Multi-scale free-discontinuity problems with soft inclusions

Andrea Braides
Dipartimento di Matematica, Università di Roma Tor Vergata
via della ricerca scientifica 1, 00133 Roma, Italy

Margherita Solci
DADU, Università di Sassari
piazza Duomo 6, 07041 Alghero (SS), Italy

1 Introduction

In this paper we study the asymptotic behaviour of free-discontinuity problems in a periodic geometry of $\mathbb{R}^n$ with "soft inclusions" represented by a periodic array of disjoint compact sets

$$E_0 = \bigcup_{i\in\mathbb{Z}^n} (i + K).$$

Homogenization problems with such a geometry are widely studied by Γ-convergence methods in the framework of integral functionals on Sobolev spaces. In that case, the prototypical energy functionals are of the form

$$G^\alpha_\varepsilon(u) = \int_{\Omega\cap\varepsilon E} |\nabla u|^2\,dx + c_\alpha\varepsilon^\alpha \int_{\Omega\cap\varepsilon E_0} |\nabla u|^2\,dx + \int_\Omega g(u)\,dx,$$

where Ω is an open subset of $\mathbb{R}^n$, $E := \mathbb{R}^n \setminus E_0$, $c_\alpha \geq 0$, $\alpha > 0$, and g is a suitable continuous function satisfying growth conditions. The limit case $c_\alpha = 0$ is the one of *perforated domains with Neumann conditions* (see e.g. Acerbi *et al.* [1], Braides and Garroni [11, 9]) while $\alpha = 2$ corresponds to *double-porosity homogenization* (see e.g. Braides, Chiadò Piat and Piatnitski [8]). In the latter a non trivial interaction between g and the "weak" term takes place.

More recently, also homogenization problems for surface energies have been studied in this geometry by Solci in [16], where functionals defined on sets of finite perimeter modeled on the prototypical case

$$H^\beta_\varepsilon(A) = \mathcal{H}^{n-1}((\Omega \setminus \varepsilon E_0) \cap \partial A) + c_\beta\varepsilon^\beta\mathcal{H}^{n-1}(\Omega \cap \varepsilon E_0 \cap \partial A) + \int_{\Omega\cap A} \psi(x)\,dx$$

have been analyzed. Note that for these energies the double-porosity phenomenon takes place for $\beta = 1$.

Free-discontinuity energies possess interacting bulk and surface parts, and their prototypical example is the Mumford-Shah functional (see e.g. Braides [5]). The corresponding "soft-inclusion" energies are then

$$\begin{aligned} F_\varepsilon^{\alpha,\beta}(u) &= \int_{\Omega\setminus\varepsilon E_0} |\nabla u|^2\,dx + c_\alpha \varepsilon^\alpha \int_{\Omega\cap\varepsilon E_0} |\nabla u|^2\,dx \\ &\quad + \mathcal{H}^{n-1}((\Omega\setminus\varepsilon E_0)\cap S(u)) + c_\beta \varepsilon^\beta \mathcal{H}^{n-1}(\Omega\cap\varepsilon E_0\cap S(u)) \\ &\quad + \int_\Omega g(u)\,dx, \end{aligned} \tag{1}$$

where $S(u)$ denotes the set of discontinuity points of u and g is a continuous function. Note that for $u \in H^1(\Omega)$ energy (1) turns into the energy $G_\varepsilon^\alpha(u)$ and for $u = \chi_A$ with A of finite perimeter we have $F_\varepsilon^{\alpha,\beta}(u) = H_\varepsilon^\beta(A)$ with $\psi(x) = g(1) - g(0)$ (up to the additive constant $g(0)|\Omega|$).

Energies (1) can be interpreted in the framework of the variational Griffith theory of fracture (see [4]) as describing a composite of brittle (linear) elastic materials with weak inclusions, whose 'weakness' derives from small elastic constants and/or from small fracture toughness. This approach may model the effect of damaged zones in an undamaged material (for this kind of problems there exists an enormous applied literature; see e.g. [3], [15], [14], etc.)

The case of Neumann boundary conditions $c_\alpha = c_\beta = 0$ and $g = 0$ has been examined by Cagnetti and Scardia [12], who proved an equicoerciveness result for the corresponding energies

$$F_\varepsilon^0(u) := \int_{\Omega\setminus\varepsilon E_0} |\nabla u|^2\,dx + \mathcal{H}^{n-1}((\Omega\setminus\varepsilon E_0)\cap S(u))$$

with respect to the convergence $u_\varepsilon \to u$ defined by

$$u_\varepsilon \chi_{\Omega\setminus\varepsilon E_0} \rightharpoonup C_E u \qquad \text{locally in } L^1(\Omega), \tag{2}$$

where $C_E = 1 - |K|$ and $u \in SBV(\Omega)$. Correspondingly, they proved a homogenization theorem showing that the Γ-limit of F_ε^0 can be written as

$$F^0(u) = \int_\Omega \langle A_0 \nabla u, \nabla u\rangle\,dx + \int_{\Omega\cap S(u)} \varphi_0(\nu_u)\,d\mathcal{H}^{n-1},$$

where A_0 is the matrix defined by the Γ-limit of G_ε^α with $c_\alpha = 0$ and $g = 0$ and φ_0 is the surface energy density defined by the Γ-limit of H_ε^β with $c_\beta = 0$ and $\psi = 0$. Their analysis provides a coerciveness result for all the families of functionals $(F_\varepsilon^{\alpha,\beta})_\varepsilon$ and a lower bound for the corresponding Γ-limit. Note that a common upper bound for all energies is given by the case $\alpha = \beta = 0$, which is treated by Braides, Defranceschi and Vitali [10], and for which the convergence in (2) reduces to ordinary strong L^1-convergence.

The description of the asymptotic behaviour of the energies $F_\varepsilon^{\alpha,\beta}$ is not a simple superposition of the corresponding analysis for the functionals G_ε^α and H_ε^β, but optimal sequences may depend on the interplay between the growth conditions and favour alternatively the introduction of large gradients or discontinuities (or both) inside the perforations.

The simplest case is the one of "very soft" inclusions, when either one of the two coefficients c_α or c_β vanishes, or we have $\alpha > 2$ or $\beta > 1$. In all cases the Γ-limit behaves as in the case of perforated domains, and is given by

$$F_g^0(u) := \int_\Omega \langle A_0 \nabla u, \nabla u\rangle \, dx + \int_{S(u)} \varphi_0(\nu_u) d\mathcal{H}^{n-1} + C_E \int_\Omega g(u)\, dx + |K| \min g.$$

Note that in the case $c_\beta = 0$ or $\beta > 1$ optimal sequences can be simply set equal to a constant value $u_{\min}$ with $g(u_{\min}) = \min g$ on the perforation (this does not influence the convergence in (2)), while some smooth cut-off argument has to be used when $c_\alpha = 0$ or $\alpha > 2$.

In the other cases when $\alpha \leq 2$ and $\beta \leq 1$ the limit actually depends on α and β through a modification of the "lower-order term", which indeed is not such for convergence (2). Indeed, for $\alpha = 2$ the gradient integral term has the same order of $g(u)$ on the perforation, while this holds for the surface part when $\beta = 1$. In general, the Γ-limit has then the form

$$F^{\alpha,\beta}(u) := \int_\Omega \langle A_0 \nabla u, \nabla u\rangle \, dx + \int_{S(u)} \varphi_0(\nu_u) d\mathcal{H}^{n-1} + C_E \int_\Omega g^{\alpha,\beta}(u)\, dx.$$

If $\alpha < 2$ and $\beta < 1$ the Γ-limit with respect to convergence (2) turns out to be equivalent to the one with respect to the strong L^1 convergence, and the term in g behaves as a continuous lower-order term, giving simply $g^{\alpha,\beta} = g$. If $\alpha < 2$ and $\beta = 1$ then optimal sequences can be taken piecewise constant on the perforation, and $g^{\alpha,\beta}$ is characterized by the problem on sets of finite perimeter

$$g^{\alpha,\beta}(z) = g(z) + \min\{c_\beta \mathcal{H}^{n-1}(\partial A) - |A|(g(z) - \min g) : A \subset K\},$$

where z has the role of a boundary datum. With this constant datum on the boundary the optimal u on K takes only the value z and $u_{\min}$, from which we deduce the minimum problem for $g^{\alpha,\beta}(z)$. Conversely, if $\alpha = 2$ and $\beta < 1$ then optimal sequences can be taken in H^1 of the perforation, and

$$g^{\alpha,\beta}(z) = C_E\, g(z) + \min\Big\{ \int_K \big(c_\alpha |\nabla v|^2 + g(v)\big)\, dx : v = z \text{ on } \partial K \Big\}$$

Finally, when $\alpha = 2$ and $\beta = 1$ both surface and bulk terms interact and give

$$g^{\alpha,\beta}(z) = C_E\, g(z) + \min\Big\{\int_K \big(c_\alpha|\nabla v|^2 + g(v)\big)\, dx + c_\beta \mathcal{H}^{n-1}(S(v)\cap K) : v = z \text{ outside } K\Big\}.$$

We have proved all our results for the simplest case of the Mumford-Shah functional in order to highlight the role of the different energy terms on the perforation without overburdening the notation, but general free-discontinuity energies can also be treated. The case when $\mathbb{R}^n \setminus E_0$ has more than one infinite connected component (which is possible for $n \geq 3$) requires a more complex treatment, both as the limit is defined on N functions (N being the number of disjoint connected components), and as it is not possible to reduce the definition of $g^{\alpha,\beta}$ to a single minimum problem. We refer to the works of Solci [17] and Braides, Chiadò Piat and Piatnitski [8] for the statements of the results and the modifications of the proofs. The main new technical part of the present paper is the possibility of reducing at "almost all" elements of the perforation to a single minimization problem with a constant boundary datum and on the correct function space (which varies with α and β in the cases mentioned above). Once that is done, the proof follows from the papers mentioned.

2 Notation and preliminaries

Basic notation. The Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ is denoted by $|E|$, and the $(n-1)$-dimensional Hausdorff measure is denoted by $\mathcal{H}^{n-1}$. For every $x \in \mathbb{R}^n$ and $\varrho > 0$, $B_\varrho(x)$ will be the open ball with centre x and radius ϱ, and S^{n-1} will be the boundary of the ball $B_1(0)$. We use standard notation for the Lebesgue spaces $L^p(\Omega)$ and the Sobolev space $H^1(\Omega)$, where Ω is an open set.

