QUASI-STATIC EVOLUTIONS IN LINEAR PERFECT PLASTICITY AS A VARIATIONAL LIMIT OF FINITE PLASTICITY: A ONE-DIMENSIONAL CASE

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Abstract. In the framework of the energetic approach to rate independent evolutions, we show that one-dimensional linear perfect plasticity can be obtained by linearization as a variational limit of a finite plasticity model with hardening proposed by A. Mielke (*SIAM J. Math. Anal.*, 2004).

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Contents

1. Introduction

Linearization in Continuum Mechanics is usually carried out under the assumption that the main quantities involved in the problem are *suitably small*, together with their derivatives. Within this framework, a Taylor expansion is employed to simplify the equations governing the various models. Checking *a posteriori* that the assumptions under which the equations were derived are actually satisfied is a hard issue, as existence of solutions is usually proved in a weak sense.

Sometimes it may happen that successful linear theories manifestly violate the assumptions under which they were derived, an example being linear elasticity in fracture mechanics. Indeed, it is well known that in a two-dimensional domain with a crack, under suitable loads at the boundary, the linear elastic displacement (or more precisely its gradient, the strain) exhibits a singularity at the crack tip. Such a singularity is the crucial property of the configuration in connection with the propagation of the crack according to Griffith's theory, so that any reasonable *classical* justification for employing linear elasticity in this context is doomed to failure.

A way to overcome the above mentioned difficulties is to resort to variational arguments. In their pioneering paper [1], Dal Maso, Negri and Percivale justify linear elasticity as a *variational* limit of nonlinear elasticity. They consider a nonlinear elastic energy of the form

$$\int_{\Omega} W(\nabla\Phi)\,dx, \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a sufficiently smooth domain, $W : \mathrm{M}^N \to [0, +\infty]$ a frame indifferent bulk energy density and $\Phi : \Omega \to \mathbb{R}^N$ an admissible elastic deformation of Ω (here M^N denotes the set of $N \times N$ matrices). Considering body forces with density εf, being $\varepsilon > 0$ a small parameter, and vanishing boundary displacements, they show, under suitable assumptions on W, that the corresponding equilibrium elastic configuration Φ_ε, written in the form $\Phi_\varepsilon = Id + \varepsilon u_\varepsilon$, is such that u_ε converges in the weak sense of Sobolev spaces to the linearized equilibrium displacement u of Ω under the action of the forces f and with vanishing boundary conditions. Moreover, the linearized theory is characterized by the elastic moduli $\mathbb{C} := \partial^2 W(Id)$. The main tool used in [1] is a Γ-convergence result for a suitable rescaling of (1.1).

In the present paper we are interested in a linearization problem arising in the study of quasi-static evolutions in plasticity theory. The main difference between the nonlinear and the linear regime is given by the way in which elastic and plastic strains are linked to the *total strain* of a configuration. At a finite strain level, a configuration is given by (Φ, F_{el}, F_{pl}), where Φ is the deformation, F_{el} the elastic strain and F_{pl} the plastic strain. They are linked through the *multiplicative decomposition*

$$\nabla\Phi = F_{el} \circ F_{pl}. \tag{1.2}$$

The elastic behaviour is described by an energy of the form (1.1) acting only on F_{el}; moreover, if plastic deformations are assumed to be isochoric, then F_{pl} takes values in $SL(N)$, the space of matrices with determinant 1. In a linearized context, configurations are described by triplets (u, e, p), where $u : \Omega \to \mathbb{R}^N$ is the displacement, $e : \Omega \to \mathrm{M}^N_{\mathrm{sym}}$ is the elastic strain, and $p : \Omega \to \mathrm{M}^N_D$ is the plastic strain. Here $\mathrm{M}^N_{\mathrm{sym}}$ is the space of symmetric matrices and M^N_D the space of symmetric and deviatoric matrices (which is the tangent space to $SL(N)$ at Id). The link between the strains and the displacement is given by the *additive decomposition*

$$Eu = e + p, \tag{1.3}$$

where Eu is the symmetrized gradient of u.

Both in the nonlinear and in the linear setting, quasi-static evolutions under time-dependent external loads are usually characterized in terms of balance equations and a flow rule. The latter involves an interplay between the stress and the plastic strain: the stress belongs to a yield region, and plastic strains are activated only when the boundary, the so called *yield surface*, is reached, the direction of flow being determined by the normal to the surface. If the yield region increases or moves depending on the developed plastic deformations, we say that the model involves *hardening*.

Decompositions (1.2) and (1.3) are classically related *via* a Taylor expansion. For a variational justification of such a link in the evolutionary case it is convenient to resort to the *variational approach to rate independent evolutions* developed by Mielke and co-authors in several papers: we refer the reader to [7] for the specific case of elasto-plasticity with finite strains, and to the survey [9] for further applications in continuum mechanics. In that framework, a quasi-static evolution $t \mapsto (\Phi(t), F_{pl}(t))$ (and the same applies also to a linearized evolution) is seen as a time-parametrized family of solutions of minimum problems, determined by the elastic and plastic properties of the material, satisfying furthermore an energy inequality which is a sort of second principle of thermodynamics. The flow rule is expressed by a suitable *dissipation distance* acting on the plastic strains. Hardening, if present, is dealt with a hardening potential.

Mielke and Stefanelli have recently proved in [10] that *in the presence of hardening*, linearized evolutions can be obtained as a variational limit of evolutions in the finite strain regime. More precisely, they show that under vanishing boundary displacements, external loads of the form $\varepsilon\Lambda(t)$, with $\varepsilon > 0$ a small parameter, and suitable assumptions for W and the other functionals governing the system, a *quasi-static evolution* $t \mapsto (\Phi^\varepsilon(t), F^\varepsilon_{pl}(t))$ at finite strain, rewritten in the form

$$\Phi^\varepsilon(t) = Id + \varepsilon u_\varepsilon(t) \qquad \text{and} \qquad F^\varepsilon_{pl}(t) = Id + \varepsilon p_\varepsilon(t), \tag{1.4}$$

converges as $\varepsilon \to 0$ in a suitable sense to an evolution $t \mapsto (u(t), p(t))$ for a linear plasticity model with (kinematic) hardening. While an existence and uniqueness result is available for the linearized evolution with hardening (see e.g. [4]), for which $(u(t), p(t)) \in H^1(\Omega; \mathbb{R}^N) \times L^2(\Omega; \mathrm{M}^N_D)$, general existence results in the finite strain regime are still not available (some results have been obtained if further strain gradient terms are introduced, see [6]). Therefore Mielke and Stefanelli *assume* the existence of quasi-static evolutions of the form $t \mapsto (\Phi^\varepsilon(t), F^\varepsilon_{pl}(t)) \in H^1(\Omega;\mathbb{R}^N) \times L^2(\Omega; SL(N))$ to perform their analysis. Actually, their result also holds for a suitable notion of *discrete in time* evolutions which is largely used in the engineering community.

The aim of the present paper is to address the problem of a variational justification of linear *perfect plasticity* starting from a finite strain theory. Contrary to the case with hardening, in perfect plasticity the yield region does not change in time, and concentrations of the plastic strain along *shear bends* can happen. Existence results for quasi-static evolutions in linear perfect plasticity have been first obtained by Suquet [11] and more recently by Dal Maso, De Simone and Mora [2] in a variational framework. In order to capture concentrations of the plastic strains, the functional framework employed in these results require that

$$(u(t), e(t), p(t)) \in BD(\Omega) \times L^2(\Omega; \mathrm{M}^N_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega; \mathrm{M}^N_D),$$

i.e., $p(t)$ is modeled as a bounded Radon measure, while the displacements, in view of the additive decomposition (1.3), belong to the space of *functions with bounded deformation* $BD(\Omega)$ (see [12]). The low regularity of the displacements and of the plastic strains entails several technical difficulties in the mathematical analysis of the problem: a particular care is required to deal with boundary conditions (as the trace operator is not continuous, even in a weak sense), and also with external loads, which have to satisfy a condition usually referred to as *safe load condition*.

In this paper we restrict our attention to the *one-dimensional setting*, the general case presenting several technical difficulties which will be hopefully addressed in a future study. In order to rely on a rigorous existence result at the finite strain level, we concentrate on a particular model proposed by Mielke (see [8, Sections 5 and 6]) concerning one-dimensional quasi-static evolutions with hardening modeled through internal variables. To our knowledge, this is the only existence result in standard finite plasticity available in the literature.

A configuration of $\Omega = (0, \ell)$ is given by the triplet (φ, F_{pl}, ζ). The internal variable ζ describes the hardening, while the plastic strain F_{pl} is assumed to take values in $\mathbb{R}^+$, avoiding incompressibility constraints which would be too severe in one dimension. The precise form of the bulk energy and of the dissipation distance governing the system are specified in Section 3 (see (3.7) and (3.8)).

In order to obtain perfect plasticity by linearization, we force the hardening to vanish by suitably choosing the constants in the dissipation distance (see Remark 3.4) depending on a small parameter $\varepsilon > 0$. Considering external loads of the type $\varepsilon\Lambda(t)$, with $\Lambda(t)$ satisfying the safe load condition (4.17), and employing the decomposition (1.4), we show in Theorem 4.4 that $t \mapsto (u_\varepsilon(t), p_\varepsilon(t))$ determines in the limit a quasi-static evolution for a model in perfect plasticity, whose elastic modulus and yield region are set by the finite strain model (see (4.14)).

Clearly the one-dimensional setting simplifies some aspects of the analysis. On the one hand, displacements belong to the space BV of functions with bounded variation, rather than to BD, and the additive decomposition (1.3) does not represent a severe restriction in the constructions required by our analysis (see in particular Step 2 in Subsection 5.3). On the other hand, invariance requirements for the nonlinear elastic energy W become trivial, since the set of rotations reduce to the identity map: some compactness issues are then handled more easily. Nevertheless, in spite of the mentioned advantages, the transition from the multiplicative to the additive decomposition, the problem of boundary conditions and the issue of the safe load condition still require a careful analysis.

The paper is organized as follows. In Section 2 we recall some basic facts about Radon measures and BV functions which are used throughout the paper. In Section 3 we recall Mielke's model in finite strain elasto-plasticity, formulated in the framework of the energetic approach, together with his existence result for quasi-static evolutions. The last part of the section contains an informal linearization of the model, which leads to perfect plasticity. The heuristic argument is formalized in Section 4: the precise rescaling assumptions on the external loads and on the dissipation distance are specified in Subsection 4.1, while the mathematical description of linearized evolutions is contained in Subsection 4.2; the rigorous formulation of the linearization result, which employs a safe load condition for the external loads, is given in Theorem 4.4. The entire Section 5 is devoted to the proof of Theorem 4.4. A careful analysis is needed in order to exploit suitable *a priori* bounds for both the elastic and plastic strains and the internal variable: we make use of a delicate interplay between the dissipation potential and the external loads which takes advantage of the safe load condition (see Subsection 5.1). The compactness results involving the transition from the multiplicative to the additive decomposition are contained in Subsection 5.2. Finally the proof of the convergence result is given in Subsection 5.3.

2. Notation and preliminaries

In this Section we fix the notation and collect some basic facts about Radon measures and BV functions which are used to deal with quasi-static evolutions in perfect plasticity.

General notation. $L^p(a,b)$ with $p \in [1,\infty[$ will denote the space of p-summable functions on the interval $[a,b] \subseteq \mathbb{R}$. $\|f\|_\infty$ will stand for the sup norm of f. Finally if $A \subseteq \mathbb{R}$, 1_A will denote the characteristic function of A, *i.e.*, $1_A(x) = 1$ if $x \in A$, $1_A(x) = 0$ if $x \notin A$.

Measures. If $E \subseteq \mathbb{R}$ is locally compact, $\mathcal{M}_b(E)$ will denote the space of finite Radon measures on E. For $\mu \in \mathcal{M}_b(E)$, we denote by $|\mu|$ its total variation. We say that

$$\mu_n \overset{*}{\rightharpoonup} \mu \qquad \text{weakly}^* \text{ in } \mathcal{M}_b(E)$$

provided that

$$\lim_n \int_E \varphi \, d\mu_n = \int_E \varphi \, d\mu$$

for every $\varphi \in C^0(E)$ which "vanish at the boundary", *i.e.*, such that for every $\varepsilon > 0$ there exists a compact set $K \subseteq E$ with $|\varphi(x)| < \varepsilon$ for $x \notin K$. Sequences in $\mathcal{M}_b(E)$ with bounded total variation always admit a weakly* convergent subsequence.

Functions with bounded variation. For $[a,b] \subseteq \mathbb{R}$, we denote by $BV(a,b)$ the space of functions $u \in L^1(a,b)$ such that $u' \in \mathcal{M}_b(]a,b[)$. $BV(a,b)$ is a Banach space with respect to the norm

$$\|u\|_{BV} := \|u\|_{L^1(a,b)} + |u'|(]a,b[).$$

Functions in $BV(a,b)$ admit boundary values at the extremes of $[a,b]$ such that the usual integration by parts formula

$$\int_{]a,b[} \varphi' u \, dx = [\varphi(b)u(b) - \varphi(a)u(a)] - \int_{]a,b[} \varphi \, du'$$

holds for every $\varphi \in C^1([a,b])$. If $c \in]a,b[$, then $u'(\{c\}) = u(c+) - u(c-)$, where $u(c+)$ and $u(c-)$ are the traces at c of the restriction of u on $]c,b[$ and $]a,c[$ respectively.

