FLOW BY MEAN CURVATURE INSIDE A MOVING AMBIENT SPACE

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ABSTRACT. We show some computations related to the motion by mean curvature flow of a submanifold inside an ambient Riemannian manifold evolving by Ricci or backward Ricci flow. Special emphasis is given to the possible generalization of Huisken's monotonicity formula and its connection with the validity of some Li–Yau–Hamilton differential Harnack–type inequalities in a moving Riemannian manifold.

1. INTRODUCTION

In this paper we present some computations concerning the mean curvature flow of a submanifold inside a moving Riemannian manifold. We are particularly interested in finding analogues of Huisken's monotonicity formula. We will see that in some special situations, notably, when the ambient is a *gradient Ricci soliton*, such a monotonicity actually holds (see Section 3). We will analyze in detail the cases when the ambient Riemannian manifold evolves by Ricci or backward Ricci flow and we will discuss the connection between the monotonicity of Huisken's integral and the validity of some Li–Yau–Hamilton differential Harnack–type inequalities in the moving manifold.

Some of the computations here are mentioned by Ni [11]. A very closely related paper is the one by Lott [9]. Moreover, the work of Ecker [3] and the discussion in Section 3.10, Chapter 11 of the book by Chow, Lu and Ni [2] also deal with the subject of coupling the Ricci flow with the mean curvature flow.

We recall the fundamental Huisken's monotonicity formula (see [7]) for the mean curvature flow (from now on MCF) in the Euclidean space.
Let us assume that we have a smooth, compact, n–dimensional submanifold N immersed in $\mathbb{R}^m$ evolving by the MCF. That is, the flow is described by a smooth map $\varphi : N\times[0,T)\to\mathbb{R}^m$ with

$$\partial_t\varphi(p,t) = \mathrm{H}(p,t)$$

where the map $\varphi_t = \varphi(\cdot,t) : N\to\mathbb{R}^m$ is an immersion for every $t\in[0,T)$. Here $\mathrm{H}(p,t)$ is the vector valued mean curvature of the submanifold at time t and point p.
The immersion φ_t induces (by pull–back of the standard scalar product of $\mathbb{R}^m$) a metric h_t on N at every time t, turning (N, h_t) into a Riemannian manifold with a canonically associated Riemannian volume measure μ_t.
We then consider the *backward* heat kernel $\rho_{x_0,T}(x,t)$ on $\mathbb{R}^m$ centered at some point $x_0\in\mathbb{R}^m$ and with maximal time $T>0$, that is,

$$\rho_{x_0,T}(x,t) = \frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{m/2}}\,.$$

Theorem 1.1 (Huisken's Monotonicity Formula [7])**.** *For every $x_0\in\mathbb{R}^m$ and $T>0$, there holds*

$$\begin{aligned}\frac{d}{dt}\left\{[4\pi(T-t)]^{\frac{m-n}{2}}\int_N\rho_{x_0,T}\,d\mu_t\right\} &= \frac{d}{dt}\int_N\frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\,d\mu_t \qquad (1.1)\\ &= -\int_N\left|\mathrm{H}+\frac{(x-x_0)^\perp}{2(T-t)}\right|^2\frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\,d\mu_t\\ &= -\,[4\pi(T-t)]^{\frac{m-n}{2}}\int_N\left|\mathrm{H}-\nabla^\perp\log\rho_{x_0,T}\right|^2\,\rho_{x_0,T}\,d\mu_t\end{aligned}$$

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in the time interval $[0,T)$, where $\nabla^\perp$ denotes the projection on the normal space to N of the gradient in $\mathbb{R}^m$ of a function.
Hence, the integral $\int_N \frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\,d\mu_t$ is nonincreasing during the flow in $[0,T)$.

Following Hamilton [6], we can consider, more generally, the flow by mean curvature of an n–dimensional, smooth, compact submanifold N of a Riemannian manifold (M,g) in a time interval $[0,T)$ and a positive solution $u : M\times[0,T)\to\mathbb{R}$ of the backward heat equation $u_t = -\Delta^M u$ in the "ambient" space.
Making use of the formula

$$\Delta^N u = \Delta^M u - g^{\alpha\beta}\nabla^2_{\alpha\beta}u + \langle\nabla^M u\,|\,\mathrm{H}\rangle\,,$$

where we denoted the "normal" indices with Greek letters (this means that the intermediate term on the right hand side is the "trace" of the 2–form $\nabla^2 u$, restricted only to the normal space to N), we compute

$$\begin{aligned}
&\frac{d}{dt}\left\{[4\pi\,(T-t)]^{\frac{m-n}{2}}\int_N u\,d\mu_t\right\}\\
&\quad= -\,[4\pi(T-t)]^{\frac{m-n}{2}}\int_N\Big(-u_t + |\mathrm{H}|^2u - \langle\nabla^M u\,|\,\mathrm{H}\rangle + \frac{(m-n)}{2(T-t)}u\Big)\,d\mu_t\\
&\quad= -\,[4\pi(T-t)]^{\frac{m-n}{2}}\int_N\Big(\Delta^M u + |\mathrm{H}|^2u - \langle\nabla^M u\,|\,\mathrm{H}\rangle + \frac{(m-n)}{2(T-t)}u\Big)\,d\mu_t\\
&\quad= -\,[4\pi(T-t)]^{\frac{m-n}{2}}\int_N\Big(\Delta^N u + g^{\alpha\beta}\nabla^2_{\alpha\beta}u + |\mathrm{H}|^2u - 2\langle\nabla^M u\,|\,\mathrm{H}\rangle + \frac{(m-n)}{2(T-t)}u\Big)\,d\mu_t\,.
\end{aligned}$$

As the integral of $\Delta^N u$ is zero and, "completing the square" by adding and subtracting the term $\frac{|\nabla^\perp u|^2}{u}$ inside the integral, we get the formula

$$\begin{aligned}
\frac{d}{dt}\left\{[4\pi(T-t)]^{\frac{m-n}{2}}\int_N u\,d\mu_t\right\} = &-\,[4\pi(T-t)]^{\frac{m-n}{2}}\int_N|\mathrm{H}-\nabla^\perp\log u|^2u\,d\mu_t\\
&-[4\pi(T-t)]^{\frac{m-n}{2}}\int_N\Big(\nabla^2_{\alpha\beta}\log u + \frac{g_{\alpha\beta}}{2(T-t)}\Big)g^{\alpha\beta}u\,d\mu_t\,,
\end{aligned}$$

for every $t\in[0,T)$, where $\nabla^\perp$ denotes the projection on the normal space to N of the gradient in M of a function.

Remark 1.2. In the special case of $M=\mathbb{R}^m$ and u equal to the backward heat kernel $\rho_{x_0,T}$, the last term vanishes because of the special choice of u and we have the "classical" Huisken's monotonicity formula.

The right hand side of this formula consists of a nonpositive quantity (minus the integral of a perfect square times u, which is positive) and a term which could be nonpositive in case the two form $\nabla^2\log u + \frac{g}{2(T-t)}$ were nonnegative definite.