Functions of bounded variation. For the general theory on this topic we refer to [5]; here we recall some definitions and properties used in the sequel. Let Ω be a bounded open subset of $\mathbb{R}^n$ and $u\colon \Omega \to \mathbb{R}$ be a Borel function. We say that $z \in \mathbb{R}$ is the approximate limit of u in x (denoted by $\text{ap-lim}_{y\to x} u(y)$) if for every $\varepsilon > 0$

$$\lim_{\varrho\to 0^+} \frac{|\{y \in B_\varrho(x) \cap \Omega : |u(y) - z| > \varepsilon\}|}{\varrho^n} = 0.$$

The subset $S(u)$ of Ω where the approximate limit of u does not exist turns out to be a Borel set with $|S(u)| = 0$.

The function u is *approximately differentiable* in x if there exists $L \in \mathbb{R}^n$ such that

$$\text{ap-}\lim_{y\to x} \frac{u(y) - u(x) - L\cdot(y - x)}{|y - x|} = 0;$$

if u is approximately differentiable in x, then the unique L satisfying the equality is the approximate gradient of u in x, denoted by $\nabla u(x)$. A function $u \in L^1(\Omega)$

is of bounded variation ($u \in BV(\Omega)$) if its distributional derivatives $D_i u$ are Radon measures with finite total variation in Ω. We use Du to indicate the vector-valued measure $(D_1 u, \dots, D_n u)$. If $u \in BV(\Omega)$, then $S(u)$ is countably $(n-1)$-rectifiable, i.e. $S(u) = \bigcup_{i \in \mathbb{N}} K_i \cup N$, where $\mathcal{H}^{n-1}(N) = 0$ and each K_i is a compact set contained in a C^1-manifold of dimension $n-1$. Moreover, there exist Borel functions $\nu_u \colon S(u) \to S^{n-1}$ and $u^+, u^- \colon S(u) \to \mathbb{R}$ such that for $\mathcal{H}^{n-1}$-a.e. $x \in S(u)$

$$\lim_{\varrho \to 0^+} \varrho^{-n} \int_{B_\varrho^+(x) \cap \Omega} |u(y) - u^+(x)| dy = 0, \quad \lim_{\varrho \to 0^+} \varrho^{-n} \int_{B_\varrho^-(x) \cap \Omega} |u(y) - u^-(x)| dy = 0,$$

where $B_\varrho^\pm(x) = B_\varrho^\pm(x, \nu_u(x))$. The triple $(u^+(x), u^-(x), \nu_u(x))$ is uniquely determined up to a change of sign of $\nu_u(x)$ and an interchange between $u^+(x)$ and $u^-(x)$. The vector ν_u is normal to $S(u)$ in the sense that, representing $S(u)$ as above, then $\nu_u(x)$ is normal to the hypersurface Γ_i for a.a. $x \in K_i$. The approximate gradient $\nabla u(x)$ exists for a.a. $x \in \Omega$, and ∇u is the density of the absolutely continuous part of the measure Du with respect to the Lebesgue measure. We say that a function $u \in BV(\Omega)$ is a special function of bounded variation if the singular part (with respect to the Lebesgue measure) of Du is concentrated on $S(u)$; it is given by $(u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner S(u)$, i.e.

$$Du = \nabla u \mathcal{L}^n + (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner S(u).$$

The space of special functions of bounded variation is denoted by $SBV(\Omega)$; for the properties of $SBV(\Omega)$ we refer to [5]. For $p > 1$, we say that a function $u \colon \Omega \to \mathbb{R}$ belongs to the space $SBV^p(\Omega)$ if $u \in SBV(\Omega)$, $\nabla u \in L^p(\Omega; \mathbb{R}^n)$, and $\mathcal{H}^{n-1}(S(u)) < +\infty$.

Γ-*convergence.* We recall the notion of Γ-convergence (we refer to [6, 7, 13] for a complete analysis of the subject). Let (X, d) be a metric space, $F_\varepsilon \colon X \to \overline{\mathbb{R}}$ ($\varepsilon > 0$) a family of functionals, and $F \colon X \to \overline{\mathbb{R}}$. We say that $\{F_\varepsilon\}$ Γ-converges to F at $x \in X$ as $\varepsilon \to 0$ if:

i) for every infinitesimal sequence $\{\varepsilon_j\}$ and for every sequence $\{x_j\}$ converging to x in X, we have $F(x) \leq \liminf_{j \to \infty} F_{\varepsilon_j}(x_j)$;

ii) for every infinitesimal sequence $\{\varepsilon_j\}$ there exists a sequence $\{x_j\}$ converging to x in X such that $F(x) = \lim_{j \to \infty} F_{\varepsilon_j}(x_j)$.

The condition *ii*) can be replaced by the following

ii)′ for every $\eta > 0$ and for every infinitesimal sequence $\{\varepsilon_j\}$ there exists a sequence $\{x_j\}$ converging to x such that $F(x) \geq \limsup_{j \to \infty} F_{\varepsilon_j}(x_j) - \eta$.

If *i*) and *ii*) (or *ii*)′) hold for every $x \in X$ we say that $\{F_\varepsilon\}$ Γ-converges to F in X, and $F = \Gamma\text{-}\lim_{\varepsilon \to 0} F_\varepsilon$.

3 Setting of the problem and main result

Let $Q = (-1/2, 1/2)^n$ and $K \subset Q$ be a compact set of class C^2. We define the set E as

$$E = \mathbb{R}^n \setminus \bigcup_{i \in \mathbb{Z}^n} (i + K).$$

Let Ω be an open bounded subset of $\mathbb{R}^n$ such that $|\partial\Omega| = 0$, and let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous functions with the property that for any $t > 0$ there exist $T^+ \geq t$ and $T^- \leq -t$ such that

$$g(T^+) = \min\{g(s) : s \geq T^+\} \quad \text{and} \quad g(T^-) = \min\{g(s) : s \leq T^-\}. \tag{3}$$

This clearly implies that g is bounded below; note that (3) holds e.g. if $\lim_{t\to+\infty} g(t) = \lim_{t\to-\infty} g(t) = +\infty$.

For $\varepsilon > 0$ we consider the functional

$$\begin{aligned} F_\varepsilon^{\alpha,\beta}(u) = &\int_{\Omega\cap\varepsilon E} |\nabla u|^2\,dx + \mathcal{H}^{n-1}(S(u)\cap\varepsilon E) \\ &+ c_\alpha\,\varepsilon^\alpha \int_{\Omega\setminus\varepsilon E} |\nabla u|^2\,dx + c_\beta\,\varepsilon^\beta \mathcal{H}^{n-1}(S(u)\setminus\varepsilon E) + \int_\Omega g(u)\,dx \end{aligned} \tag{4}$$

defined for $u \in SBV^2(\Omega) \cap L^2(\Omega)$, where $c_\alpha, c_\beta \in [0, +\infty)$ and $\alpha, \beta \in [0, +\infty)$.

We are interested in the description of the asymptotic behaviour of the sequence $(F_\varepsilon^{\alpha,\beta})$. To that end, we introduce the following notion of convergence in $SBV^2(\Omega) \cap L^2(\Omega)$; given $(u_\varepsilon) \subset SBV^2(\Omega) \cap L^2(\Omega)$ and $u \in SBV^2(\Omega) \cap L^2(\Omega)$ we say that $u_\varepsilon \to u$ if

$$\chi_{\varepsilon E} u_\varepsilon \rightharpoonup C_K u \quad \text{in} \quad L^2(\Omega) \tag{5}$$

where $C_K = 1 - |K|$. Moreover, for $u \in SBV^2(\Omega) \cap L^2(\Omega)$ we set

$$F_\varepsilon^0(u) = \int_{\Omega\cap\varepsilon E} |\nabla u|^2\,dx + \mathcal{H}^{n-1}(S(u)\cap\varepsilon E) \tag{6}$$

which corresponds to $F_\varepsilon^{\alpha,\beta}$ in the case $c_\alpha = c_\beta = 0$ and $g = 0$.

For the sequence (F_ε^0) the following Γ-convergence result has been proven in [12].

Theorem 1 (Homogenization of Neumann problems [12, Th. 7.2]). *The family (F_ε^0) defined in (6) Γ-converges with respect to the strong topology of $L^2(\Omega)$ to the functional*

$$F^0(u) = \int_\Omega \langle A_0 \nabla u, \nabla u\rangle\,dx + \int_{S(u)} \varphi_0(\nu_u)\,d\mathcal{H}^{n-1}$$

with

$$\langle A_0\xi,\xi\rangle = \min\Big\{\int_{Q\cap E}|\nabla u+\xi|^2\,dx : u\in H^1(Q) \text{ with periodic boundary values}\Big\}$$

and

$$\begin{aligned}\varphi_0(\nu) \quad = \quad \lim_{T\to+\infty}\frac{1}{T^{n-1}}\min\Big\{&\mathcal{H}^{n-1}(S(u)\cap E\cap TQ^\nu) : u\in SBV(TQ^\nu),\\ &\nabla u=0, u=u^\nu \text{ on } \partial TQ^\nu\Big\},\end{aligned}$$

where Q^ν stands for any unit cube centered in 0 with two faces orthogonal to ν, and

$$u^\nu(x)=\begin{cases}1 & \text{if } \langle x,\nu\rangle\geq 0\\ 0 & \text{if } \langle x,\nu\rangle<0.\end{cases}$$

A key point in the proof of Theorem 1 ([12, Th. 7.2]) is the following extension lemma, which we will use in the proof of the general case.