We will say that

$$u_n \overset{*}{\rightharpoonup} u \qquad \text{weakly}^* \text{ in } BV(a,b)$$

if $u_n \to u$ strongly in $L^1(a,b)$ and $u_n' \overset{*}{\rightharpoonup} u'$ weakly* in $\mathcal{M}_b(]a,b[)$. Bounded sequences in BV always admit weakly* convergent subsequences. Notice that the boundary values are not stable under weak* convergence: this will be a source of difficulty for our analysis.

3. Quasi-static evolutions in one-dimensional finite plasticity

In this section we recall the homogeneous one-dimensional model for quasi-static evolution in finite plasticity proposed by Mielke in [8] and based on the energetic approach to rate independent evolutions [9]. In the last part of the section the model is studied from a classical viewpoint: arguing on the associated balance equations and flow rule, a suitable rescaling is found which *formally* leads to linear perfect plasticity.

Let the elasto-plastic body be given by the one-dimensional open interval $(0,\ell)$ with $\ell > 0$. We assume that the body is constrained at $x = 0$, and that body forces on $(0,\ell)$ and a traction force at $x = \ell$ are applied.

A configuration is given by a triplet

$$(\varphi, F_{pl}, \zeta)$$

where

- the deformation φ belongs to the family

$$\mathcal{F} := \{\varphi : [0,\ell] \to \mathbb{R} \,:\, \varphi(0) = 0,\ \varphi \text{ is absolutely continuous and } \varphi' > 0 \text{ a.e. in } (0,\ell)\}; \tag{3.1}$$

- $F_{pl} : (0,\ell) \to \mathbb{R}^+$ denotes the *plastic strain*;
- $\zeta : (0,\ell) \to \mathbb{R}$ is a measurable field of *internal variables*, which keeps track of the irreversible plastic processes taking place during a loading history.

The elements of the set

$$\mathcal{Z} := \{z = (F_{pl}, \zeta) : (0,\ell) \to \mathbb{R}^+ \times \mathbb{R} \,:\, z \text{ is measurable}\} \tag{3.2}$$

are called *admissible internal states*. The sign condition on the total strain φ' prevents interpenetration of matter, and the same holds true for the plastic deformation associated to F_{pl}. Notice that the requirement that plastic deformations be isochoric, which is usually considered in two or three dimensional plasticity, is a too severe constraint in the one dimensional setting.

Finite elasto-plasticity is based on the multiplicative decomposition

$$\varphi' = F_{el}\, F_{pl}, \tag{3.3}$$

where F_{el} is referred to as the *elastic strain* of the configuration. The elastic properties of $(0,\ell)$ are given in terms of a stored elastic energy

$$\mathcal{W}(\varphi, F_{pl}, \zeta) := \int_0^\ell W(F_{el}(x), \zeta)\, dx = \int_0^\ell W\left(\frac{\varphi'(x)}{F_{pl}(x)}, \zeta\right) dx,$$

where $W : \mathbb{R} \times \mathbb{R} \to [0,+\infty]$. The irreversibility associated to plastic deformations is given in terms of a dissipation distance defined as follows: considering $\Delta : (0,\ell) \times \mathbb{R} \times \mathbb{R} \to [0,+\infty]$ convex and positively homogeneous of degree one in the last variable, the *dissipation distance* on $\mathbb{R}^+ \times \mathbb{R}$ at x is given by

$$D(x, z_1, z_2) := \inf\left\{\int_0^1 \Delta(x, z(s), \dot{z}(s))\, ds \,:\, z \in C^1([0,1]; \mathbb{R}^+ \times \mathbb{R}),\ z(0) = z_0, z(1) = z_1\right\}.$$

The *total dissipation* between two internal states $z_j \in \mathcal{Z}$ is obtained by the dissipation distance integrating on $(0,\ell)$:

$$\mathcal{D}(z_0, z_1) := \int_0^\ell D(x, z_0(x), z_1(x))\, dx.$$

Finally, the external loads on the time interval $[0,T]$ are given by

$$\langle \Lambda(t), \varphi \rangle := g(t)\varphi(\ell) + \int_0^\ell f(t,x)\varphi(x)\, dx$$

for some

$$f \in C^1([0,T]; C^0([0,\ell])) \qquad \text{and} \qquad g \in C^1([0,T]). \tag{3.4}$$

The notion of quasi-static evolution for the elasto-plastic system is the following.

Definition 3.1 (**Quasi-static evolution).** *We say that a map*

$$t \in [0,T] \mapsto (\varphi(t), F_{pl}(t), \zeta(t)) \in \mathcal{F} \times \mathcal{Z}$$

is a quasi-static evolution relative to the loading history $t \mapsto \Lambda(t)$ provided that the following facts hold for every $t \in [0,T]$.

(a) *Global stability: for every* $(\psi, Q, \xi) \in \mathcal{F} \times \mathcal{Z}$

$$\mathcal{W}(\varphi(t), F_{pl}(t), \zeta(t)) - \langle \Lambda(t), \varphi(t) \rangle \le \mathcal{W}(\psi, Q, \xi) - \langle \Lambda(t), \psi \rangle + \mathcal{D}((Q, \xi), (F_{pl}(t), \zeta(t))). \tag{3.5}$$

(b) *Energy inequality: setting for every* $[a, b] \subseteq [0, T]$

$$Diss(a, b; (F_{pl}, \zeta)) := \inf \left\{ \sum_{i=1}^{N} \mathcal{D}\Big((F_{pl}(t_{i-1}), \zeta(t_{i-1})), (F_{pl}(t_i), \zeta(t_i))\Big) : a = t_0 < \cdots < t_N = b \right\}$$

the following inequality holds

$$\begin{aligned} \mathcal{W}(\varphi(t), F_{pl}(t), \zeta(t)) - \langle \Lambda(t), \varphi(t) \rangle + Diss(0, t; (F_{pl}, \zeta)) \\ \le \mathcal{W}(\varphi(0), F_{pl}(0), \zeta(0)) - \langle \Lambda(0), \varphi(0) \rangle - \int_0^t \langle \dot{\Lambda}(\tau), \varphi(\tau) \rangle \, d\tau. \end{aligned} \tag{3.6}$$

In order to obtain a rigorous existence result for a quasi-static evolution, Mielke considered in [8] the elastic energy density $W : \mathbb{R} \to [0, +\infty]$ (independent of the internal variable)

$$W(F) := \begin{cases} \frac{1}{\alpha} \left(F^{\alpha} + F^{-\alpha} - 2 \right) & \text{for } F > 0 \\ +\infty & \text{otherwise,} \end{cases} \tag{3.7}$$

for some $\alpha > 0$, and the dissipation distance (independent of x)

$$D((a_1, b_1), (a_2, b_2)) := \begin{cases} c(e^{\alpha b_2} - e^{\alpha b_1}) & \text{for } b_2 \ge b_1 + |\ln \frac{a_2}{a_1}| \\ +\infty & \text{otherwise.} \end{cases} \tag{3.8}$$

Notice that W is finite only for positive elastic strains, and diverges as $F \to 0^+$ or $F \to +\infty$ as usual in finite elasticity. Moreover, the dissipation distance is obtained by choosing

$$\Delta((F_{pl}, \zeta), (\dot{F}_{pl}, \dot{\zeta})) := \begin{cases} c\alpha e^{\alpha\zeta} \dot{\zeta} & \text{for } \dot{\zeta} \ge \left| \frac{\dot{F}_{pl}}{F_{pl}} \right| \\ +\infty & \text{otherwise.} \end{cases}$$

The existence result proved by Mielke in [8, Sections 5 and 6] is the following.

Theorem 3.2 (**Existence of a quasi-static evolution**). *Let* $\alpha > 3$ *and* $\varphi^0 \in \mathcal{F}$, $(F^0_{pl}, \zeta^0) \in C^0([0, \ell]; \mathbb{R}^2)$ *with* $F^0_{pl} > 0$ *on* $[0, \ell]$ *and such that* $(\varphi^0, F^0_{pl}, \zeta^0)$ *is globally stable according to* (3.5). *Then there exists a quasi-static evolution*

$$t \in [0, T] \mapsto (\varphi(t), F_{pl}(t), \zeta(t)) \in \mathcal{F} \times \mathcal{Z}$$

with $(\varphi(0), F_{pl}(0), \zeta(0)) = (\varphi^0, F^0_{pl}, \zeta^0)$ *and such that for every* $t \in [0, T]$

$$\varphi(t) \in W^{1,\alpha/3}(0, \ell), \qquad F_{pl}(t) \in L^{\alpha}(0, \ell), \qquad e^{\zeta(t)} \in L^{\alpha}(0, \ell).$$

Moreover, the maps $t \mapsto \varphi(t)$, $t \mapsto F_{pl}(t)$ *are continuous from* $[0, T]$ *to* $W^{1,\alpha/3}(0, \ell)$ *and* $L^{\alpha}(0, \ell)$ *respectively. Finally, the energy inequality* (3.6) *holds as an equality.*

Remark 3.3 (**Classical formulation**). Let $t \mapsto (\varphi(t), F_{pl}(t), \zeta(t))$ be a quasi-static evolution. Assuming enough regularity, it can be proved that the quasi-static evolution is described in more classical terms by the following conditions which involve the nonlinear stress

$$\Sigma(t, x) := \frac{1}{F_{pl}(t, x)} W' \left(\frac{\varphi'(t, x)}{F_{pl}(t, x)} \right) :$$

(a) for every $t \in [0, T]$

$$\begin{cases} -\Sigma'(t) = f(t) & \text{on } (0, \ell) \\ \Sigma(t, \ell) = g(t); \end{cases}$$

(b) for every $t \in [0, T]$ and $x \in (0, \ell)$

$$\zeta(t, x) = \int_0^t \frac{|\dot{F}_{pl}(\tau, x)|}{F_{pl}(\tau, x)} \, d\tau;$$

(c) for every $t \in [0,T]$ and $x \in (0,\ell)$

$$|\varphi'(t,x)\Sigma(t,x)| \le c\alpha e^{\alpha\zeta(t,x)}; \tag{3.9}$$

(d) for every $t \in [0,T]$ and $x \in (0,\ell)$

$$|\varphi'(t,x)\Sigma(t,x)| < c\alpha e^{\alpha\zeta(t,x)} \Longrightarrow \dot{F}_{pl}(t,x) = 0$$

and

$$|\varphi'(t,x)\Sigma(t,x)| = c\alpha e^{\alpha\zeta(t,x)} \Longrightarrow sign(\dot{F}_{pl}(t,x)) = sign\left(\varphi'(t,x)\Sigma(t,x)\right).$$

Item (a) represents a *balance equation* for the nonlinear stress $\Sigma(t,x)$, while (c) can be interpreted as a *confinement condition*: it indeed expresses that $\Sigma(t,x)$, multiplied by $\varphi'(t,x)$, belongs to a *yield surface* depending on the internal variable $\zeta(t,x)$, which is related to the plastic deformation history by (b). Item (d) corresponds to a *flow rule* for the plastic strain $F_{pl}(t,x)$: plastic deformations occur if and only if the yield surface is reached, and they flow along the normal to the surface (in the one-dimensional setting, this amounts in a condition on the sign of the deformation). Finally, notice that the confinement condition (3.9) gives a hardening effect which depends on the effective accumulated plastic deformation

$$\int_0^t \frac{|\dot{F}_{pl}(\tau,x)|}{F_{pl}(\tau,x)}\, d\tau.$$

We conclude this section with some naive considerations concerning the linearization of the quasi-static evolution given in Theorem 3.2, which will be the focus of the analysis of the subsequent sections.

