho\big(\mathbf{M}(S \llcorner K + X) + \mathbf{M}(S \llcorner K)\big).$$

This inequality can be rewritten as (note that $\varrho \le \varrho_o = 1/(2\lambda)$)

$$\mathbf{M}(S \llcorner K) \le \frac{1 + \lambda\varrho}{1 - \lambda\varrho}\, \mathbf{M}(S \llcorner K + X) \le \big(1 + 4\lambda\varrho\big)\, \mathbf{M}(S \llcorner K + X).$$

Adding $\mathbf{M}(S \llcorner (\mathbb{R}^{n+k} \setminus K))$ to both sides of the previous inequality then yields the second assertion of the lemma. □

¿From [9, Lemma 2.2, Remark 2.4] we have the following

Lemma 5.4. *Suppose that $S \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with $\partial S = 0$ is a λ-minimizing current. Then, the following assertions hold:*

(i) *If $x_o \in \operatorname{spt} S$, then $(0,1) \ni \varrho \mapsto \varrho^{-(n-1)} e^{\lambda\varrho}\mathbf{M}(S \llcorner B_\varrho(x_o))$ is nondecreasing.*

(ii) *If $x_o \in \operatorname{spt} S$, then the $(n-1)$-dimensional density satisfies $\Theta^{n-1}(\|S\|, x_o) \ge 1$ and moreover there holds*

$$\omega_{n-1} e^{-\lambda} \le \frac{\mathbf{M}(S \llcorner B_\varrho(x_o))}{\varrho^{n-1}} \le e^{\lambda}\mathbf{M}(S) \qquad \text{for any } \varrho \in (0,1).$$

(iii) *The density function $x \mapsto \Theta^{n-1}(\|S\|, x)$ is upper semicontinuous on $\operatorname{spt} S$, i.e.*

$$\limsup_{j\to\infty} \Theta^{n-1}(\|S\|, x_j) \le \Theta^{n-1}(\|S\|, x)$$

whenever $x_j \to x$.

The following lemma is a modification of [21, Theorem 34.5] for λ-minimizers. We state the result in a more general form for λ-minimizing currents S_j with possibly non-vanishing boundary ∂S_j. However, in the application we will have $\partial S_j = 0$. The proof follows almost verbatim along the lines of the one in [21, Theorem 34.5] and therefore we skip it.

Lemma 5.5. *Let $\lambda \ge 0$ and suppose that $S_j \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ is a sequence of λ-minimizing currents in $\mathbb{R}^{n+k}$. If $S_j \to S$ holds locally with respect to the $\mathbf{F}_{\mathrm{loc}}$-topology and $\sup_{j\in\mathbb{N}}(\mathbf{M}(S_j) + \mathbf{M}(\partial S_j)) < \infty$, then $S \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ is λ-minimizing in $\mathbb{R}^{n+k}$. Moreover, we have $\|S_j\| \to \|S\|$ in the sense of Radon measures.*

Lemma 5.6. *Suppose that in addition to the assumptions from Lemma 5.5 the currents S_j are closed, i.e. that $\partial S_j = 0$ holds true. Then, $\operatorname{spt} S_j \to \operatorname{spt} S$ in the Kuratowski convergence, that is*

(i) *if $x_j \in \operatorname{spt} S_j$ for any $j \in \mathbb{N}$, then any limit point x belongs to $\operatorname{spt} S$.*
(ii) *for every $x \in \operatorname{spt} S$ there exists a sequence $\{x_j\}_{j\in\mathbb{N}}$ with $x_j \in \operatorname{spt} S_j$ for any $j \in \mathbb{N}$ converging to x.*

Proof. For the proof of assertion (i) we consider a sequence $x_j \in \operatorname{spt} S_j$ and a limit point x. Assume that $x \notin \operatorname{spt} S$, then there exists $\varrho > 0$ such that $\overline{B}_\varrho(x) \cap \operatorname{spt} S = \emptyset$ and hence $\mathbf{M}(S \llcorner \overline{B}_\varrho(x)) = 0$. Further, there exists a subsequence of x_j, still denoted by x_j such that $x_j \to x$. Then, by Lemma 5.4 (ii) we have

$$\omega_{n-1}(\varrho/2)^{n-1}e^{-\lambda} \le \mathbf{M}\big(S_j \llcorner B_{\varrho/2}(x_j)\big) \le \mathbf{M}\big(S_j \llcorner B_\varrho(x)\big), \tag{5.9}$$

provided j is large enough to ensure that $B_{\varrho/2}(x_j) \subset B_\varrho(x)$. On the other hand, we know from Lemma 5.5 that

$$\limsup_{j\to\infty} \mathbf{M}\big(S_j \llcorner B_\varrho(x)\big) \le \mathbf{M}(S \llcorner \overline{B}_\varrho(x)) = 0$$

which contradicts (5.9). Therefore, it must hold that $x \in \operatorname{spt} S$.

In order to prove assertion (ii) we suppose that there exists $x \in \operatorname{spt} S$ and $\varrho > 0$ such that $\{j \in \mathbb{N} : B_\varrho(x) \cap \operatorname{spt} S_j = \emptyset\}$ is not finite. Together with Lemma 5.4 (ii) and the lower semi continuity of the mass this yields a contradiction, since

$$\omega_{n-1}\varrho^{n-1}e^{-\lambda} \le \mathbf{M}(S \llcorner B_\varrho(x)) \le \liminf_{j\to\infty} \mathbf{M}\big(S_j \llcorner B_\varrho(x)\big) = 0.$$

Hence, for any $x \in \operatorname{spt} S$ and any $\varrho > 0$ the set $\{j \in \mathbb{N} : B_\varrho(x) \cap \operatorname{spt} S_j = \emptyset\}$ is finite. But this means that there exists a sequence $x_j \in \operatorname{spt} S_j$ such that $x_j \to x$. $\square$

Remark 5.7. A similar reasoning shows that the set $\{j \in \mathbb{N} : H \cap \operatorname{spt} S_j \neq \emptyset\}$ is finite for every compact set $H \subset \mathbb{R}^{n+k} \setminus \operatorname{spt} S$. $\square$

Lemma 5.8. *Let $\Lambda \ge n$ and $R > 1$. Then any minimizer of the functional*

$$\mathcal{F}(T) := \mathbf{M}(T) + \Lambda|\mathbf{m}(T) - \omega_n|$$

in the class $\{T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k}) : \partial T = 0,\, \operatorname{spt} T \subset B_R\}$ is the boundary of a flat n-dimensional unit disk with support in B_R.

Proof. ¿From Lemma 5.1 applied with $C_1 = 0$ we infer the existence of a minimizer $S \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ of $\mathcal{F}$ with support in B_R. In the following we prove that S is the boundary of flat n-dimensional unit disk $[\![D_1]\!]$. By the minimality of S we have $\mathcal{F}(S) \le \mathcal{F}(\partial[\![D_1]\!])$ for any flat n-dimensional unit disk $[\![D_1]\!]$ with support in B_R, i.e.

$$\mathbf{M}(S) + \Lambda|\mathbf{m}(S) - \omega_n| \le \mathbf{M}(\partial[\![D_1]\!]) = n\omega_n. \tag{5.10}$$

Suppose that $\mathbf{m}(S) > \omega_n$, then we have $\mathbf{M}(Q(S)) = \mathbf{m}(S) > \omega_n$ for any mass minimizer $Q(S)$ subject to the boundary condition $\partial Q(S) = S$. Therefore, the isoperimetric inequality from Theorem 3.4 yields that

$$\mathbf{M}(S) \ge n\omega_n^{\frac{1}{n}}\mathbf{M}(Q(S))^{\frac{n-1}{n}} > n\omega_n$$

contradicting (5.10). Therefore, it cannot happen that $\mathbf{m}(S) > \omega_n$. Next, we assume that $\mathbf{m}(S) < \omega_n$. Then, there exists $0 < r < 1$ such that $\mathbf{M}([\![D_r]\!]) = \mathbf{m}(S)$. Since $\mathbf{m}(S) = \mathbf{M}([\![D_r]\!]) = \omega_n r^n$ inequality (5.10) can be rewritten as

$$\mathbf{M}(S) + \Lambda(1 - r^n)\omega_n \le n\omega_n.$$

On the other hand, we know from Theorem 3.4 that $\mathbf{M}(S) \ge n\omega_n r^{n-1}$ which together with the last inequality yields

$$\Lambda(1 - r^n) \le n(1 - r^{n-1}).$$

But this contradicts the assumption $\Lambda \geq n$ and therefore we must have $\mathbf{m}(S) = \omega_n$. Using the isoperimetric inequality from Theorem 3.4 we deduce $\mathcal{F}(S) = \mathbf{M}(S) \geq n\omega_n$ and equality holds if and only if $S = \partial[\![D_1]\!]$ for some flat n-dimensional unit disk $[\![D_1]\!]$ with support in B_R. □

6. Proof of the quantitative isoperimetric inequality

The first result of this section enables us to reduce the problem to a situation where we only have to consider currents with compact support. Roughly speaking the Lemma asserts that any closed current T with $\mathbf{m}(T) = \omega_n$ can be truncated in such a way that the asymmetry index $\mathbf{d}$ decreases at most by a multiplicative constant $1/C$ while the isoperimetric gap increases at most by a multiplicative constant C, where $C = C(n) \geq 1$. The result of the truncation procedure is a current with support in a ball of radius R_o which depends only on the dimension n. The content of the Lemma is the higher codimension analogue of [17, Lemma 5.1] and the arguments used here are similar to the ones used therein.

Lemma 6.1. *There exist a constant $\widetilde{C} = \widetilde{C}(n,k) \geq 1$ and a radius $R_o = R_o(n) \geq 1$ such that for every $T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with $\partial T = 0$ and $\mathbf{m}(T) = \omega_n$, we find $T' \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with $\partial T' = 0$, $\mathbf{m}(T') = \omega_n$ and $\operatorname{spt} T' \subset B_{R_o}$ satisfying*

$$\mathbf{d}(T) \leq \widetilde{C}\left(\mathbf{d}(T') + \mathbf{D}(T)\right) \quad \textit{and} \quad \mathbf{D}(T') \leq \widetilde{C}\,\mathbf{D}(T). \tag{6.1}$$

Proof. We start by assuming that $\mathbf{D}(T)$ is sufficiently small, that is $\mathbf{D}(T) \leq \mu$ where $\mu \leq \frac{1}{16}(2^{\frac{1}{n}} - 1)$ is to be chosen later. Next, we choose a mass minimizer $Q(T) \in \mathcal{R}_n(\mathbb{R}^{n+k})$ with boundary T. For $t \in \mathbb{R}$ we define the slices

$$\langle Q(T), t_- \rangle := \partial\big(Q(T) \llcorner \{x_1 < t\}\big) - T \llcorner \{x_1 < t\}$$

and

$$\langle Q(T), t_+ \rangle := -\partial\big(Q(T) \llcorner \{x_1 > t\}\big) + T \llcorner \{x_1 > t\}.$$

We note that $\langle Q(T), t_- \rangle = \langle Q(T), t_+ \rangle$ for all but countably many values of $t \in \mathbb{R}$ which are characterized by the fact that $\mathbf{M}(Q(T) \llcorner \{x_1 = t\} + \mathbf{M}(T \llcorner \{x_1 = t\} > 0$ (cf. [21, 28.6, 28.7], [13, 4.2.1, 4.3]). The common value will be denoted $\langle Q(T), t \rangle$. First of all, we observe that

$$\mathbf{M}\big(\partial\big(Q(T) \llcorner \{x_1 < t\}\big)\big) \leq \mathbf{M}\big(T \llcorner \{x_1 < t\}\big) + \mathbf{M}\big(\langle Q(T), t_- \rangle\big) \tag{6.2}$$

and

$$\mathbf{M}\big(\partial\big(Q(T) \llcorner \{x_1 > t\}\big)\big) \leq \mathbf{M}\big(T \llcorner \{x_1 > t\}\big) + \mathbf{M}\big(\langle Q(T), t_+ \rangle\big) \tag{6.3}$$

hold for any $t \in \mathbb{R}$. Next, we define the function $g\colon \mathbb{R} \to [0,1]$ by

$$g(t) := \frac{\mathbf{M}\big(Q(T) \llcorner \{x_1 < t\}\big)}{\omega_n}.$$

We note that g is non-decreasing, differentiable for a.e. $t \in \mathbb{R}$ and continuous from the left. We now set

$$a := \inf\{t \in \mathbb{R} : g(t) > 0\} \quad \text{and} \quad b := \sup\{t \in \mathbb{R} : g(t) < 1\}$$

such that $-\infty \leq a \leq b \leq \infty$ and $0 < g(t) < 1$ for any $t \in (a,b)$. In case that $a > -\infty$ this means that $g(a) = 0$, while for $a = -\infty$ we have $g(t) \downarrow 0$ as $t \to -\infty$. The same holds for the right end point b, that is $g(b) = 1$ when $b < \infty$ and $g(t) \to 1$ as $t \to \infty$ when $b = \infty$. Moreover, we define

$$N := \big\{t \in [a,b] : \mathbf{M}(Q(T) \llcorner \{x_1 = t\}) + \mathbf{M}(T \llcorner \{x_1 = t\}) > 0 \text{ or } g'(t) \text{ does not exist}\big\}.$$