Theorem 2 (Extension of SBV functions in perforated domains [12, Th. 1.3]). *Let E be a periodic, connected, open subset of $\mathbb{R}^n$, with Lipschitz boundary. Let Ω be a bounded open subset of $\mathbb{R}^n$. There exist an extension operator $T_\varepsilon\colon SBV^2(\Omega\cap\varepsilon E)\cap L^\infty(\Omega\cap\varepsilon E)\to SBV^2(\Omega)\cap L^\infty(\Omega)$ and a constant $k>0$, depending only on E and n, such that*

1. $T_\varepsilon u=u$ *a.e. in* $\Omega\cap\varepsilon E$;
2. $\|T_\varepsilon u\|_{L^\infty(\Omega)}\leq\|u\|_{L^\infty(\Omega\cap\varepsilon E)}$;
3. $$\begin{aligned}&\int_\Omega|\nabla T_\varepsilon u|^2\,dx+\mathcal{H}^{n-1}(S(T_\varepsilon u)\cap\Omega)\\ &\leq k\left(\int_{\Omega\cap\varepsilon E}|\nabla u|^2\,dx+\mathcal{H}^{n-1}(S(u)\cap\Omega\cap\varepsilon E)+\mathcal{H}^{n-1}(\partial\Omega)\right)\end{aligned}$$

for every $u\in SBV^2(\Omega\cap\varepsilon E)\cap L^\infty(\Omega\cap\varepsilon E)$.

The main result of this paper is the following Γ-convergence theorem.

Theorem 3. *Let $F_\varepsilon^{\alpha,\beta}$ be the functional defined in $SBV^2(\Omega)\cap L^2(\Omega)$ by (4). The Γ-limit of the sequence $(F_\varepsilon^{\alpha,\beta})$ as $\varepsilon\to 0$ with respect to convergence (5) is given by the functional $F^{\alpha,\beta}$ defined in $SBV^2(\Omega)\cap L^2(\Omega)$ as*

$$F^{\alpha,\beta}(u)=\int_\Omega\langle A_0\nabla u,\nabla u\rangle\,dx+\int_{S(u)}\varphi_0(\nu_u)d\mathcal{H}^{n-1}+\int_\Omega g^{\alpha,\beta}(u)\,dx$$

where A_0 and φ_0 are as in Theorem 1 and $g^{\alpha,\beta}$ is given by the following formulae depending on α,β,c_α and c_β:

1. *if* $c_\alpha = 0$ *or* $c_\beta = 0$*, or in the case* $c_\alpha, c_\beta > 0$ *with* $\alpha > 2$ *or* $\beta > 1$

$$g^{\alpha,\beta}(z) = C_K g(z) + (1 - C_K)\min g;$$

2. *in the case* $c_\alpha, c_\beta > 0$ *with* $\alpha \in (0, 2)$ *and* $\beta \in (0, 1)$

$$g^{\alpha,\beta}(z) = g(z);$$

3. *in the case* $c_\alpha, c_\beta > 0$ *with* $\alpha \in (0, 2)$ *and* $\beta = 1$

$$g^{\alpha,\beta}(z) = g(z) + \min\{c_\beta \mathcal{H}^{n-1}(\partial A) - |A|(g(z) - \min g) : A \subset K\};$$

4. *in the case* $c_\alpha, c_\beta > 0$ *with* $\alpha = 2$ *and* $\beta \in (0, 1)$

$$g^{\alpha,\beta}(z) = C_K g(z) + \min\Big\{\int_K \big(c_\alpha|\nabla v|^2 + g(v)\big)\,dx : v = z \text{ on } Q \setminus K\Big\};$$

5. *in the case* $c_\alpha, c_\beta > 0$ *with* $\alpha = 2$ *and* $\beta = 1$

$$\begin{aligned} g^{\alpha,\beta}(z) = C_K g(z) \ &+ \min\Big\{\int_K \big(c_\alpha|\nabla v|^2 + g(v)\big)\,dx \\ &+ c_\beta \mathcal{H}^{n-1}(S(v) \cap K) : v = z \text{ on } Q \setminus K\Big\}. \end{aligned}$$

The compactness result for the sequence (F^0_ε) (see [12, Th. 7.1]) and the lower boundedness of g imply the following theorem.

Theorem 4 (Compactness). *Let* $(u_\varepsilon) \subset SBV^2(\Omega) \cap L^\infty(\Omega)$ *such that* (u_ε) *is equibounded in* $L^\infty(\Omega)$ *and*

$$\sup_{\varepsilon > 0} F^{\alpha,\beta}_\varepsilon(u_\varepsilon) < +\infty.$$

Then, there exists $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$ *such that* $u_\varepsilon \to u$ *in the sense of (5).*

Remark 5. We note that

(a) it is sufficient to prove Theorem 3 for $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$;

(b) in the proof of Theorem 3 it suffices to prove the lim inf inequality for sequences (u_ε) equibounded in $L^\infty(\Omega)$.

Proof of Remark 5. (a) Suppose that Theorem 3 holds for $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$. To prove the lower bound in the general case $u \in SBV^2(\Omega) \cap L^2(\Omega)$, recalling the properties of g we can choose sequences (λ^+_k) and (λ^-_k) such that $\lambda^+_k, \lambda^-_k \to +\infty$ as $k \to +\infty$ and

$g(\lambda_k^+) \le g(t)$ for any $t \ge \lambda_k^+$, $g(-\lambda_k^-) \le g(t)$ for any $t \le -\lambda_k^-$. Given a sequence (u_ε) such that $u_\varepsilon \to u$ in $L^2(\Omega)$, we set

$$u^k = (-\lambda_k^- \vee u) \wedge \lambda_k^+ \text{ and } u_\varepsilon^k = (-\lambda_k^- \vee u_\varepsilon) \wedge \lambda_k^+.$$

It follows that $u^k \to u$ in $L^1(\Omega)$ as $k \to +\infty$; the properties of $(\lambda_k^\pm)$ and the lim inf inequality in L^∞ ensure that

$$\liminf_{\varepsilon\to 0} F_\varepsilon^{\alpha,\beta}(u_\varepsilon) \ge \liminf_{\varepsilon\to 0} F_\varepsilon^{\alpha,\beta}(u_\varepsilon^k) \ge F^{\alpha,\beta}(u^k).$$

Since $F^{\alpha,\beta}$ is semicontinuous with respect to the $L^1(\Omega)$ convergence we get

$$\liminf_{\varepsilon\to 0} F_\varepsilon^{\alpha,\beta}(u_\varepsilon) \ge \liminf_{k\to+\infty} F^{\alpha,\beta}(u^k) \ge F^{\alpha,\beta}(u).$$

As to upper bound, by density it is sufficient to prove the estimate for a function $u \in L^\infty(\Omega)$.

(b) The hypothesis on g ensures the existence of $T^+ \ge 2\|u\|_\infty$ and $T^- \le -2\|u\|_\infty$ such that $g(t) \ge g(T^+)$ for any $t \ge T^+$, and $g(t) \ge g(T^-)$ for any $t \le T^-$. We define $v_\varepsilon = (T^- \vee u_\varepsilon) \wedge T^+$ and $w_\varepsilon = (T^- \vee T_\varepsilon u_\varepsilon) \wedge T^+$; the sequence (w_ε) converges to u in $L^1(\Omega)$, so that

$$w_\varepsilon \chi_{\varepsilon E} \overset{*}{\rightharpoonup} c_E u.$$

Since $v_\varepsilon = w_\varepsilon$ in εE, it follows that $v_\varepsilon \to u$ in the sense of (5). Moreover, from the hypothesis on g we deduce

$$\int_\Omega g(u_\varepsilon)\,dx \ge \int_\Omega g(v_\varepsilon)\,dx,$$

so that $F_\varepsilon^{\alpha,\beta}(u_\varepsilon) \ge F_\varepsilon^{\alpha,\beta}(v_\varepsilon)$; this shows that we can assume (u_ε) uniformly bounded in $L^\infty(\Omega)$. □

4 Proof of Theorem 3

For any $\varepsilon > 0$ and $i \in \mathbb{Z}^n$ we define $Q_\varepsilon^i = \varepsilon i + \varepsilon Q$ and $K_\varepsilon^i = \varepsilon i + \varepsilon K$; since $|\partial\Omega| = 0$, setting $\mathcal{I}_\varepsilon = \{i \in \mathbb{Z}^n : Q_\varepsilon^i \subset \Omega\}$ it follows that $\left|\Omega \setminus \bigcup_{i\in\mathcal{I}_\varepsilon} Q_\varepsilon^i\right| \to 0$ as $\varepsilon \to 0$. Moreover, for $\varrho > 0$ fixed small enough, we set

$$K_{\varepsilon,\varrho}^i = \{x \in K_\varepsilon^i : \mathrm{dist}(x, \partial K_\varepsilon^i) > \varepsilon\varrho\}.$$

Remark 6. Let (u_ε) be a sequence in $SBV^2(\Omega) \cap L^\infty(\Omega)$ such that $\|u_\varepsilon\|_{L^\infty(\Omega\cap\varepsilon E)} \le M$ and

$$\int_{\Omega\cap\varepsilon E} |\nabla u_\varepsilon|^2\,dx + \mathcal{H}^{n-1}(S(u_\varepsilon) \cap \Omega \cap \varepsilon E) \le M.$$

For $\varrho > 0$ sufficiently small, we define the set of indices

$$\mathcal{I}^*_\varepsilon(\varrho) = \{i \in \mathbb{Z}^n : T_\varepsilon u_\varepsilon \in H^1(K^i_{\varepsilon,\varrho})\}; \tag{7}$$

from the proof of Theorem 2 (see [12, Th. 1.3]) we deduce that

$$\#\left(\mathcal{I}_\varepsilon \setminus \mathcal{I}^*_\varepsilon(\varrho)\right) \le \frac{c}{\varepsilon^{n-1}} \tag{8}$$

where c depends only on n, ϱ, and the uniform bound M.

The following remark ensures that in the proof of the lower bound we can apply the lim inf inequality for the sequence (F^0_ε), which is shown for sequences converging with respect to the strong convergence in $L^2(\Omega)$.