If $\Lambda(t) \equiv 0$, and if the initial conditions are given by

$$\varphi^0(x) = x, \qquad F^0_{pl}(x) = 1, \qquad \zeta^0(x) = 0,$$

then it is readily seen that

$$(\varphi(t,x), F_{pl}(t,x), \zeta(t,x)) = (x,1,0) \qquad \text{for every } t \in [0,T] \tag{3.10}$$

is an admissible evolution. It is natural to expect that small external loads would produce an evolution which is *near* configuration (3.10), and that the corresponding evolution *can be described in terms of linear plasticity.*

In order to formalize this intuition, let us consider external loads of the form $\Lambda_\varepsilon(t) := \varepsilon\Lambda(t)$, with $\varepsilon > 0$. Moreover, let us assume that the constant c in the dissipation distance (3.8) is replaced by $c_\varepsilon := d\varepsilon$ for some $d > 0$. We write the associated quasi-static evolution $t \mapsto (\varphi^\varepsilon(t), F^\varepsilon_{pl}(t), \zeta^\varepsilon(t))$ in the form

$$\varphi^\varepsilon(t,x) = x + \varepsilon u_\varepsilon(t,x), \qquad F^\varepsilon_{pl}(t,x) = 1 + \varepsilon p_\varepsilon(t,x), \qquad \zeta^\varepsilon(t,x) = \varepsilon z_\varepsilon(t,x).$$

The balance equations and the confinement condition outlined in Remark 3.3 can be rewritten in terms of $(u_\varepsilon, p_\varepsilon, z_\varepsilon)$ as

$$\begin{cases} -\left(\dfrac{1}{1+\varepsilon p_\varepsilon(t)} W'\left(\dfrac{1+\varepsilon u'_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)}\right)\right)' = \varepsilon f(t) \quad \text{on } (0,\ell) \\ \dfrac{1}{1+\varepsilon p_\varepsilon(t,\ell)} W'\left(\dfrac{1+\varepsilon u'_\varepsilon(t,\ell)}{1+\varepsilon p_\varepsilon(t,\ell)}\right) = \varepsilon g(t) \end{cases}$$

and

$$\left| W'\left(\frac{1+\varepsilon u'_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)}\right) \frac{1+\varepsilon u'_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)} \right| \le d\varepsilon\alpha e^{\alpha\varepsilon z_\varepsilon(t)} \qquad \text{on } (0,\ell).$$

In view of the confinement condition, the flow rule can be expressed as

$$W'\left(\frac{1+\varepsilon u'_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)}\right) \frac{1+\varepsilon u'_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)} \dot{p}_\varepsilon(t) = d\varepsilon\alpha e^{\alpha\varepsilon z_\varepsilon(t)} |\dot{p}_\varepsilon(t)| \qquad \text{on } (0,\ell).$$

If for $\varepsilon \to 0$ we have, in a *suitable sense*

$$u_\varepsilon(t) \to u(t), \qquad p_\varepsilon(t) \to p(t), \qquad z_\varepsilon(t) \to z(t),$$

then, taking into account that $W'(1)=0$ and setting

$$\sigma(t) := c_W(u'(t) - p(t)),$$

where $c_W := W''(1)$, we obtain the balance equation

$$\begin{cases} -\sigma'(t) = f(t) & \text{on } (0,\ell) \\ \sigma(t,\ell) = g(t), \end{cases}$$

the confinement condition

$$|\sigma(t)| \leq S_Y \qquad \text{on } (0,\ell) \tag{3.11}$$

and the flow rule

$$\sigma(t)\dot{p}(t) = S_Y|\dot{p}(t)| \qquad \text{on } (0,\ell), \tag{3.12}$$

where $S_Y := d\alpha$. We can thus interpret $u(t)$ and $p(t))$ as the displacement and the plastic strain for a linearly elastic-perfectly plastic material. The multiplicative decomposition is replaced by the additive decomposition

$$u'(t) = e(t) + p(t).$$

The constant c_W can be interpreted as the elastic constant of the "new" material, so that we obtain the usual constitutive equation $\sigma(t) = c_W e(t)$ for the Cauchy stress. Relations (3.11) and (3.12) express the yield condition and the flow rule of perfect plasticity with *yield constant* S_Y.

The aim of the present paper is to provide a rigorous justification of the previous linearization argument.

Remark 3.4. Notice that we rescaled the yield constant in (3.9) to $c_\varepsilon := d\varepsilon$. This is somehow mandatory since otherwise the nonlinear stress $\varphi'(t)\Sigma(t)$, which should be almost null at every x in the present linearization, would never reach the yield surface. Consequently plastic deformations could not activate, and we fall in a purely elastic deformation process. We conclude that a linearized plasticity description of quasi-static evolutions under ε-small external loads requires that also the yield surface for the nonlinear stress is assumed to be ε-small.

4. The linearization result

In this section we formulate in rigorous terms the intuitive linearization argument outlined at the end of the previous section. In Subsection 4.1 we exploit the precise rescaling concerning the external loads and the dissipation distance in terms of a small parameter $\varepsilon > 0$ adopted in our analysis. The quasi-static evolution in the finite deformation regime is rewritten (see Proposition 4.1) in terms of "linearized quantities" $(u_\varepsilon, p_\varepsilon, z_\varepsilon)$ which will be studied as $\varepsilon \to 0$. Subsection 4.2 is devoted to the mathematical description of the linearized evolution in perfect plasticity involving Radon measures and BV functions. Finally, the rigorous linearization result is stated in Subsection 4.3.

4.1. **Setting of the problem.** Given $\varepsilon > 0$, let us consider as above the elastic energy density

$$W(F) := \begin{cases} \frac{1}{\alpha}\left(F^\alpha + F^{-\alpha} - 2\right) & \text{for } F > 0 \\ +\infty & \text{otherwise,} \end{cases} \qquad \alpha > 3, \tag{4.1}$$

and the (rescaled) dissipation distance

$$D_\varepsilon((a_1,b_1),(a_2,b_2)) := \begin{cases} d\varepsilon(e^{\alpha b_2} - e^{\alpha b_1}) & \text{for } b_2 \geq b_1 + |\ln \frac{a_2}{a_1}| \\ +\infty & \text{otherwise,} \end{cases} \qquad d > 0. \tag{4.2}$$

Notice that D_ε depends on the yield constant $c_\varepsilon := d\varepsilon$ (see Remark 3.4).

Given $f \in C^1([0,T]; C^0([0,\ell]))$ and $g \in C^1([0,T])$ with

$$g(0) = 0 \qquad \text{and} \qquad f(0,\cdot) = 0 \text{ on } [0,\ell], \tag{4.3}$$

we consider external loads $\Lambda_\varepsilon(t)$ determined by $f_\varepsilon := \varepsilon f$ and $g_\varepsilon := \varepsilon g$, *i.e.*,

$$\langle \Lambda_\varepsilon(t), \varphi \rangle := \varepsilon \langle \Lambda(t), \varphi \rangle \tag{4.4}$$

with

$$\langle \Lambda(t), \varphi \rangle := g(t)\varphi(\ell) + \int_0^\ell f(t,x)\varphi(x)\,dx. \tag{4.5}$$

According to Theorem 3.2, let us consider a quasi-static evolution

$$t \mapsto (\varphi^\varepsilon(t), F^\varepsilon_{pl}(t), \zeta^\varepsilon(t))$$

associated with the choices (4.1) for the elastic energy, (4.2) for the dissipation functional, (4.4) for the external loads, and with initial conditions

$$\varphi^\varepsilon(0,x) = x, \qquad F^\varepsilon_{pl}(0,x) := 1, \qquad \zeta^\varepsilon(0,x) := 0.$$

In the study of the asymptotic behavior of this evolution for $\varepsilon \to 0$, following the considerations of the previous section it is useful to write

$$\varphi^\varepsilon(t,x) := x + \varepsilon u_\varepsilon(t,x), \qquad F^\varepsilon_{pl}(t,x) := 1 + \varepsilon p_\varepsilon(t,x), \qquad \zeta^\varepsilon(t,x) := \varepsilon z_\varepsilon(t,x) \tag{4.6}$$

and to concentrate on the evolution

$$t \mapsto (u_\varepsilon(t), p_\varepsilon(t), z_\varepsilon(t)).$$

Now consider the spaces

$$\mathcal{F}_\varepsilon := \{v : [0,\ell] \to \mathbb{R} \,:\, v \text{ is absolutely continuous and } 1 + \varepsilon v' > 0 \text{ a.e. in } (0,\ell)\}$$

and

$$\mathcal{Z}_\varepsilon := \{(q,\xi) : (0,\ell) \to \mathbb{R}^2 \text{ measurable and such that } 1 + \varepsilon q > 0 \text{ a.e. in } (0,\ell)\}.$$

These spaces are a rewriting of the spaces $\mathcal{F}$ and $\mathcal{Z}$ given in (3.1) and (3.2) in terms of the *linearized structure* $\varphi(x) = x + \varepsilon v(x)$ and $F_{pl}(x) = 1 + \varepsilon q(x)$ for the admissible deformations and plastic strains.

For $(u,p,z) \in \mathcal{F}_\varepsilon \times \mathcal{Z}_\varepsilon$ let us set

$$\mathcal{W}_\varepsilon(u,p) := \frac{1}{\varepsilon^2} \int_0^\ell W\left(1 + \varepsilon \frac{u' - p}{1 + \varepsilon p}\right) dx$$

and, given $(q,\xi) \in \mathcal{Z}_\varepsilon$,

$$\mathcal{H}_\varepsilon((p,z),(q,\xi)) := \frac{d}{\varepsilon} \int_0^\ell \widetilde{D}_\varepsilon((p,z),(q,\xi))\,dx,$$

where

$$\widetilde{D}_\varepsilon((a_1,b_1),(a_2,b_2)) := \begin{cases} e^{\alpha\varepsilon b_2} - e^{\alpha\varepsilon b_1} & \text{if } b_2 \geq b_1 + \dfrac{1}{\varepsilon}\left|\ln \dfrac{1+\varepsilon a_2}{1+\varepsilon a_1}\right| \\ +\infty & \text{otherwise.} \end{cases} \tag{4.7}$$

By interpreting the global stability condition and the energy *equality* in Definition 3.1 for $(\varphi^\varepsilon, F^\varepsilon_{pl}, \zeta^\varepsilon)$ in terms of the functions $(u_\varepsilon, p_\varepsilon, z_\varepsilon)$ given in (4.6), we readily obtain the following result.

Proposition 4.1. *Assume* (4.1), (4.2), (4.4) *and let*

$$t \in [0,T] \mapsto (\varphi^\varepsilon(t), F^\varepsilon_{pl}(t), \zeta^\varepsilon(t))$$

be a quasi-static evolution according to Theorem 3.2 such that $(\varphi^\varepsilon(0,x), F^\varepsilon_{pl}(0,x), \zeta^\varepsilon(0,x)) = (x,1,0)$ *for* $x \in (0,\ell)$.

Let

$$t \in [0,T] \mapsto (u_\varepsilon(t), p_\varepsilon(t), z_\varepsilon(t))$$

be the evolution given by (4.6). *Then the following items hold true.*

(a) *The functions* $t \mapsto u_\varepsilon(t)$ *and* $t \mapsto p_\varepsilon(t)$ *are absolutely continuous from* $[0,T]$ *to* $W^{1,\alpha/3}(0,\ell)$ *and* $L^\alpha(0,\ell))$ *respectively. Moreover* $z_\varepsilon(t) : (0,\ell) \to \mathbb{R}$ *is measurable for every* $t \in [0,T]$ *and*

$$z_\varepsilon(t_2) \geq z_\varepsilon(t_1) + \frac{1}{\varepsilon}\left|\ln \frac{1+\varepsilon p_\varepsilon(t_2)}{1+\varepsilon p_\varepsilon(t_1)}\right| \qquad \text{a.e. in } (0,\ell) \qquad \text{for every } 0 \leq t_1 \leq t_2 \leq T. \tag{4.8}$$

Finally $(u_\varepsilon(0), p_\varepsilon(0), z_\varepsilon(0)) = (0,0,0)$.

(b) *Global stability: for every* $t \in [0,T]$ *and* $(v,q,\xi) \in \mathcal{F}_\varepsilon \times \mathcal{Z}_\varepsilon$

$$\mathcal{W}_\varepsilon(u_\varepsilon(t), p_\varepsilon(t)) - \langle \Lambda(t), u_\varepsilon(t)\rangle \le \mathcal{W}_\varepsilon(v,q) - \langle \Lambda(t), v\rangle + \mathcal{H}_\varepsilon((p_\varepsilon(t), z_\varepsilon(t)), (q,\xi)).$$

(c) *Energy equality: for every* $t \in [0,T]$

$$\mathcal{W}_\varepsilon(u_\varepsilon(t), p_\varepsilon(t)) - \langle \Lambda(t), u_\varepsilon(t)\rangle + Diss_\varepsilon(0,t;(p_\varepsilon, z_\varepsilon)) = -\int_0^t \langle \dot\Lambda(\tau), u_\varepsilon(\tau)\rangle \, d\tau, \tag{4.9}$$

where

$$Diss_\varepsilon(0,t;(p_\varepsilon, z_\varepsilon)) := \frac{d}{\varepsilon}\int_0^\ell \left(e^{\alpha\varepsilon z_\varepsilon(t)} - 1\right) dx. \tag{4.10}$$

Notice that the expression (4.10) for the dissipation on $[0,t]$ is readily computed from (4.7).

4.2. **Quasi-static evolutions for linearly elastic-perfectly plastic materials.** As discussed at the end of Section 3, we expect that $t \in [0,T] \mapsto (u_\varepsilon(t), p_\varepsilon(t), z_\varepsilon(t))$ converges as $\varepsilon \to 0$ to a quasi-static evolution for a linearly elastic-perfectly plastic material whose elastic modulus and yield constant are given respectively by

$$c_W := W''(1) \qquad \text{and} \qquad S_Y := d\alpha, \tag{4.11}$$

and under external loads $\Lambda(t)$ given in (4.5). Following the energetic approach adopted in [2], we detail in this subsection the mathematical setting required to describe such an evolution.