Setting $v(p,s)=u(p,T-s)$, the function $v : M\times(0,T]\to\mathbb{R}$ is a positive solution of the standard *forward* heat equation on (M,g) and setting $t=T-s$, we have $\nabla^2\log u + \frac{g}{2(T-t)} = \nabla^2\log v + \frac{g}{2s}$. In particular, its trace (the standard *full* trace) is given by $\Delta^M\log v + \frac{m}{2s}$ which is exactly the Li–Yau quantity for positive solutions of the heat equation on a compact manifold (M,g). Actually, in the paper [8], Li and Yau showed that if the Ricci tensor of M is nonnegative, then the differential Harnack inequality $\Delta^M\log v + \frac{m}{2(T-t)}\geq 0$ holds. In the spirit of this result, in [5] Hamilton (see also [10]) generalized this inequality to a matrix version, showing that under the assumptions that (M,g) has parallel Ricci tensor ($\nabla\mathrm{Ric}=0$) and nonnegative sectional curvatures, the 2–form $\nabla^2\log v + g/(2s)$ is nonnegative definite (*Hamilton's matrix Li–Yau Harnack differential inequality*).

As a consequence, under these hypotheses the two form

$$\nabla^2\log u + \frac{g}{2(T-t)} = \nabla^2\log v + \frac{g}{2s}$$

is nonnegative definite and we get Hamilton's generalization of Huisken's monotonicity formula.

Theorem 1.3 (Huisken's Monotonicity Formula – Hamilton's Extension [6]). *A smooth, compact, n–dimensional submanifold N of a Riemannian manifold (M, g) moves by mean curvature in the time interval $[0, T)$ and $u : M \times [0, T) \to \mathbb{R}$ is a positive smooth solution of the backward heat equation $u_t = -\Delta^M u$.*
Then, if the manifold (M, g) has nonnegative sectional curvatures and satisfies $\nabla^M \mathrm{Ric} = 0$ the quantity $[4\pi(T-t)]^{\frac{m-n}{2}} \int_N u\, d\mu_t$ is nonincreasing during the flow in $[0, T)$.

Remark 1.4. All this discussion in the static ambient situation provides a first example of the connection of the monotonicity of the "coupled" integral $[4\pi(T-t)]^{\frac{m-n}{2}} \int_N u\, d\mu_t$ with the validity of a Li–Yau–Hamilton Harnack differential inequality.

2. Moving Ambient Spaces

Let us now assume that the metric of the ambient space evolves according to $\partial_t g = -2\mathrm{Q}$ (if $\mathrm{Q} = \mathrm{Ric}$ we have the Ricci flow) and modify the backward heat equation as follows

$$u_t = -\Delta^M u + \mathrm{K}u$$

for some function K.
If we repeat the previous computations in this new setting, we get two extra terms. The first comes from the modified equation for u and the second from the effect of the motion of the ambient space on the time derivative of the measure μ_t induced on N. Indeed, the associated metric h_t on N is affected not just by the motion of the submanifold but also by the evolution of the ambient metric $g(t)$ on M. After some computations, we have

$$\frac{d}{dt}\mu_t = (-\mathrm{H}^2 - g^{ij}\mathrm{Q}_{ij})\mu_t = (-\mathrm{H}^2 - \mathrm{tr}\,\mathrm{Q} + g^{\alpha\beta}\mathrm{Q}_{\alpha\beta})\mu_t$$

where, as before (and in the rest of the paper), the Greek letters $\alpha, \beta, \ldots$ denote the indices associated to the coordinates which are normal to N and with $i, j, k, \ldots$ the indices for the coordinates on N.
With this notation, we get

$$\begin{aligned}
\frac{d}{dt}\Big(\tau^{\frac{m-n}{2}} \int_N u\, d\mu_t\Big) &= -\tau^{\frac{m-n}{2}} \int_N \Big|\mathrm{H} - \frac{\nabla^\perp u}{u}\Big|^2 u\, d\mu_t \\
&\quad - \tau^{\frac{m-n}{2}} \int_N \Big(\frac{\nabla^2_{\alpha\beta}u}{u} - \frac{\nabla_\alpha u \nabla_\beta u}{u^2} + \frac{g_{\alpha\beta}}{2\tau}\Big) g^{\alpha\beta} u\, d\mu_t \\
&\quad + \tau^{\frac{m-n}{2}} \int_N (\mathrm{K} - \mathrm{tr}\,\mathrm{Q} + g^{\alpha\beta}\mathrm{Q}_{\alpha\beta})\, u\, d\mu_t \\
&= -\tau^{\frac{m-n}{2}} \int_N |\mathrm{H} + \nabla^\perp f|^2 e^{-f}\, d\mu_t \\
&\quad + \tau^{\frac{m-n}{2}} \int_N \Big(\nabla^2_{\alpha\beta} f + \mathrm{Q}_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau}\Big) g^{\alpha\beta} e^{-f}\, d\mu_t \\
&\quad + \tau^{\frac{m-n}{2}} \int_N (\mathrm{K} - \mathrm{tr}\,\mathrm{Q})\, e^{-f}\, d\mu_t\,,
\end{aligned}$$

where we substituted $\tau = T - t$ and $f = -\log u$, hence, $f_t = -\Delta^M f + |\nabla f|^2 - \mathrm{K}$.

We will concentrate on the following situations: $\mathrm{Q} = \mathrm{Ric}$ or $\mathrm{Q} = -\mathrm{Ric}$, that is, the metric g on M evolves either by the Ricci flow or by the backward Ricci flow and we will choose $\mathrm{K} = 0$ or $\mathrm{K} = \mathrm{tr}\,\mathrm{Q}$. In this latter case the last term in the formula above clearly vanishes and we obtain

$$\begin{aligned}
\frac{d}{dt}\Big(\tau^{\frac{m-n}{2}} \int_N u\, d\mu_t\Big) &= -\tau^{\frac{m-n}{2}} \int_N |\mathrm{H} + \nabla^\perp f|^2 e^{-f}\, d\mu_t \\
&\quad + \tau^{\frac{m-n}{2}} \int_N \Big(\nabla^2_{\alpha\beta} f + \mathrm{Q}_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau}\Big) g^{\alpha\beta} e^{-f}\, d\mu_t\,.
\end{aligned}$$

Moreover, notice that with the choice $\mathrm{K} = \mathrm{tr}\,\mathrm{Q}$, we also have

$$\frac{d}{dt}\int_M u = \int_M (u_t - \mathrm{tr}\,\mathrm{Q}u) = \int_M -\Delta^M u = 0\,,$$

when M is compact, hence the "ambient" integral $\int_M u = \int_M e^{-f}$ is constant during the flow.

A family of metrics $g(t)$ on a manifold M for $t \in [0, T)$, evolves by the *Ricci flow* if $\partial_t g = -2\mathrm{Ric}_{g(t)}$. Moreover, we say that $g(t)$ evolves by the *backward Ricci flow* if $\partial_t g = 2\mathrm{Ric}_{g(t)}$ (in the following the subscript $g(t)$ will be always understood).
Under the Ricci flow, the Christoffel symbols of the evolving Levi–Civita connection, the Ricci tensor and the scalar curvature evolve according to

$$\begin{aligned}\partial_t \Gamma_{ij}^k &= -g^{kl}(\nabla_i \mathrm{R}_{jl} + \nabla_j \mathrm{R}_{il} - \nabla_l \mathrm{R}_{ij})\,,\\ \partial_t \mathrm{R}_{ij} &= \Delta \mathrm{R}_{ij} + 2\mathrm{R}^{pq}\mathrm{R}_{ipjq} - 2g^{pq}\mathrm{R}_{ip}\mathrm{R}_{qj}\,,\\ \partial_t \mathrm{R} &= \Delta \mathrm{R} + 2|\mathrm{Ric}|^2\,.\end{aligned}$$

The analogous evolution equations for the *backward* Ricci flow (inverting the time direction) are simply the same with a minus sign in front of the right hand sides.