The preceding arguments show that N is a set of measure zero, i.e. $\mathcal{L}^1(N) = 0$. Moreover, by [21, 28.9] we have

$$\mathbf{M}\big(\langle Q(T), t \rangle\big) \leq \omega_n g'(t) \qquad \text{for any } t \in [a,b] \setminus N. \tag{6.4}$$

¿From the definition of g we infer for any $t \in (a,b)$ that

$$\mathbf{M}\Big(g(t)^{-\frac{1}{n}}\big(Q(T)\llcorner\{x_1<t\}\big)\Big) = g(t)^{-1}\mathbf{M}\big(Q(T)\llcorner\{x_1<t\}\big) = \omega_n = \mathbf{M}([\![D_1]\!])$$

which by the isoperimetric inequality from Theorem 3.4 implies that

$$n\omega_n \le \mathbf{M}\Big(\partial\big[g(t)^{-\frac{1}{n}}\big(Q(T)\llcorner\{x_1<t\}\big)\big]\Big) = g(t)^{-\frac{n-1}{n}}\mathbf{M}\big(\partial[Q(T)\llcorner\{x_1<t\}]\big).$$

Here and in the following we write for simplicity λS instead of $(\mu_\lambda)_\# S$, where $\mu_\lambda(x) = \lambda x$ denotes the homothety. Joining this with (6.2) and assuming that $t \in (a,b)\setminus N$ we find that

$$n\omega_n g(t)^{1-\frac{1}{n}} \le \mathbf{M}\big(\partial[Q(T)\llcorner\{x_1<t\}]\big) \le \mathbf{M}\big(T\llcorner\{x_1<t\}\big) + \mathbf{M}\big(\langle Q(T),t\rangle\big). \tag{6.5}$$

Our next aim is to infer a similar estimate from below for $\mathbf{M}(T\llcorner\{x_1>t\})$ instead of $\mathbf{M}(T\llcorner\{x_1<t\})$. ¿From the definition of g and the fact that the mass is additive on Borel sets we infer for any $t\in(a,b)\setminus N$ that

$$\begin{aligned}
\mathbf{M}\Big((1-g(t))^{-\frac{1}{n}}\big(Q(T)\llcorner\{x_1>t\}\big)\Big) &= (1-g(t))^{-1}\mathbf{M}\big(Q(T)\llcorner\{x_1>t\}\big)\\
&= (1-g(t))^{-1}\Big(\mathbf{M}\big(Q(T)\big) - \mathbf{M}\big(Q(T)\llcorner\{x_1\le t\}\big)\Big)\\
&= (1-g(t))^{-1}\Big(\omega_n - \mathbf{M}\big(Q(T)\llcorner\{x_1<t\}\big) - \mathbf{M}\big(Q(T)\llcorner\{x_1=t\}\big)\Big)\\
&= (1-g(t))^{-1}\Big(\omega_n - \mathbf{M}\big(Q(T)\llcorner\{x_1<t\}\big)\Big)\\
&= (1-g(t))^{-1}\big(\omega_n - \omega_n g(t)\big) = \omega_n = \mathbf{M}([\![D_1]\!]).
\end{aligned}$$

The isoperimetric inequality from Theorem 3.4 therefore ensures that

$$n\omega_n \le \mathbf{M}\Big(\partial\big[(1-g(t))^{-\frac{1}{n}}\big(Q(T)\llcorner\{x_1>t\}\big)\big]\Big) = (1-g(t))^{-\frac{n-1}{n}}\mathbf{M}\big(\partial[Q(T)\llcorner\{x_1>t\}]\big)$$

which together with (6.3) yields for any $t\in(a,b)\setminus N$ that

$$\begin{aligned}
n\omega_n(1-g(t))^{1-\frac{1}{n}} &\le \mathbf{M}\big(\partial[Q(T)\llcorner\{x_1>t\}]\big)\\
&\le \mathbf{M}\big(T\llcorner\{x_1>t\}\big) + \mathbf{M}\big(\langle Q(T),t\rangle\big).
\end{aligned} \tag{6.6}$$

Adding the inequalities (6.5) and (6.6) and taking into account that $\mathbf{M}(T\llcorner\{x_1=t\})=0$ for $t\in(a,b)\setminus N$ we find that

$$\begin{aligned}
n\omega_n&\Big(g(t)^{1-\frac{1}{n}} + (1-g(t))^{1-\frac{1}{n}}\Big)\\
&\le \mathbf{M}\big(T\llcorner\{x_1<t\}\big) + \mathbf{M}\big(T\llcorner\{x_1>t\}\big) + 2\mathbf{M}\big(\langle Q(T),t\rangle\big)\\
&= \mathbf{M}(T) + 2\mathbf{M}\big(\langle Q(T),t\rangle\big).
\end{aligned}$$

Recalling the definition of $\mathbf{D}(T)$, i.e. the fact that $\mathbf{M}(T) = n\omega_n(1+\mathbf{D}(T))$ we can rewrite the preceding inequality as follows:

$$\mathbf{M}\big(\langle Q(T),t\rangle\big) \ge \tfrac{1}{2}n\omega_n\big(\psi(g(t)) - \mathbf{D}(T)\big) \qquad \text{for any } t\in(a,b)\setminus N, \tag{6.7}$$

where the function $\psi\colon [0,1]\to[0,2^{\frac{1}{n}}-1]$ is defined by

$$\psi(t) := t^{1-\frac{1}{n}} + (1-t)^{1-\frac{1}{n}} - 1.$$

We note that $\psi(0)=\psi(1)=0$, that $\psi(1/2) = 2^{\frac{1}{n}}-1$ is the maximum, and that ψ is concave, so that

$$\psi(t) \ge 2\big(2^{\frac{1}{n}}-1\big)t \qquad \text{for any } t\in[0,\tfrac{1}{2}]. \tag{6.8}$$

Next, we define $\delta_o := 2\mathbf{D}(T)$ and set

$$t_1 := \sup\{t\in[a,b] : \psi(g(t))\le\delta_o, g(t)\le 1/2\}$$

and

$$t_2 := \inf\{t \in [a,b] : \psi(g(t)) \le \delta_o, g(t) \ge 1/2\}.$$

We first note that t_1 is well defined since $g(a) = \psi(g(a)) = 0$ if $a > -\infty$ and $g(t), \psi(g(t)) \downarrow 0$ as $t \downarrow -\infty$ when $a = -\infty$. Similarly, t_2 is well defined since $g(b) = 1$ and $\psi(g(b)) = 0$ if $b < \infty$ and $g(t) \uparrow 1$ and $\psi(g(t)) \downarrow 0$ as $t \uparrow \infty$ when $b = \infty$. ¿From the choice of t_1 and t_2 and the left continuity of g we infer that $\psi(g(t_1)) \le \delta_o$ and $\psi(g(t_2+1)) \le \delta_o$ which together with (6.8) and the fact that $(2^{\frac{1}{n}}-1)^{-1} \le n/\log 2 \le 2n$ implies that

$$g(t_1) \le \frac{\delta_o}{2(2^{\frac{1}{n}}-1)} \le n\delta_o \qquad \text{and} \qquad 1 - g(t_2+1) \le \frac{\delta_o}{2(2^{\frac{1}{n}}-1)} \le n\delta_o. \tag{6.9}$$

The choice of t_1 and t_2 also implies that

$$\psi(g(t)) \ge \delta_o \qquad \text{for any } t \in (t_1, t_2).$$

By (6.7) and the definition of δ_o we therefore have for any $t \in (t_1, t_2) \setminus N$ that

$$\begin{aligned} \mathbf{M}\big(\langle Q(T), t\rangle\big) &\ge \tfrac{1}{2} n\omega_n \big(\psi(g(t)) - \mathbf{D}(T)\big) \\ &= \tfrac{1}{4} n\omega_n \psi(g(t)) + \tfrac{1}{4} n\omega_n \big(\psi(g(t)) - 2\mathbf{D}(T)\big) \\ &\ge \tfrac{1}{4} n\omega_n \psi(g(t)) + \tfrac{1}{4} n\omega_n \big(\delta_o - 2\mathbf{D}(T)\big) \\ &= \tfrac{1}{4} n\omega_n \psi(g(t)). \end{aligned} \tag{6.10}$$

We now define

$$H(t) := \int_0^t \frac{ds}{\psi(s)} = \int_0^t \frac{ds}{s^{1-\frac{1}{n}} + (1-s)^{1-\frac{1}{n}} - 1} \qquad \text{for } t \in [0,1].$$

Note that H is $C^1((0,1)) \cap C^0([0,1])$ and

$$H(1) = \int_0^1 \frac{ds}{\psi(s)} =: \alpha(n) \in (0, \infty).$$

¿From the definition of H, (6.4) and (6.10) we infer that

$$2\frac{d}{dt} H(g(t)) = \frac{2g'(t)}{\psi(g(t))} \ge \frac{n}{2} \ge 1 \qquad \text{for any } t \in (t_1, t_2) \setminus N$$

which after integration over (t_1, t_2) implies

$$t_2 - t_1 \le 2\big(H(g(t_2)) - H(g(t_1))\big) = 2\int_{g(t_1)}^{g(t_2)} \frac{ds}{\psi(s)} \le 2\alpha(n). \tag{6.11}$$

Next, we use [21, 28.10], the definition of the function g, (6.8) and the fact that $\psi(g(t_1)) \le \delta_o = 2\mathbf{D}(T)$ to compute

$$\begin{aligned} \int_{t_1-8n}^{*t_1} \mathbf{M}\big(\langle Q(T), t_\pm\rangle\big)\, dt &\le \mathbf{M}\big(Q(T) \llcorner \{x_1 < t_1\}\big) = \omega_n g(t_1) \\ &\le \frac{\omega_n}{2(2^{\frac{1}{n}}-1)}\, \psi(g(t_1)) \le \frac{\omega_n}{2^{\frac{1}{n}}-1}\, \mathbf{D}(T) \le 2n\omega_n \mathbf{D}(T). \end{aligned}$$

Denoting by

$$S_\pm := \big\{t \in [t_1 - 8n, t_1] : \mathbf{M}\big(\langle Q(T), t_\pm\rangle\big) > \omega_n \mathbf{D}(T)\big\}$$

the sets of those t in which the slices $\langle Q(T), t_\pm\rangle$ have mass at least $\omega_n \mathbf{D}(T)$, we infer from the preceding inequality that

$$\omega_n \mathbf{D}(T) |S_\pm| \le \int_{t_1-8n}^{*t_1} \mathbf{M}\big(\langle Q(T), t_\pm\rangle\big)\, dt \le 2n\omega_n \mathbf{D}(T),$$

which means that $|S_\pm| \le 2n$ and therefore

$$|[t_1 - 8n, t_1] \setminus (S_+ \cup S_- \cup N)| \ge 4n.$$

Therefore, we can find $\tau_1 \in [t_1 - 8n, t_1] \setminus (S_+ \cup S_- \cup N)$. By the definition of $S_\pm$ this means that we have

$$\mathbf{M}\big(\langle Q(T), \tau_1\rangle\big) \le \omega_n \mathbf{D}(T). \tag{6.12}$$