As explained in the Introduction, in linear perfect elasto-plasticity the plastic strains are modeled as Radon measures in order to capture possible concentrations. As a consequence of the one-dimensional context, displacements turn out to be functions of bounded variation. More precisely, we adopt the following

Definition 4.2 (**Admissible configurations).** *The family* $\mathcal{A}$ *of admissible configuration is given by the triplets*

$$(u,e,p) \in BV(0,\ell) \times L^2(0,\ell) \times \mathcal{M}_b([0,\ell[)$$

such that

$$u' = e + p \qquad \textit{on } (0,\ell) \tag{4.12}$$

and

$$p(\{0\}) = u(0), \tag{4.13}$$

where $u(0)$ *is the trace of* u *at* $x = 0$.

The function u denotes the displacement of the body, e is the associated elastic strain and p is the corresponding plastic strain. In the linear context, the multiplicative decomposition (3.3) is replaced by the *additive decomposition* (4.12) of the total strain u' into elastic and plastic parts.

Condition (4.13) is a relaxed version of the Dirichlet boundary condition at $x = 0$: if $u(0) \neq 0$, then we assume that a plastic strain has been created at $x = 0$ in the form of a Dirac delta with coefficient $u(0)$: this plastic strain will be penalized in terms of dissipation along the evolution, so that it is not convenient for the system to violate the boundary condition.

For every $(u,e,p) \in \mathcal{A}$ we set

$$\mathcal{Q}(e) := c_W \int_0^\ell e^2 \, dx \qquad \text{and} \qquad \mathcal{H}(p) := S_Y |p|([0,\ell[), \tag{4.14}$$

where c_W and S_Y are the constants given in (4.11). $\mathcal{Q}(e)$ stands for the elastic energy of the configuration, while the functional $\mathcal{H} : \mathcal{M}_b([0,\ell[) \to [0;\infty[$ is the associated dissipation potential.

For $t \in [0,T] \mapsto p(t) \in \mathcal{M}_b([0,\ell[)$ and $[a,b] \subseteq [0,T]$ the dissipation on $[a,b]$ is given by

$$Diss(a,b;p) := \sup\left\{ \sum_{i=1}^N \mathcal{H}(p(t_i) - p(t_{i-1})) \, : \, a = t_0 < t_1 < \cdots < t_N = b \right\}.$$

We are now in a position to define rigorously the linearized evolution.

Definition 4.3 (Quasi-static evolution in linear perfect plasticity). *We say that a map*

$$t \in [0,T] \mapsto (u(t), e(t), p(t)) \in \mathcal{A}, \qquad (u(0), e(0), p(0)) = (0,0,0)$$

is a quasi-static evolution relative to the external loads Λ *given in* (4.5) *and satisfying* $\Lambda(0) = 0$ *provided that the following conditions hold true.*

(a) *Global stability: for every* $t \in [0,T]$ *and* $(v, \eta, q) \in \mathcal{A}$

$$\mathcal{Q}(e(t)) - \langle \Lambda(t), u(t) \rangle \le \mathcal{Q}(\eta) - \langle \Lambda(t), v \rangle + \mathcal{H}(q - p). \tag{4.15}$$

(b) *Energy equality: for every* $t \in [0,T]$

$$\mathcal{Q}(e(t)) - \langle \Lambda(t), u(t) \rangle + Diss(0,t;p) = -\int_0^t \langle \dot{\Lambda}(\tau), u(\tau) \rangle \, d\tau. \tag{4.16}$$

4.3. **The main result.** As mentioned in the Introduction, existence of quasi-static evolutions in linear perfect plasticity can be proved by means of variational methods under an additional assumption on the external loads, the so called *safe load condition.*

We say that the external loads $t \mapsto \Lambda(t)$ given in (4.5) satisfy the safe load condition if there exists $\rho \in C^1([0,T] \times [0,\ell])$ such that for every $t \in [0,T]$

$$\begin{cases} -\rho'(t) = f(t) & \text{in } (0,\ell), \\ \rho(t,\ell) = g(t) \end{cases} \qquad S_Y - \|\rho\|_\infty =: \beta > 0, \tag{4.17}$$

where S_Y is given in (4.11). It is readily seen that for every $u \in \mathcal{F}_\varepsilon$

$$\langle \Lambda(t), u \rangle = \int_0^\ell \rho(t,x) u'(x) \, dx, \tag{4.18}$$

and that for every $(u, e, p) \in \mathcal{A}$

$$\langle \Lambda(t), u \rangle = \int_0^\ell \rho(t,x) e(x) \, dx + \int_{[0,\ell[} \rho(t,x) \, dp(x) =: \langle \rho(t), e \rangle + \langle \rho(t), p \rangle, \tag{4.19}$$

with

$$\mathcal{H}(p) - \langle \rho(t), p \rangle \ge \beta |p|([0,\ell[), \tag{4.20}$$

where $\mathcal{H}$ is given in (4.14).

We can now state precisely the main result of the paper.

Theorem 4.4 (The linearization result). *Let* $f \in C^1([0,T]; C^0([0,\ell]))$ *and* $g \in C^1([0,T])$ *with*

$$g(0) = 0 \qquad \text{and} \qquad f(0,x) = 0 \text{ on } (0,\ell)$$

satisfy the safe load condition (4.17) *for some* $\rho \in C^1([0,T] \times [0,\ell])$.

For every $\varepsilon > 0$, *let* $t \in [0,T] \mapsto (\varphi^\varepsilon(t), F^\varepsilon_{pl}(t), \zeta^\varepsilon(t))$ *be a quasi-static evolution according to Theorem 3.2 such that* $(\varphi^\varepsilon(0,x), F^\varepsilon_{pl}(0,x), \zeta^\varepsilon(0,x)) = (x, 1, 0)$ *for every* $x \in (0,\ell)$, *under the choices* (4.1) *for the elastic energy density,* (4.2) *for the dissipation distance,* (4.4) *for the external loads, and let*

$$t \in [0,T] \mapsto (u_\varepsilon(t), p_\varepsilon(t), z_\varepsilon(t))$$

be the evolution defined through relation (4.6).

Then there exist a quasi-static evolution in linear perfect plasticity

$$t \in [0,T] \mapsto (u(t), e(t), p(t))$$

according to Definition 4.3 and a sequence $\varepsilon_n \to 0$ *such that, setting*

$$u_n := u_{\varepsilon_n}, \qquad p_n := p_{\varepsilon_n}, \qquad z_n := z_{\varepsilon_n},$$

then for every $t \in [0,T]$

$$u_n(t) \overset{*}{\rightharpoonup} u(t) \qquad \text{weakly}^* \text{ in } BV(0,\ell), \tag{4.21}$$

$$u_n(t) - p_n(t) \rightharpoonup e(t) \qquad \text{weakly in } L^1(0,\ell), \tag{4.22}$$

$$p_n(t) \overset{*}{\rightharpoonup} p(t) \qquad \text{weakly}^* \text{ in } \mathcal{M}_b([0,\ell[), \tag{4.23}$$

and

$$\lim_n \mathcal{W}_{\varepsilon_n}(u_n(t), p_n(t)) = \mathcal{Q}(e(t)). \tag{4.24}$$

The proof of Theorem 4.4 will be given in the next section.

5. Proof of the linearization result

This section contains the proof of Theorem 4.4. Given $\varepsilon > 0$, let $t \in [0,T] \mapsto (\varphi^\varepsilon(t), F^\varepsilon_{pl}(t), \zeta^\varepsilon(t))$ be a quasi-static evolution for the finite plasticity model, and let

$$t \in [0,T] \mapsto (u_\varepsilon(t), p_\varepsilon(t), z_\varepsilon(t))$$

be the evolution defined through (4.6).

As a first step, in Proposition 5.1 we derive suitable a priori bounds satisfied by the evolution. As a second step, in Proposition 5.6 we show how configurations in linear perfect plasticity arise as $\varepsilon \to 0$. Finally Subsection 5.3 is devoted to the proof that the linearized evolution satisfies the properties of a quasi-static evolution in perfect plasticity.

5.1. A priori bounds.

This section is devoted to the proof of the following

Proposition 5.1. *There exists $C > 0$ such that for every $\varepsilon > 0$ and $t \in [0,T]$*

$$\left\| \frac{u'_\varepsilon(t) - p_\varepsilon(t)}{1 + \varepsilon p_\varepsilon(t)} \right\|^2_{L^2(0,\ell)} + \frac{1}{\varepsilon} \int_0^\ell \left(e^{\alpha \varepsilon z_\varepsilon(t)} - 1 \right) dx + \mathcal{V}_{L^1}(0, t; p_\varepsilon) \leq C, \tag{5.1}$$

where $\mathcal{V}_{L^1}(a, b; p_\varepsilon)$ denotes the total variation on $[a, b] \subseteq [0, T]$ of $t \mapsto p_\varepsilon(t)$ as a map from $[0,T]$ to $L^1(0,\ell)$.

In order to prove Proposition 5.1, we need two technical lemmas.

Lemma 5.2. *The following inequality holds: for every $x, y > -1$ and $\alpha > 1$*

$$e^{\alpha|\ln(1+x)|} \left[e^{\alpha \left| \ln \frac{1+y}{1+x} \right|} - 1 \right] \geq \alpha |y - x|.$$

Proof. Setting

$$\ln(1 + x) = v, \qquad \ln(1 + y) = t,$$

we have to prove that

$$\forall v, t \in \mathbb{R} : e^{\alpha|v|} \left[e^{\alpha|t-v|} - 1 \right] \geq \alpha |e^t - e^v|.$$

Consider the real functions

$$g_1(x) := e^{\alpha x} - \alpha e^x, \qquad g_2(x) := e^{-\alpha x} + \alpha e^x;$$

being $\alpha > 1$, it is easy to see that $g_{1,2}$ have global minimum in 0 and $g_1(0) = 1 - \alpha$, $g_2(0) = 1 + \alpha$. Hence we have

$$e^{\alpha x} - 1 \geq \alpha(e^x - 1), \quad e^{-\alpha x} - 1 \geq \alpha(1 - e^x).$$

Moreover, it is clear that $e^{\alpha|x|} \geq e^{\pm \alpha x}$, so that

$$e^{\alpha|x|} - 1 \geq \alpha |e^x - 1|.$$

Now take $x := t - v$ and multiply the left-hand side by $e^{\alpha|v|}$ and the right-side by e^v. Since $e^{\alpha|v|} \geq e^v$ and all the terms are positive, it follows that

$$e^{\alpha|v|} \left[e^{\alpha|t-v|} - 1 \right] \geq e^v \alpha |e^{t-v} - 1| = \alpha |e^t - e^v|,$$

which completes the proof. ∎

Lemma 5.3. *Let $\rho \in C^1([0,T]\times[0,\ell])$ be the function associated to the safe load condition (4.17) satisfied by the external loads. For every $t\in[0,T]$, $\varepsilon>0$, and S subdivision of $[0,t]$, there exists a refinement of S given by*

$$0 = t_0 < t_1 < \cdots < t_N = t,$$

with $\max_{i=1,\dots,N}(t_i - t_{i-1}) < \varepsilon$ and such that

$$\int_0^t \langle \dot\rho(\tau), p_\varepsilon(\tau)\rangle d\tau = \langle \rho(t), p_\varepsilon(t)\rangle - \sum_{i=1}^{N} \langle \rho(t_i), p_\varepsilon(t_i) - p_\varepsilon(t_{i-1})\rangle + \sum_{i=0}^{N-1} \langle r^i_\varepsilon(t), p_\varepsilon(t_i)\rangle + r_\varepsilon(t),$$

where $|r_\varepsilon(t)| < \varepsilon$ and $r^i_\varepsilon(t) : (0,\ell) \to \mathbb{R}$ are such that $\|r^i_\varepsilon(t)\|_\infty < \varepsilon(t_{i+1}-t_i)$ for $i = 0,\dots,N-1$.