Here R_{ijkl} are the components of the $(4, 0)$–Riemann tensor Riem (with the convention that for the standard sphere $\mathbb{S}^n$ we have $\mathrm{Riem}(v, w, v, w) > 0$), Ric is the Ricci tensor with components $\mathrm{R}_{ik} = g^{jl}\mathrm{R}_{ijkl}$ and finally $\mathrm{R} = g^{ik}\mathrm{R}_{ik}$ is the scalar curvature.

2.1. **$\mathbf{RF_0}$ – Ricci Flow and** $\mathrm{K} = 0$**.** We assume $\partial_t g = -2\mathrm{Ric}$ and $u_t = -\Delta^M u$, then

$$\begin{aligned}\frac{d}{dt}\Big(\tau^{\frac{m-n}{2}}\int_N u\,d\mu_t\Big) =& -\tau^{\frac{m-n}{2}}\int_N |\mathrm{H} + \nabla^\perp f|^2 e^{-f}\,d\mu_t\\ &+\tau^{\frac{m-n}{2}}\int_N \Big(\nabla^2_{\alpha\beta} f + \mathrm{R}_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau}\Big)g^{\alpha\beta}e^{-f}\,d\mu_t\\ &-\tau^{\frac{m-n}{2}}\int_N \mathrm{R}\,e^{-f}\,d\mu_t\\ =& -\tau^{\frac{m-n}{2}}\int_N |\mathrm{H} + \nabla^\perp f|^2 e^{-f}\,d\mu_t\\ &+\tau^{\frac{m-n}{2}}\int_N \Big(\nabla^2_{\alpha\beta} f + \mathrm{R}_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau} - \frac{\mathrm{R}g_{\alpha\beta}}{m-n}\Big)g^{\alpha\beta}e^{-f}\,d\mu_t\,,\end{aligned}$$

with $f = -\log u$, hence, $f_t = -\Delta^M f + |\nabla f|^2$.

2.2. **$\mathbf{BRF_0}$ – Back–Ricci Flow and** $\mathrm{K} = 0$**.** We assume $\partial_t g = 2\mathrm{Ric}$ and $u_t = -\Delta^M u$, then

$$\begin{aligned}\frac{d}{dt}\Big(\tau^{\frac{m-n}{2}}\int_N u\,d\mu_t\Big) =& -\tau^{\frac{m-n}{2}}\int_N |\mathrm{H} + \nabla^\perp f|^2 e^{-f}\,d\mu_t\\ &+\tau^{\frac{m-n}{2}}\int_N \Big(\nabla^2_{\alpha\beta} f - \mathrm{R}_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau}\Big)g^{\alpha\beta}e^{-f}\,d\mu_t\\ &+\tau^{\frac{m-n}{2}}\int_N \mathrm{R}\,e^{-f}\,d\mu_t\\ =& -\tau^{\frac{m-n}{2}}\int_N |\mathrm{H} + \nabla^\perp f|^2 e^{-f}\,d\mu_t\\ &+\tau^{\frac{m-n}{2}}\int_N \Big(\nabla^2_{\alpha\beta} f - \mathrm{R}_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau} + \frac{\mathrm{R}g_{\alpha\beta}}{m-n}\Big)g^{\alpha\beta}e^{-f}\,d\mu_t\,,\end{aligned}$$

with $f = -\log u$, hence, $f_t = -\Delta^M f + |\nabla f|^2$.

2.3. **RF – Ricci Flow and** $\mathrm{K} = \mathrm{tr}\,\mathrm{Q} = \mathrm{R}$**.** We assume $\partial_t g = -2\mathrm{Ric}$ and $u_t = -\Delta^M u + \mathrm{R}u$, then

$$\begin{aligned}\frac{d}{dt}\Big(\tau^{\frac{m-n}{2}}\int_N u\,d\mu_t\Big) =& -\tau^{\frac{m-n}{2}}\int_N |\mathrm{H} + \nabla^\perp f|^2 e^{-f}\,d\mu_t\\ &+\tau^{\frac{m-n}{2}}\int_N \Big(\nabla^2_{\alpha\beta} f + \mathrm{R}_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau}\Big)g^{\alpha\beta}e^{-f}\,d\mu_t\,,\end{aligned}$$

with $f = -\log u$, hence $f_t = -\Delta^M f + |\nabla f|^2 - \mathrm{R}$.

Monotonicity of $\tau^{\frac{m-n}{2}}\int_N u\,d\mu_t$ is then related to the nonpositivity of the Li–Yau–Hamilton quantity

$$\Big(\nabla^2_{\alpha\beta} f + \mathrm{R}_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau}\Big)g^{\alpha\beta}\,.$$

Notice that the same conclusion holds also if $u_t \leq -\Delta^M u + \mathrm{R}u$.

We emphasize that in the $\mathbf{RF_0}$ case, the same nonpositivity property clearly implies the monotonicity when R is always nonnegative.

2.4. **BRF – Back–Ricci Flow and** $\mathrm{K} = \operatorname{tr}\mathrm{Q} = -\mathrm{R}$**.** We assume $\partial_t g = 2\mathrm{Ric}$ and $u_t = -\Delta^M u - \mathrm{R}u$, then

$$\frac{d}{dt}\Big(\tau^{\frac{m-n}{2}}\int_N u\,d\mu_t\Big) = -\,\tau^{\frac{m-n}{2}}\int_N |\mathrm{H}+\nabla^\perp f|^2 e^{-f}\,d\mu_t + \tau^{\frac{m-n}{2}}\int_N \Big(\nabla^2_{\alpha\beta}f - \mathrm{R}_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau}\Big)g^{\alpha\beta}e^{-f}\,d\mu_t\,,$$

with $f = -\log u$, hence $f_t = -\Delta^M f + |\nabla f|^2 + \mathrm{R}$.

Monotonicity of $\tau^{\frac{m-n}{2}}\int_N u\,d\mu_t$ is then related to the nonpositivity of

$$\Big(\nabla^2_{\alpha\beta}f - \mathrm{R}_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau}\Big)g^{\alpha\beta}\,.$$

Notice that the same conclusion holds also if $u_t \leq -\Delta^M u - \mathrm{R}u$.

3. Ricci Solitons

We choose now $\mathrm{Q} = \mathrm{Ric}$, that is, the metric g on M evolves by the Ricci flow in some time interval $I \subset \mathbb{R}$ and we set $\mathrm{K} = \mathrm{R}$ to be the scalar curvature of (M, g).
By the previous computations in the **RF** case, we get

$$\frac{d}{dt}\Big(\tau^{\frac{m-n}{2}}\int_N u\,d\mu_t\Big) = -\,\tau^{\frac{m-n}{2}}\int_N |\mathrm{H}+\nabla^\perp f|^2 e^{-f}\,d\mu_t + \tau^{\frac{m-n}{2}}\int_N \Big(\nabla^2_{\alpha\beta}f + \mathrm{R}_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau}\Big)g^{\alpha\beta}e^{-f}\,d\mu_t\,, \tag{3.1}$$

for a positive solution of the conjugate heat equation

$$u_t = -\Delta^M u + \mathrm{R}u \tag{3.2}$$

and $f = -\log u$, $\tau = T - t$, for $t \in I$ with $t < T \in \mathbb{R}$.