Remark 7. Thanks to Remark 5, in the proof of the lower bound we can restrict our attention to $u \in L^\infty(\Omega)$ and (u_ε) uniformly bounded in $L^\infty(\Omega)$. Now, if we consider a sequence $(u_\varepsilon) \subset SBV^2(\Omega) \cap L^\infty(\Omega)$ converging to u as in (5), with $(F^0_\varepsilon(u_\varepsilon))$ and $\|u_\varepsilon\|_{L^\infty(\Omega)}$ uniformly bounded, then the sequence of the extensions $T_\varepsilon u_\varepsilon$ given by Theorem 2 converges strongly in $L^1(\Omega)$ to u. Indeed, $(T_\varepsilon u_\varepsilon)$ is equibounded in $SBV(\Omega)$, and by compactness we can extract a subsequence converging to $w \in SBV(\Omega)$ strongly in $L^1(\Omega)$. This implies $\chi_{\varepsilon E} u_\varepsilon \stackrel{*}{\rightharpoonup} C_K w$ in $L^\infty(\Omega)$; since $\chi_{\varepsilon E} u_\varepsilon \stackrel{*}{\rightharpoonup} C_K u$ in $L^\infty(\Omega)$, it follows that $w = u$ and $T_\varepsilon u_\varepsilon \to u$ strongly in $L^1(\Omega)$. Since in addition the sequence (u_ε) is equibounded in $L^\infty(\Omega)$, the convergence of $T_\varepsilon u_\varepsilon$ is strong in $L^2(\Omega)$. Thus, we can use the lower bound inequality for the sequence $(T_\varepsilon u_\varepsilon)$ in Theorem 1 to prove the lower bound inequality in our case.

Proof of Theorem 3 in the case $c_\alpha = 0$ or $c_\beta = 0$. The lower bound follows from the lim inf inequality in Theorem 1. Indeed, given $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$ and a sequence $(u_\varepsilon) \subset SBV^2(\Omega) \cap L^\infty(\Omega)$ such that $u_\varepsilon \to u$ as in (5) and (u_ε) uniformly bounded in $L^\infty(\Omega)$ (see Remark 5), Remark 7 ensures that we can apply Theorem 1 to the sequence $(T_\varepsilon u_\varepsilon)$ which coincides with (u_ε) in $\Omega \cap \varepsilon E$ and obtain

$$\begin{aligned}\liminf_{\varepsilon\to 0} \int_{\Omega\cap\varepsilon E} |\nabla u_\varepsilon|^2\,dx + \mathcal{H}^{n-1}(S(u_\varepsilon)\cap\varepsilon E)&\\ \ge \int_\Omega \langle A_0\nabla u, \nabla u\rangle\,dx + \int_{S(u)} \varphi_0(\nu_u)\,d\mathcal{H}^{n-1}&.\end{aligned} \tag{9}$$

The continuity of g and the uniform bound on (u_ε) in $L^\infty(\Omega)$ allow to apply the Dominated Convergence Theorem, getting

$$\lim_{\varepsilon\to 0} \int_{\Omega\cap\varepsilon E} g(T_\varepsilon u_\varepsilon)\,dx = C_K \int_\Omega g(u)\,dx,$$

so that

$$\begin{aligned}\liminf_{\varepsilon\to 0} F_\varepsilon^{\alpha,\beta}(u_\varepsilon) &\geq \int_\Omega \langle A_0\nabla u,\nabla u\rangle\,dx + \int_{S(u)}\varphi_0(\nu_u)\,d\mathcal{H}^{n-1}\\ &\qquad +C_K\int_\Omega g(u)\,dx + (1-C_K)|\Omega|\min g\\ &= \int_\Omega \langle A_0\nabla u,\nabla u\rangle\,dx + \int_{S(u)}\varphi_0(\nu_u)\,d\mathcal{H}^{n-1} + \int_\Omega g^{\alpha,\beta}(u)\,dx\end{aligned}$$

since $\chi_{\Omega\setminus\varepsilon E} \overset{*}{\rightharpoonup} 1-C_K$ in $L^\infty(\Omega)$.

As for the upper bound, given $u\in SBV^2(\Omega)\cap L^\infty(\Omega)$ let (u_ε) be a recovery sequence for the functionals F_ε^0 (from Theorem 1); that is,

$$\limsup_{\varepsilon\to 0} F_\varepsilon^0(u_\varepsilon) \leq \int_\Omega \langle A_0\nabla u,\nabla u\rangle\,dx + \int_{S(u)}\varphi_0(\nu_u)\,d\mathcal{H}^{n-1}. \tag{10}$$

The extension result Theorem 2 ensures that we can assume $\mathcal{H}^{n-1}(S(u_\varepsilon)\cap\Omega)$ uniformly bounded.

Now, we divide the proof in two cases.

- Case $c_\alpha = 0$. We modify u_ε inside each K_ε^i with $i\in\mathcal{I}_\varepsilon$ by setting

$$\widetilde{u}_\varepsilon(x) = \varphi_\rho^\varepsilon u_\varepsilon + (1-\varphi_\rho^\varepsilon)\overline{u}, \tag{11}$$

where $g(\overline{u}) = \min g$ and

$$\varphi_\rho^\varepsilon(x) = \Big(1-\frac{1}{\varepsilon\rho}\operatorname{dist}(x,\varepsilon E)\Big)^+.$$

With this definition

$$\begin{aligned}\mathcal{H}^{n-1}(S(\widetilde{u}_\varepsilon)\setminus\varepsilon E) &= \sum_{i\in\mathcal{I}_\varepsilon}\mathcal{H}^{n-1}(S(\widetilde{u}_\varepsilon)\cap K_\varepsilon^i) + o(1)_{\varepsilon\to 0}\\ &\leq \mathcal{H}^{n-1}(S(u_\varepsilon)\cap\Omega) + o(1)_{\varepsilon\to 0}.\end{aligned}$$

Then, recalling (10), the properties of g and the uniform bound on $\mathcal{H}^{n-1}(S(u_\varepsilon)\cap\Omega)$ imply

$$\begin{aligned}\limsup_{\varepsilon\to 0} F_\varepsilon^{\alpha,\beta}(\widetilde{u}_\varepsilon) &\leq F^0(u) + \lim_{\varepsilon\to 0} c_\beta\varepsilon^\beta\mathcal{H}^{n-1}(S(u_\varepsilon)\cap\Omega) + C_K\int_\Omega g(u)\,dx\\ &\qquad + \lim_{\varepsilon\to 0}\int_{\Omega\setminus\varepsilon E}\min g\,dx\\ &= \int_\Omega \langle A_0\nabla u,\nabla u\rangle\,dx + \int_{S(u)}\varphi_0(\nu_u)\,d\mathcal{H}^{n-1} + \int_\Omega g^{\alpha,\beta}(u)\,dx.\end{aligned}$$

• Case $c_\beta = 0$. Let $\overline{u} \in \operatorname{argmin}(g)$. Setting

$$\widetilde{u}_\varepsilon(x) = \begin{cases} u_\varepsilon(x) & \text{in } \Omega \cap \varepsilon E \\ \overline{u} & \text{in } \Omega \setminus \varepsilon E \end{cases} \tag{12}$$

we get

$$\begin{aligned} \limsup_{\varepsilon\to 0} F_\varepsilon^{\alpha,\beta}(\widetilde{u}_\varepsilon) &\leq F^0(u) + C_K \int_\Omega g(u)\,dx + \lim_{\varepsilon\to 0}\int_{\Omega\setminus\varepsilon E} \min g\,dx \\ &= \int_\Omega \langle A_0\nabla u, \nabla u\rangle\,dx + \int_{S(u)} \varphi_0(\nu_u)\,d\mathcal{H}^{n-1} + \int_\Omega g^{\alpha,\beta}(u)\,dx \end{aligned}$$

as desired. □

Proof of Theorem 3 in the case $c_\alpha, c_\beta > 0$ and $\alpha > 2$. The lower estimate follows immediately from the previous case $c_\alpha = 0$ or $c_\beta = 0$. Indeed

$$F_\varepsilon^{\alpha,\beta}(u) \geq F_\varepsilon^0(u) + \int_\Omega g(u)\,dx.$$

Given $u \in SBV^2(\Omega)\cap L^\infty(\Omega)$, let (u_ε) be a recovery sequence for F_ε^0, that is $u_\varepsilon \to u$ strongly in $L^2(\Omega)$ and the estimate (10) holds. The extension result Theorem 2 allows to assume $\|\nabla u_\varepsilon\|_{L^2(\Omega)}$ and $\mathcal{H}^{n-1}(S(u_\varepsilon)\cap\Omega)$ uniformly bounded. Moreover, we note that it is not restrictive to assume (u_ε) uniformly bounded in $L^\infty(\Omega)$, by considering $(-2\|u\|_\infty \vee u_\varepsilon)\wedge 2\|u\|_\infty$.

We define $\widetilde{u}_\varepsilon$ as in (11), getting

$$\begin{aligned} F_\varepsilon^{\alpha,\beta}(\widetilde{u}_\varepsilon) &\leq \int_{\Omega\cap\varepsilon E} |\nabla u_\varepsilon|^2\,dx + \mathcal{H}^{n-1}(S(u_\varepsilon)\cap\varepsilon E) \\ &\quad + c_\alpha\varepsilon^\alpha \int_{\Omega\setminus\varepsilon E} |\nabla\widetilde{u}_\varepsilon|^2\,dx + c_\beta\,\varepsilon^\beta\mathcal{H}^{n-1}(S(\widetilde{u}_\varepsilon)\setminus\varepsilon E) + \int_\Omega g(\widetilde{u}_\varepsilon)\,dx \\ &\leq \int_\Omega \langle A_0\nabla u, \nabla u\rangle\,dx + \int_{S(u)} \varphi_0(\nu(u))d\mathcal{H}^{n-1} \\ &\quad + 2c_\alpha\varepsilon^\alpha \int_{\Omega\setminus\varepsilon E}\Big(\frac{1}{\varepsilon^2\rho^2}|u_\varepsilon - \overline{u}|^2 + |\nabla u_\varepsilon|^2\Big)\,dx + c_\beta\,\varepsilon^\beta\mathcal{H}^{n-1}(S(u_\varepsilon)\cap\Omega) \\ &\quad + C_K\int_\Omega g(u)\,dx + \int_{(\Omega\setminus\varepsilon E)\cap\{\operatorname{dist}(x,\varepsilon E)<\varepsilon\rho\}} g(\widetilde{u}_\varepsilon)\,dx \\ &\quad + \big|\Omega\cap\{\operatorname{dist}(x,\varepsilon E) > \varepsilon\rho\}\big|\min g + o(1)_{\varepsilon\to 0} \\ &\leq \int_\Omega \langle A_0\nabla u, \nabla u\rangle\,dx + \int_{S(u)} \varphi_0(\nu(u))d\mathcal{H}^{n-1} + \int_\Omega g^{\alpha,\beta}(u)\,dx \\ &\quad + C\varepsilon^{\alpha-2}\frac{1}{\rho^2} + C\rho + o(1)_{\varepsilon\to 0} \end{aligned}$$

where C does not depend on ε and ϱ. Since $\alpha > 2$ and ρ is arbitrary we have the thesis. □

Proof of Theorem 3 in the case $c_\alpha, c_\beta > 0$ *and* $\beta > 1$. As in the previous case, the lower estimate follows immediately from the case $c_\alpha = 0$ or $c_\beta = 0$.