Proof. Since by Proposition 4.1 the map $\tau \mapsto \langle \dot\rho(\tau), p_\varepsilon(\tau)\rangle$ is continuous on $[0,T]$, we can find a refinement $0 = t_0 < t_1 < \cdots < t_N = t$ of S with $\max_{i=1,\dots,N}(t_i - t_{i-1}) < \varepsilon$ and such that

$$\int_0^t \langle \dot\rho(\tau), p_\varepsilon(\tau)\rangle d\tau = \sum_{i=0}^{N-1} (t_{i+1}-t_i)\langle \dot\rho(t_i), p_\varepsilon(t_i)\rangle + r_\varepsilon(t), \tag{5.2}$$

where $|r_\varepsilon(t)| < \varepsilon$. Thanks to the regularity of ρ, we can assume that the subdivision is so fine that for every $i = 0,\dots,N-1$

$$\left\| \frac{\rho(t_{i+1}) - \rho(t_i)}{t_{i+1}-t_i} - \dot\rho(t_i) \right\|_\infty < \varepsilon.$$

Being $p_\varepsilon(0) = 0$, we obtain

$$\begin{aligned}
&\sum_{i=0}^{N-1} (t_{i+1}-t_i)\langle \dot\rho(t_i), p_\varepsilon(t_i)\rangle \\
&= \sum_{i=0}^{N-1} \langle \rho(t_{i+1}) - \rho(t_i), p_\varepsilon(t_i)\rangle + \sum_{i=0}^{N-1} \langle r^i_\varepsilon(t), p_\varepsilon(t_i)\rangle \\
&= \sum_{i=1}^{N} \langle \rho(t_i), p_\varepsilon(t_{i-1})\rangle - \sum_{i=0}^{N-1} \langle \rho(t_i), p_\varepsilon(t_i)\rangle + \sum_{i=0}^{N-1} \langle r^i_\varepsilon(t), p_\varepsilon(t_i)\rangle \\
&= \langle \rho(t), p_\varepsilon(t_{N-1})\rangle - \sum_{i=1}^{N-1} \langle \rho(t_i), p_\varepsilon(t_i) - p_\varepsilon(t_{i-1})\rangle + \sum_{i=0}^{N-1} \langle r^i_\varepsilon(t), p_\varepsilon(t_i)\rangle \\
&= \langle \rho(t), p_\varepsilon(t)\rangle - \sum_{i=1}^{N} \langle \rho(t_i), p_\varepsilon(t_i) - p_\varepsilon(t_{i-1})\rangle + \sum_{i=0}^{N-1} \langle r^i_\varepsilon(t), p_\varepsilon(t_i)\rangle,
\end{aligned} \tag{5.3}$$

where $r^i_\varepsilon(t) := (t_{i+1}-t_i)\dot\rho(t_i) - \rho(t_{i+1}) + \rho(t_i)$ is such that $\|r^i_\varepsilon(t)\|_\infty < \varepsilon(t_{i+1}-t_i)$. The result follows by combining (5.2) and (5.3). □

We are now in a position to prove Proposition 5.1.

Proof of Proposition 5.1. Let $t \in [0,T]$ and $\varepsilon > 0$ be fixed, and let $S := \{0 = s_0 < \cdots < s_M = t\}$ be a subdivision of $[0,t]$ such that

$$\mathcal{V}_{L^1}(0,t;p_\varepsilon) \le \varepsilon + \sum_{j=1}^{M} \|p_\varepsilon(s_j) - p_\varepsilon(s_{j-1})\|_{L^1(0,\ell)}. \tag{5.4}$$

Let $0 = t_0 < \cdots < t_N = t$ be the refinement of S given by Lemma 5.3.

In view of the energy equality (4.9), the representation (4.18) for the external load, and Lemma 5.3, we get for every $t \in [0,T]$

$$\frac{1}{\varepsilon^2}\int_0^\ell W\left(1+\varepsilon\frac{u_\varepsilon'(t)-p_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)}\right)\,dx \\ +\frac{d}{\varepsilon}\int_0^\ell\left(e^{\alpha\varepsilon z_\varepsilon(t)}-1\right)\,dx-\sum_{i=1}^N\langle\rho(t_i),p_\varepsilon(t_i)-p_\varepsilon(t_{i-1})\rangle \\ =\langle\rho(t),u_\varepsilon'(t)-p_\varepsilon(t)\rangle-\int_0^t\langle\dot\rho(\tau),u_\varepsilon'(\tau)-p_\varepsilon(\tau)\rangle d\tau-\sum_{i=0}^{N-1}\langle r_\varepsilon^i(t),p_\varepsilon(t_i)\rangle-r_\varepsilon(t),$$

where $|r_\varepsilon(t)|<\varepsilon$ and $\|r_\varepsilon^i(t)\|_\infty<\varepsilon(t_{i+1}-t_i)$ for $i=0,\dots,N-1$.

We proceed now in three steps, estimating both sides of the previous equality and drawing the conclusion.

Step 1: Estimate of the left-hand side. Since

$$\int_0^\ell\left(e^{\alpha\varepsilon z_\varepsilon(t)}-1\right)\,dx=\sum_{i=1}^N\int_0^\ell\left(e^{\alpha\varepsilon z_\varepsilon(t_i)}-e^{\alpha\varepsilon z_\varepsilon(t_{i-1})}\right)\,dx \\ =\sum_{i=1}^N\int_0^\ell e^{\alpha\varepsilon z_\varepsilon(t_{i-1})}\left[e^{\alpha\varepsilon[z_\varepsilon(t_i)-z_\varepsilon(t_{i-1})]}-1\right]\,dx \\ \geq\sum_{i=1}^N\int_0^\ell e^{\alpha|\ln(1+\varepsilon p_\varepsilon(t_{i-1}))|}\left[e^{\alpha\left|\ln\frac{1+\varepsilon p_\varepsilon(t_i)}{1+\varepsilon p_\varepsilon(t_{i-1})}\right|}-1\right]\,dx,$$

where the last inequality is a consequence of the relation (4.8) linking the internal variable $\varepsilon z_\varepsilon$ to the plastic strain $1+\varepsilon p_\varepsilon$, by Lemma 5.2 we can write for $d=d_1+d_2$

$$\frac{d}{\varepsilon}\int_0^\ell\left(e^{\alpha\varepsilon z_\varepsilon(t)}-1\right)\,dx\geq\frac{d_1}{\varepsilon}\int_0^\ell\left(e^{\alpha\varepsilon z_\varepsilon(t)}-1\right)\,dx+\alpha d_2\sum_{i=1}^N\int_0^\ell|p_\varepsilon(t_i)-p_\varepsilon(t_{i-1})|\,dx.$$

Taking into account the safe load condition (4.17), we can assume that $d_1>0$ is so small that

$$\alpha d_2-\|\rho\|_\infty>\frac{\beta}{2}>0.$$

In view of (5.4) we conclude that

$$\frac{d}{\varepsilon}\int_0^\ell\left(e^{\alpha\varepsilon z_\varepsilon(t)}-1\right)\,dx-\sum_{i=1}^N\langle\rho(t_i),p_\varepsilon(t_i)-p_\varepsilon(t_{i-1})\rangle \\ >\frac{d_1}{\varepsilon}\int_0^\ell\left(e^{\alpha\varepsilon z_\varepsilon(t)}-1\right)\,dx+(\alpha d_2-\|\rho\|_\infty)\sum_{i=1}^N\|p_\varepsilon(t_i)-p_\varepsilon(t_{i-1})\|_{L^1(0,\ell)} \\ \geq\frac{d_1}{\varepsilon}\int_0^\ell\left(e^{\alpha\varepsilon z_\varepsilon(t)}-1\right)\,dx+(\alpha d_2-\|\rho\|_\infty)\sum_{j=1}^m\|p_\varepsilon(s_j)-p_\varepsilon(s_{j-1})\|_{L^1(0,\ell)} \\ >\frac{d_1}{\varepsilon}\int_0^\ell\left(e^{\alpha\varepsilon z_\varepsilon(t)}-1\right)\,dx+\frac{\beta}{2}\left(\mathcal{V}_{L^1}(0,t;p_\varepsilon)-\varepsilon\right).$$

On the other hand, by the very definition of W, there exists $\beta_1>0$ such that

$$\frac{1}{\varepsilon^2}\int_0^1 W\left(1+\varepsilon\frac{u_\varepsilon'(t)-p_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)}\right)\,dx\geq\beta_1\left\|\frac{u_\varepsilon'(t)-p_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)}\right\|_{L^2(0,\ell)}^2,$$

so that we infer

$$(5.5)\quad \frac{1}{\varepsilon^2}\int_0^\ell W\left(1+\varepsilon\frac{u_\varepsilon'(t)-p_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)}\right)dx + \frac{d}{\varepsilon}\int_0^\ell\left(e^{\alpha\varepsilon z_\varepsilon(t)}-1\right)dx - \sum_{i=1}^N \langle\rho(t_i),p_\varepsilon(t_i)-p_\varepsilon(t_{i-1})\rangle$$
$$> \beta_1\left\|\frac{u_\varepsilon'(t)-p_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)}\right\|^2_{L^2(0,\ell)} + \frac{d_1}{\varepsilon}\int_0^\ell\left(e^{\alpha\varepsilon z_\varepsilon(t)}-1\right)dx + \frac{\beta}{2}\left(\mathcal{V}_{L^1}(0,t;p_\varepsilon)-\varepsilon\right).$$

Step 2: Estimate of the right-hand side. Notice that since $\alpha > 3$

$$\|1+\varepsilon p_\varepsilon(t)\|_{L^2(0,\ell)} = \sqrt{\int_0^\ell(1+\varepsilon p_\varepsilon(t))^2\,dx} \le \sqrt{\int_0^\ell e^{2|\ln(1+\varepsilon p_\varepsilon(t))|}\,dx}$$
$$\le \sqrt{\int_0^\ell e^{\alpha|\ln(1+\varepsilon p_\varepsilon(t))|}\,dx} \le \sqrt{\int_0^\ell\left[e^{\alpha|\ln(1+\varepsilon p_\varepsilon(t))|}-1\right]dx} + \sqrt{\ell}.$$

As a consequence

$$(5.6)\quad |\langle\rho(t),u_\varepsilon'(t)-p_\varepsilon(t)\rangle| \le \|\rho\|_\infty\|u_\varepsilon'(t)-p_\varepsilon(t)\|_{L^1(0,\ell)}$$
$$\le \|\rho\|_\infty\left\|\frac{u_\varepsilon'(t)-p_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)}\right\|_{L^2(0,\ell)}\|1+\varepsilon p_\varepsilon(t)\|_{L^2(0,\ell)}$$
$$\le \|\rho\|_\infty\left\|\frac{u_\varepsilon'(t)-p_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)}\right\|_{L^2(0,\ell)}\left[\sqrt{\int_0^\ell\left[e^{\alpha|\ln(1+\varepsilon p_\varepsilon(t))|}-1\right]dx}+\sqrt{\ell}\right].$$

By Cauchy-Schwartz inequality we can write for every $\hat{C} > 0$

$$\|\rho\|_\infty\left\|\frac{u_\varepsilon'(t)-p_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)}\right\|_{L^2(0,\ell)}\sqrt{\int_0^\ell\left[e^{\alpha|\ln(1+\varepsilon p_\varepsilon(t))|}-1\right]dx}$$
$$\le \frac{\varepsilon\|\rho\|^2_\infty}{2\hat{C}}\left\|\frac{u_\varepsilon'(t)-p_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)}\right\|^2_{L^2(0,\ell)} + \frac{\hat{C}}{2\varepsilon}\int_0^\ell\left[e^{\alpha|\ln(1+\varepsilon p_\varepsilon(t))|}-1\right]dx,$$

so that from (5.6) we infer

$$(5.7)\quad |\langle\rho(t),u_\varepsilon'(t)-p_\varepsilon(t)\rangle| \le \|\rho\|_\infty\sqrt{\ell}\left\|\frac{u_\varepsilon'(t)-p_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)}\right\|_{L^2(0,\ell)}$$
$$+\frac{\varepsilon\|\rho\|^2_\infty}{2\hat{C}}\left\|\frac{u_\varepsilon'(t)-p_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)}\right\|^2_{L^2(0,\ell)} + \frac{\hat{C}}{2\varepsilon}\int_0^\ell\left[e^{\alpha|\ln(1+\varepsilon p_\varepsilon(t))|}-1\right]dx.$$

In a similar way, for every $0 \le \tau \le t$ we obtain the following estimate

$$|\langle\dot{\rho}(\tau),(u_\varepsilon'(\tau)-p_\varepsilon(\tau))\rangle| \le \|\dot{\rho}\|_\infty\sqrt{\ell}\left\|\frac{u_\varepsilon'(\tau)-p_\varepsilon(\tau)}{1+\varepsilon p_\varepsilon(\tau)}\right\|_{L^2(0,\ell)}$$
$$+\frac{\varepsilon\|\dot{\rho}\|^2_\infty}{2\hat{C}}\left\|\frac{u_\varepsilon'(\tau)-p_\varepsilon(\tau)}{1+\varepsilon p_\varepsilon(\tau)}\right\|^2_{L^2(0,\ell)} + \frac{\hat{C}}{2\varepsilon}\int_0^\ell\left[e^{\alpha|\ln(1+\varepsilon p_\varepsilon(\tau))|}-1\right]dx.$$

Since $|\ln(1+\varepsilon p_\varepsilon(\tau))| \le \varepsilon z_\varepsilon(\tau) \le \varepsilon z_\varepsilon(t)$ on $(0,\ell)$ thanks to (4.8), we get for every $\hat{C} > 0$

$$\left|\int_0^t \langle \dot\rho(\tau), u_\varepsilon'(\tau) - p_\varepsilon(\tau)\rangle\, d\tau\right| \le t\|\dot\rho\|_\infty \sqrt{\ell} \sup_{0\le\tau\le t} \left\| \frac{u_\varepsilon'(\tau) - p_\varepsilon(\tau)}{1+\varepsilon p_\varepsilon(\tau)} \right\|_{L^2(0,\ell)} + \frac{\varepsilon t \|\dot\rho\|_\infty^2}{2\hat{C}} \sup_{0\le\tau\le t} \left\| \frac{u_\varepsilon'(\tau) - p_\varepsilon(\tau)}{1+\varepsilon p_\varepsilon(\tau)} \right\|^2_{L^2(0,\ell)} + \frac{\hat{C} t}{2\varepsilon} \int_0^\ell \left[e^{\alpha\varepsilon z_\varepsilon(t)} - 1\right] dx. \tag{5.8}$$