Let us assume that $(M, g(t))$ is a gradient soliton (a self–similar solution) of Ricci flow and $F : M \times I \to \mathbb{R}$ its "potential" function, namely,

- *Shrinking Soliton:* the flow is defined on $I = (-\infty, T_{\max})$, the metric g and the function F satisfy $\nabla^2 F + \mathrm{Ric} = g/2(T_{\max} - t)$.
- *Steady Soliton:* the flow is "eternal", $I = \mathbb{R}$, the metric g and the function F satisfy $\nabla^2 F + \mathrm{Ric} = 0$.
- *Expanding Soliton:* the flow is defined on $I = (T_{\min}, +\infty)$, the metric g and the function F satisfy $\nabla^2 F + \mathrm{Ric} = g/2(T_{\min} - t)$.

Then we analyze these three situations separately.

- Shrinking Solitons: It can be seen that the function $u = e^{-F}/(T_{\max} - t)^{m/2}$ satisfies the conjugate heat equation (3.2) (see [10, Section 1.5], for instance, for this and the next cases). Then, letting $f = -\log u = F + \frac{m}{2}\log(T_{\max} - t)$ and substituting inside equation (3.1), we get

$$\frac{d}{dt}\Big(\tau^{\frac{m-n}{2}}\int_N \frac{e^{-F}}{(T_{\max}-t)^{m/2}}\,d\mu_t\Big) = -\,\frac{(T-t)^{\frac{m-n}{2}}}{(T_{\max}-t)^{m/2}}\int_N |\mathrm{H}+\nabla^\perp F|^2 e^{-F}\,d\mu_t + \frac{(T-t)^{\frac{m-n}{2}}}{(T_{\max}-t)^{m/2}}\int_N \frac{m-n}{2}\Big(\frac{1}{T_{\max}-t} - \frac{1}{T-t}\Big)\,e^{-F}\,d\mu_t\,,$$

which is nonpositive for every $t \in (-\infty, \min\{T, T_{\max}\})$, if $T \leq T_{\max}$.
Actually, the right side is always negative if $T < T_{\max}$ and in the particular case of $T = T_{\max}$, we have the neat formula

$$\frac{d}{dt}\int_N \frac{e^{-F}}{(T_{\max}-t)^{n/2}}\,d\mu_t = -\int_N |\mathrm{H}+\nabla^\perp F|^2\frac{e^{-F}}{(T_{\max}-t)^{n/2}}\,d\mu_t \leq 0\,,$$

with equality if and only if the submanifold N satisfies $\mathrm{H} + \nabla^\perp F = 0$ at every point, for some time t.

An almost trivial example of this situation is a "static" maximal sphere $\mathbb{S}^n$ in the sphere $\mathbb{S}^m$ evolving by Ricci flow. Indeed, this latter "generates" a gradient,

shrinking Ricci soliton with a constant in space potential function F and the maximal sphere $\mathbb{S}^n$ satisfies $\mathrm{H} = 0$.
Another example is given by the flat $\mathbb{R}^m$ with potential function $F(x,t) = \frac{|x-x_0|^2}{4(T_{\max}-t)}$ which is called the *Gaussian* shrinking soliton, for some $x_0 \in \mathbb{R}^m$. Substituting in the last equation above, one recovers the "classical" Huisken's monotonicity formula (1.1).
Notice anyway that the family of cylinders $(\mathbb{S}^2 \times \mathbb{R}, g(t))$ with the evolving metric $g(t) = -2t(g^{\mathbb{S}^2}_{\mathrm{can}} + dr^2)$ in the halfline $t \in (-\infty, 0)$, is a gradient, shrinking, Ricci soliton with $T_{\max} = 0$ and potential function $F : \mathbb{S}^2 \times \mathbb{R} \times (-\infty, 0) \to \mathbb{R}$ given by $F(\theta, r, t) = -\frac{(r-r_0)^2}{4t}$, for some $r_0 \in \mathbb{R}$. Any 2–sphere $\mathbb{S}^2 \times \{\overline{r}\}$ inside $\mathbb{S}^2 \times \mathbb{R}$ is actually "static" during its flow by mean curvature, since its second fundamental form (hence, its mean curvature) is zero, but the Huisken's integral is not constant, unless $\overline{r} = r_0$ (it holds only for a single 2–sphere of the whole family fibering the cylinder). This follows easily as the vector $\nabla^\perp F = -\frac{(r-r_0)}{2t}\partial_r$ must be zero in such case.

- Steady Solitons: The function $u = e^{-F}$ satisfies the conjugate heat equation (3.2) hence, letting $f = -\log u = F$ in equation (3.1) we have

$$\frac{d}{dt}\Big(\tau^{\frac{m-n}{2}}\int_N e^{-F}\,d\mu_t\Big) = -\tau^{\frac{m-n}{2}}\int_N |\mathrm{H} + \nabla^\perp F|^2 e^{-F}\,d\mu_t - \tau^{\frac{m-n-2}{2}}\frac{m-n}{2}\int_N e^{-F}\,d\mu_t\,,$$

which is always negative for every $t \in (-\infty, T)$.
Notice that in this case, it follows

$$\frac{d}{dt}\int_N e^{-F}\,d\mu_t = -\int_N |\mathrm{H} + \nabla^\perp F|^2 e^{-F}\,d\mu_t\,,$$

for every $t \in \mathbb{R}$.

- Expanding Solitons: In this case the function $u = e^{-F}/(t - T_{\min})^{m/2}$ satisfies the conjugate heat equation (3.2), then, letting $f = -\log u = F + \frac{m}{2}\log(t - T_{\min})$ and substituting inside equation (3.1), we get

$$\begin{aligned}\frac{d}{dt}\Big(\tau^{\frac{m-n}{2}}\int_N \frac{e^{-F}}{(t-T_{\min})^{m/2}}\,d\mu_t\Big) = &-\frac{(T-t)^{\frac{m-n}{2}}}{(t-T_{\min})^{m/2}}\int_N |\mathrm{H} + \nabla^\perp F|^2 e^{-F}\,d\mu_t\\ &+\frac{(T-t)^{\frac{m-n}{2}}}{(t-T_{\min})^{m/2}}\int_N \frac{m-n}{2}\Big(\frac{1}{T_{\min}-t} - \frac{1}{T-t}\Big)e^{-F}\,d\mu_t\,,\end{aligned}$$

which is always negative for every $t \in (T_{\min}, T)$ (notice that in this case $T \leq T_{\min}$ has no meaning).

Proposition 3.1. *If $(M, g(t))$ is an m–dimensional, shrinking, gradient Ricci soliton in the interval $(-\infty, T_{\max})$ and F its potential function, then, the Huisken's integral $\tau^{\frac{m-n}{2}}\int_N u\,d\mu_t$, with $u = e^{-F}/(T_{\max} - t)^{m/2}$, $\tau = T - t$ and $T \leq T_{\max}$, of an n–dimensional submanifold N moving by mean curvature inside $(M, g(t))$ is monotone nonincreasing for every $t \in (-\infty, T)$.*
It is actually monotone decreasing, unless $T = T_{\max}$ and at some time the submanifold N satisfies $\mathrm{H} + \nabla^\perp F = 0$ at every point.
If $(M, g(t))$ is an m–dimensional steady or expanding, gradient Ricci soliton with potential function F in the interval $(T_{\min}, +\infty)$, then, the Huisken's integral $\tau^{\frac{m-n}{2}}\int_N u\,d\mu_t$, with $u = e^{-F}$ or $u = e^{-F}/(t - T_{\min})^{m/2}$ respectively, $\tau = T - t$, $T > T_{\min}$ and N as above, is monotone decreasing for every $t \in (T_{\min}, T)$.
Moreover, in the steady case, the integral $\int_N e^{-F}\,d\mu_t$ in monotone nonincreasing for every $t \in \mathbb{R}$ and actually decreasing unless the submanifold N satisfies $\mathrm{H} + \nabla^\perp F = 0$ at every point.