Here we used that $\langle Q(T), \tau_1\rangle = \langle Q(T), (\tau_1)_+\rangle = \langle Q(T), (\tau_1)_-\rangle$ by the choice of τ_1. A similar reasoning as before, i.e. using [21, 28.10], the definition of g, (6.8), the symmetry of ψ and the fact that $\psi(g(t_2+1)) \le \delta_o = 2\mathbf{D}(T)$, we can estimate

$$\begin{aligned}
\int^{*\,t_2+8n}_{t_2+1} \mathbf{M}\big(\langle Q(T), t_\pm\rangle\big)\,dt &\le \mathbf{M}\big(Q(T) \llcorner \{x_1 > t_2+1\}\big)\\
&= \mathbf{M}\big(Q(T)\big) - \mathbf{M}\big(Q(T)\llcorner\{x_1 < t_2+1\}\big) - \mathbf{M}\big(Q(T)\llcorner\{x_1 = t_2+1\}\big)\\
&\le \mathbf{M}\big(Q(T)\big) - \mathbf{M}\big(Q(T)\llcorner\{x_1 < t_2+1\}\big)\\
&= \omega_n(1 - g(t_2+1)) \le \frac{\omega_n}{2(2^{\frac{1}{n}}-1)}\,\psi(g(t_2+1)) \le 2n\omega_n\mathbf{D}(T).
\end{aligned}$$

The arguments from above now yield the existence of $\tau_2 \in [t_2+1, t_2+8n] \setminus N$ such that

$$\mathbf{M}\big(\langle Q(T), \tau_2\rangle\big) \le \omega_n \mathbf{D}(T). \tag{6.13}$$

At this stage we define $\widetilde{Q} := Q(T)\llcorner\{\tau_1 \le x_1 \le \tau_2\}$. ¿From (6.11) and the definition of τ_1 and τ_2 we have the bound $\tau_2 - \tau_1 \le 2\alpha + 16n$. Moreover, from the definitions of g and δ_o and (6.9) we obtain

$$\begin{aligned}
\mathbf{M}(\widetilde{Q}) &= \mathbf{M}\big(Q(T)\big) - \mathbf{M}\big(Q(T)\llcorner\{x_1 < \tau_1\}\big) - \mathbf{M}\big(Q(T)\llcorner\{x_1 > \tau_2\}\big)\\
&= \omega_n\big(1 - g(\tau_1)\big) - \omega_n\big(1 - g(\tau_2)\big) + \mathbf{M}\big(Q(T)\llcorner\{x_1 = \tau_2\}\big)\\
&= \omega_n\big(1 - g(\tau_1)\big) - \omega_n\big(1 - g(\tau_2)\big)\\
&\ge \omega_n\big(1 - g(t_1)\big) - \omega_n\big(1 - g(t_2+1)\big)\\
&\ge \omega_n(1 - 2n\delta_o) = \omega_n\big(1 - 4n\mathbf{D}(T)\big).
\end{aligned}$$

Next, we define $\widetilde{T} := \partial\widetilde{Q}$. From the choices of τ_1 and τ_2 we infer that

$$\begin{aligned}
\mathbf{M}(\widetilde{T}) &\le \mathbf{M}\big(T\llcorner\{\tau_1 < x_1 < \tau_2\}\big) + \mathbf{M}\big(\langle Q(T), \tau_1\rangle\big) + \mathbf{M}\big(\langle Q(T), \tau_2\rangle\big)\\
&\le \mathbf{M}(T) + 2\omega_n\mathbf{D}(T).
\end{aligned} \tag{6.14}$$

We now define

$$T' := \sigma\widetilde{T} \qquad \text{where } \sigma := \Big(\frac{\omega_n}{\mathbf{M}(\widetilde{Q})}\Big)^{\frac{1}{n}} \le \big(1 - 4n\mathbf{D}(T)\big)^{-\frac{1}{n}} \in [1, 2^{\frac{1}{n}}].$$

Then, $\operatorname{spt} T'$ is contained in a strip $[\tau_1', \tau_2'] \times \mathbb{R}^{N-1}$ of width $\tau_2' - \tau_1' \le 2^{\frac{1}{n}}(2\alpha + 16n)$. Moreover, we have

$$\mathbf{M}\big(Q(T')\big) = \mathbf{M}\big(Q(\sigma\widetilde{T})\big) = \mathbf{M}\big(\sigma Q(\widetilde{T})\big) = \sigma^n\mathbf{M}\big(Q(\widetilde{T})\big) = \omega_n.$$

At this stage it remains to prove (6.1). Recalling the definition of T', using (6.14) and the fact that $\mathbf{M}(T) = n\omega_n(1 + \mathbf{D}(T))$ we get

$$\mathbf{M}(T') = \sigma^{n-1}\mathbf{M}(\widetilde{T}) \le \sigma^{n-1}\big(\mathbf{M}(T) + n\omega_n\mathbf{D}(T)\big) = n\omega_n\sigma^{n-1}\big(1 + 2\mathbf{D}(T)\big).$$

At this point we use the definition of σ and the assumption $\mathbf{D}(T) \le \frac{1}{16}(2^{\frac{1}{n}}-1) \le \frac{1}{8n}$ to compute

$$\begin{aligned}\sigma^{n-1}\big(1+2\mathbf{D}(T)\big) &\le \frac{1+2\mathbf{D}(T)}{(1-4n\mathbf{D}(T))^{\frac{n-1}{n}}} \le \frac{1+2\mathbf{D}(T)}{1-4n\mathbf{D}(T)}\\ &\le 1+\frac{5n\mathbf{D}(T)}{1-4n\mathbf{D}(T)} \le 1+10n\mathbf{D}(T).\end{aligned}$$

Inserting this above yields

$$\mathbf{M}(T') \le n\omega_n\big(1+10n\mathbf{D}(T)\big)$$

which proves the second estimate in (6.1). Finally, the first assertion in (6.1) can be achieved as follows: Using the bound $\mathbf{m}(T-\widetilde{T}) \le \mathbf{M}(Q(T)-\widetilde{Q})$ (note that $\partial(Q(T)-\widetilde{Q}) = T-\partial\widetilde{Q} = T-\widetilde{T}$) together with (6.12), (6.13) and (6.9) we obtain

$$\begin{aligned}\mathbf{m}(T-\widetilde{T}) &\le \mathbf{M}\big(Q(T)\llcorner\{x_1<\tau_1\}\big)+\mathbf{M}\big(Q(T)\llcorner\{x_1>\tau_2\}\big)\\ &= \omega_n\big(g(\tau_1)+1-g(\tau_2)\big) \le \omega_n\big(g(t_1)+1-g(t_2+1)\big)\\ &\le 2n\omega_n\delta_o = 4n\omega_n\mathbf{D}(T).\end{aligned}$$

We now let $[\![D_{1/\sigma}]\!]$ be the disk with radius $1/\sigma$ in $\mathbb{R}^{n+k}$ which realizes $\mathbf{d}(\widetilde{T})$ up to an error $\varepsilon>0$, i.e. $\mathbf{m}(\widetilde{T}-\partial[\![D_{1/\sigma}]\!]) < \mathbf{d}(\widetilde{T})+\varepsilon$ (recall that $\mathbf{M}(\widetilde{Q}) = \omega_n/\sigma^n$) and let $[\![D_1]\!]$ be the disk of radius 1 lying in the same n-dimensional plane as $[\![D_{1/\sigma}]\!]$ and having the same center. Then, we get

$$\begin{aligned}\mathbf{d}(T) &\le \mathbf{m}(T-\partial[\![D_1]\!]) \le \mathbf{m}(T-\widetilde{T})+\mathbf{m}(\widetilde{T}-\partial[\![D_{1/\sigma}]\!])+\mathbf{m}(\partial[\![D_{1/\sigma}]\!]-\partial[\![D_1]\!])\\ &\le 4\omega_n\mathbf{D}(T)+\mathbf{d}(\widetilde{T})+\varepsilon+\omega_n(1-\sigma^{-n}) \le \mathbf{d}(T')+C\,\mathbf{D}(T).\end{aligned}$$

In the last line we used $d(\widetilde{T}) = \mathbf{d}(T')$ and

$$1-\sigma^{-n} = 1-(1-4n\mathbf{D}(T)) = 4n\mathbf{D}(T).$$

This proves also the first inequality in (6.1). Starting from T' we repeat the same construction with respect to x_2 provided that $\mathbf{D}(T') \le 10n\mathbf{D}(T) \le 10n\mu \le \frac{1}{16}(2^{\frac{1}{n}}-1)$. Thus we get a new current $T'' \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ still satisfying (6.1) with a new constant and with $\operatorname{spt} T''$ now contained in $[\tau_1',\tau_2'] \times [\tau_1'',\tau_2''] \times \mathbb{R}^{n+k-2}$ with $\tau_2'-\tau_1'$ and $\tau_2''-\tau_1''$ bounded by a universal constant. Thus, the assertion follows by repeating the argument with respect to all the remaining coordinate directions and assuming μ sufficiently small.

Finally, if $\mathbf{D}(T) > \mu$ then the result is easily obtained by taking T' equal to a unit disk with support in B_{R_o}. □

In the final proof of Theorem 2.1 we shall also need the following regularity theorem which can be viewed as the higher codimension version of [23], see also [22].

Theorem 6.2 (Regularity). *Suppose $S_j \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ is a sequence of closed rectifiable $(\mathbf{M},\omega)$-minimizing currents in $\mathbb{R}^{n+k}$ for a modulus $\omega(\varrho) \equiv C_\omega\varrho$ and with $\varrho_o = 2/C_\omega$. Furthermore, suppose that $\|S_j\| \to \|\partial[\![D]\!]\|$ in the sense of Radon measures and that $\operatorname{spt} S_j \to \operatorname{spt}\partial[\![D]\!]$ in the Kuratowski convergence as $j\to\infty$ and that $\operatorname{spt} S_j \subset B_{R_o}$ for some $R_o>0$. Then there exists $j_o \in \mathbb{N}$ such that for any $j \ge j_o$ there exist maps $u_j \in C^{1,\frac12}(S^{n-1})$ and $v_j \in C^{1,\frac12}(S^{n-1},\mathbb{R}^k)$ such that the S_j admit the spherical graph representation*

$$S_j = X_{j\#}[\![S^{n-1}]\!],$$

where the maps $X_j\colon S^{n-1}\to\mathbb{R}^{n+k}$ are defined for $x \in S^{n-1}$ by

$$X_j(x) := (1+u_j(x))(x,0)+(0,v_j(x)) \in \mathbb{R}^{n+k}.$$

Moreover, the representing maps u_j, v_j satisfy for any $\alpha \in (0, \frac{1}{2})$:

$$\lim_{j\to\infty} \Big(\|u_j\|_{C^{1,\alpha}(S^{n-1})} + \|v_j\|_{C^{1,\alpha}(S^{n-1},\mathbb{R}^k)} \Big) = 0. \tag{6.15}$$

Proof. The proof will be divided into several steps. Before starting with certain geometric constructions we recall that the Kuratowski convergence of $\operatorname{spt} S_j \to S^{n-1} \times \{0\}$ and the fact that $\operatorname{spt} S_j \subset B_{R_o}$ ensure that for any $\varepsilon \in (0, \frac{1}{2}]$ the inclusion

$$\operatorname{spt} S_j \Subset \big\{ z \in \mathbb{R}^{n+k} \colon \operatorname{dist}(z, S^{n-1} \times \{0\}) < \varepsilon \big\}$$

holds true for all but finitely many $j \in \mathbb{N}$.