Let us come to the third term. Since $p_\varepsilon(0) = 0$, for every $i = 0, \dots, N$

$$\|p_\varepsilon(t_i)\|_{L^1(0,\ell)} \le \sum_{j=1}^N \|p_\varepsilon(t_j) - p_\varepsilon(t_{j-1})\|_{L^1(0,\ell)} \le \mathcal{V}_{L^1}(0,t;p_\varepsilon),$$

and we deduce, recalling that $\|r_\varepsilon^i(t)\|_\infty < \varepsilon(t_{i+1} - t_i)$,

$$\left|\sum_{i=0}^{N-1} \langle r_\varepsilon^i(t), p_\varepsilon(t_i)\rangle\right| \le \mathcal{V}_{L^1}(0,t;p_\varepsilon) \sum_{i=0}^{N-1} \|r_\varepsilon^i(t)\|_\infty \le \varepsilon t \mathcal{V}_{L^1}(0,t;p_\varepsilon). \tag{5.9}$$

In view of (5.7), (5.8), (5.9), and since $|r_\varepsilon(t)| < \varepsilon$, we obtain the following estimate for the right-hand side:

$$\begin{aligned} &\langle \rho(t), u_\varepsilon'(t) - p_\varepsilon(t)\rangle - \int_0^t \langle \dot\rho(\tau), u_\varepsilon'(\tau) - p_\varepsilon(\tau)\rangle\, d\tau - \sum_{i=0}^{N-1} \langle r_\varepsilon^i(t), p_\varepsilon(t_i)\rangle - r_\varepsilon(t) \\ &\le C_1 \sup_{0\le\tau\le t} \left\| \frac{u_\varepsilon'(\tau) - p_\varepsilon(\tau)}{1+\varepsilon p_\varepsilon(\tau)} \right\|_{L^2(0,\ell)} + \varepsilon C_2 \sup_{0\le\tau\le t} \left\| \frac{u_\varepsilon'(\tau) - p_\varepsilon(\tau)}{1+\varepsilon p_\varepsilon(\tau)} \right\|^2_{L^2(0,\ell)} \\ &\quad + \frac{C_3}{\varepsilon} \int_0^\ell \left[e^{\alpha\varepsilon z_\varepsilon(t)} - 1\right] dx + \varepsilon t \mathcal{V}_{L^1}(0,t;p_\varepsilon) + \varepsilon, \end{aligned} \tag{5.10}$$

for suitable $C_1, C_2, C_3 > 0$. Moreover C_3 can be chosen arbitrarily small.

Step 3: Conclusion. Combining (5.5) and (5.10) we deduce that for every $t \in [0,T]$

$$\begin{aligned} &\beta_1 \left\| \frac{u_\varepsilon'(t) - p_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)} \right\|^2_{L^2(0,\ell)} + \frac{d_1}{\varepsilon} \int_0^\ell \left(e^{\alpha\varepsilon z_\varepsilon(t)} - 1\right) dx + \frac{\beta}{2} \left(\mathcal{V}_{L^1}(0,t;p_\varepsilon) - \varepsilon\right) \\ &\le C_1 \sup_{0\le\tau\le t} \left\| \frac{u_\varepsilon'(\tau) - p_\varepsilon(\tau)}{1+\varepsilon p_\varepsilon(\tau)} \right\|_{L^2(0,\ell)} + \varepsilon C_2 \sup_{0\le\tau\le t} \left\| \frac{u_\varepsilon'(\tau) - p_\varepsilon(\tau)}{1+\varepsilon p_\varepsilon(\tau)} \right\|^2_{L^2(0,\ell)} \\ &\quad + \frac{C_3}{\varepsilon} \int_0^\ell \left[e^{\alpha\varepsilon z_\varepsilon(t)} - 1\right] dx + \varepsilon T\, \mathcal{V}_{L^1}(0,T;p_\varepsilon) + \varepsilon, \end{aligned}$$

where $d_1, C_1, C_2, C_3 > 0$, and with C_3 which can be chosen arbitrarily small. By choosing $C_3 < d_1$, and using the fact that z_ε is increasing in time (see (4.8)), we deduce that there exists $C > 0$ such that for ε small enough

$$\sup_{0\le\tau\le T} \left\| \frac{u_\varepsilon'(\tau) - p_\varepsilon(\tau)}{1+\varepsilon p_\varepsilon(\tau)} \right\|^2_{L^2(0,\ell)} + \frac{1}{\varepsilon} \int_0^\ell \left[e^{\alpha\varepsilon z_\varepsilon(T)} - 1\right] dx + \mathcal{V}_{L^1}(0,T;p_\varepsilon) \le C,$$

so that (5.1) is proved. □

Remark 5.4. Following the computation at the beginning of Step 2, and in view of the a priori bound (5.1), we can prove that there exists $C > 0$ such that for every $\varepsilon > 0$ and $t \in [0,T]$

$$\|1 + \varepsilon p_\varepsilon(t)\|_{L^\alpha(0,\ell)} \le C.$$

Moreover there exists $C' > 0$ such that for every $\varepsilon > 0$ and $t \in [0,T]$

$$\|u_\varepsilon'(t) - p_\varepsilon(t)\|_{L^1(0,\ell)} \le C'. \tag{5.11}$$

Indeed

$$\int_0^\ell |u'_\varepsilon(t) - p_\varepsilon(t)|\, dx = \int_0^\ell \left| \frac{u'_\varepsilon(t) - p_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)} \right| (1+\varepsilon p_\varepsilon(t))\, dx$$
$$\leq \left\| \frac{u'_\varepsilon(t) - p_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)} \right\|_{L^2(0,\ell)} \|1+\varepsilon p_\varepsilon(t)\|_{L^2(0,\ell)} \leq C'.$$

Finally, following the computation of Step 1, for every subdivision $0 = s_0 < s_1 < \cdots < s_m = t$, the following inequality holds

$$Diss_\varepsilon(0,t;(p_\varepsilon, z_\varepsilon)) \geq \alpha d \sum_{i=1}^m \int_0^\ell |p_\varepsilon(s_j) - p_\varepsilon(s_{j-1})|\, dx = \sum_{i=1}^m \mathcal{H}(p_\varepsilon(s_j) - p_\varepsilon(s_{j-1})). \tag{5.12}$$

In view of the a priori bound (5.1), we can obtain the following variant of Lemma 5.3 which will be useful in the asymptotic analysis as $\varepsilon \to 0$.

Lemma 5.5. *Let $\rho \in C^1([0,T]\times[0,\ell])$ be the function associated with the safe load condition (4.17) satisfied by the external loads. Then for every $t \in [0,T]$ and every subdivision $S = \{0 = s_0 < s_1 < \cdots < s_m = t\}$ of $[0,t]$ we have*

$$\int_0^t \langle \dot\rho(\tau), p_\varepsilon(\tau)\rangle d\tau = \langle \rho(t), p_\varepsilon(t)\rangle - \sum_{j=1}^m \langle \rho(s_j), p_\varepsilon(s_j) - p_\varepsilon(s_{j-1})\rangle + o_{\varepsilon,\delta}(t), \tag{5.13}$$

where

$$\delta := \max_{j=1,\dots,m} (s_j - s_{j-1})$$

and

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} |o_{\varepsilon,\delta}(t)| = 0. \tag{5.14}$$

Proof. Let us consider the refinement $\{0 = t_0 < t_1 < \cdots < t_N = t\}$ of S given by Lemma 5.3. Then in view of the a priori bound on $\mathcal{V}_{L^1}(0,t;p_\varepsilon)$ and since $p_\varepsilon(0) = 0$ we have

$$\int_0^t \langle \dot\rho(\tau), p_\varepsilon(\tau)\rangle d\tau = \langle \rho(t), p_\varepsilon(t)\rangle - \sum_{i=1}^N \langle \rho(t_i), p_\varepsilon(t_i) - p_\varepsilon(t_{i-1})\rangle + \hat r_\varepsilon(t),$$

where $\lim_{\varepsilon\to 0} \hat r_\varepsilon(t) = 0$. Replacing $\rho(t_i)$ with $\rho(s_j)$ on those intervals $[t_{i-1}, t_i]$ which lie in the same interval $[s_{j-1}, s_j]$, we get in view of the regularity of ρ

$$\int_0^t \langle \dot\rho(\tau), p_\varepsilon(\tau)\rangle d\tau = \langle \rho(t), p_\varepsilon(t)\rangle - \sum_{j=1}^m \langle \rho(s_j), p_\varepsilon(s_j) - p_\varepsilon(s_{j-1})\rangle + \tilde r_{\varepsilon,\delta}(t) + \hat r_\varepsilon(t),$$

where

$$|\tilde r_{\varepsilon,\delta}(t)| \leq \delta \|\dot\rho\|_\infty \mathcal{V}_{L^1}(0,t;p_\varepsilon).$$

Then (5.13) and (5.14) follow by setting $o_{\varepsilon,\delta}(t) := \tilde r_{\varepsilon,\delta}(t) + \hat r_\varepsilon(t)$. □

5.2. **Compactness.** In this subsection we prove that $t \mapsto (u_\varepsilon(t), p_\varepsilon(t), z_\varepsilon(t))$ determines as $\varepsilon \to 0$ a map $t \mapsto (u(t), e(t), p(t)) \in \mathcal{A}$, where $\mathcal{A}$ is the family of admissible configurations in linear perfect plasticity introduced in Definition 4.2.

In order to handle the boundary condition at $x = 0$, we extend $u_\varepsilon(t)$ and $p_\varepsilon(t)$ to $]-\ell, 0[$ by setting

$$u_\varepsilon(t) = p_\varepsilon(t) = 0.$$

From Proposition 5.1 we can say that there exists $C > 0$ such that for every $t \in [0,T]$ and $\varepsilon > 0$

$$\left\| \frac{u'_\varepsilon(t) - p_\varepsilon(t)}{1+\varepsilon p_\varepsilon(t)} \right\|^2_{L^2(-\ell,\ell)} + \frac{1}{\varepsilon}\int_0^\ell \left(e^{\alpha\varepsilon z_\varepsilon(t)} - 1 \right) dx + \tilde{\mathcal{V}}_{L^1}(0,t;p_\varepsilon) \leq C,$$

where $\tilde{\mathcal{V}}_{L^1}(a,b;p_\varepsilon)$ denotes the total variation on $[a,b] \subseteq [0,T]$ of $t \mapsto p_\varepsilon(t)$ as a map from $[0,T]$ to $L^1(-\ell,\ell)$.

Proposition 5.6. *The following properties hold true for every $t \in [0,T]$.*

(a) *There exist $p \in BV(0,T;\mathcal{M}_b(]-\ell,\ell[))$ and a sequence $\varepsilon_n \to 0$ such that setting $p_n(t) := p_{\varepsilon_n}(t)$, for every $t \in [0,T]$*

$$p_n(t) \overset{*}{\rightharpoonup} p(t) \qquad \text{weakly}^* \text{ on } \mathcal{M}_b(]-\ell,\ell[). \tag{5.15}$$

In particular $p(t) = 0$ on $]-\ell,0[$.

(b) *Setting $u_n(t) := u_{\varepsilon_n}(t)$, there exist $e(t) \in L^2(-\ell,\ell)$ and $u(t) \in BV(-\ell,\ell)$ with $e(t) = u(t) = 0$ on $]-\ell,0[$ and a subsequence n_k depending possibly on t such that*

$$u'_{n_k}(t) - p_{n_k}(t) \rightharpoonup e(t) \qquad \text{weakly in } L^1(-\ell,\ell), \tag{5.16}$$

and

$$u_{n_k}(t) \overset{*}{\rightharpoonup} u(t) \qquad \text{weakly}^* \text{ in } BV(-\ell,\ell) \tag{5.17}$$

with

$$u'(t) = e(t) + p(t) \qquad \text{on }]-\ell,\ell[. \tag{5.18}$$

Finally, setting for every $M > 1$

$$A^n_M := \{x \in]-\ell,\ell[:\ 1 + \varepsilon_n p_n(t,x) \le M\},$$

then

$$(u'_{n_k}(t) - p_{n_k}(t))1_{A^{n_k}_M} \rightharpoonup e(t) \qquad \text{weakly in } L^2(-\ell,\ell). \tag{5.19}$$

(c) *Setting $z_n(t) := z_{\varepsilon_n}(t)$, we have*

$$e^{\alpha\varepsilon_n z_n(t)} \to 1 \qquad \text{strongly in } L^1(0,\ell). \tag{5.20}$$

Proof. Since $\tilde{\mathcal{V}}_{L^1}(0,T;p_\varepsilon) \le C$ for every $\varepsilon > 0$, by the generalized version of Helly's theorem (see [5, Theorem 3.2]) we deduce that there exist

$$p \in BV(0,T;\mathcal{M}_b(]-\ell,\ell[))$$

and a sequence $\varepsilon_n \to 0$ such that, setting

$$(u_n(t), p_n(t), z_n(t)) := (u_{\varepsilon_n}(t), p_{\varepsilon_n}(t), z_{\varepsilon_n}(t)),$$

for every $t \in [0,T]$

$$p_n(t) \overset{*}{\rightharpoonup} p(t) \qquad \text{weakly}^* \text{ on } \mathcal{M}_b(]-\ell,\ell[).$$

Hence (5.15) holds true. Moreover, since $p_n(t) = 0$ on $]-\ell,0[$, we obtain $p(t) = 0$ on $]-\ell,0[$, and property (a) is proved.