4. Computations I – Ricci Flow and LYH Matrix Harnack Inequalities

In this section we will deal with the **RF** case, that is, we will assume that $(M, g(t))$ is an m–dimensional Riemannian manifold evolving by the Ricci flow $\partial_t g = -2\mathrm{Ric}$ and the smooth function $u : M \times [0, T) \to \mathbb{R}$ is a positive solution of $u_t = -\Delta u + \mathrm{R}u$. Under these assumptions, considering a compact n–submanifold N moving by mean curvature, we have seen that, setting $\tau = T - t$, the monotonicity of the Huisken's integral

$$\tau^{\frac{m-n}{2}} \int_N u \, d\mu_t$$

is implied by the nonpositivity of the expression

$$\Big(\nabla^2_{\alpha\beta} f + \mathrm{R}_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau}\Big) g^{\alpha\beta} \, ,$$

with $f = -\log u$ which hence satisfies $f_t = -\Delta f + |\nabla f|^2 - \mathrm{R}$. This would be a straightforward consequence of the nonpositivity (along the flow) of the full 2–form

$$\nabla^2_{ij} f + \mathrm{R}_{ij} - \frac{g_{ij}}{2\tau} \, ,$$

which is clearly a stronger property.
Equivalently, if we had chosen $f = \log u$, we would be interested in the nonnegativity of

$$\nabla^2_{ij} f - \mathrm{R}_{ij} + \frac{g_{ij}}{2\tau} \, , \tag{4.1}$$

for $f = \log u$ satisfying

$$f_t = -\Delta f - |\nabla f|^2 + \mathrm{R} \, ,$$

which is an analogue of Li–Yau–Hamilton differential matrix Harnack inequality in a moving ambient space.

We set $L_{ij} = \nabla^2_{ij} f - \mathrm{R}_{ij}$, $H_{ij} = \tau L_{ij} + g_{ij}/2 = \tau[\nabla^2_{ij} f - \mathrm{R}_{ij}] + g_{ij}/2$ and we compute the evolution equation of the form H, whose nonnegativity is trivially equivalent to the one of the form (4.1). In normal coordinates, using the following commutation rule between the Laplacian and the second covariant derivatives of a function $f : M \to \mathbb{R}$ that can be obtained interchanging repeatedly the covariant derivatives and using the II Bianchi identity

$$\begin{aligned}
\nabla^2_{ij} \Delta f - \Delta \nabla^2_{ij} f = & - (\nabla_i \mathrm{R}_{jk} + \nabla_j \mathrm{R}_{ik} - \nabla_k \mathrm{R}_{ij}) \nabla^k f \\
& - g^{pq} \mathrm{R}_{jp} \nabla^2_{qi} f - g^{pq} \mathrm{R}_{ip} \nabla^2_{qj} f + 2 g^{pr} g^{qs} \mathrm{R}_{ipjq} \nabla^2_{rs} f \, ,
\end{aligned}$$

we have

$$\begin{aligned}
(\partial_t + \Delta) H_{ij} = & - L_{ij} - \mathrm{R}_{ij} \\
& + \tau[\Delta \nabla^2_{ij} f + \nabla^2_{ij} f_t + (\nabla_i \mathrm{R}_{jk} + \nabla_j \mathrm{R}_{ik} - \nabla_k \mathrm{R}_{ij}) \nabla_k f] \\
& - \tau[\partial_t \mathrm{R}_{ij} + \Delta \mathrm{R}_{ij}] \\
= & - L_{ij} - \mathrm{R}_{ij} \\
& + \tau[\Delta \nabla^2_{ij} f - \nabla^2_{ij} \Delta f - \nabla^2_{ij} |\nabla f|^2 \\
& + (\nabla_i \mathrm{R}_{jk} + \nabla_j \mathrm{R}_{ik} - \nabla_k \mathrm{R}_{ij}) \nabla_k f] \\
& - \tau[2\Delta \mathrm{R}_{ij} + 2\mathrm{R}_{pq} \mathrm{R}_{ipjq} - 2\mathrm{R}_{ip} \mathrm{R}_{pj} - \nabla^2_{ij} \mathrm{R}] \\
= & - L_{ij} - \mathrm{R}_{ij} \\
& + \tau[(\nabla_i \mathrm{R}_{jk} + \nabla_j \mathrm{R}_{ik} - \nabla_k \mathrm{R}_{ij}) \nabla_k f \\
& + \mathrm{R}_{jp} \nabla^2_{ip} f + \mathrm{R}_{ip} \nabla^2_{pj} f + 2\mathrm{R}_{ikpj} \nabla^2_{kp} f \\
& - \nabla^2_{ij} |\nabla f|^2 + (\nabla_i \mathrm{R}_{jk} + \nabla_j \mathrm{R}_{ik} - \nabla_k \mathrm{R}_{ij}) \nabla_k f] \\
& - \tau[2\Delta \mathrm{R}_{ij} + 2\mathrm{R}_{pq} \mathrm{R}_{ipjq} - 2\mathrm{R}_{ip} \mathrm{R}_{pj} - \nabla^2_{ij} \mathrm{R}] \\
= & - L_{ij} - \mathrm{R}_{ij} \\
& + \tau[\mathrm{R}_{jp} \nabla^2_{ip} f + \mathrm{R}_{ip} \nabla^2_{pj} f + 2\mathrm{R}_{ikpj} \nabla^2_{kp} f \\
& - 2\nabla^2_{ip} f \nabla^2_{jp} f - 2\nabla^3_{ijk} f \nabla_k f + 2(\nabla_i \mathrm{R}_{jk} + \nabla_j \mathrm{R}_{ik} - \nabla_k \mathrm{R}_{ij}) \nabla_k f] \\
& - \tau[2\Delta \mathrm{R}_{ij} + 2\mathrm{R}_{pq} \mathrm{R}_{ipjq} - 2\mathrm{R}_{ip} \mathrm{R}_{pj} - \nabla^2_{ij} \mathrm{R}] \, .
\end{aligned}$$

Commuting the covariant derivatives of the term containing the third derivatives of f, that is, $\nabla^3_{ijk}f = \nabla^3_{kij}f + \mathrm{R}_{ikjp}\nabla_p f$, we get

$$\begin{aligned}(\partial_t + \Delta)H_{ij} = &- L_{ij} - \mathrm{R}_{ij}\\ &+ \tau[\mathrm{R}_{jp}\nabla^2_{ip}f + \mathrm{R}_{ip}\nabla^2_{pj}f - 2\nabla^2_{ip}f\nabla^2_{jp}f - 2\nabla^3_{kij}f\nabla_k f]\\ &- \tau[2\Delta\mathrm{R}_{ij} + 2\mathrm{R}_{pq}\mathrm{R}_{ipjq} - 2\mathrm{R}_{ip}\mathrm{R}_{pj} - \nabla^2_{ij}\mathrm{R}]\\ &+ \tau[2(\nabla_i\mathrm{R}_{jk} + \nabla_j\mathrm{R}_{ik} - \nabla_k\mathrm{R}_{ij})\nabla_k f\\ &- 2\mathrm{R}_{ikjp}\nabla^2_{pk}f - 2\mathrm{R}_{ikjp}\nabla_p f\nabla_k f]\,.\end{aligned}$$