Given $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$, let (u_ε) be such that $u_\varepsilon \to u$ strongly in $L^2(\Omega)$, the estimate (10) holds and $\|u_\varepsilon\|_{L^\infty(\Omega)}$, $\|\nabla u_\varepsilon\|_{L^2(\Omega)}$ and $\mathcal{H}^{n-1}(S(u_\varepsilon) \cap \Omega)$ are uniformly bounded. We define $\widetilde{u}_\varepsilon$ as in (12). Since $\beta > 1$ we have

$$\lim_{\varepsilon \to 0} c_\beta \varepsilon^\beta \mathcal{H}^{n-1}(S(\widetilde{u}_\varepsilon) \setminus \varepsilon E) \le \lim_{\varepsilon \to 0} c_\beta \varepsilon^\beta \ \#\mathcal{I}_\varepsilon \ \varepsilon^{n-1} \mathcal{H}^{n-1}(\partial K) = 0.$$

Thus, we can conclude as in the case $c_\beta = 0$. □

Proof of Theorem 3 in the case $0 < \alpha \le 2$, $0 < \beta \le 1$. *Lower bound.* As noticed in Remark 5, it is sufficient to prove the result with the hypothesis $u \in L^\infty(\Omega)$. Given $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$, let (u_ε) be a sequence in $SBV^2(\Omega) \cap L^\infty(\Omega)$ such that $u_\varepsilon \to u$ and $\sup_{\varepsilon > 0} F_\varepsilon^{\alpha,\beta}(u_\varepsilon) < +\infty$. Remark 5 ensures that we can assume (u_ε) uniformly bounded in $L^\infty(\Omega)$, so that as noticed in Remark 7 the sequence of extensions $(T_\varepsilon u_\varepsilon)$ given by Theorem 2 converges to u in $L^2(\Omega)$.

Now we modify the extensions $T_\varepsilon u_\varepsilon$ to obtain a sequence of functions which are constant in the holes, except for a small neighborhood of the boundary. Fixed $\varrho > 0$, for any $i \in \mathcal{I}_\varepsilon^*(\varrho)$ (see (7)) we define

$$u_\varepsilon^i = \frac{1}{|K_{\varepsilon,\varrho}^i \setminus K_{\varepsilon,2\varrho}^i|} \int_{K_{\varepsilon,\varrho}^i \setminus K_{\varepsilon,2\varrho}^i} T_\varepsilon u_\varepsilon \, dx \tag{13}$$

and we modify $T_\varepsilon u_\varepsilon$ by setting

$$\widetilde{T}_\varepsilon u_\varepsilon = \begin{cases} u_\varepsilon^i & \text{in } K_{\varepsilon,2\varrho}^i \ \text{ for } \ i \in \mathcal{I}_\varepsilon^*(\varrho) \\ \varphi_\varepsilon^\varrho T_\varepsilon u_\varepsilon + (1 - \varphi_\varepsilon^\varrho) u_\varepsilon^i & \text{in } K_{\varepsilon,\varrho}^i \setminus K_{\varepsilon,2\varrho}^i \ \text{ for } \ i \in \mathcal{I}_\varepsilon^*(\varrho) \\ T_\varepsilon u_\varepsilon & \text{otherwise in } \Omega \end{cases}$$

where with an abuse of notation we set in this case

$$\varphi_\varepsilon^\varrho(x) = \min\left\{ \left(2 - \frac{\operatorname{dist}(x, \varepsilon E)}{\varepsilon\varrho}\right)^+, 1 \right\}. \tag{14}$$

For $i \in \mathcal{I}^*_\varepsilon(\varrho)$ we get, applying Poincaré's inequality to $T_\varepsilon u_\varepsilon \in H^1(K^i_{\varepsilon,\varrho} \setminus K^i_{\varepsilon,2\varrho})$

$$\begin{aligned}
\int_{K^\varepsilon_i} |\nabla \widetilde{T}_\varepsilon u_\varepsilon|^2\,dx &\leq \int_{K^\varepsilon_i} |\nabla T_\varepsilon u_\varepsilon|^2\,dx + 2\int_{K^i_{\varepsilon,\varrho}\setminus K^i_{\varepsilon,2\varrho}} |\nabla \widetilde{\varphi}^\varrho_\varepsilon (T_\varepsilon u_\varepsilon - u^i_\varepsilon)|^2\,dx \\
&\quad +2\int_{K^\varepsilon_i} |\widetilde{\varphi}^\varrho_\varepsilon \nabla T_\varepsilon u_\varepsilon|^2\,dx \\
&\leq 3\int_{K^\varepsilon_i} |\nabla T_\varepsilon u_\varepsilon|^2\,dx + \frac{2}{\varepsilon^2\varrho^2}\int_{K^i_{\varepsilon,\varrho}\setminus K^i_{\varepsilon,2\varrho}} |T_\varepsilon u_\varepsilon - u^i_\varepsilon|^2\,dx \\
&\leq 3\int_{K^\varepsilon_i} |\nabla T_\varepsilon u_\varepsilon|^2\,dx + \frac{2}{\varrho^2}\int_{K^i_{\varepsilon,\varrho}\setminus K^i_{\varepsilon,2\varrho}} |\nabla T_\varepsilon u_\varepsilon|^2\,dx \\
&\leq \frac{C}{\varrho^2}\int_{K^\varepsilon_i} |\nabla T_\varepsilon u_\varepsilon|^2\,dx
\end{aligned}$$

where C does not depend on ε and ϱ.

Now, for each $i \in \mathcal{I}^*_\varepsilon(\varrho)$ we consider the function defined on Q^i_ε by $w^i_\varepsilon = u_\varepsilon - \widetilde{T}_\varepsilon u_\varepsilon + u^i_\varepsilon$. Note that the outer trace of w^i_ε on ∂K^i_ε is equal to u^i_ε, and that $w^i_\varepsilon = u_\varepsilon$ on $K^i_{\varepsilon,2\rho}$.

We prove that we can estimate the functional by considering the terms depending on w_ε instead of u_ε. Fixed $\eta > 0$, the estimate on $\int_{K^i_\varepsilon} |\nabla \widetilde{T}_\varepsilon u_\varepsilon|^2\,dx$ gives

$$\int_{K^\varepsilon_i} |\nabla u_\varepsilon|^2\,dx \geq (1-\eta)\int_{K^\varepsilon_i} |\nabla w_\varepsilon|^2\,dx - \frac{C_\eta}{\varrho^2}\int_{K^\varepsilon_i} |\nabla T_\varepsilon u_\varepsilon|^2\,dx.$$

Moreover, we have

$$\begin{aligned}
\mathcal{H}^{n-1}(K^\varepsilon_i \cap S(u_\varepsilon)) &\geq \mathcal{H}^{n-1}(K^\varepsilon_i \cap S(w_\varepsilon)) - \mathcal{H}^{n-1}(K^\varepsilon_i \cap S(Tu_\varepsilon)) \\
\int_{K^\varepsilon_i} g(u_\varepsilon)\,dx &\geq \int_{K^\varepsilon_i} g(w_\varepsilon) - C\varepsilon^n\varrho
\end{aligned}$$

where C depends only on $\mathcal{H}^{n-1}(\partial K)$ and g. As a consequence,

$$\begin{aligned}
&\sum_{i\in\mathcal{I}^*_\varepsilon(\varrho)} \Big(c_\alpha\varepsilon^\alpha \int_{K^\varepsilon_i} |\nabla u_\varepsilon|^2\,dx + c_\beta\varepsilon^\beta\mathcal{H}^{n-1}(K^\varepsilon_i \cap S(u_\varepsilon)) + \int_{K^\varepsilon_i} g(u_\varepsilon)\,dx\Big) \\
\geq\ & (1-\eta)\sum_{i\in\mathcal{I}^*_\varepsilon(\varrho)} \Big(c_\alpha\varepsilon^\alpha \int_{K^\varepsilon_i} |\nabla w_\varepsilon|^2\,dx + c_\beta\varepsilon^\beta\mathcal{H}^{n-1}(K^\varepsilon_i \cap S(w_\varepsilon)) + \int_{K^\varepsilon_i} g(w_\varepsilon)\,dx\Big) \\
&-C|\Omega|\varrho - C_\eta c_\alpha\varepsilon^\alpha\int_\Omega |\nabla Tu_\varepsilon|^2\,dx - c_\beta\varepsilon^\beta\mathcal{H}^{n-1}(S(Tu_\varepsilon)\cap\Omega).
\end{aligned}$$

Note that the last two terms tend to 0 as $\varepsilon \to 0$. This shows that we may estimate only the term depending on w_ε. Setting

$$\psi^{\alpha,\beta}_\varepsilon(z) = \min\Big\{c_\alpha\varepsilon^{\alpha-2}\int_K |\nabla v|^2\,dx + c_\beta\varepsilon^{\beta-1}\mathcal{H}^{n-1}(K\cap S(v)) + \int_K g(v)\,dx\Big\} \qquad (15)$$

where the minimum is taken on all v with outer trace on ∂K equal to z, we have

$$c_\alpha \varepsilon^\alpha \int_{K_i^\varepsilon} |\nabla w_\varepsilon|^2 \, dx + c_\beta \varepsilon^\beta \mathcal{H}^{n-1}(K_i^\varepsilon \cap S(w_\varepsilon)) + \int_{K_i^\varepsilon} g(w_\varepsilon) \, dx \geq \varepsilon^n \psi_\varepsilon^{\alpha,\beta}(u_\varepsilon^i).$$

The sequence $(\psi_\varepsilon^{\alpha,\beta})$ is increasing as $\varepsilon \to 0$, then, fixed $\delta > 0$, for any $0 < \varepsilon < \delta$ we have

$$\sum_{i \in \mathcal{I}_\varepsilon^*(\varrho)} \Big(c_\alpha \varepsilon^\alpha \int_{K_i^\varepsilon} |\nabla u_\varepsilon|^2 \, dx + c_\beta \varepsilon^\beta \mathcal{H}^{n-1}(K_i^\varepsilon \cap S(u_\varepsilon)) + \int_{K_i^\varepsilon} g(u_\varepsilon) \, dx \Big)$$
$$\geq (1 - \eta) \int_\Omega \chi_\varepsilon \psi_\delta^{\alpha,\beta}(\overline{u}_\varepsilon) \, dx - C|\Omega|\varrho + o(1)_{\varepsilon \to 0} \qquad (16)$$

where $\overline{u}_\varepsilon$ is the piecewise constant function defined by

$$\overline{u}_\varepsilon = \sum_{i \in \mathcal{I}_\varepsilon} \chi_{Q_\varepsilon^i} u_\varepsilon^i$$

and χ_ε is the characteristic function of the set $\bigcup_{i \in \mathcal{I}_\varepsilon^*(\varrho)} Q_\varepsilon^i$. Note that, recalling (8), $\chi_\varepsilon \to 1$ in $L^1(\Omega)$.