Step 1: Geometric simplifications. Points in $\mathbb{R}^{n+k}$ are denoted again by $z = (x, y)$. For a (relatively) open subset $U \subset S^{n-1} \times \{0\} \subset \mathbb{R}^{n+k}$ and $0 < s \le \frac{1}{2}$ we consider sets of the form

$$\mathcal{N}_U(s) := \bigcup_{z\in U} \big\{ z + v \in \mathbb{R}^{n+k} \colon |v| < s,\, v \perp T_z(S^{n-1} \times \{0\}) \big\}.$$

Then, $\mathcal{N}_{S^{n-1}\times\{0\}}(s)$ is the tubular neighborhood of $S^{n-1} \times \{0\}$ in $\mathbb{R}^{n+k}$ of width s on which the nearest point retraction $\pi \colon \mathcal{N}_{S^{n-1}\times\{0\}}(s) \to S^{n-1} \times \{0\}$ is well defined. For $z \in S^{n-1} \times \{0\}$, the vectors $\{z, e_{n+1}, \dots e_{n+k}\}$ are an orthonormal basis of $T_z^\perp(S^{n-1} \times \{0\})$. Hence, for points $z = (x, y) \in \mathcal{N}_{S^{n-1}\times\{0\}}(s)$ the nearest point retraction is given by $\pi(x, y) = \big(\frac{x}{|x|}, 0\big)$. The normal component $z^\perp$ of z has the form

$$z^\perp := z - \pi(z) = (x, y) - \big(\tfrac{x}{|x|}, 0\big) = \big(\tfrac{x}{|x|}(|x| - 1), y\big).$$

Now, if $\varphi \colon S^{n-1} \times \{0\} \supset U \to W \subset \mathbb{R}^{n-1}$ is a local coordinate chart then

$$\Phi(x, y) := \big(\varphi(\pi(x, y)), |x| - 1, y\big)$$

is a trivialization of $\mathcal{N}_U(s)$. The image $\Phi(\mathcal{N}_U(s))$ is the set $W \times B_s^{1+k}(0)$. Denoting by $\psi \colon W \to U$ the inverse of φ, the inverse $\Psi := \Phi^{-1} \colon W \times B_s^{1+k}(0) \to \mathcal{N}_U(s)$ is

$$\Psi(\xi', \xi_n, y) := \Phi^{-1}(\xi', \xi_n, y) = (1 + \xi_n)\psi(\xi') + (0, y),$$

whenever $\xi' \in W$ and $(\xi_n, y) \in B_s^{1+k}(0)$. We note that Ψ maps a fiber $\{\xi'\} \times B_s^{1+k}(0)$ with $\xi' \in W$ isometrically onto $\psi(\xi') + \{v \in T^\perp_{\psi(\xi')}(S^{n-1} \times \{0\}) \colon |v| < s\}$.

Without loss of generality we assume that $W \equiv B_\varrho^{n-1}(0) \subset \mathbb{R}^{n-1}$ for some $\varrho \in (0, 1]$, $\psi(0) = e_n \in \mathbb{R}^{n+k}$ and $D\psi(0) = \mathbb{I}_{n-1}$. This can be achieved by a rotation in $\mathbb{R}^{n+k}$ keeping $\{0\} \times \mathbb{R}^k$ fixed and a particular choice of the coordinate chart φ, for example by choosing $\psi \colon B_\varrho^{n-1}(0) \to \mathbb{R}^{n+k}$ as $\psi(\xi') := (\xi', \sqrt{1 - |\xi'|^2}, 0)$. We first compute the derivative of Ψ. With $\sigma = (\tau', \tau_n, w) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^k$ and $z = (\xi', \xi_n, y)$ with $\xi' \in B_\varrho^{n-1}(0)$ and $(\xi_n, y) \in B_s^{1+k}(0)$ we have

$$D\Psi(z)\sigma = D\Psi(\xi', \xi_n, y)(\tau', \tau_n, w) = \underbrace{(1 + \xi_n)D\psi(\xi')(\tau', 0)}_{\in T_{\psi(\xi')}(S^{n-1}\times\{0\})} + \underbrace{\psi(\xi')(0, \tau_n, 0) + (0, w)}_{\in T^\perp_{\psi(\xi')}(S^{n-1}\times\{0\})}.$$

Therefore, taking into account that $0 < s \le \frac{1}{2}$, we have

$$\begin{aligned}|D\Psi(z)\sigma| &\le (1 + s)|\tau'| \sup_{B_\varrho^{n-1}(0)} \|D\psi\| + |\tau_n| + |w| \\ &\le 4 \sup_{B_\varrho^{n-1}(0)} \|D\psi\| |\sigma| = C(\psi)\, |\sigma|.\end{aligned}$$

This allows us, whenever $z = (\xi', \xi_n, y)$, $\tilde z = (\tilde\xi', \tilde\xi_n, \tilde y) \in B_\varrho^{n-1}(0) \times B_s^{1+k}(0)$, to estimate

$$\begin{aligned}&|D\Psi(z)\sigma - D\Psi(\tilde z)\sigma| \\ &\quad\le \big|\big((1 + \xi_n)(D\psi(\xi') - D\psi(\tilde\xi')) + (\xi_n - \tilde\xi_n)D\psi(\tilde\xi')\big)\tau'\big| + \big|\psi(\xi') - \psi(\tilde\xi')\big||\tau_n| \\ &\quad\le |1 + \xi_n|\|D\psi(\xi') - D\psi(\tilde\xi')\||\tau'| + \|D\psi(\tilde\xi')\||\xi_n - \tilde\xi_n||\tau'| + |\psi(\xi') - \psi(\tilde\xi')||\tau_n|\end{aligned}$$

$$\le (1+s) \sup_{B_\varrho^{n-1}(0)} \|D^2\psi\| |\xi' - \tilde{\xi}'||\tau'| + \sup_{B_\varrho^{n-1}(0)} \|D\psi\| \big(|\tilde{\xi}_n - \tilde{\xi}_n||\tau'| + |\xi' - \tilde{\xi}'||\tau_n|\big)$$
$$\le \Big((1+s) \sup_{B_\varrho^{n-1}(0)} \|D^2\psi\| + \sup_{B_\varrho^{n-1}(0)} \|D\psi\|\Big)\big(|\xi' - \tilde{\xi}'| + |\tilde{\xi}_n - \tilde{\xi}_n|\big)\big(|\tau'| + |\tau_n|\big)$$
$$\le 4\Big(\sup_{B_\varrho^{n-1}(0)} \|D^2\psi\| + \sup_{B_\varrho^{n-1}(0)} \|D\psi\|\Big)|(\xi', \xi_n) - (\tilde{\xi}', \tilde{\xi}_n)||(\tau', \tau_n)|$$
$$= C(\psi)\, |(\xi', \xi_n) - (\tilde{\xi}', \tilde{\xi}_n)||(\tau', \tau_n)|.$$

The preceding estimate yields the Lipschitz continuity of $D\Psi$, that is

$$\|D\Psi(z) - D\Psi(\tilde{z})\| \le C(\psi)\, |(\xi', \xi_n) - (\tilde{\xi}', \tilde{\xi}_n)|,$$

whenever $z = (\xi', \xi_n, y)$, $\tilde{z} = (\tilde{\xi}', \tilde{\xi}_n, \tilde{y}) \in B_\varrho^{n-1}(0) \times B_s^{1+k}(0)$. A straightforward computation now shows that also the map $z \in B_\varrho^{n-1}(0) \times B_s^{1+k}(0) \mapsto \bigwedge_{n-1} D\Psi(z) \in \mathcal{L}(\bigwedge_{n-1} \mathbb{R}^{n+k}, \bigwedge_{n-1} \mathbb{R}^{n+k})$ is Lipschitz continous, i.e. we have

$$\Big\| \bigwedge\nolimits_{n-1} D\Psi(z) - \bigwedge\nolimits_{n-1} D\Psi(\tilde{z})\Big\| \le C(n, \psi)\, |(\xi', \xi_n) - (\tilde{\xi}', \tilde{\xi}_n)|. \tag{6.16}$$

Taking into account that $D\psi(0) = \mathbb{I}_{n-1}$ where $\mathbb{I}_{n-1} \colon \mathbb{R}^{n-1} \to \mathbb{R}^{n+k}$ denotes the embedding of $\mathbb{R}^{n-1}$ into $\mathbb{R}^{n+k}$ via the inclusion $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^{n+k}$, we obtain that $D\Psi(0) = \mathbb{I}_{n+k}$. Hence, from (6.16) we infer the following bounds:

$$1 - C(n, \psi)(\varrho + s) \le \Big\| \bigwedge\nolimits_{n-1} D\Psi(z)\Big\| \le 1 + C(n, \psi)(\varrho + s). \tag{6.17}$$

Step 2: Estimates for the parametric integrand. We define a parametric integrand

$$F \colon B_\varrho^{n-1}(0) \times B_s^{1+k}(0) \times \bigwedge\nolimits_{n-1} \mathbb{R}^{n+k} \to [0, \infty)$$

by letting

$$F(z, \zeta) := \Big| \bigwedge\nolimits_{n-1} D\Psi(z)\zeta\Big|.$$

Then, apart from the fact that the constant in the bound from below is not equal to one, (6.17) corresponds to the hypothesis [10, (1.1)]. Moreover, the assumption [10, (1.5)] with the Lipschitz modulus $\kappa(t) = Ct$ follows from estimate (6.16). The remaining hypotheses [10, (1.2), (1.3), (1.6)] can be easily verified to hold. We omit the straightforward computations and state only the corresponding estimates:

$$\big\|D_{(2)}F(z, \zeta)\big\| \le C$$
$$\big\|D^2_{(2)}F(z, \zeta)\big\| \le \frac{C}{1 - C(\varrho + s)}\, |\zeta|^{-1}$$
$$\big\|D_{(2)}F(z, \zeta) - D_{(2)}F(\tilde{z}, \zeta)\big\| \le \frac{C}{1 - C(\varrho + s)}\, |z - \tilde{z}|$$

for all $z, \tilde{z} \in B_\varrho^{n-1}(0) \times B_s^{1+k}(0)$ and $0 \ne \zeta \in \bigwedge_{n-1} \mathbb{R}^{n+k}$. Here the constant C depends only on n and ψ. Finally, the quantitative continuity of $\zeta \mapsto D^2_{(2)}F(x, \zeta)$ follows from the fact that $D^2_{(2)}F(x, \zeta)$ is continuous on $\bigwedge_{n-1} \mathbb{R}^{n+k} \setminus \{0\}$. Therefore, we have

$$\big\|D_{(2)}F(z, \zeta) - D_{(2)}F(z, \eta)\big\| \le \nu(|\zeta - \eta|)$$

whenever $\zeta, \eta \in \bigwedge_{n-1} \mathbb{R}^{n+k}$ with $|\zeta| = 1 = |\eta|$. Actually, a direct computation shows that $\nu(t) = \frac{C\,t}{1 - C(\varrho + s)}$.

Step 3: Reduction to $(\mathbb{F}, \omega)$-minimizing currents. Now, let $s \in (0, \frac{1}{2}]$. From the Kuratowski convergence $\operatorname{spt} S_j \to S^{n-1} \times \{0\}$ we conclude that the currents S_j have compact support in $\mathcal{N}_{S^{n-1} \times \{0\}}(s)$ for $j \in \mathbb{N}$ large enough. Since $\partial(\pi_\# S_j) = \pi_\#(\partial S_j) = 0$, by the constancy theorem [13, 4.1.7] we find $m_j \in \mathbb{Z}$ such that $\pi_\# S_j = m_j [\![S^{n-1} \times$

$\{0\}]\!]$. We claim that $m_j = 1$. But this follows from the weak convergence $S_j \to [\![S^{n-1} \times \{0\}]\!]$, because

$$[\![S^{n-1} \times \{0\}]\!] = \pi_\# [\![S^{n-1} \times \{0\}]\!] = \lim_{j\to\infty} \pi_\# S_j = \lim_{j\to\infty} m_j [\![S^{n-1} \times \{0\}]\!]$$

implies that $m_j = 1$ for j large. Therefore, discarding finitely many indices $j \in \mathbb{N}$ if necessary, we can assume that

$$\pi_\# S_j = [\![S^{n-1} \times \{0\}]\!] \quad \text{and} \quad \operatorname{spt} S_j \Subset \mathcal{N}_{S^{n-1}\times\{0\}}(s)$$

for all $j \in \mathbb{N}$. This allows us to define a *global excess functional* by

$$\mathbb{E}(S_j) := \mathbf{M}(S_j) - \mathbf{M}(\pi_\# S_j) = \mathbf{M}(S_j) - n\omega_n = n\omega_n \mathbf{D}(S_j).$$

Note that $\mathbf{M}(S_j) \to n\omega_n$ as $j \to \infty$ since $\|S_j\| \to \|[\![S^{n-1} \times \{0\}]\!]\|$ in the sense of Radon measures.