For every $M > 1$ let us also set

$$B^n_M := \{x \in]-\ell,\ell[:\ 1 + \varepsilon_n p_n(t,x) > M\}.$$

Since

$$\frac{1}{\varepsilon_n}\int_0^\ell \left(e^{\alpha\varepsilon_n z_n(t)} - 1\right) dx \le C,$$

and $\varepsilon_n z_n(t) \ge \ln(1 + \varepsilon_n p_n(t))$ on $(0,\ell)$ by (4.8), keeping into account that $B^n_M \subseteq [0,\ell[$ we infer

$$|B^n_M| \le \frac{C\varepsilon_n}{M^\alpha - 1}. \tag{5.21}$$

As

$$C \ge \int_{-\ell}^{\ell} \left|\frac{u'_n(t) - p_n(t)}{1 + \varepsilon_n p_n(t)}\right|^2 dx \ge \frac{1}{M^2}\int_{A^n_M} |u'_n(t) - p_n(t)|^2\, dx,$$

using a diagonal argument we deduce that there exists a subsequence $(n_k)_{k\in\mathbb{N}}$, depending possibly on t, such that for every $M \in \mathbb{N}, M > 1$,

$$(u'_{n_k}(t) - p_{n_k}(t))1_{A^{n_k}_M} \rightharpoonup e_M(t) \qquad \text{weakly in } L^2(-\ell,\ell), \tag{5.22}$$

as $k \to \infty$, where $e_M(t) \in L^2(-\ell,\ell)$ with $e_M(t) = 0$ on $]-\ell,0[$.

We claim that $e_M(t) =: e(t)$ does not depend on M, and that

$$u'_{n_k}(t) - p_{n_k}(t) \rightharpoonup e(t) \qquad \text{weakly in } L^1(-\ell,\ell).$$

Indeed

$$u'_{n_k}(t) - p_{n_k}(t) = (u'_{n_k}(t) - p_{n_k}(t))1_{A_M^{n_k}} + (u'_{n_k}(t) - p_{n_k}(t))1_{B_M^{n_k}},$$

and in view of Remark 5.4 (recall that $\alpha > 3$) and of (5.21)

$$\begin{aligned}\int_{-\ell}^{\ell} |u'_{n_k}(t) - p_{n_k}(t)| 1_{B_M^{n_k}}\, dx &= \int_{-\ell}^{\ell} \left| \frac{u'_{n_k}(t) - p_{n_k}(t)}{1 + \varepsilon_{n_k} p_{n_k}(t)} \right| (1 + \varepsilon_{n_k} p_{n_k}(t)) 1_{B_M^{n_k}}\, dx \\ &\le \left\| \frac{u'_{n_k}(t) - p_{n_k}(t)}{1 + \varepsilon_{n_k} p_{n_k}(t)} \right\|_{L^2(-\ell,\ell)} \|(1 + \varepsilon_{n_k} p_{n_k}(t)) 1_{B_M^{n_k}}\|_{L^2(-\ell,\ell)} \\ &\le C \left\| \frac{u'_{n_k}(t) - p_{n_k}(t)}{1 + \varepsilon_{n_k} p_{n_k}(t)} \right\|_{L^2(-\ell,\ell)} \|(1 + \varepsilon_{n_k} p_{n_k}(t))\|_{L^\alpha(-\ell,\ell)}^{\frac{2}{\alpha}} |B_M^{n_k}|^{\frac{\alpha-2}{\alpha}} \to 0.\end{aligned}$$

Then taking into account (5.22), the claim follows. In particular we deduce that (5.16) holds true. Moreover, we can let the truncation level M vary in $\mathbb{R}$, obtaining thus (5.19).

Finally, in view of (5.16) and (5.15), we deduce that there exists $u(t) \in BV(-\ell,\ell)$ with $u(t) = 0$ on $]-\ell, 0[$ and such that (5.17) is satisfied. Moreover, (5.18) is obtained taking the limit as $k \to \infty$ in

$$u'_{n_k}(t) = (u'_{n_k}(t) - p_{n_k}(t)) + p_{n_k}(t),$$

so that (b) is now completely proved.

Coming to property (c), from

$$\frac{1}{\varepsilon_n} \int_0^\ell \left(e^{\alpha \varepsilon_n z_n(t)} - 1 \right) dx \le C$$

and recalling that $z_n(t) \ge 0$ on $(0,\ell)$, we immediately deduce that (5.20) holds, and the proof is concluded. □

5.3. **Proof of the linearization result.** We now prove Theorem 4.4.

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$,

$$p \in BV(0,T; \mathcal{M}_b(]-\ell,\ell[))$$

and

$$t \mapsto (u(t), e(t)) \in BV(-\ell,\ell) \times L^2(-\ell,\ell)$$

be given by Proposition 5.6.

Let us restrict $u(t), e(t)$ to $(0,\ell)$, and $p(t)$ to $[0,\ell[$. Recalling that $u(t) = e(t) = p(t) = 0$ on $]-\ell, 0[$, in view of (5.18) we get

$$u'(t) = e(t) + p(t) \qquad \text{on }]0,\ell[$$

and

$$p(t)(\{0\}) = u(t,0),$$

being $u(t,0)$ the trace of $u(t)$ at $x = 0$. We thus deduce that for every $t \in [0,T]$

$$(u(t), e(t), p(t)) \in \mathcal{A}.$$

In order to complete the proof of Theorem 4.4, we need to show that $t \in [0,T] \mapsto (u(t), e(t), p(t))$ satisfies the conditions qualifying a quasi-static evolution in linear perfect plasticity, and that convergences (4.21) and (4.22) hold along the entire sequence ε_n (and not only along ε_{n_k} as in (5.17) and (5.16)). Finally we need to prove that the convergence for the elastic energy (4.24) holds true.

We proceed in four steps.

Step 1: Lower semicontinuity for the elastic energy. First of all we prove that for every $t \in [0,T]$, being n_k the subsequence given by Proposition 5.6,

$$\mathcal{Q}(e(t)) \le \liminf_k \mathcal{W}_{\varepsilon_{n_k}}(u_{n_k}(t), p_{n_k}(t)). \tag{5.23}$$

Indeed for every $\tilde{c} < c_W := W''(1)$ there exists a neighborhood A of 0 such that

$$\forall y \in A \,:\, W(1+y) \geq \frac{\tilde{c}}{2} y^2.$$

Since $\frac{u'_{n_k}(t)-p_{n_k}(t)}{1+\varepsilon_{n_k} p_{n_k}(t)}$ is bounded in $L^2(0,\ell)$ in view of the a priori bound (5.1), we infer

$$\varepsilon_{n_k} \frac{u'_{n_k}(t)-p_{n_k}(t)}{1+\varepsilon_{n_k} p_{n_k}(t)} \to 0 \qquad \text{strongly in } L^2(0,\ell).$$

Up to a subsequence (not relabeled), the convergence is almost uniform, hence for every $\eta > 0$ there exists $E_\eta \subseteq [0,\ell]$ with $|E_\eta| < \eta$ and such that for $x \notin E_\eta$ and for k large enough

$$\varepsilon_{n_k} \frac{u'_{n_k}(t,x)-p_{n_k}(t,x)}{1+\varepsilon_{n_k} p_{n_k}(t,x)} \in A.$$

In particular, taking into account (5.19), for every $M > 1$ we deduce that

$$\begin{aligned}\liminf_k \frac{1}{\varepsilon_{n_k}^2} \int_0^\ell W\left(1+\varepsilon_{n_k} \frac{u'_{n_k}(t)-p_{n_k}(t)}{1+\varepsilon_{n_k} p_{n_k}(t)}\right) dx &\geq \liminf_k \frac{\tilde{c}}{2} \int_{[0,\ell]\setminus E_\eta} \left| \frac{u'_{n_k}(t)-p_{n_k}(t)}{1+\varepsilon_{n_k} p_{n_k}(t)} \right|^2 dx \\ &\geq \liminf_k \frac{\tilde{c}}{2M^2} \int_{[0,\ell]\setminus E_\eta} \left| (u'_{n_k}(t)-p_{n_k}(t)) 1_{A_M^{n_k}} \right|^2 dx \geq \frac{\tilde{c}}{2M^2} \int_{[0,\ell]\setminus E_\eta} |e(t)|^2 \, dx.\end{aligned}$$

Letting $\tilde{c} \to c_W$, $M \to 1$ and $\eta \to 0$, inequality (5.23) follows.

Step 2: Global stability. Let us show that $(u(t), e(t), p(t)) \in \mathcal{A}$ is a globally stable configuration according to (4.15) for every $t \in [0,T]$.

Given $(v,\eta,q) \in \mathcal{A}$, let us consider $\xi_j \in C^\infty([0,\ell])$ such that

$$\xi_j \overset{*}{\rightharpoonup} q - p(t) \qquad \text{weakly}^* \text{ in } \mathcal{M}_b([0,\ell[) \tag{5.24}$$

with

$$\lim_{j\to\infty} \|\xi_j\|_{L^1(0,\ell)} = |q-p(t)|([0,\ell[). \tag{5.25}$$

Let moreover $\eta_j \in C^\infty([0,\ell])$ be such that

$$\eta_j \to \eta \qquad \text{strongly in } L^2(0,\ell). \tag{5.26}$$

Notice that η_j can be constructed by a regularization via convolution, while ξ_j can be obtained by translating the measure $q-p(t)$ infinitesimally to the right, regularizing by convolution, and employing a diagonal argument.

We construct a configuration $(v_{n_k}, q_{n_k}, \zeta_{n_k}) \in \mathcal{F}_{\varepsilon_{n_k}} \times \mathcal{Z}_{\varepsilon_{n_k}}$ in the following way.

(1) Let $q_{n_k} :]0,\ell[\to \mathbb{R}$ be such that

$$\frac{1+\varepsilon_{n_k} q_{n_k}}{1+\varepsilon_{n_k} p_{n_k}(t)} = 1 + \varepsilon_{n_k} \xi_j, \tag{5.27}$$

that is

$$q_{n_k} := \xi_j + p_{n_k}(t) + \varepsilon_{n_k} p_{n_k}(t) \xi_j. \tag{5.28}$$

Since ξ_j is smooth on $[0,\ell]$, for k large enough we have

$$1 + \varepsilon_{n_k} q_{n_k} > 0 \qquad \text{a.e. in }]0,\ell[. \tag{5.29}$$

(2) For k large enough let

$$\zeta_{n_k} := z_{n_k}(t) + \frac{1}{\varepsilon_{n_k}} \left| \ln \frac{1+\varepsilon_{n_k} q_{n_k}}{1+\varepsilon_{n_k} p_{n_k}(t)} \right|, \tag{5.30}$$

so that in view of (5.29)

$$(q_{n_k}, \zeta_{n_k}) \in \mathcal{Z}_{\varepsilon_{n_k}}.$$

(3) Let finally $v_{n_k} : [0,\ell] \to \mathbb{R}$ be such that

$$\frac{v'_{n_k} - q_{n_k}}{1+\varepsilon_{n_k} q_{n_k}} = \eta_j, \tag{5.31}$$

that is

$$v'_{n_k} := q_{n_k} + \eta_j + \varepsilon_{n_k} q_{n_k} \eta_j, \qquad v_{n_k}(0) = 0. \tag{5.32}$$

Since

$$1 + \varepsilon_{n_k} v'_{n_k} = (1+\varepsilon_{n_k} q_{n_k})(1+\varepsilon_{n_k}\eta_j),$$

we deduce in view of (5.29) that $v_{n_k} \in \mathcal{F}_{n_k}$ for k large enough.

Thanks to (5.31) we get

$$\begin{aligned} \lim_k \mathcal{W}_{\varepsilon_{n_k}}(v_{n_k}, q_{n_k}) &= \lim_k \frac{1}{\varepsilon_{n_k}^2} \int_0^\ell W\left(1 + \varepsilon_{n_k} \frac{v'_{n_k} - q_{n_k}}{1+\varepsilon_{n_k} q_{n_k}}\right) dx \\ &= \lim_k \frac{1}{\varepsilon_{n_k}^2} \int_0^\ell W\left(1+\varepsilon_{n_k}\eta_j\right) dx = \frac{W''(1)}{2} \int_0^\ell |\eta_j|^2 \, dx = \mathcal{Q}(\eta_j). \end{aligned} \tag{5.33}$$

Moreover, taking into account (5.30) and (5.27), we obtain

$$\begin{aligned} \lim_k \mathcal{H}_{\varepsilon_{n_k}}\Big((p_{n_k}(t), z_{n_k}(t)), (q_{n_k}, \zeta_{n_k})\Big) &= \lim_k \frac{d}{\varepsilon_{n_k}} \int_0^\ell e^{\alpha \varepsilon_{n_k} z_{n_k}(t)} \left[e^{\alpha |\ln(1+\varepsilon_{n_k}\xi_j)|} - 1\right] dx \\ &= \lim_k d \int_0^\ell e^{\alpha\varepsilon_{n_k} z_{n_k}(t)} \left[\frac{e^{\alpha|\ln(1+\varepsilon_{n_k}\xi_j)|} - 1}{\varepsilon_{n_k}}\right] dx = \alpha d \int_0^\ell |\xi_j| \, dx = \mathcal{H}(\xi_j) \end{aligned} \tag{5.34}$$

where we used (5.20) to compute the limit.