Now, we have to prove the strong convergence of the sequence $(\overline{u}_\varepsilon)$ to the function u in order to apply the Fatou Lemma and obtain an estimate of the lim inf. We denote the set $K_{\varepsilon,\varrho}^i \setminus K_{\varepsilon,2\varrho}^i$ by $D_\varepsilon^i = \varepsilon i + \varepsilon D$, omitting the dependence on ϱ, and define the piecewise constant function

$$\widetilde{u}_\varepsilon = \sum_{i \in \mathcal{I}_\varepsilon} \chi_{Q_\varepsilon^i} \frac{1}{|Q_\varepsilon^i \setminus D_\varepsilon^i|} \int_{Q_\varepsilon^i \setminus D_\varepsilon^i} T_\varepsilon u_\varepsilon \, dx.$$

Since $T_\varepsilon u_\varepsilon \to u$ in $L^2(\Omega)$, then

$$\overline{u}_\varepsilon \rightharpoonup u \;\text{ and }\; \widetilde{u}_\varepsilon \rightharpoonup u \;\text{ weakly in } L^2(\Omega). \qquad (17)$$

Now, we show that

$$\liminf_{\varepsilon \to 0} \|\overline{u}_\varepsilon\|_{L^2(\Omega)} = \liminf_{\varepsilon \to 0} \|\widetilde{u}_\varepsilon\|_{L^2(\Omega)} = \|u\|_{L^2(\Omega)};$$

the weak convergence and the convergence of the norm imply the required strong convergence in $L^2(\Omega)$ of the sequences $(\overline{u}_\varepsilon)$ and $(\widetilde{u}_\varepsilon)$. The weak lower semicontinuity of the norm ensures that

$$\begin{aligned}
2\|u\|^2_{L^2(\Omega)} &\le \liminf_{\varepsilon\to 0}(\|\overline{u}_\varepsilon\|^2_{L^2(\Omega)}+\|\widetilde{u}_\varepsilon\|^2_{L^2(\Omega)})\\
&\le \liminf_{\varepsilon\to 0}\sum_{i\in\mathcal{I}_\varepsilon}\varepsilon^n\Big(\frac{1}{\varepsilon^n|D|}\int_{D^i_\varepsilon}T_\varepsilon u_\varepsilon\,dx\Big)^2+\varepsilon^n\Big(\frac{1}{\varepsilon^n(1-|D|)}\int_{Q^i_\varepsilon\setminus D^i_\varepsilon}T_\varepsilon u_\varepsilon\,dx\Big)^2\\
&\le \liminf_{\varepsilon\to 0}\Big(\sum_{i\in\mathcal{I}_\varepsilon}\Big(\frac{1}{|D|}\int_{D^i_\varepsilon}|T_\varepsilon u_\varepsilon|^2\,dx+\frac{1}{1-|D|}\int_{Q^i_\varepsilon\setminus D^i_\varepsilon}|T_\varepsilon u_\varepsilon|^2\,dx\Big)\\
&\qquad -2\|T_\varepsilon u_\varepsilon\|^2_{L^2(\Omega)}+2\|T_\varepsilon u_\varepsilon\|^2_{L^2(\Omega)}\Big)\\
&= \liminf_{\varepsilon\to 0}\Big(\sum_{i\in\mathcal{I}_\varepsilon}\int_{Q^i_\varepsilon}\Big(\frac{1}{|D|}\chi_{D^i_\varepsilon}+\frac{1}{|Q\setminus D|}\chi_{Q^i_\varepsilon\setminus D^i_\varepsilon}-\frac{2}{|Q|}\chi_{Q^i_\varepsilon}\Big)|T_\varepsilon u_\varepsilon|^2\,dx\\
&\qquad +2\|T_\varepsilon u_\varepsilon\|^2_{L^2(\Omega)}\Big).
\end{aligned}$$

Since $\chi_{\bigcup_i D^i_\varepsilon}\overset{*}{\rightharpoonup}|D|$ and $\chi_{\bigcup_i(Q^i_\varepsilon\setminus D^i_\varepsilon)}\overset{*}{\rightharpoonup}|Q\setminus D|$ weak-$*$ in $L^\infty(\Omega)$ and $T_\varepsilon u_\varepsilon\to u$ in $L^2(\Omega)$ it follows that

$$2\|u\|^2_{L^2(\Omega)}\le\liminf_{\varepsilon\to 0}(\|\overline{u}_\varepsilon\|^2_{L^2(\Omega)}+\|\widetilde{u}_\varepsilon\|^2_{L^2(\Omega)})\le\liminf_{\varepsilon\to 0}2\|T_\varepsilon u_\varepsilon\|^2_{L^2(\Omega)}=2\|u\|^2_{L^2(\Omega)};$$

recalling (17), this implies the strong convergence $\overline{u}_\varepsilon\to u$ and $\widetilde{u}_\varepsilon\to u$ in $L^2(\Omega)$.

Since the function $\psi^{\alpha,\beta}_\delta$ introduced in (15) is continuous (in particular lower semicontinuous) and it is bounded from below thanks to the hypothesis on g, an application of the Fatou Lemma gives

$$\liminf_{\varepsilon\to 0}\int_\Omega\chi_\varepsilon\psi^{\alpha,\beta}_\delta(\overline{u}_\varepsilon)\,dx\ge\int_\Omega\liminf_{\varepsilon\to 0}\chi_\varepsilon\psi^{\alpha,\beta}_\delta(\overline{u}_\varepsilon)\,dx\ge\int_\Omega\psi^{\alpha,\beta}_\delta(u)\,dx. \tag{18}$$

The $\liminf$ inequality in Theorem 1 implies that

$$\liminf_{\varepsilon\to 0}F^0_\varepsilon(u_\varepsilon)\ge\int_\Omega\langle A_0\nabla u,\nabla u\rangle\,dx+\int_{S(u)}\varphi_0(\nu(u))d\mathcal{H}^{n-1};$$

then we get from (16) the estimate

$$\begin{aligned}
\liminf_{\varepsilon\to 0}F^{\alpha,\beta}_\varepsilon(u_\varepsilon) &\ge \int_\Omega\langle A_0\nabla u,\nabla u\rangle\,dx+\int_{S(u)}\varphi_0(\nu(u))d\mathcal{H}^{n-1}\\
&\quad +C_K\int_\Omega g(u)\,dx+(1-\eta)\int_\Omega\psi^{\alpha,\beta}_\delta(u)\,dx-C\varrho
\end{aligned}$$

for any $\eta,\varrho,\delta>0$ small enough. Taking the limit for $\varrho,\eta\to 0$ and the sup for $\delta>0$, we deduce

$$\begin{aligned}
\liminf_{\varepsilon\to 0} F_\varepsilon^{\alpha,\beta}(u_\varepsilon) &\geq \int_\Omega \langle A_0\nabla u, \nabla u\rangle\, dx + \int_{S(u)} \varphi_0(\nu(u))d\mathcal{H}^{n-1} \\
&\quad + C_K \int_\Omega g(u)\, dx + \sup_{\delta>0}\int_\Omega \psi_\delta^{\alpha,\beta}(u)\, dx \\
&= \int_\Omega \langle A_0\nabla u, \nabla u\rangle\, dx + \int_{S(u)} \varphi_0(\nu(u))d\mathcal{H}^{n-1} \\
&\quad + \int_\Omega \Big(C_K\, g(u) + \sup_\delta \psi_\delta^{\alpha,\beta}(u)\Big)\, dx.
\end{aligned}$$

The last equality follows from the dominated convergence theorem since the sequence $(\psi_\delta^{\alpha,\beta})$ is increasing as $\delta\to 0$ and $|\psi_\delta^{\alpha,\beta}(u)| \leq |K|\max\{|\min g|, g(\|u\|_\infty)\}$ a.e. in Ω.

Since the sequence of functions $(\psi_\delta^{\alpha,\beta})$ is increasing as δ goes to 0, we have

$$\begin{aligned}
\sup_\delta \psi_\delta^{\alpha,\beta}(z) &= \min\Big\{\sup_\delta\Big(c_\alpha \delta^{\alpha-2}\int_K |\nabla v|^2\, dx + c_\beta\delta^{\beta-1}\mathcal{H}^{n-1}(K\cap S(v)) \\
&\qquad + \int_K g(v)\Big) : v = z \text{ on } Q\setminus K\Big\}
\end{aligned}$$

(see e.g. [6, Remark 1.40]). This allows to show that

$$C_K\, g(z) + \sup_\delta \psi_\delta^{\alpha,\beta}(z) = g^{\alpha,\beta}(z) \tag{19}$$

concluding the proof of the lower bound.

1. *Case* $\alpha < 2$, $\beta < 1$. Since $\delta^{\alpha-2}$ and $\delta^{\beta-1}$ go to $+\infty$ as $\delta\to 0$, the minimum is attained for $v = z$ in K, then

$$C_K\, g(z) + \sup_\delta \psi_\delta^{\alpha,\beta}(z) = C_K\, g(z) + |K|\, g(z) = g(z) = g^{\alpha,\beta}(z).$$

2. *Case* $\alpha < 2$, $\beta = 1$. In this case $\delta^{\alpha-2}\to+\infty$, so that we can consider the minimum on piecewise constant functions, getting

$$\begin{aligned}
C_K\, g(z) + \sup_\delta \psi_\delta^{\alpha,\beta}(z) &= g(z) + \min\{c_\beta \mathcal{H}^{n-1}(A) - |A|(g(z) - \min g) : A\subset K\} \\
&= g^{\alpha,\beta}(z).
\end{aligned}$$

Note that this minimum problem is related with the theory of Cheeger sets (see [16]).