We now fix $s, \varrho \in (0, \frac{1}{2}]$ small enough to have

$$C(n, \psi)(\varrho + s) \le \tfrac{1}{2},$$

where $\psi\colon B_\varrho^{n-1}(0) \to S^{n-1} \times \{0\}$ is the local parametrization from the Step 2. ¿From now on we omit in our notation the center 0 and write $B_\varrho^{n-1} \times B_s^{1+k}$ for short. We set $S_j' := S_j \llcorner \mathcal{N}_{\psi(B_\varrho^{n-1})}(s)$ and $S_j'' := \Phi_\# S_j'$. Then $S_j'' \in \mathcal{R}_{n-1}(B_\varrho^{n-1} \times B_s^{1+k})$ and $S_j' = \Psi_\# S_j''$. We have

$$\mathbf{M}(S_j') = \mathbf{M}(\Psi_\# S_j'') = \int \big|(D\Psi)_\# \vec{S_j''}\big|\, d\|S_j''\| =: \int F(z, \vec{S_j''})\, d\|S_j''\| =: \mathbb{F}(S_j'').$$

Here the associated elliptic integrand is defined by

$$F(z,\zeta) := \big|(D\Psi(z))_\# \zeta\big| = \Big|\bigwedge\nolimits_{n-1} D\Psi(z)\zeta\Big|$$

whenever $z \in B_\varrho^{n-1} \times B_s^{1+k}$ and $\zeta \in \bigwedge_{n-1} \mathbb{R}^{n+k}$. Note that F is homogeneous of degree one in the second variable. We now consider a compact set K which is contained in a ball $B_r^{n+k}(z_o) \subset B_\varrho^{n-1} \times B_s^{1+k}$. For the radius r we assume that the smallness condition $C(\psi) r \le 2/C_\omega$ holds true. Then, $\Psi(B_r^{n+k}(z_o))$ is contained in a ball $B^{n+k}_{C(\psi)r}(\Psi(z_o))$. We now consider $X \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with $\partial X = 0$ and $\operatorname{spt} X \subset K$. By the $(\mathbf{M}, \omega)$ minimality of S_j (applied with the comparison current $\Psi_\# X$, the compact set $\Psi(K)$ which is contained in the ball $B^{n+k}_{C(\psi)r}(\Psi(z_o))$) we obtain that

$$\begin{aligned} \mathbb{F}(S_j'') = \mathbf{M}(S_j') &\le \mathbf{M}(S_j' + \Psi_\# X) + C(\psi) C_\omega r \mathbf{M}\big(S_j' \llcorner \Psi(K) + \Psi_\# X\big) \\ &= \mathbb{F}(S_j'' + X) + C(\psi) C_\omega r\, \mathbb{F}(S_j'' \llcorner K + X) \\ &\le \mathbb{F}(S_j'' + X) + \tfrac{3}{2} C(\psi) C_\omega r\, \mathbf{M}(S_j'' \llcorner K + X). \end{aligned} \tag{6.18}$$

In the last line we used the bound from above for the integrand F, i.e. the fact that $F(z,\zeta) \le \frac{3}{2}$. Note here, that we have chosen ϱ, s small enough. Hence, S_j'' is $(\mathbb{F}, \omega)$-minimizing in $B_\varrho^{n-1} \times B_s^{1+k}$ for the modulus $\omega(r) := \frac{3}{2} C(\psi) C_\omega r$. Moreover, since $\operatorname{spt} S_j \subset \mathcal{N}_{S^{n-1}\times\{0\}}(s)$, we have $\operatorname{spt} \partial S_j'' \subset \partial B_\varrho^{n-1} \times B_s^{1+k}$. Actually, we can assume that $\operatorname{spt} S_j'' \subset B_\varrho^{n-1} \times B_{\mu s}^{1+k}$, $\operatorname{spt} \partial S_j'' \subset \partial B_\varrho^{n-1} \times B_{\mu s}^{1+k}$ for a given fixed $0 < \mu \le 1$ which is still at our disposal, and moreover that S_j'' is $(\mathbb{F}, \omega)$-minimizing in $C_\varrho := B_\varrho^{n-1} \times \mathbb{R}^{1+k}$. For this we only need to discard finitely many j from our sequence. At this stage we keep in mind that $\|S_j''\| \to \|[\![B_\varrho^{n-1} \times \{0\}]\!]\|$ in the sense of Radon measures on C_ϱ and in the sense of Kuratowski convergence.

Step 4: Regularity. In this step we want to apply the ε-regularity theorem from [10] to the currents S_j'' for large $j \in \mathbb{N}$. Therefore we need to check that hypothesis (1.18) – (1.20) of [10] hold true. We first note that $S_j'' \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ and that $S_j'' = S_j'' \llcorner C_\varrho$. Moreover, we have $\partial S_j'' \llcorner C_\varrho = 0$. We denote by $\mathbf{p}\colon \mathbb{R}^{n+k} \to \mathbb{R}^{n-1}$ and $\mathbf{q}\colon \mathbb{R}^{n+k} \to \mathbb{R}^{1+k}$ the orthogonal projections of $\mathbb{R}^{n+k}$ on $\mathbb{R}^{n-1}$, respectively on $\mathbb{R}^{1+k}$, i.e. for $z = (\xi, \eta) \in \mathbb{R}^{n-1} \times \mathbb{R}^{1+k}$ we have $\mathbf{p}(z) = \xi$ and $\mathbf{q}(z) = \eta$. Then, $\mathbf{p}_\#(S''_j) \in \mathcal{R}_{n-1}(\mathbb{R}^{n-1})$ has no boundary in B^{n-1}_ϱ. By the constancy theorem [13, 4.1.7] there exist $m_j \in \mathbb{Z}$ such that $\mathbf{p}_\#(S''_j) = m_j [\![B^{n-1}_\varrho]\!]$. ¿From the weak convergence of $S''_j \to [\![B^{n-1}_\varrho \times \{0\}]\!]$ we easily see that

$$[\![B^{n-1}_\varrho]\!] = \mathbf{p}_\#([\![B^{n-1}_\varrho \times \{0\}]\!]) = \lim_{j\to\infty} \mathbf{p}_\# S''_j = \lim_{j\to\infty} m_j [\![B^{n-1}_\varrho]\!],$$

and this implies that $m_j = 1$ for large j and therefore $\mathbf{p}_\# S''_j = [\![B^{n-1}_\varrho]\!]$. Hence (1.18) – (1.20) of [10] are fulfilled except from the fact that we can at this stage not ensure that $0 \in \operatorname{spt} S''_j$. At this point we note that the weak convergence in the sense of Radon measures implies

$$\begin{aligned} \mathbb{E}(S''_j, \varrho) &:= \varrho^{1-n}\big[\mathbf{M}\big(S''_j \llcorner C_\varrho\big) - \mathbf{M}\big(\mathbf{p}_\#\big(S''_j \llcorner C_\varrho\big)\big)\big] \\ &= \varrho^{1-n}\big[\mathbf{M}\big(S''_j \llcorner C_\varrho\big) - \mathbf{M}\big([\![B^{n-1}_\varrho \times \{0\}]\!]\big)\big] \to 0, \end{aligned} \tag{6.19}$$

as $j \to \infty$. Next, we claim that there exist $z_j = (0, \eta_j) \in \operatorname{spt} S''_j$ with $|\eta_j| \to 0$. Indeed, if such η_j would not exist, then $0 \notin \mathbf{p}_\# S''_j$ and $\eta_j \to 0$ follows from the Kuratowski convergence of $\operatorname{spt} S''_j \to B^{n-1}_\varrho \times \{0\}$. Instead of S''_j we now consider $T_j := S''_j - z_j = \boldsymbol{\tau}^{-1}_{z_j \#} S''_j$, where $\boldsymbol{\tau}_{z_j}(z) := z + z_j$ denotes the translation in $\mathbb{R}^{n+k}$. For the projection of T_j onto $\mathbb{R}^{n-1}$ we obtain $\mathbf{p}_\# T_j = [\![B^{n-1}_\varrho]\!]$. Moreover, we have $0 \in \operatorname{spt} T_j$ and also $T_j = T_j \llcorner C_\varrho$. Finally, we obtain

$$\partial T_j \llcorner C_\varrho = \partial\big(\boldsymbol{\tau}^{-1}_{z_j \#} S''_j\big) \llcorner C_\varrho = \boldsymbol{\tau}^{-1}_{z_j \#} \partial S''_j \llcorner C_\varrho = \boldsymbol{\tau}^{-1}_{z_j \#}\big(\partial S''_j \llcorner C_\varrho\big) = 0.$$

This proves that (1.18) – (1.20) of [10] are fulfilled by the currents T_j and it remains to show that they are also $(\widetilde{\mathbb{F}}_j, \omega)$-minimizing in C_ϱ for an elliptic integrand $\widetilde{\mathbb{F}}_j$ and a modulus ω. For $z \in B^{n-1}_\varrho \times B^{1+k}_{(1-\mu)s}$ and and $\zeta \in \bigwedge_{n-1} \mathbb{R}^{n+k}$ we define the integrand $\widetilde{F}_j$ by

$$\widetilde{F}_j(z, \zeta) := F(z + z_j, \zeta)$$

and the corresponding parametric integral $\widetilde{\mathbb{F}}_j$ by

$$\widetilde{\mathbb{F}}_j(T) := \int \widetilde{F}_j(z, \vec{T}(z))\, d\|T\|$$

whenever $T \in \mathcal{R}_{n-1}\big(B^{n-1}_\varrho \times B^{1+k}_{(1-\mu)s}\big)$. Since $|\xi| \le \varrho$ and $|\eta| \le \mu s$ for any $z = (\xi, \eta) \in \operatorname{spt} S''_j$ we infer that $|\xi| \le \varrho$ and $|\eta| \le 2\mu s$ for any $z = (\xi, \eta) \in \operatorname{spt} T_j$. In order to have $\operatorname{spt} T_j \subset \operatorname{clos}\big(B^{n-1}_\varrho \times B^{1+k}_{(1-\mu)s}\big)$ we need that $0 < \mu < \frac{1}{3}$, which we will assume from now on. At this stage it is straightforward to check that T_j is $(\widetilde{\mathbb{F}}_j, \omega)$-minimizing in $B^{n-1}_\varrho \times B^{1+k}_{(1-\mu)s}$. To be more precise: Let K be a compact set which is contained in a ball $B^{n+k}_r(z_o) \subset B^{n-1}_\varrho \times B^{1+k}_{(1-\mu)s}$ and $X \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with $\partial X = 0$ and $\operatorname{spt} X \subset K$ and $C(\psi) r \le 2/C_\omega$. Then, from (6.18) we deduce that

$$\widetilde{\mathbb{F}}_j(T_j) \le \widetilde{\mathbb{F}}_j(T_j + X) + \tfrac{3}{2} C(\psi) C_\omega r \mathbf{M}(T_j \llcorner K + X)\,.$$

Hence, the currents T_j are $(\widetilde{\mathbb{F}}_j, \omega)$-minimizing in $B^{n-1}_\varrho \times B^{n-1}_{(1-\mu)s}$ for the modulus $\omega(r) := \frac{3}{2} C(\psi) C_\omega r$. The elliptic integrands $\widetilde{F}_j$ fulfill the assumptions (1.1) – (1.6) of [10] with $(C(n,\psi), C(n,\psi)t, C(n,k,\psi))$ instead of (Λ, κ, ν) and with the modulus $\omega(r)$. We note that $\omega(r) = C(\psi, C_\omega) r$. In particular the functions $\mathsf{K}(r)$ and $\Omega(r)$ introduced in [10, (1.15)] are given by $\mathsf{K}(r) = C(n,\psi) r$ and $\Omega(r) = C(\psi, C_\omega) r$. Finally, (6.19) yields also that

$$\mathbb{E}(T_j, \varrho) := \varrho^{1-n}\big[\mathbf{M}\big(T_j \llcorner C_\varrho\big) - \mathbf{M}\big(\mathbf{p}_\#\big(T_j \llcorner C_\varrho\big)\big)\big] \to 0 \quad \text{as } j \to \infty.$$