Since $(u_{n_k}(t), p_{n_k}(t), z_{n_k}(t))$ is globally stable we can write

$$\begin{aligned} &\mathcal{W}_{\varepsilon_{n_k}}(u_{n_k}(t), p_{n_k}(t)) - \langle \Lambda(t), u_{n_k}(t)\rangle \\ &\qquad \le \mathcal{W}_{\varepsilon_{n_k}}(v_{n_k}, q_{n_k}) - \langle \Lambda(t), v_{n_k}\rangle + \mathcal{H}_{\varepsilon_{n_k}}\Big((p_{n_k}(t), z_{n_k}(t)), (q_{n_k}, \zeta_{n_k})\Big). \end{aligned} \tag{5.35}$$

In view of (5.32) and of (5.28) we get

$$v'_{n_k} = \eta_j + \xi_j + p_{n_k}(t) + \alpha_k$$

where $\alpha_k \to 0$ strongly in $L^1(0,\ell)$ as $k \to \infty$. Then

$$\langle \Lambda(t), v_{n_k}\rangle = \langle \rho(t), \eta_j + \xi_j + p_{n_k}(t)\rangle + o_k(t),$$

where $o_k(t) \to 0$ as $k \to \infty$. Erasing the terms $\langle \rho(t), p_{n_k}(t)\rangle$ from both sides, we can rewrite (5.35) as

$$\begin{aligned} &\mathcal{W}_{\varepsilon_{n_k}}(u_{n_k}(t), p_{n_k}(t)) - \langle \rho(t), e_{n_k}(t)\rangle \\ &\qquad \le \mathcal{W}_{\varepsilon_{n_k}}(v_{n_k}, q_{n_k}) - \langle \rho(t), \eta_j + \xi_j\rangle + \mathcal{H}_{\varepsilon_{n_k}}\Big((p_{n_k}(t), z_{n_k}(t)), (q_{n_k}, \zeta_{n_k})\Big) + o_k(t). \end{aligned}$$

Taking the limit as $k \to \infty$, and keeping into account (5.23), (5.16), (5.33) and (5.34), we obtain

$$\mathcal{Q}(e(t)) - \langle \rho(t), e(t)\rangle \le \mathcal{Q}(\eta_j) - \langle \rho(t), \eta_j + \xi_j\rangle + \mathcal{H}(\xi_j).$$

Let now $j \to \infty$; in view of (5.24), (5.25) and (5.26) we deduce that

$$\mathcal{Q}(e(t)) - \langle \rho(t), e(t)\rangle \le \mathcal{Q}(\eta) - \langle \rho(t), \eta\rangle - \int_{[0,\ell[} \rho(t) \, d(q - p(t)) + \mathcal{H}(q - p(t))$$

(recall that (5.24) and (5.25) entail *strict convergence* of ξ_j to the measure $q - p(t)$ on $[0,\ell[$, so that integrals on $[0,\ell[$ of bounded and continuous functions, like $\rho(t)$, pass to the limit). The global stability (4.15) now easily follows.

Step 3: Consequences of the global stability. We claim that $(u(t), e(t))$ are uniquely determined by $p(t)$, so that (5.16) and (5.17) hold without passing to a subsequence n_k depending on

t, and the lower semicontinuity (5.23) for the elastic energy holds along the entire sequence ε_n (in particular (4.21) and (4.22) will follow).

Since the map

$$(v, \eta, q) \in \mathcal{A} \mapsto \mathcal{Q}(\eta) - \langle \Lambda(t), v \rangle + \mathcal{H}(q)$$

is convex, the analysis of the first-order optimality condition entails that the global stability of $(u(t), e(t), p(t))$ is equivalent to the inequalities

$$\forall (v, \eta, q) \in \mathcal{A} \,:\, -\mathcal{H}(q) \le c_W \int_0^\ell e(t)\eta \, dx - \langle \Lambda(t), v \rangle \le \mathcal{H}(-q). \tag{5.36}$$

Assume that $(\tilde{u}(t), \tilde{e}(t))$ is a limit point of $(u_n(t), u_n'(t) - p_n(t))$. By Step 2, $(\tilde{u}(t), \tilde{e}(t), p(t)) \in \mathcal{A}$ is globally stable, and thanks to (5.36) with $(v, \eta, q) = (\tilde{u}(t) - u(t), \tilde{e}(t) - e(t), 0)$ we can write

$$\int_0^\ell e(t)(\tilde{e}(t) - e(t))\, dx = 0 \qquad \text{and} \qquad \int_0^\ell \tilde{e}(t)(\tilde{e}(t) - e(t))\, dx = 0,$$

so that $\mathcal{Q}(\tilde{e}(t) - e(t)) = 0$, hence $\tilde{e}(t) = e(t)$.

Since $\tilde{u}'(t) = \tilde{e}(t) + p(t) = e(t) + p(t) = u'(t)$, and $\tilde{u}(t, 0) = p(t)(\{0\}) = u(t, 0)$, we deduce that $\tilde{u}(t) = u(t)$, so that the claim follows.

Step 4: Conclusion. In view of the above steps, in order to conclude the proof we need to show that $t \in [0, T] \mapsto (u(t), e(t), p(t))$ satisfies the energy equality (4.16) and that the convergence (4.24) holds true.

For every $\delta > 0$, let $S = \{0 = s_0 < s_1 < \cdots < s_m = t\}$ be a subdivision of $[0, t]$ with $\max_{i=1,\dots,m}(s_i - s_{i-1}) < \delta$ and such that

$$\begin{aligned}\sum_{j=1}^m [\mathcal{H}(p(s_j) - p(s_{j-1})) - \langle \rho(s_j), p(s_j) - p(s_{j-1}) \rangle] \\ \ge Diss(0, t; p) - \langle \rho(t), p(t) \rangle + \int_0^t \langle \dot{\rho}(\tau), p(\tau) \rangle \, d\tau - \delta.\end{aligned} \tag{5.37}$$

Such a subdivision can be found by choosing a first subdivision $S' = \{0 = s_0' < s_1' < \cdots < s_m' = t\}$ with

$$\sum_{j=1}^m \mathcal{H}(p(s_j') - p(s_{j-1}')) \ge Diss(0, t; p) - \delta/2.$$

Then we refine S' in order to approximate the integral $\int_0^t \langle \dot{\rho}(\tau), p(\tau) \rangle \, d\tau$ following the arguments of Lemma 5.3. Notice that $p(t)$ is now a measure, rather than an L^1-function, which is uniformly bounded in $\mathcal{M}_b([0, \ell[)$ as $t \in [0, T]$. Hence $\tau \mapsto \langle \rho(\tau), p(\tau) \rangle$ is only measurable and bounded: for the approximation of the associated Lebesgue integral by means of Riemann sums we refer the reader to [3, Lemma 4.12].

Writing the energy equality with the help of the function $\rho(t)$ as

$$\begin{aligned}\mathcal{W}_n(u_n(t), p_n(t)) + Diss_n(0, t; (p_n, z_n)) - \langle \rho(t), p_n(t) \rangle + \int_0^t \langle \dot{\rho}(\tau), p_n(\tau) \rangle \, d\tau \\ = -\int_0^t \langle \dot{\rho}(\tau), u_n'(\tau) - p_n(\tau) \rangle \, d\tau + \langle \rho(t), u_n'(t) - p_n(t) \rangle,\end{aligned}$$

in view of Lemma 5.5 and of (5.12), we obtain

$$\begin{aligned}\mathcal{W}_n(u_n(t), p_n(t)) + \sum_{j=1}^m [\mathcal{H}(p_n(s_j) - p_n(s_{j-1})) - \langle \rho(s_j), p_n(s_j) - p_n(s_{j-1}) \rangle] \\ \le -\int_0^t \langle \dot{\rho}(\tau), u_n'(\tau) - p_n(\tau) \rangle \, d\tau + \langle \rho(t), u_n'(t) - p_n(t) \rangle + o_{n,\delta}(t),\end{aligned}$$

where $o_{n,\delta}(t)$ is such that

$$\lim_{\delta \to 0} \limsup_{n \to \infty} |o_{n,\delta}(t)| = 0.$$

Letting $\psi_\eta \in C^\infty([0,\ell])$ be such that

$$\begin{cases} 0 \le \psi_\eta \le 1 & \text{on } [0,\ell] \\ \psi_\eta = 1 & \text{on } [0,\ell-\eta] \\ \psi_\eta = 0 & \text{on } [l-\eta/2,\ell], \end{cases}$$

we can write for every j in view of the safe load condition (4.17)

$$\mathcal{H}(p_n(s_j) - p_n(s_{j-1})) - \langle \rho(s_j), p_n(s_j) - p_n(s_{j-1})\rangle \\ \ge \mathcal{H}(\psi_\eta(p_n(s_j) - p_n(s_{j-1}))) - \langle \rho(s_j), \psi_\eta(p_n(s_j) - p_n(s_{j-1}))\rangle.$$

Hence

$$\mathcal{W}_n(u_n(t), p_n(t)) + \sum_{j=1}^{m} [\mathcal{H}(\psi_\eta(p_n(s_j) - p_n(s_{j-1}))) - \langle \rho(s_j), \psi_\eta(p_n(s_j) - p_n(s_{j-1}))\rangle] \\ \le -\int_0^t \langle \dot\rho(\tau), u_n'(\tau) - p_n(\tau)\rangle\, d\tau + \langle \rho(t), u_n'(t) - p_n(t)\rangle + o_{n,\delta}(t).$$

Now take the limit as $n \to \infty$ and then as $\eta \to 0$: since $\psi_\eta \rho(t)$ has compact support in $[0,\ell[$, we get in view of (5.15)

$$\lim_n \langle \rho(s_j), \psi_\eta(p_n(s_j) - p_n(s_{j-1}))\rangle = \lim_n \int_0^\ell \rho(s_j)\psi_\eta(p_n(s_j) - p_n(s_{j-1}))\, dx \\ = \int_{[0,\ell[} \rho(s_j)\psi_\eta \, d(p(s_j) - p(s_{j-1})).$$

Taking into account (5.23) (along the entire sequence in view of Step 3), (5.37), and (4.22) together with (5.11), we obtain

$$\mathcal{Q}(e(t)) + Diss(0,t;p) - \langle \rho(t), p(t)\rangle + \int_0^t \langle \dot\rho(\tau), p(\tau)\rangle\, d\tau - \delta \\ \le -\int_0^t \langle \dot\rho(\tau), u'(\tau) - p(\tau)\rangle\, d\tau + \langle \rho(t), e(t)\rangle + \tilde o_\delta(t), \tag{5.38}$$

where $\tilde o_\delta(t) \to 0$ as $\delta \to 0$. Letting $\delta \to 0$, we obtain the energy inequality

$$\mathcal{Q}(e(t)) - \langle \Lambda(t), u(t)\rangle + Diss(0,t;p) \le -\int_0^t \langle \dot\Lambda(\tau), u(\tau)\rangle\, d\tau. \tag{5.39}$$

The reverse inequality is a standard consequence of the global stability condition (4.15): it suffices to set $s_m^i := \frac{i}{m}t$ for $i = 0, \dots, m$, to test the minimality of the configuration $(u(s_i), e(s_i), p(s_i))$ by $(u(s_{i+1}), e(s_{i+1}), p(s_{i+1}))$, summing over i, and to let $m \to \infty$. We refer the reader to [2, Theorem 4.7] for the details. We conclude that

$$\mathcal{Q}(e(t)) - \langle \Lambda(t), u(t)\rangle + Diss(0,t;p) = -\int_0^t \langle \dot\Lambda(\tau), u(\tau)\rangle\, d\tau$$

so that $t \in [0,T] \mapsto (u(t), e(t), p(t))$ is a quasi-static evolution according to Definition 4.3.

In order to complete the proof, we need to show the convergence (4.24) for the elastic energies. Note that by lower semicontinuity (see Step 1), any limit point of $(\mathcal{W}_{\varepsilon_n}(u_n(t), p_n(t)))_{n\in\mathbb{N}}$ is greater than $\mathcal{Q}(e(t))$. If by contradiction one of them, achieved along a subsequence $(n_k)_{k\in\mathbb{N}}$, is strictly greater than $\mathcal{Q}(e(t))$, then performing the argument above along that sequence we would get a strict inequality in (5.38), which would be maintained in (5.39): this is against energy equality. □

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