3. *Case* $\alpha = 2$, $\beta < 1$. We can consider the minimum problem restricted to $H^1(K)$; hence

$$\begin{aligned}
C_K\, g(z) + \sup_\delta \psi_\delta^{\alpha,\beta}(z) = C_K\, g(z) + \min\Big\{\int_K \big(c_\alpha|\nabla v|^2 + g(v)\big)\, dx : \\
v = z \text{ on } Q\setminus K\Big\} = g^{\alpha,\beta}(z).
\end{aligned}$$

4. Case $\alpha = 2$, $\beta = 1$. In this case we get

$$C_K\, g(z) + \sup_\delta \psi_\delta^{\alpha,\beta}(z) = C_K\, g(z) + \min\Big\{ \int_K \big(c_\alpha |\nabla v|^2 + g(v)\big)\, dx + c_\beta \mathcal{H}^{n-1}(S(v) \cap K) : v = z \text{ on } Q \setminus K \Big\} = g^{\alpha,\beta}(z).$$

This concludes the proof of the lim inf inequality:

$$\liminf_{\varepsilon \to 0} F_\varepsilon^{\alpha,\beta}(u_\varepsilon) \geq \int_\Omega \langle A_0 \nabla u, \nabla u\rangle\, dx + \int_{S(u)} \varphi_0(\nu(u)) d\mathcal{H}^{n-1} + \int_\Omega g^{\alpha,\beta}(u)\, dx.$$

Upper bound. Given $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$, let (u_ε) be a recovery sequence for the functional F^0 (see Theorem 1), that is $u_\varepsilon \to u$ in $L^2(\Omega)$ and the inequality (10) holds. We recall that the sequence (u_ε) can be chosen such that $\|\nabla u_\varepsilon\|_{L^2(\Omega)}$ and $\mathcal{H}^{n-1}(S(u_\varepsilon) \cap \Omega)$ are uniformly bounded. Moreover, it is not restrictive to assume (u_ε) uniformly bounded in $L^\infty(\Omega)$, by considering $(-2\|u\|_\infty \vee u_\varepsilon) \wedge 2\|u\|_\infty$.

1. *Case $\alpha < 2$, $\beta < 1$.* The strong convergence $u_\varepsilon \to u$ in $L^2(\Omega)$ and the hypotheses on g imply the convergence

$$\int_\Omega g(u_\varepsilon)\, dx \to \int_\Omega g(u)\, dx.$$

Then, the lim sup inequality follows immediately from (10), recalling that $\alpha, \beta > 0$ and that $\|\nabla u_\varepsilon\|_{L^2(\Omega)}$ and $\mathcal{H}^{n-1}(S(u_\varepsilon) \cap \Omega)$ are uniformly bounded.

2. *Case $\alpha < 2$, $\beta = 1$.* We modify the sequence (u_ε) in $\Omega \setminus \varepsilon E$ defining $\widehat{u}_\varepsilon = \sum_{i \in \mathcal{I}_\varepsilon} \widehat{u}_\varepsilon^i \chi_{Q_\varepsilon^i}$, where $\widehat{u}_\varepsilon^i$ stands for the integral average of u_ε in Q_ε^i. Now, let $A(z)$ be a solution of the minimum problem

$$\min\{c_\beta \mathcal{H}^{n-1}(\partial A) - |A|(g(z) - \min g) : A \subset K\}$$

and define $A_\varepsilon^i(z) = \varepsilon i + \varepsilon A(z)$. We pose

$$\widetilde{u}_\varepsilon = \begin{cases} u_\varepsilon & \text{in } \Omega \setminus \bigcup_i A_\varepsilon^i(\widehat{u}_\varepsilon^i) \\ \min g & \text{otherwise.} \end{cases}$$

Since $\displaystyle\int_\Omega |g(u_\varepsilon) - g(\widehat{u}_\varepsilon)|\, dx \to 0$ and $(\|\nabla u_\varepsilon\|_{L^2(\Omega)})$ is equibounded, applying (10) it follows that

$$\begin{aligned} F_\varepsilon^{\alpha,\beta}(\widetilde{u}_\varepsilon) &\leq F_\varepsilon^0(u_\varepsilon) + \varepsilon^\alpha c_\alpha \|\nabla u_\varepsilon\|_{L^2(\Omega)} + \int_{\Omega \cap \varepsilon E} g(u_\varepsilon)\, dx \\ &\quad + \sum_i \big(c_\beta \varepsilon \mathcal{H}^{n-1}(\partial A_\varepsilon^i(\widehat{u}_\varepsilon^i)) + |A_\varepsilon^i(\widehat{u}_\varepsilon^i)| \min g\big) + \sum_i \int_{K_\varepsilon^i \setminus A_\varepsilon^i(\widehat{u}_\varepsilon^i)} g(u_\varepsilon)\, dx \\ &= F_\varepsilon^0(u_\varepsilon) + \int_{\Omega \cap \varepsilon E} g(u)\, dx + \int_{\Omega \setminus \varepsilon E} g(u_\varepsilon)\, dx \\ &\quad + \sum_i \big(c_\beta \varepsilon \mathcal{H}^{n-1}(\partial A_\varepsilon^i(\widehat{u}_\varepsilon^i)) - |A_\varepsilon^i|(g(\widehat{u}_\varepsilon^i) - \min g)\big) + o(1)_{\varepsilon \to 0}. \end{aligned}$$

Recalling the definition of $A(z)$ and the strong convergence $\widehat{u}_\varepsilon \to u$, we get

$$\limsup_{\varepsilon\to 0} F_\varepsilon^{\alpha,\beta}(\widetilde{u}_\varepsilon) \le \int_\Omega \langle A_0\nabla u, \nabla u\rangle\, dx + \int_{S(u)} \varphi_0(\nu_u) d\mathcal{H}^{n-1} + \int_\Omega g^{\alpha,\beta}(u)\, dx.$$

3. *Case* $\alpha = 2$, $\beta < 1$. Note that for $\varrho > 0$ small enough the sequence (u_ε) can be assumed to be such that $u_\varepsilon \in H^1(K^i_{\varepsilon,\varrho})$ for $i \in \mathcal{I}^*_\varepsilon$. We recall that $\#\,(\mathcal{I}_\varepsilon \setminus \mathcal{I}^*_\varepsilon) \le c\varepsilon^{1-n}$.

Let $v_z \in H^1(K)$ be a solution of the minimum problem

$$\min\Big\{\int_K \big(c_\alpha|\nabla v|^2 + g(v)\big)\, dx : v = z \text{ on } Q\setminus K\Big\}.$$

The hypotheses on g ensures that v_z belongs to $L^\infty(K)$. We show that, up to a term which is infinitesimal with ϱ, this minimum is greater than the minimum of the corresponding problem with K substituted by $K(2\varrho) = \{x \in K : \operatorname{dist}(x, \partial K) \ge 2\varrho\}$. Since K is of class C^2 there exists $\sigma > 0$ such that, setting

$$K(\sigma) = \{x \in K : \operatorname{dist}(x, \partial K) \ge \sigma\},$$

the normal projection $\pi\colon K \setminus K(\sigma) \to \partial K$ is well defined, and for any $x \in K\setminus K(\sigma)$ there exist unique $y \in \partial K(\sigma)$ and unique $t \in (0, \sigma]$ such that $x = y + t\nu$, ν being the inner normal to ∂K. Now, for $\varrho < \sigma/2$ we define in the set $K(2\varrho)$ the function

$$v^\varrho(x) = v_z(\phi^\varrho(x))$$

where

$$\phi^\varrho(x) = \begin{cases} x & \text{in } K(\sigma) \\ y + \dfrac{\sigma}{\sigma - 2\varrho} t\nu & \text{for } y \in \partial K(\sigma) \text{ and } t \in (0, \sigma - 2\varrho]. \end{cases}$$

Noting that $|\nabla\phi^\varrho| \le 1 + c\varrho$ with c depending only on K and σ, we get

$$\begin{aligned}\int_{K(2\varrho)} \big(|\nabla v^\varrho|^2 + g(v^\varrho)\big)\, dx &\le (1 + 2c\varrho)\int_{K(2\varrho)} \big(|\nabla v_z|^2 + g(v_z)\big)\, dx \\ &\le (1 + 2c\varrho)\int_K \big(|\nabla v_z|^2 + g(v_z)\big)\, dx + c'\varrho\end{aligned}$$

so that

$$\begin{aligned}&\min\Big\{\int_{K(2\varrho)} \big(c_\alpha|\nabla v|^2 + g(v)\big)\, dx : v = z \text{ on } Q \setminus K(2\varrho)\Big\} \\ &\qquad \le \min\Big\{\int_K \big(c_\alpha|\nabla v|^2 + g(v)\big)\, dx : v = z \text{ on } Q\setminus K\Big\} + C\varrho\end{aligned} \tag{20}$$

where C depends only on K and g. Now, we modify the sequence (u_ε) in the holes. Let $v^{\varrho,z}$ be a solution of the minimum problem in $K(2\varrho)$, and set $v^i_\varepsilon(x) = v^{\varrho,u^i_\varepsilon}(x/\varepsilon - i)$, where u^i_ε stands for the integral average of u_ε in $K^i_{\varepsilon,\varrho} \setminus K^i_{\varepsilon,2\varrho}$, and define

$$\widetilde{u}_\varepsilon = \begin{cases} u_\varepsilon & \text{in } \Omega \setminus \bigcup_{i\in\mathcal{I}^*_\varepsilon} K^i_{\varepsilon,\varrho} \\ \varphi^\varrho_\varepsilon u_\varepsilon + (1-\varphi^\varrho_\varepsilon) u^i_\varepsilon & \text{in } K^i_{\varepsilon,\varrho} \setminus K^i_{\varepsilon,2\varrho} \text{ for each } i \in \mathcal{I}^*_\varepsilon \\ v^i_\varepsilon & \text{in } K^i_{\varepsilon,2\varrho} \text{ for each } i \in \mathcal{I}^*_\varepsilon, \end{cases} \tag{21}$$