It remains to check that $\widetilde{F}_j$ is an elliptic integrand in the sense that [10, (1.12)] holds true. This means that there exists a positive constant C such that the inequality

$$(\widetilde{\mathbb{F}}_j)_{z_o}(T) - (\widetilde{\mathbb{F}}_j)_{z_o}(S) \geq C\,\big[\mathbf{M}(T) - \mathbf{M}(S)\big]$$

holds true whenever $S, T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with the same boundary $\partial S = \partial T$ and S is represented by an $\mathcal{H}^{n-1}$ measurable subset of some $(n-1)$-dimensional subspace in $\mathbb{R}^{n+k}$ with constant orientation $(n-1)$-vector field and $\mathcal{H}^{n-1}$ summable positive integer valued multiplicity. ¿From [13, 5.1.3] we infer that this property is implied in case that

$$\begin{aligned}\big\langle D^2_{(2)}F(z_o,\zeta)\eta,\eta\big\rangle &= \frac{1}{|\bigwedge_{n-1}D\Psi(z)\zeta\,|}\left[\big|\bigwedge\nolimits_{n-1}D\Psi(z)\eta\,\big|^2 - \frac{\langle\bigwedge_{n-1}D\Psi(z)\zeta,\eta\rangle^2}{|\bigwedge_{n-1}D\Psi(z)\zeta\,|^2}\right]\\ &\geq \frac{C}{|\zeta|}\left[|\eta|^2 - \frac{\langle\zeta,\eta\rangle^2}{|\zeta|^2}\right]\end{aligned}$$

holds true for some constant $C > 0$. Again, from [13, 5.1.3] we see that any choice of $0 < C < \|L^{-1}\|^{-4}\|L\|^3$ suffices, where $L := \bigwedge_{n-1}D\Psi(z)\colon \bigwedge_{n-1}\mathbb{R}^{n+k} \to \bigwedge_{n-1}\mathbb{R}^{n+k}$. Now, (6.17) yields that C can be chosen in dependence of n and ψ. Altogether we have shown that the hypotheses of the interior ε-regularity theorem [10, Theorem 6.1] hold true in the present situation. Therefore from [10, Theorem 6.1] we infer the existence of a C^1-map $g_j\colon B^{n-1}_{\varrho/34} \to \mathbb{R}^{1+k}$ such that

$$T_j \llcorner \big(B^{n-1}_{\varrho/34} \times \mathbb{R}^{1+k}\big) = [\![\mathrm{graph}(g_j)]\!].$$

Moreover, the derivative of g_j has the modulus of continuity

$$|Dg_j(\xi) - Dg_j(\xi')| \leq C\big[\mathbb{E}(T_j,\varrho) + 1\big]^{\frac12}\sqrt{|x-x'|} \leq C\sqrt{|x-x'|}$$

whenever $\xi, \xi' \in B^{n-1}_{\varrho/34}$, where $C = C(n,k,\psi)$. Here we used in the last step the fact that $\mathbb{E}(T_j,\varrho) \to 0$ as $j \to \infty$. Therefore, also the original currents S''_j admit local graph representations. In fact, we have

$$S''_j \llcorner B^{n-1}_{\varrho/34} \times \mathbb{R}^{1+k} = [\![\eta_j + \mathrm{graph}(g_j)]\!].$$

Note that $\eta_j \to 0$ as $j \to \infty$. Thus, we obtained a uniform $C^{1,\frac12}$ bound for the local graph representations of S''_j. Therefore, by the Arzelà & Ascoli theorem a subsequence of $(\eta_j + g_j)_{j\gg 1}$ converges in $C^{1,\alpha}$ to a $C^{1,\alpha}$ map g for any $\alpha \in (0,\frac12)$. But from the Kuratowski convergence we must have $g \equiv 0$. Therefore, the subsequence converges to 0 in $C^{1,\alpha}$, and since the limit does not depend on the subsequence (as seen before it is uniquely identified as $g \equiv 0$) the whole sequence converges in $C^{1,\alpha}$ to 0. Now, from the local graph representation of S''_j the spherical graph representation on $\mathcal{N}_{\psi(B^{n-1}_{\varrho/34})}(s)$ of the original sequence S_j follows. Indeed

$$S_j \llcorner \mathcal{N}_{\psi(B^{n-1}_{\varrho/34})}(s) = \Psi_\#[\![\eta_j + \mathrm{graph}(g_j)]\!].$$

Since finitely many sets of the form $\mathcal{N}_{\psi(B^{n-1}_{\varrho/34})}(s)$ cover the tubular neighborhood $\mathcal{N}_{S^{n-1}\times\{0\}}(s)$ we obtain the desired spherical graph representation. □

Proof of Theorem 2.1. The proof is divided into several steps. First, since the asymmetry index $\mathbf{d}(T)$ and the isoperimetric gap $\mathbf{D}(T)$ are scaling invariant, we may assume without loss of generality that $\mathbf{m}(T) = \omega_n$. In this case the quantitative isoperimetric inequality (2.1) reduces to

$$\mathbf{D}(T) \equiv \frac{\mathbf{M}(T) - n\omega_n}{n\omega_n} \geq C\,\mathbf{d}_1^2(T).$$

Step 1: Reduction to currents with uniform bounded support and small isoperimetric gap. Here, we establish that it is sufficient to prove the quantitative isoperimetric inequality in the following form: There exists a constant $\delta_o > 0$ such that whenever

$T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ fulfills $\partial T = 0$, $\mathbf{m}(T) = \omega_n$, $\operatorname{spt} T \subset B_{R_o}$ and $\mathbf{D}(T) \leq \delta_o$ then the quantitative isoperimetric inequality

$$\mathbf{D}(T) \geq C_1\, \mathbf{d}_1(T)^2 \tag{6.20}$$

holds true with a universal constant $C_1 = C_1(n,k)$. Here, $R_o = R_o(n)$ denotes the radius from Lemma 6.1. Assume for the moment that such $\delta_o > 0$ exists. Then, for $T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ satisfying $\partial T = 0$, $\mathbf{m}(T) = \omega_n$ and $\mathbf{D}(T) > \delta_o/\widetilde{C}$, where $\widetilde{C} = \widetilde{C}(n,k)$ is the constant from Lemma 6.1, we have

$$\mathbf{d}_1^2(T) \leq 4\omega_n^2 < \frac{4\omega_n^2 \widetilde{C}}{\delta_o}\, \mathbf{D}(T),$$

i.e. the quantitative isoperimetric inequality with the constant $4\omega_n^2\widetilde{C}/\delta_o$. Here, we used the fact that $\mathbf{d}_1(T) \leq \mathbf{m}(T) + \omega_n = 2\omega_n$. Now, if $\mathbf{D}(T) \leq \delta_o/\widetilde{C}$ then Lemma 6.1 ensures the existence of $T' \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ satisfying $\partial T' = 0$, $\mathbf{m}(T') = \omega_n$ and $\operatorname{spt} T' \subset B_{R_o}$ such that $\mathbf{d}_1(T) \leq \widetilde{C}(\mathbf{d}_1(T') + \mathbf{D}(T))$ and $\mathbf{D}(T') \leq \widetilde{C}\mathbf{D}(T) \leq \delta_o$ hold true. Therefore, we can apply (6.20) to T' in order to have

$$\begin{aligned}
\mathbf{d}_1(T)^2 &\leq 2\widetilde{C}^2\big(\mathbf{d}_1(T')^2 + \mathbf{D}(T)^2\big) \\
&\leq 2\widetilde{C}^2\big(\tfrac{1}{C_1}\mathbf{D}(T') + \tfrac{\delta_o}{\widetilde{C}}\mathbf{D}(T)\big) \leq 2\widetilde{C}^2\big(\tfrac{\widetilde{C}}{C_1} + \tfrac{\delta_o}{\widetilde{C}}\big)\mathbf{D}(T),
\end{aligned}$$

and this yields the quantitative isoperimetric inequality with the constant $[2\widetilde{C}^2(\frac{\widetilde{C}}{C_1} + \frac{\delta_o}{\widetilde{C}})]^{-1}$.

Step 2: The contradiction assumption. In the following we argue by contradiction assuming (6.20) to be false. Then, there exists a sequence of $(n-1)$-dimensional currents $T_j \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with $\partial T_j = 0$, $\mathbf{m}(T_j) = \omega_n$ and $\operatorname{spt} T_j \subset B_{R_o}$ satisfying

$$\delta_j := \mathbf{D}(T_j) \equiv \frac{\mathbf{M}(T_j) - n\omega_n}{n\omega_n} \to 0 \quad \text{as } j \to \infty,$$

and

$$\delta_j < C_1\, \mathbf{d}_1^2(T_j). \tag{6.21}$$

Step 3: Convergence to a flat n-dimensional unit disk. We choose mass minimizers $Q(T_j) \in \mathcal{R}_n(\mathbb{R}^{n+k})$ with $\partial Q(T_j) = T_j$ such that $\omega_n = \mathbf{m}(T_j) = \mathbf{M}(Q(T_j))$. We note that since $\operatorname{spt} T_j \subset \overline{B}_{R_o}$ these mass minimizers can be chosen to have also support in $\overline{B}_{R_o}$, i.e. $\operatorname{spt} Q(T_j) \subset \overline{B}_{R_o}$. Since $\mathbf{M}(Q(T_j)) + \mathbf{M}(T_j) = \omega_n + \mathbf{M}(T_j) \to (1+n)\omega_n$ in the limit $j \to \infty$, we have a uniform bound $\sup_{j\in\mathbb{N}}(\mathbf{M}(Q(T_j)) + \mathbf{M}(T_j)) < \infty$ and therefore we can apply the compactness Theorem 3.2 to infer the existence of a current $Q \in \mathcal{R}_n(\mathbb{R}^{n+k})$ with support in $\overline{B}_{R_o}$ and a (not relabeled) subsequence such that $Q(T_j) \to Q$ with respect to the $\mathbf{F}_{\mathrm{loc}}$-topology. In particular, we have $\mathbf{F}_{B_{2R_o}}(Q(T_j) - Q) \to 0$ and $\mathbf{F}_{B_{2R_o}}(T_j - \partial Q) \to 0$ in the limit $j \to \infty$, because $\operatorname{spt} T_j, \operatorname{spt} Q(T_j) \subset \overline{B}_{R_o}$ for any $j \in \mathbb{N}$.

Next, we claim the the limit Q is an n-dimensional flat unit disk in $\mathbb{R}^{n+k}$. Applying Lemma 3.3 we find that

$$\mathbf{m}(T_j - \partial Q) \leq \big[c(n)\,\mathbf{M}(T_j - \partial Q) + 1\big]\mathbf{F}_{B_{2R_o}}(T_j - \partial Q) \to 0$$

in the limit $j \to \infty$. Note that this implies $\mathbf{m}(T_j) \to \mathbf{m}(\partial Q)$ as $j \to \infty$. Using also the lower semicontinuity of the mass with respect to weak convergence, i.e. the fact that $\mathbf{M}(Q) \leq \liminf_{j\to\infty} \mathbf{M}(Q(T_j)) = \omega_n$, we obtain

$$\omega_n = \lim_{j\to\infty} \mathbf{M}(Q(T_j)) = \lim_{j\to\infty} \mathbf{m}(T_j) = \mathbf{m}(\partial Q) \leq \mathbf{M}(Q) \leq \omega_n.$$

Hence $\mathbf{M}(Q) = \omega_n$. Therefore, by the optimal isoperimetric inequality from Theorem 3.4 we must have $\mathbf{M}(\partial Q) \geq n\omega_n$. On the other hand, by the weak convergence $T_j \to \partial Q$ the lower semicontinuity of the mass together with the convergence $\mathbf{M}(T_j) \to n\omega_n$ implies

$$\mathbf{M}(\partial Q) \le \liminf_{j\to\infty} \mathbf{M}(T_j) \le n\omega_n.$$

Therefore, we have $\mathbf{M}(\partial Q) = n\omega_n$ and $\mathbf{M}(Q) = \omega_n$ which implies that in the isoperimetric inequality we have equality, so that $Q = [\![D]\!]$ for some n-dimensional flat unit disk $[\![D]\!] \subset \mathbb{R}^{n+k}$. Hence, we know that $Q(T_j) \to [\![D]\!]$ and $T_j \to \partial[\![D]\!]$ with respect to the $\mathbf{F}_{\rm loc}$-topology and also with respect to the weak topology. This implies in particular that $\mathbf{d}_1(T_j) \to 0$ when $j \to \infty$.