Finally, substituting $L_{ij} = [H_{ij} - g_{ij}/2]/\tau$ and $\nabla^2_{ij}f = [H_{ij} - g_{ij}/2]/\tau + \mathrm{R}_{ij}$, we obtain

$$\begin{aligned}(\partial_t + \Delta)H_{ij} = &[H_{ij} - 2H^2_{ij}]/\tau - 2\nabla_k H_{ij}\nabla_k f\\ &- [\mathrm{R}_{ik}H_{jk} + \mathrm{R}_{jk}H_{ik} + 2\mathrm{R}_{ikjp}H_{pk}]\\ &- \tau[2\Delta\mathrm{R}_{ij} - 2\mathrm{R}_{jp}\mathrm{R}_{ip} + 4\mathrm{R}_{pq}\mathrm{R}_{ipjq} - \nabla^2_{ij}\mathrm{R} - \mathrm{R}_{ij}/\tau]\\ &+ \tau[2(\nabla_i\mathrm{R}_{jk} + \nabla_j\mathrm{R}_{ik} - 2\nabla_k\mathrm{R}_{ij})\nabla_k f]\\ &- 2\tau\mathrm{R}_{ikjp}\nabla_p f\nabla_k f\,,\end{aligned}$$

which in a generic coordinate system reads

$$\begin{aligned}(\partial_t + \Delta)H_{ij} = &[H_{ij} - 2H^2_{ij}]/\tau - 2\nabla_k H_{ij}\nabla^k f - g^{pq}\mathrm{R}_{ip}H_{jq} - g^{pq}\mathrm{R}_{jp}H_{iq} - 2\mathrm{R}_{ipjq}H^{pq}\\ &- \tau\Big[2\Delta\mathrm{R}_{ij} - \nabla^2_{ij}\mathrm{R} - 2g^{pq}\mathrm{R}_{ip}\mathrm{R}_{jq} + 4\mathrm{R}^{pq}\mathrm{R}_{ipjq} - \mathrm{R}_{ij}/\tau\\ &- 2(\nabla_i\mathrm{R}_{jk} + \nabla_j\mathrm{R}_{ik} - 2\nabla_k\mathrm{R}_{ij})\nabla^k f + 2\mathrm{R}_{ipjq}\nabla^p f\nabla^q f\Big]\\ = &[H_{ij} - 2H^2_{ij}]/\tau - 2\nabla_k H_{ij}\nabla^k f - \mathrm{R}^k_i H_{kj} - \mathrm{R}^k_j H_{ki} - 2\mathrm{R}_{ipjq}H^{pq} - \tau W_{ij}\,,\end{aligned}$$

where we set

$$\begin{aligned}W_{ij} = &2\Delta\mathrm{R}_{ij} - \nabla^2_{ij}\mathrm{R} - 2g^{pq}\mathrm{R}_{ip}\mathrm{R}_{jq} + 4\mathrm{R}^{pq}\mathrm{R}_{ipjq} - \mathrm{R}_{ij}/\tau\\ &- 2(\nabla_i\mathrm{R}_{jk} + \nabla_j\mathrm{R}_{ik} - 2\nabla_k\mathrm{R}_{ij})\nabla^k f + 2\mathrm{R}_{ipjq}\nabla^p f\nabla^q f\,.\end{aligned}$$

Notice that when $t > T$ (not our case) this form W is the Hamilton's Harnack quadratic, defined in [4], contracted with ∇f, (this term with the "wrong time" also appears in the computations about the *reduced length* in Perelman's paper [12]). This quantity vanishes on a shrinking, gradient Ricci soliton with $T = T_{\max}$ when f is equal to minus its potential function F, so sometimes it is called Hamilton's matrix Harnack quadratic *for shrinkers* (the original Hamilton's Harnack quadratic is instead zero on expanders).

Arguing as in Hamilton [5] by means of his matrix maximum principle, if there is a sequence $t_i \to T$ such that the form H is positive definite, the Riemann curvature operator and the form W are nonnegative definite in $M \times [0, T)$, then it follows that the form H is nonnegative definite in the whole $M \times [0, T)$ which is what we need to get the monotonicity of the Huisken's integral.

Unfortunately, the form W is not, in general, nonnegative definite for every function f, even if the Ricci flow is *ancient* and the Riemann curvature operator is nonnegative, in contrast to the nicely behaved original Hamilton's Harnack quadratic. Anyway, when the flow is a gradient, shrinking Ricci soliton with nonnegative Riemann operator and $\tau = T_{\max} - t$, the form W is nonnegative definite for every function f, indeed, there hold (by the soliton equation, see [2, Chapter 8, Section 5])

$$2\Delta\mathrm{R}_{ij} - \nabla^2_{ij}\mathrm{R} - 2g^{pq}\mathrm{R}_{ip}\mathrm{R}_{jq} + 4\mathrm{R}^{pq}\mathrm{R}_{ipjq} - \mathrm{R}_{ij}/\tau = 2(\nabla_k\mathrm{R}_{ij} - 2\nabla_j\mathrm{R}_{ik})\nabla^k F$$

and

$$\nabla_j\mathrm{R}_{ki} - \nabla_k\mathrm{R}_{ji} = -\mathrm{R}_{jkip}\nabla^p F\,,$$

hence, the equality

$$\begin{aligned}W_{ij} = &2\mathrm{R}_{jkip}\nabla^p F\nabla^k F + 2(\mathrm{R}_{jkip}\nabla^p F + \mathrm{R}_{ikjp}\nabla^p F)\nabla^k f + 2\mathrm{R}_{ikjp}\nabla^p f\nabla^k f\\ = &2\mathrm{R}_{jkip}\nabla^p(F + f)\nabla^k(F + f)\,,\end{aligned}$$

implies the claim, by the curvature assumption. Then, if a solution u of the conjugate heat equation (3.2) satisfies

$$\nabla^2 \log u(\cdot, t_i) - \mathrm{Ric}(\cdot, t_i) + \frac{g(\cdot, t_i)}{2(T_{\max} - t)} \geq 0\,,$$

for a sequence of times $t_i \to T_{\max}$ on the whole M, the monotonicity of the Huisken's integral follows in the interval $[0, T_{\max})$.
With an analogous argument, it can be shown that if the flow is a gradient, steady Ricci soliton with nonnegative Riemann operator and the function u satisfies the same condition as before, then the monotonicity of the Huisken's integral holds in the interval $[0, T)$ (in this case the interval can also be $\mathbb{R}$ and $T = +\infty$).

5. Computations II – Backward Ricci Flow

We deal now with the **BRF** case, that is, $(M, g(t))$ is an m–dimensional Riemannian manifold evolving by the backward Ricci flow $\partial_t g = 2\mathrm{Ric}$ and the smooth function $u : M \times [0, T) \to \mathbb{R}$ is a positive solution of $u_t = -\Delta u - \mathrm{R}u$. Then, we have seen that if $\tau = T - t$ the monotonicity of the Huisken's integral

$$\tau^{\frac{m-n}{2}} \int_N u\, d\mu_t\,,$$

where N is a compact n–submanifold moving by mean curvature, is implied by the non-positivity of the expression

$$\Big(\nabla^2_{\alpha\beta} f - \mathrm{R}_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau}\Big) g^{\alpha\beta}\,,$$

with $f = -\log u$ which hence satisfies $f_t = -\Delta f + |\nabla f|^2 + \mathrm{R}$. Choosing instead $f = \log u$ which then satisfies

$$f_t = -\Delta f - |\nabla f|^2 - \mathrm{R}$$

the above monotonicity would be a consequence of the stronger statement that the full 2–form

$$\nabla^2_{ij} f + \mathrm{R}_{ij} + \frac{g_{ij}}{2\tau}$$

is nonnegative definite.