where $\varphi^\varrho_\varepsilon$ is the function introduced in (14) and $\mathcal{I}^*_\varepsilon$ is as in (7). It follows that

$$\begin{aligned} c_\alpha \varepsilon^2 \sum_{i\in\mathcal{I}^*_\varepsilon} \int_{K^i_{\varepsilon,\varrho}\setminus K^i_{\varepsilon,2\varrho}} |\nabla \widetilde{u}_\varepsilon|^2\,dx &\leq \frac{2c_\alpha\varepsilon^2}{\varepsilon^2\varrho^2} \sum_{i\in\mathcal{I}^*_\varepsilon} \int_{K^i_{\varepsilon,\varrho}\setminus K^i_{\varepsilon,2\varrho}} |u_\varepsilon - u^i_\varepsilon|^2\,dx \\ &\quad + 2c_\alpha\varepsilon^2 \sum_{i\in\mathcal{I}^*_\varepsilon} \int_{K^i_{\varepsilon,\varrho}\setminus K^i_{\varepsilon,2\varrho}} |\nabla u_\varepsilon|^2\,dx \\ &\leq \frac{c\varepsilon^2}{\varrho^2} \int_\Omega |\nabla u_\varepsilon|^2\,dx \end{aligned}$$

where we applied the Poincaré inequality in $K^i_{\varepsilon,\varrho} \setminus K^i_{\varepsilon,2\varrho}$. Since $|\nabla u_\varepsilon|$ is uniformly bounded in $L^2(\Omega)$, we get

$$\begin{aligned} & c_\alpha\varepsilon^2 \sum_{i\in\mathcal{I}^*_\varepsilon} \int_{K^i_{\varepsilon,2\varrho}} |\nabla\widetilde{u}_\varepsilon|^2\,dx + \sum_{i\in\mathcal{I}^*_\varepsilon} \int_{K^i_{\varepsilon,2\varrho}} g(v^i_\varepsilon)\,dx \\ &\qquad = c_\alpha\varepsilon^n \sum_{i\in\mathcal{I}^*_\varepsilon} \int_{K_{2\varrho}} |\nabla v^{\varrho,u^i_\varepsilon}(x)|^2\,dx + \varepsilon^n \sum_{i\in\mathcal{I}^*_\varepsilon} \int_{K_{2\varrho}} g(v^{\varrho,u^i_\varepsilon})\,dx \\ &\qquad = \varepsilon^n \sum_{i\in\mathcal{I}^*_\varepsilon} \min\Big\{ \int_{K(2\varrho)} \big(c_\alpha|\nabla v|^2 + g(v)\big)\,dx : v = u^i_\varepsilon \text{ on } Q\setminus K(2\varrho)\Big\} \\ &\qquad = \sum_{i\in\mathcal{I}^*_\varepsilon} \int_{Q^i_\varepsilon} \gamma(\overline{u}_\varepsilon, K(2\varrho))\,dx \end{aligned}$$

where $\overline{u}_\varepsilon$ is the piecewise constant function defined by $\overline{u}_\varepsilon = \sum_{i\in\mathcal{I}^*_\varepsilon} \chi_{Q^i_\varepsilon} u^i_\varepsilon$ and for any $z\in\mathbb{R}$ and V compact subset of Q

$$\gamma(z,V) = \min\Big\{ \int_V \big(c_\alpha|\nabla v|^2 + g(v)\big)\,dx : v = z \text{ on } Q\setminus V\Big\}.$$

Recalling (20), we get

$$\sum_{i\in\mathcal{I}^*_\varepsilon} \int_{Q^i_\varepsilon} \gamma(\overline{u}_\varepsilon, K(2\varrho)) \leq \sum_{i\in\mathcal{I}^*_\varepsilon} \int_{Q^i_\varepsilon} \gamma(\overline{u}_\varepsilon, K) + C|\Omega|\varrho.$$

The uniform bounds on $\|\widetilde{u}_\varepsilon\|_{L^\infty(\Omega)}$, $\|\nabla u_\varepsilon\|_{L^2(\Omega)}$ and $\mathcal{H}^{n-1}(S(u_\varepsilon)\cap\Omega)$ allow to deduce from the previous inequalities

$$c_\alpha\varepsilon^2\int_{\Omega\setminus\varepsilon E}|\nabla\widetilde{u}_\varepsilon|^2\,dx+c_\beta\varepsilon^\beta\mathcal{H}^{n-1}(S(\widetilde{u}_\varepsilon)\setminus\varepsilon E)+\int_\Omega g(\widetilde{u}_\varepsilon)\,dx$$
$$\leq\sum_{i\in\mathcal{I}_\varepsilon^*}\int_{Q_\varepsilon^i}\gamma(\overline{u}_\varepsilon,K)\,dx+\int_{\Omega\cap\varepsilon E}g(u_\varepsilon)\,dx+C|\Omega|\varrho+o(1)_{\varepsilon\to0}$$

Then, the strong convergence of $\overline{u}_\varepsilon$, the continuity of γ and the estimate (10) give

$$\begin{aligned}\limsup_{\varepsilon\to0}F_\varepsilon^{\alpha,\beta}(\widetilde{u}_\varepsilon)\ &\leq\ \int_\Omega\langle A_0\nabla u,\nabla u\rangle\,dx+\int_{S(u)}\varphi_0(\nu_u)d\mathcal{H}^{n-1}\\&\quad+\int_\Omega g^{\alpha,\beta}(u)\,dx+C|\Omega|\varrho.\end{aligned}$$

Taking the limit for $\varrho\to0$ we get the lim sup inequality.

4. *Case* $\alpha=2$, $\beta=1$. For any $z\in\mathbb{R}$ and V compact subset of Q, we set

$$\gamma'(z,V)=\min\Big\{\int_V\big(c_\alpha|\nabla v|^2+g(v)\big)\,dx+c_\beta\mathcal{H}^{n-1}(S(v)\cap V):v=z\text{ on }Q\setminus V\Big\}.$$

The same construction of the previous case allows to deduce that for any z

$$\gamma'(z,K(2\varrho))\leq\gamma'(z,K)+C\varrho,\tag{22}$$

with C depending only on K and g. Indeed, if v_z realizes the minimum in K then the jump set of the function $v^\varrho=v_z(\phi^\varrho)$ defined as above satisfies $\mathcal{H}^{n-1}(S(v^\varrho)\cap K(2\varrho))\leq(1+c\varrho)\mathcal{H}^{n-1}(S(v_z)\cap K)$, with c depending only on K and σ. Thanks to the estimate (22), defining $\widetilde{u}_\varepsilon$ as in (21) the lim sup inequality follows as in the previous case since for the jump set in the holes we have

$$\begin{aligned}&c_\beta\varepsilon\sum_{i\in\mathcal{I}_\varepsilon^*}\mathcal{H}^{n-1}(S(\widetilde{u}_\varepsilon)\cap K_{\varepsilon,\varrho}^i)\\&\qquad\leq c_\beta\sum_{i\in\mathcal{I}_\varepsilon^*}\mathcal{H}^{n-1}(S(v_z(\phi^\varrho))\cap K(2\varrho))+\varepsilon\mathcal{H}^{n-1}(S(u_\varepsilon))\\&\qquad=c_\beta\sum_{i\in\mathcal{I}_\varepsilon^*}\mathcal{H}^{n-1}(S(v_z(\phi^\varrho))\cap K(2\varrho))+o(1)_{\varepsilon\to0}.\end{aligned}$$

The proof is thus complete. □

References

[1] E. Acerbi, V. Chiadò Piat, G. Dal Maso and D. Percivale. An extension theorem from connected sets, and homogenization in general periodic domains. *Nonlinear Anal.* **18** (1992), 481–496.

[2] L. Ambrosio, N. Fusco and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, Oxford, 2000.

[3] I. Babuška, B. Andersson, P.J. Smith, and K. Levin. Damage analysis of fiber composites Part I: Statistical analysis on fiber scale. *Computer Methods in Applied Mechanics and Engineering* **172** (1999), 27–77

[4] B. Bourdin, G.A. Francfort, and J.-J. Marigo. *The Variational Approach to Fracture. Journal of Elasticity* **91** (2008), 5–148,

[5] A. Braides. *Approximation of Free-Discontinuity Problems.* Lecture Notes in Math. **1694**, Springer Verlag, Berlin, 1998.

[6] A. Braides. Γ*-convergence for Beginners*, Oxford University Press, Oxford, 2002.

[7] A. Braides. A handbook of Γ-convergence. In *Handbook of Differential Equations. Stationary Partial Differential Equations, Volume* 3 (M. Chipot and P. Quittner, eds.), Elsevier, 2006.

[8] A. Braides, V. Chiadò Piat and A. Piatnitski. A variational approach to double-porosity problems. *Asymptotic Anal.* **39** (2004), 281–308.

[9] A. Braides and A. Defranceschi. *Homogenization of Multiple Integrals.* Oxford University Press, Oxford, 1998.

[10] A. Braides, A. Defranceschi and E. Vitali. Homogenization of free discontinuity problems. *Arch. Rational Mech. Anal.* **135** (1996), 297–356.

[11] A. Braides and A. Garroni. Homogenization of periodic nonlinear media with soft and stiff inclusions. *Math. Meth. Mod. Appl. Sci.* **5** (1995), 543–564

[12] F. Cagnetti and L. Scardia. An extension theorem in SBV and an application to the homogenization of the Mumford-Shah functional in perforated domains. *J. Math. Pures Appl.* **95** (2011), 349–381

[13] G. Dal Maso. *An Introduction to* Γ*-convergence*, Birkhauser, Boston, 1993.

[14] A. DeSimone, J.J. Marigo, and L. Teresi, A damage mechanics approach to stress softening and its application to rubber. *European Journal of Mechanics A* **20** (2001), 873–892.

[15] G.J. Dvorak. Composite materials: Inelastic behavior, damage, fatigue and fracture. *International Journal of Solids and Structures* **37** (2000), 155–170

[16] M. Solci. Double-porosity homogenization for perimeter functionals. *Math. Meth. Appl. Sci.* **32** (2009), 1971–2002.

[17] M. Solci. Multi-phase double-porosity homogenization for perimeter functionals. *Math. Meth. Appl. Sci.* (2012), in print.