Step 4: Penalization. Let $\Lambda > 2n$. For $j \in \mathbb{N}$ we define penalized variational functionals $\mathcal{F}_j\colon \mathcal{R}_{n-1}(\mathbb{R}^{n+k}) \to [0,\infty)$ by

$$\mathcal{F}_j(T) := \mathbf{M}(T) + C_1|\mathbf{d}_1(T) - \mathbf{d}_1(T_j)| + \Lambda|\mathbf{m}(T) - \omega_n|.$$

Here, $C_1 > 0$ is fixed and will be chosen later on in a universal way in dependence on n and k. ¿From Lemma 5.1 we infer the existence of $S_j \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with support $\operatorname{spt} S_j \subset \overline{B}_{R_o}$ minimizing the functional $\mathcal{F}_j$ amongst all closed $T \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ satisfying $\operatorname{spt} T \subset \overline{B}_{R_o}$. By the convex hull property we can choose mass minimizing currents $Q(S_j) \in \mathcal{R}_n(\mathbb{R}^{n+k})$ with boundary $\partial Q(S_j) = S_j$ and support in $\overline{B}_{R_o}$. Note that $\mathbf{m}(S_j) = \mathbf{M}(Q(S_j))$. Since S_j is $\mathcal{F}_j$-minimizing we have

$$\mathcal{F}_j(S_j) \le \mathcal{F}_j(T_j) = \mathbf{M}(T_j).$$

On the other hand, the following bound from below holds:

$$\mathcal{F}_j(S_j) \ge \mathbf{M}(S_j) + \Lambda(\mathbf{m}(S_j) - \omega_n).$$

The two preceding estimates imply the following mass bound

$$\mathbf{M}(S_j) + \mathbf{m}(S_j) \le \mathbf{M}(S_j) + \Lambda\mathbf{m}(S_j) \le \mathcal{F}_j(S_j) + \Lambda\omega_n \le \mathbf{M}(T_j) + \Lambda\omega_n,$$

yielding a uniform mass bound for the sequences $(S_j)_{j\in\mathbb{N}}$ and $(Q(S_j))_{j\in\mathbb{N}}$. ¿From Theorem 3.2 we infer the existence of a mass minimizing current $Q_\infty \in \mathcal{R}_n(\mathbb{R}^{n+k})$ (mass minimizing with respect to its own boundary ∂Q_∞) such that (up to a subsequence) $Q(S_j) \to Q_\infty$ with respect to the $\mathbf{F}_{\rm loc}$-topology. We also have $\partial Q(S_j) = S_j \to \partial Q_\infty$ in the $\mathbf{F}_{\rm loc}$-topology (and therefore also in the sense of weak convergence of currents). Next, we define the functional $\mathcal{F}_\infty\colon \mathcal{R}_{n-1}(\mathbb{R}^{n+k}) \to [0,\infty)$ by

$$\mathcal{F}_\infty(T) := \mathbf{M}(T) + \Lambda|\mathbf{m}(T) - \omega_n|.$$

¿From Lemma 5.8 we infer that the boundary $\partial[\![D]\!]$ of a flat n-dimensional unit disk with support in $\overline{B}_{R_o}$ minimizes $\mathcal{F}_\infty$. Using the minimality of S_j and $\partial[\![D]\!]$ and the definition of δ_j we obtain

$$\begin{aligned}\mathcal{F}_j(S_j) &\le \mathcal{F}_j(T_j) = \mathbf{M}(T_j) = n\omega_n(1+\delta_j) = \mathbf{M}(\partial[\![D]\!]) + n\omega_n\delta_j\\ &= \mathcal{F}_\infty(\partial[\![D]\!]) + n\omega_n\delta_j \le \mathcal{F}_\infty(S_j) + n\omega_n\delta_j.\end{aligned}$$

By the definitions of $\mathcal{F}_j$ and $\mathcal{F}_\infty$ and (6.21) the preceding inequality can be rewritten in the form

$$C_1|\mathbf{d}_1(S_j) - \mathbf{d}_1(T_j)| \le n\omega_n\delta_j < n\omega_n C_1\,\mathbf{d}_1^2(T_j). \tag{6.22}$$

Now, since $\mathbf{d}_1(T_j) \to 0$ as $j \to \infty$ we also have $\mathbf{d}_1(S_j) \to 0$ as $j \to \infty$. Therefore, by the definition of $\mathbf{d}_1$ for any $j \in \mathbb{N}$ we can choose a flat n-dimensional unit disk $[\![D_j]\!]$ such that $\mathbf{m}(S_j - \partial[\![D_j]\!]) < \mathbf{d}_1(S_j) + \frac{1}{j}$. Therefore, $S_j - \partial[\![D_j]\!] \to 0$ as $j \to \infty$ in the flat metric (and also weakly). Now, since $S_j \to \partial Q_\infty$ we also have $\partial[\![D_j]\!] \to \partial Q_\infty$. But this implies $[\![D_j]\!] \to [\![D]\!]$ for some flat n-dimensional unit disk with support in $\overline{B}_{R_o}$; the latter holds because $\operatorname{spt} Q_\infty \subset \overline{B}_{R_o}$. Therefore we have $\partial Q_\infty = \partial[\![D]\!]$. Since Q_∞ is mass minimizing subject to the boundary $\partial[\![D]\!]$ we have $Q_\infty = [\![D]\!]$. Here we use the convex hull property (cf. [21, Remark 34.2 (2)] and the constancy theorem [13, 4.1.7]). Thus we have shown that $Q(S_j) \to [\![D]\!]$ as $j \to \infty$. Using again the minimality of S_j and (6.21) we further get

$$\begin{aligned} \mathbf{M}(S_j) + \Lambda|\mathbf{m}(S_j) - \omega_n| &\le \mathcal{F}_j(S_j) \le \mathcal{F}_j(T_j) = \mathbf{M}(T_j) \\ &= n\omega_n(1+\delta_j) < n\omega_n\big(1 + C_1\,\mathbf{d}_1^2(T_j)\big). \end{aligned} \tag{6.23}$$

Step 5: λ-mass minimality and almost minimality of S_j. By Lemma 5.3 we know that the currents S_j are λ-minimizing in $\mathbb{R}^{n+k}$ with $\lambda := C_1 + \Lambda$, that is for any $P \in \mathcal{R}_n(\mathbb{R}^{n+k})$ it holds

$$\mathbf{M}(S_j) \le \mathbf{M}(S_j + \partial P) + \lambda \mathbf{M}(P).$$

Moreover, they are $(\mathbf{M}, \omega)$-minimizing for the modulus $\omega(\varrho) := 4\lambda\varrho$ and with $\varrho_o = 1/(2\lambda)$, in the sense that there holds

$$\mathbf{M}(S_j) \le \mathbf{M}(S_j + X) + 4\lambda\varrho\,\mathbf{M}(S_j \llcorner K + X)$$

whenever $X \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with $\partial X = 0$ and support contained in a compact set K which is contained in a ball of radius $\varrho \le 1/(2\lambda)$. Recalling the uniform mass bound $\sup_{j\in\mathbb{N}} \mathbf{M}(S_j) < \infty$ and the convergence $S_j \to \partial[\![D]\!]$ in the $\mathbf{F}_{\mathrm{loc}}$-topology we can conclude by Lemma 5.5 that $\|S_j\| \to \|\partial[\![D]\!]\|$ in the sense of Radon measures and, moreover by Lemma 5.6 that $\operatorname{spt} S_j \to \operatorname{spt} \partial[\![D]\!]$ in the Kuratowski convergence. Moreover, for the mass minimizers $Q(S_j)$ we can conclude (by the same arguments) that $\|Q(S_j)\| \to \|[\![D]\!]\|$ in the sense of Radon measures.

Step 6: Adjusting the mass constraint by rescaling. Here, we rescale S_j in order to have for the rescaled currents S_j' the mass constraint $\mathbf{m}(S_j') = \omega_n$. We set

$$S_j' := \lambda_j S_j \qquad \text{where } \lambda_j := \left(\frac{\omega_n}{\mathbf{m}(S_j)}\right)^{\frac{1}{n}}$$

such that $\mathbf{m}(S_j') = \mathbf{M}(Q(S_j')) = \lambda_j^n \mathbf{m}(S_j) = \omega_n$. Here, $Q(S_j')$ is the mass minimizing current obtained by scaling the mass minimizing current $Q(S_j)$ by λ_j, that is $Q(S_j') = Q(\lambda_j S_j) := \lambda_j Q(S_j)$. From [11, Chapter 1.9, Theorem 1] and the weak convergence of Radon measures $\|Q(S_j)\| \to \|[\![D]\!]\|$ we infer that $\mathbf{m}(S_j) = \mathbf{M}(Q(S_j)) \to \omega_n$. Using this and $\mathbf{d}_1(T_j) \to 0$ in (6.23) we see that $\limsup_{j\to\infty} \mathbf{M}(S_j) \le n\omega_n$. Combining this with $n\omega_n = \mathbf{M}(\partial[\![D]\!])$ and the lower semicontinuity of the mass with respect to weak convergence (note that $S_j \to \partial[\![D]\!]$) we obtain that

$$\lim_{j\to\infty} \mathbf{M}(S_j) = n\omega_n.$$

Since $\sup_{j\in\mathbb{N}} \mathbf{M}(S_j) < \infty$ and $\lambda_j \to 1$ the rescaled currents S_j' also converge to $\partial[\![D]\!]$ in the $\mathbf{F}_{\mathrm{loc}}$-topology, weakly as currents and in the Kuratowski convergence. Further, $\|S_j'\| \to \|\partial[\![D]\!]\|$ in the sense of Radon measures. Finally, since $\lambda_j \in [\frac{1}{2}, 2]$ (for j large enough) the rescaled currents S_j' are $(\mathbf{M}, \omega)$-minimizing in the sense that

$$\mathbf{M}(S_j') \le \mathbf{M}(S_j' + X) + 8\lambda\varrho\,\mathbf{M}(S_j' \llcorner K + X) \tag{6.24}$$

holds true for any $X \in \mathcal{R}_{n-1}(\mathbb{R}^{n+k})$ with $\partial X = 0$ and support contained in a compact set $K \subset B_\varrho(x_o)$ where $\varrho \in (0, 1/(4\lambda)]$. Since $\mathbf{M}(S_j)/\mathbf{m}(S_j) \to n$ as $j \to \infty$ and $\Lambda > 2n$ we may assume for j large enough that $\mathbf{M}(S_j) < \frac{1}{2}\Lambda\mathbf{m}(S_j)$. Therefore, we have

$$\begin{aligned} |\mathbf{M}(S_j') - \mathbf{M}(S_j)| &= |\lambda_j^{n-1} - 1|\,\mathbf{M}(S_j) \le \tfrac{1}{2}\Lambda|\lambda_j^{n-1} - 1|\,\mathbf{m}(S_j) \\ &\le \tfrac{1}{2}\Lambda|\lambda_j^n - 1|\,\mathbf{m}(S_j) = \tfrac{1}{2}\Lambda|\omega_n - \mathbf{m}(S_j)|. \end{aligned}$$

Note that from (6.22) it follows for j large $\mathbf{d}_1(T_j) \le 2\mathbf{d}_1(S_j)$. Therefore, the previous inequality, together with (6.23), yields

$$\begin{aligned} \mathbf{M}(S_j') - n\omega_n &\le \mathbf{M}(S_j) + \tfrac{1}{2}\Lambda|\mathbf{m}(S_j) - \omega_n| - n\omega_n \\ &\le n\omega_n C_1\,\mathbf{d}_1^2(T_j) - \tfrac{1}{2}\Lambda|\mathbf{m}(S_j) - \omega_n| \end{aligned}$$

$$\le 4n\omega_n C_1\, \mathbf{d}_1^2(S_j) - \tfrac{1}{2}\Lambda|\mathbf{m}(S_j) - \omega_n|. \tag{6.25}$$