We then set $L_{ij} = \nabla^2_{ij} f + \mathrm{R}_{ij}$, $H_{ij} = \tau L_{ij} + g_{ij}/2 = \tau[\nabla^2_{ij} f + \mathrm{R}_{ij}] + g_{ij}/2$ and we compute the evolution equation of the form H (as before) whose nonnegativity is trivially equivalent to the one of the form above. In normal coordinates, we have (along the same line of the **RF** case)

$$\begin{aligned}
(\partial_t + \Delta) H_{ij} =& - L_{ij} + \mathrm{R}_{ij}\\
&+ \tau[\Delta\nabla^2_{ij} f + \nabla^2_{ij} f_t - (\nabla_i \mathrm{R}_{jk} + \nabla_j \mathrm{R}_{ik} - \nabla_k \mathrm{R}_{ij})\nabla_k f]\\
&+ \tau[\partial_t \mathrm{R}_{ij} + \Delta \mathrm{R}_{ij}]\\
=& - \nabla^2_{ij} f + \tau[\Delta\nabla^2_{ij} f - \nabla^2_{ij}\Delta f - \nabla^2_{ij}|\nabla f|^2 - (\nabla_i \mathrm{R}_{jk} + \nabla_j \mathrm{R}_{ik} - \nabla_k \mathrm{R}_{ij})\nabla_k f]\\
&- \tau[2\mathrm{R}_{pq}\mathrm{R}_{ipjq} - 2\mathrm{R}_{ip}\mathrm{R}_{pj} + \nabla^2_{ij}\mathrm{R}]\\
=& - \nabla^2_{ij} f + \tau[(\nabla_i \mathrm{R}_{jk} + \nabla_j \mathrm{R}_{ik} - \nabla_k \mathrm{R}_{ij})\nabla_k f + \mathrm{R}_{jp}\nabla^2_{ip} f + \mathrm{R}_{ip}\nabla^2_{pj} f - 2\mathrm{R}_{ipjq}\nabla^2_{pq} f]\\
&+ \tau[-\nabla^2_{ij}|\nabla f|^2 - (\nabla_i \mathrm{R}_{jk} + \nabla_j \mathrm{R}_{ik} - \nabla_k \mathrm{R}_{ij})\nabla_k f]\\
&- \tau[2\mathrm{R}_{pq}\mathrm{R}_{ipjq} - 2\mathrm{R}_{ip}\mathrm{R}_{pj} + \nabla^2_{ij}\mathrm{R}]\\
=& - \nabla^2_{ij} f + \tau[\mathrm{R}_{jp}\nabla^2_{ip} f + \mathrm{R}_{ip}\nabla^2_{pj} f - 2\mathrm{R}_{ipjq}\nabla^2_{pq} f - \nabla^2_{ij}|\nabla f|^2]\\
&- \tau[2\mathrm{R}_{pq}\mathrm{R}_{ipjq} - 2\mathrm{R}_{ip}\mathrm{R}_{pj} + \nabla^2_{ij}\mathrm{R}]\\
=& - \nabla^2_{ij} f + \tau[\mathrm{R}_{jp}\nabla^2_{ip} f + \mathrm{R}_{ip}\nabla^2_{pj} f - 2\mathrm{R}_{ipjq}\nabla^2_{pq} f]\\
&- \tau[2\mathrm{R}_{pq}\mathrm{R}_{ipjq} - 2\mathrm{R}_{ip}\mathrm{R}_{pj} + \nabla^2_{ij}\mathrm{R}]\\
&- \tau[2\nabla^2_{ip} f\nabla^2_{jp} f + 2\nabla^3_{ijk} f\nabla_k f]\\
=& - \nabla^2_{ij} f + \tau[\mathrm{R}_{jp}\nabla^2_{ip} f + \mathrm{R}_{ip}\nabla^2_{pj} f - 2\mathrm{R}_{ipjq}\nabla^2_{pq} f]\\
&- \tau[2\mathrm{R}_{pq}\mathrm{R}_{ipjq} - 2\mathrm{R}_{ip}\mathrm{R}_{pj} + \nabla^2_{ij}\mathrm{R}]\\
&- \tau[2\nabla^2_{ip} f\nabla^2_{jp} f + 2\nabla^3_{kij} f\nabla_k f + 2\mathrm{R}_{ipjq}\nabla_p f\nabla_q f]\,.
\end{aligned}$$

Substituting now $L_{ij} = [H_{ij} - g_{ij}/2]/\tau$ and $\nabla^2_{ij} f = [H_{ij} - g_{ij}/2]/\tau - \mathrm{R}_{ij}$, we get

$$\begin{aligned}
(\partial_t + \Delta) H_{ij} = &- H_{ij}/\tau + g_{ij}/2\tau + \mathrm{R}_{ij}\\
&- \tau[\mathrm{R}_{jp}\mathrm{R}_{ip} + \mathrm{R}_{ip}\mathrm{R}_{pj} - 2\mathrm{R}_{ipjq}\mathrm{R}_{pq}]\\
&+ [\mathrm{R}_{jp}H_{ip} + \mathrm{R}_{ip}H_{pj} - 2\mathrm{R}_{ipjq}H_{pq}]\\
&- \tau[2\mathrm{R}_{pq}\mathrm{R}_{ipjq} - 2\mathrm{R}_{ip}\mathrm{R}_{pj} + \nabla^2_{ij}\mathrm{R}]\\
&- 2\tau[H^2_{ij}/\tau^2 - H_{ij}/\tau^2 + g_{ij}/4\tau^2 + \mathrm{R}_{ik}\mathrm{R}_{kj} - \mathrm{R}_{ik}H_{jk}/\tau - \mathrm{R}_{jk}H_{ik}/\tau + \mathrm{R}_{ij}/\tau]\\
&- 2\tau[\nabla_k H_{ij}\nabla_k f/\tau - \nabla_k \mathrm{R}_{ij}\nabla_k f + \mathrm{R}_{ipjq}\nabla_p f \nabla_q f]\\
= &[H_{ij} - 2H^2_{ij}]/\tau - \mathrm{R}_{ij}\\
&+ [3\mathrm{R}_{jp}H_{ip} + 3\mathrm{R}_{ip}H_{pj} - 2\mathrm{R}_{ipjq}H_{pq}]\\
&- \tau\nabla^2_{ij}\mathrm{R} - 2\tau\mathrm{R}_{ik}\mathrm{R}_{kj}\\
&- 2\tau[\nabla_k H_{ij}\nabla_k f/\tau - \nabla_k\mathrm{R}_{ij}\nabla_k f + \mathrm{R}_{ipjq}\nabla_p f\nabla_q f]\\
= &[H_{ij} - 2H^2_{ij}]/\tau - 2\nabla_k H_{ij}\nabla_k f + 3\mathrm{R}_{jp}H_{ip} + 3\mathrm{R}_{ip}H_{pj} - 2\mathrm{R}_{ipjq}H_{pq}\\
&- \tau[\nabla^2_{ij}\mathrm{R} + 2\mathrm{R}_{ik}\mathrm{R}_{kj} + \mathrm{R}_{ij}/\tau - 2\nabla_k\mathrm{R}_{ij}\nabla_k f + 2\mathrm{R}_{ipjq}\nabla_p f\nabla_q f]\,.
\end{aligned}$$

Thus, getting back to generic coordinates

$$\begin{aligned}
(\partial_t + \Delta) H_{ij} = &[H_{ij} - 2H^2_{ij}]/\tau - 2\nabla_k H_{ij}\nabla^k f + 3g^{pq}\mathrm{R}_{ip}H_{jq} + 3g^{pq}\mathrm{R}_{jp}H_{iq} - 2\mathrm{R}_{ipjq}H^{pq}\\
&- \tau[\nabla^2_{ij}\mathrm{R} + 2g^{pq}\mathrm{R}_{ip}\mathrm{R}_{jq} + \mathrm{R}_{ij}/\tau - 2\nabla_k\mathrm{R}_{ij}\nabla^k f + 2\mathrm{R}_{ipjq}\nabla^p f\nabla^q f]\\
= &[H_{ij} - 2H^2_{ij}]/\tau - 2\nabla_k H_{ij}\nabla^k f + 3g^{pq}\mathrm{R}_{ip}H_{jq} + 3g^{pq}\mathrm{R}_{jp}H_{iq} - 2\mathrm{R}_{ipjq}H^{pq}\\
&- \tau Z_{ij}\,,
\end{aligned}$$

where we set

$$Z_{ij} = \nabla^2_{ij}\mathrm{R} + 2g^{pq}\mathrm{R}_{ip}\mathrm{R}_{jq} + \mathrm{R}_{ij}/\tau - 2\nabla_k\mathrm{R}_{ij}\nabla^k f + 2\mathrm{R}_{ipjq}\nabla^p f\nabla^q f\,.$$

Then, arguing now as in the **RF** case, assuming that the Riemann curvature operator is nonnegative (such a condition is not preserved in general under the backward Ricci flow), if the form Z is nonnegative definite we can conclude that if there is a sequence $t_i \to T$ such that the form $H(\cdot, t_i)$ is positive definite, then the form H is nonnegative definite on the whole $M \times (0, T]$ and the monotonicity of the Huisken's integral follows.
Notice that the trace of Z_{ij},

$$g^{ij}Z_{ij} = \Delta\mathrm{R} + 2|\mathrm{Ric}|^2 + \mathrm{R}/\tau - 2\nabla_k\mathrm{R}\nabla^k f + 2\mathrm{R}_{pq}\nabla^p f\nabla^q f$$

coincides with the trace of the original Hamilton's Harnack quadratic

$$\begin{aligned}
&2\,\Delta\mathrm{R}_{ij} - \nabla^2_{ij}\mathrm{R} - 2g^{pq}\mathrm{R}_{ip}\mathrm{R}_{jq} + 4\mathrm{R}^{pq}\mathrm{R}_{ipjq} + \mathrm{R}_{ij}/\tau\\
&\quad - 2(\nabla_i\mathrm{R}_{jk} + \nabla_j\mathrm{R}_{ik} - 2\nabla_k\mathrm{R}_{ij})\nabla^k f + 2\mathrm{R}_{ipjq}\nabla^p f\nabla^q f\,,
\end{aligned}$$

after changing the sign of the function f. Hence, one can ask himself if under the backward Ricci flow of a manifold with nonnegative definite Riemann curvature operator, the 2–forms

$$Z^U_{ij} = \nabla^2_{ij}\mathrm{R} + 2\mathrm{R}^2_{ij} + \mathrm{R}_{ij}/\tau - 2\nabla_k\mathrm{R}_{ij}U^k + 2\mathrm{R}_{ipjq}U^pU^q$$

are all nonnegative definite, for every vector $U = \{U^i\}$ (see Ni [11, Remark 6.4]).
Unfortunately, this does not hold even in dimension two, indeed, in such case we have $\mathrm{R}_{ij} = \mathrm{R}g_{ij}/2$ and $\mathrm{R}_{ijkl} = \mathrm{R}(g_{ik}g_{jl} - g_{il}g_{jk})/2$, hence, the expression for Z^U_{ij} becomes

$$Z^U_{ij} = \nabla^2_{ij}\mathrm{R} + \frac{\mathrm{R}^2}{2}g_{ij} + \frac{\mathrm{R}}{2\tau}g_{ij} - \langle\nabla\mathrm{R}\,|\,U\rangle g_{ij} + \mathrm{R}|U|^2 g_{ij} - \mathrm{R}U_iU_j\,.$$

Checking the 2–form $Z^{\widetilde{U}}_{ij}$, where $\widetilde{U} = \lambda U$ for $\lambda \in \mathbb{R}$, against the vector U we get

$$Z^{\widetilde{U}}_{ij}U^iU^j = \Big[\nabla^2_{ij}\mathrm{R}U^iU^j + \Big(\frac{\mathrm{R}^2}{2} + \frac{\mathrm{R}}{2\tau}\Big)|U|^2\Big] - \lambda\langle\nabla\mathrm{R}\,|\,U\rangle|U|^2\,.$$

Therefore, if R is not constant, choosing $U = \nabla\mathrm{R}$, when $\lambda > 0$ is large enough this expression is negative somewhere.

5.1. **A Very Special Case.** In dimension 2, for a surface with positive scalar curvature, the function $u = \mathrm{R} > 0$ satisfies

$$u_t = -\Delta u - \mathrm{R}u\,.$$

Indeed, under the backward Ricci flow, we have

$$\partial_t \mathrm{R} = -\Delta \mathrm{R} - \mathrm{R}^2\,,$$

hence, the scalar curvature is a solution of the conjugate heat equation in dimension two (under the backward Ricci flow).
Then, for a closed curve γ evolving by its curvature inside a surface moving by backward Ricci flow, we have

$$\frac{d}{dt}\left(\sqrt{\tau}\int_\gamma \mathrm{R}\,d\mu_t\right) = -\sqrt{\tau}\int_\gamma \left|\mathrm{H} - \nabla^\perp \log \mathrm{R}\right|^2 \mathrm{R}\,d\mu_t - \sqrt{\tau}\int_\gamma \left(\nabla^2_{\nu\nu}\log \mathrm{R} + \frac{\mathrm{R}}{2} + \frac{1}{2\tau}\right)\mathrm{R}\,d\mu_t\,,$$

where ν is the unit normal to the curve.
In this situation, the Li–Yau quadratic

$$\nabla^2_{\nu\nu}\log \mathrm{R} + \frac{\mathrm{R}}{2} + \frac{1}{2\tau}$$

is nonnegative, being exactly the "special" form of Hamilton's Harnack inequality for surfaces with bounded positive scalar curvature (see [1, Proposition 15.10]) evaluated on the pair of vectors (ν, ν).

Proposition 5.1. *If $(M, g(t))$ is a family of surfaces with bounded positive scalar curvature R moving by backward Ricci flow and γ is a curve moving by its curvature inside $(M, g(t))$, we have*

$$\frac{d}{dt}\left(\sqrt{\tau}\int_\gamma \mathrm{R}\,d\mu_t\right) \leq -\sqrt{\tau}\int_\gamma \left|\mathrm{H} - \nabla^\perp \log \mathrm{R}\right|^2 \mathrm{R}\,d\mu_t\,.$$

The inequality becomes an equality if and only if M is a gradient, expanding Ricci soliton with $\mathrm{R} > 0$ and $\mathrm{k} = \nabla^\perp \log \mathrm{R}$ (see [1, Chapter 15]).

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