For $j \in \mathbb{N}$ we now define the homotopy $h\colon [0,1]\times\mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ by $h(s,x) := (1-s)x + s\lambda_j x$. Then, $h(0,\cdot) = \mathrm{id}$ and $h(1,\cdot) = \eta_{\lambda_j}$, where $\eta_{\lambda_j}(x) := \lambda_j x$. Therefore, by the homotopy formula we have

$$\begin{aligned}\partial h_\#\big([\![0,1]\!]\times S_j\big) &= h_\#\partial\big([\![0,1]\!]\times S_j\big) = h_\#\big(\{1\}\times S_j - \{0\}\times S_j - [\![0,1]\!]\times \partial S_j\big)\\ &= h_\#\big(\{1\}\times S_j - \{0\}\times S_j\big) = (\eta_{\lambda_j})_\# S_j - \mathrm{id}_\# S_j = S_j' - S_j.\end{aligned}$$

and therefore by [21, 26.23], the facts that $\operatorname{spt} S_j \subset \overline{B}_{R_o}$ and $\lambda_j \to 1$ (especially that $\lambda_j \le 2$ for j large enough) and $\mathbf{M}(S_j) < \frac{1}{2}\Lambda\mathbf{m}(S_j)$ we obtain

$$\begin{aligned}\mathbf{m}(S_j' - S_j) &\le \mathbf{M}\big(h_\#([\![0,1]\!]\times S_j)\big) \le \sup_{x\in\operatorname{spt} S_j}|x-\lambda_j x|(1+\lambda_j)^{n-1}\mathbf{M}(S_j)\\ &\le 3^{n-1}R_o|\lambda_j - 1|\,\mathbf{M}(S_j) \le \tfrac{1}{2}3^{n-1}\Lambda R_o|\lambda_j^n - 1|\,\mathbf{m}(S_j)\\ &= \tfrac{1}{2}3^{n-1}\Lambda R_o\,|\mathbf{m}(S_j) - \omega_n|.\end{aligned}$$

To proceed further we denote by $[\![D_j]\!]$ a flat n-dimensional unit disk realizing $\mathbf{d}_1(S_j')$ up to an error $\varepsilon > 0$, i.e. $\mathbf{m}(S_j' - \partial[\![D_j]\!]) < \mathbf{d}_1(S_j') + \varepsilon$. Moreover, since $\mathbf{m}(S_j) \to \omega_n$ we may assume that $|\mathbf{m}(S_j) - \omega_n| \le (2\cdot 3^{2n-1}n\omega_n C_1\Lambda R_o^2)^{-1}$ for j large enough. We therefore obtain

$$\begin{aligned}\mathbf{d}_1^2(S_j) &\le \mathbf{m}^2(S_j - \partial[\![D_j]\!]) \le \big(\mathbf{m}(S_j - S_j') + \mathbf{m}(S_j' - \partial[\![D_j]\!])\big)^2\\ &< \big(\mathbf{m}(S_j' - S_j) + \mathbf{d}_1(S_j') + \varepsilon\big)^2 \le 3\mathbf{m}^2(S_j' - S_j) + 3\mathbf{d}_1^2(S_j') + 3\varepsilon^2\\ &\le \tfrac{1}{4}3^{2n-1}\Lambda^2R_o^2|\mathbf{m}(S_j) - \omega_n|^2 + 3\mathbf{d}_1^2(S_j') + 3\varepsilon^2\\ &\le \tfrac{\Lambda}{8n\omega_n C_1}|\mathbf{m}(S_j) - \omega_n| + 3\mathbf{d}_1^2(S_j') + 3\varepsilon^2.\end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily small we can pass to the limit $\varepsilon \downarrow 0$ and obtain

$$\mathbf{d}_1^2(S_j) \le 3\mathbf{d}_1^2(S_j') + \tfrac{\Lambda}{8n\omega_n C_1}|\mathbf{m}(S_j) - \omega_n|,$$

whenever j is large enough. Using this to estimate to bound the right-hand side in (6.25) from above we find that

$$\mathbf{M}(S_j') - n\omega_n \le 12n\omega_n C_1\,\mathbf{d}_1^2(S_j'). \tag{6.26}$$

Step 7: Adjusting the barycenter condition. Here we establish that we can assume without loss of generality that the barycenter of S_j' is the origin in $\mathbb{R}^{n+k}$, i.e.

$$\operatorname{bar}(S_j') := \frac{1}{\mathbf{M}(S_j')}\int z\, d\|S_j'\| = 0$$

holds true for all $j \in \mathbb{N}$. First of all the barycenter of S_j' is well defined since $\operatorname{spt} S_j' \subset \overline{B}_{R_o}$. Moreover, since $\|S_j'\| \to \|\partial[\![D]\!]\| = \mathcal{H}^{n-1}\llcorner(S^{n-1}\times\{0\})$ as $j\to\infty$ in the sense of Radon measures we have

$$\int z\, d\|S_j'\| \to \int z\, d\mathcal{H}^{n-1} = 0 \quad \text{and} \quad \mathbf{M}(S_j') \to n\omega_n$$

in the limit $j \to \infty$, and this implies $\operatorname{bar}(S_j') \to 0$. Therefore we can replace S_j' by $S_j'' := S_j' - \operatorname{bar}(S_j')$. The new sequence now fulfills the barycenter condition $\operatorname{bar}(S_j'') = 0$ and also $\|S_j''\| \to \|\partial[\![D]\!]\| = \mathcal{H}^{n-1}\llcorner(S^{n-1}\times\{0\})$. Finally, the currents S_j'' have support in $\overline{B}_{R_o}$ and satisfy (6.26), that is we have

$$\mathbf{M}(S_j'') - n\omega_n \le 12n\omega_n C_1\,\mathbf{d}_1^2(S_j''). \tag{6.27}$$

Step 8: Adjusting the mixed second order moments. We define the *second order moments* of S_j'' by

$$M_{S_j''} := \omega_n^{-1}\int z\otimes z\, d\|S_j''\|.$$

Note that $M_{S_j''}$ is well defined since $\operatorname{spt} S_j'' \subset \overline{B}_{R_o}$. Since $\|S_j''\| \to \|\partial[\![D]\!]\|$ in the sense of Radon measures the second order moments of S_j'' converge to the second order moments of the unit sphere $S^{n-1} \times \{0\}$, i.e.

$$\lim_{j\to\infty} M_{S_j''} = M_{S^{n-1}\times\{0\}} := \omega_n^{-1} \int_{S^{n-1}\times\{0\}} z \otimes z \, d\mathcal{H}^{n-1} = \mathbb{I}_n,$$

where $\mathbb{I}_n \colon \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ is defined by $\mathbb{I}_n(x, y) := (x, 0)$. Therefore, we have

$$\lim_{j\to\infty} \|M_{S_j''} - \mathbb{I}_n\| = 0,$$

and this allows us to apply Lemma 4.2 for $j \in \mathbb{N}$ large enough, to be precise for those j for which $\|M_{S_j''} - \mathbb{I}_n\| < \varepsilon_o$ holds true, where $\varepsilon_o = \varepsilon_o(n, k) > 0$ is the constant from Lemma 4.2. Hence, we find $R_j \in S\mathbb{O}(n + k)$ satisfying

$$\|R_j - \mathbb{I}\| \le c(n, k)\|M_{S_j''} - \mathbb{I}_n\|$$

such that for the second order moments of the tilted currents $S_j''' := (R_j^{-1})_\# S_j''$, i.e. for

$$M_{S_j'''} := \omega_n^{-1} \int z \otimes z \, d\|S_j'''\|,$$

the mixed moments are zero, i.e. for $i = 1, \dots, n$ and $\alpha = 1, \dots, k$ we have

$$\big(M_{S_j'''}\big)_{i,n+\alpha} \equiv \int x_i y_\alpha \, d\|S_j'''\| = 0,$$

and moreover

$$\|M_{S_j'''} - \mathbb{I}_n\| \le c(n, k)\|M_{S_j''} - \mathbb{I}_n\|.$$

The tilted currents are of course again $(\mathbf{M}, \omega)$-minimizing and, since $S_j'' \to \|\partial[\![D]\!]\|$, they also converge in the sense of Radon measures to $\|\partial[\![D]\!]\|$. Moreover, the barycenter condition also holds true for the tilted currents. Furthermore, we have $\mathbf{M}(S_j''') = \mathbf{M}((R_j^{-1})_\# S_j'') = \mathbf{M}(S_j'')$ and $\mathbf{m}(S_j''') = \mathbf{m}((R_j^{-1})_\# S_j'') = \mathbf{m}(S_j'')$. Since $\mathbf{d}_1$ is invariant by rotations, we have $\mathbf{d}_1(S_j''') = \mathbf{d}_1(S_j'')$. But this shows, that also (6.27) holds true for the tilted currents S_j''', that is we have

$$\mathbf{M}(S_j''') - n\omega_n \le 12n\omega_n C_1 \, \mathbf{d}_1^2(S_j'''). \tag{6.28}$$

To avoid an overburdened notation, from now on we write S_j instead of S_j''', but we keep in mind that $S_j \to \partial[\![D]\!]$ in the $\mathbf{F}_{\mathrm{loc}}$-topology, weakly as currents and in the Kuratowski convergence and $\|S_j\| \to \|\partial[\![D]\!]\|$ in the sense of Radon measures. Further, the S_j are $(\mathbf{M}, \omega)$-minimizing in the sense that (6.24) holds true for S_j. For the associated mass minimizing currents we have $\mathbf{m}(S_j) = \omega_n$.

Step 9: Regularity and conclusion. We recall that the flat n-dimensional unit disk $[\![D]\!]$ is the closed unit disk centered at the origin in $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+k}$. We write $[\![S^{n-1}]\!]$ for the boundary of $\mathbf{E}^n \llcorner B_1^n(0)$. At this stage we apply the regularity theorem to our sequence S_j which is build up by $(\mathbf{M}, \omega)$-minimizing currents for the modulus $\omega(\varrho) = 8\lambda\varrho$ (meaning that we have $C_\omega = 8\lambda = 8(C_1 + \Lambda)$ in Theorem 6.2). The application of Theorem 6.2 yields for j large enough spherical graph representations $S_j = (X_j)_\# [\![S^{n-1}]\!]$ with maps u_j, v_j on S^{n-1} of class $C^{1,\frac{1}{2}}$. The supports $\Gamma_j := \operatorname{spt} S_j = X(S^{n-1})$ are $C^{1,\frac{1}{2}}$ submanifolds of $\mathbb{R}^{n+k}$. Since the currents S_j fulfill the barycenter condition and have vanishing mixed second order moments also the spherical graphs Γ_j have their barycenter in the origin and vanishing mixed second order moments. By construction also the mass constaint $\mathbf{m}(S_j) = \mathbf{m}([\![\Gamma_j]\!]) = \omega_n$ is satisfied. Finally, by (6.15) and $\mathbf{D}(S_j) \to 0$ we can apply the higher codimension version of the Fuglede's Theorem for spherical graphs for j large enough, i.e. Theorem 4.1 is applicable since all hypotheses hold true. Thus we have

$$\frac{\mathbf{M}(S_j) - n\omega_n}{n\omega_n} \equiv \frac{\mathcal{H}^{n-1}(\Gamma_j) - n\omega_n}{n\omega_n} \ge C_o \, \mathbf{d}_1^2([\![\Gamma_j]\!]) \equiv C_o \, \mathbf{d}_1^2(S_j).$$

But this contradicts (6.28), provided we choose $0 < C_1 < \frac{1}{3}C_o$. Here, we used $\mathcal{H}^{n-1}(\Gamma_j) = \mathbf{M}(S_j)$ and $\mathbf{d}_1([\![\Gamma_j]\!]) = \mathbf{d}_1(S_j)$, since $S_j = X_\#[\![S^{n-1}]\!]$. This is the contradiction we were looking for and therefore finishes the proof of Theorem 2.1. $\square$

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