A SYMMETRY RESULT FOR THE ORNSTEIN-UHLENBECK OPERATOR

ANNALISA CESARONI, MATTEO NOVAGA, AND ENRICO VALDINOCI

Abstract. In 1978 E. De Giorgi formulated a conjecture concerning the one-dimensional symmetry of bounded solutions to the elliptic equation $\Delta u = F'(u)$, which are monotone in some direction. In this paper we prove the analogous statement for the equation $\Delta u - \langle x, \nabla u\rangle u = F'(u)$, where the Laplacian is replaced by the Ornstein-Uhlenbeck operator. Our theorem holds without any restriction on the dimension of the ambient space, and this allows us to obtain a similar result in infinite dimensions, by a limit procedure.

1. Introduction

A celebrated conjecture by De Giorgi [6] asks if bounded entire solutions to the equation

$$\Delta u = u^3 - u \tag{1.1}$$

which are strictly increasing in some direction are one-dimensional, in the sense that the level sets $\{u = \lambda\}$ are hyperplanes, at least if $n \leq 8$. This conjecture has been proved by Ghoussoub and Gui [14] in dimension $n = 2$, and by Ambrosio and Cabré [2] in dimension $n = 3$, and a counterexample has been given by del Pino, Kowalczyk and Wei in [7] for $n \geq 9$. While the conjecture is still open for $4 \leq n \leq 8$, a very nice proof has been presented by O. Savin [17] under the additional assumption that u connects -1 to 1 along the direction where it increases. See also [4] for another proof in dimension $n = 2$ and [12] for a review on the subject.

In this paper, we are interested in a variant of (1.1) where the Laplacian Δ is substituted by the Ornstein-Uhlenbeck operator $\Delta - \langle x, \nabla\rangle$. Namely, we consider the semilinear elliptic equation

$$\Delta u - \langle x, \nabla u\rangle + f(u) = 0 \tag{1.2}$$

and show the one-dimensional symmetry of bounded entire solutions which are monotone in some direction.

Let us state our main result.

Theorem 1.1. *Let $n \in \mathbb{N}$, $\alpha \in (0,1)$. Let $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a solution of*

$$\Delta u - \langle x, \nabla u\rangle + f(u) = 0 \qquad in\ \mathbb{R}^n,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function. Assume that

$$\langle \nabla u(x), w\rangle > 0 \qquad for\ any\ x \in \mathbb{R}^n \tag{1.3}$$

Supported by the *Progetto CaRiPaRo "Nonlinear Partial Differential Equations: models, analysis, and control-theoretic problems"* and the *ERC grant ϵ "Elliptic Pde's and Symmetry of Interfaces and Layers for Odd Nonlinearities"*.

for some $w \in \mathbb{R}^n$. Then, u is one-dimensional, i.e. there exist $U : \mathbb{R} \to \mathbb{R}$ and $\omega \in \mathbb{R}^n$ such that

$$u(x) = U(\langle \omega, x \rangle)$$

for any $x \in \mathbb{R}^n$.

Notice that (1.2) can be regarded as the analog of (1.1) in the so-called Gauss space, that is, in $\mathbb{R}^n$ endowed with the Gaussian instead of the Lebesgue measure. Indeed, while the Pde in (1.1) is the Euler-Lagrange equation of the Allen-Cahn Energy

$$\int_{\mathbb{R}^n} \left(\frac{|\nabla u|^2}{2} + \frac{(u^2-1)^2}{4} \right) dx \,, \tag{1.4}$$

the Pde in (1.2) is the Euler-Lagrange equation of the functional

$$\int_{\mathbb{R}^n} \left(\frac{|\nabla u|^2}{2} + F(u) \right) d\gamma(x) \,, \tag{1.5}$$

where $F' = -f$ and

$$d\gamma(x) = \gamma(x) dx = \frac{e^{-|x|^2/2}}{(2\pi)^{n/2}}\, dx \tag{1.6}$$

is the standard Gaussian probability measure. It is interesting to remark that Theorem 1.1 holds for general type of nonlinearities, as it happens for the conjecture of De Giorgi when $n \le 3$ (see [1], and this is a major difference with respect to the techniques in [17]).

As in the case of the Laplacian, Theorem 1.1 is closely related to the Bernstein problem in the Gauss space, which asks for flatness of entire minimal surfaces which are graphs in some direction. We point out that minimal surfaces in the Gauss space are interesting geometric objects, since they correspond to self-similar shrinkers of the mean curvature flow (see for instance [8]), and satisfy the equation

$$\kappa = \langle x, \nu \rangle \tag{1.7}$$

where κ is the mean curvature at x and ν is the normal vector. In this context, the analog of the Bernstein Theorem has been proved by Ecker and Huisken [8], under a polynomial growth assumption on the volume of the minimal surface, and more recently by Wang in [20] without any further assumption. We point out that, differently from the Euclidean case, the result holds without any restriction on the dimension of the ambient space, and in fact there is no such restriction also in Theorem 1.1. This is due to the exponential decay of the Gaussian measure associated to the Ornstein-Uhlenbeck operator which allows for better estimates than the corresponding Euclidean ones.

Since Theorem 1.1 holds in any dimension and the Gauss space $(\mathbb{R}^n, \gamma)$ formally converges to a Wiener space (X, H, γ) (see Section 2.1 for a precise definition) as $n \to \infty$, one may expect that an analogous result holds in such infinite dimensional setting. Indeed, in this paper we confirm this expectation and show the infinite dimensional extension of Theorem 1.1:

Theorem 1.2. *Let $u \in C^1(X) \cap L^\infty(X)$ satisfy*

$$\Delta_\gamma u = f(u) \tag{1.8}$$

where $f : \mathbb{R} \to \mathbb{R}$ *is a locally Lipschitz function. Assume that*

$$\partial_i \partial_j u \in C(X) \qquad \text{for all } i, j \in \mathbb{N} \tag{1.9}$$

and

$$\inf_{x \in B_R} [\nabla u(x), w] > 0 \tag{1.10}$$

for all $x \in X$*, for all* $R > 0$ *and for some* $w \in H$*. Then,* u *is one-dimensional, in the sense that there exist* $U : \mathbb{R} \to \mathbb{R}$ *and* $\omega \in X^*$ *such that*

$$u(x) = U(\langle \omega, x \rangle) \qquad for \ all \ x \in X. \tag{1.11}$$

Notice that Theorem 1.1 can be recovered as a corollary of Theorem 1.2, when the function u depends only on finitely many variables. As far as we know, Theorem 1.1 is the first result of De Giorgi conjecture type in an infinite dimensional setting. The proof that we perform exploits and generalizes some geometric ideas of [18, 19, 10, 11].

2. Notation

We denote by $(\mathbb{R}^n, \gamma)$ the n-dimensional Gauss space, where γ is the standard Gaussian measure on $\mathbb{R}^n$ defined in (1.6).

2.1. **The Wiener space.** An abstract Wiener space is defined as a triple (X, γ, H) where X is a separable Banach space, endowed with the norm $\|\cdot\|_X$, γ is a nondegenerate centered Gaussian measure, and H is the Cameron–Martin space associated to the measure γ, that is, H is a separable Hilbert space densely embedded in X, endowed with the inner product $[\cdot, \cdot]_H$ and with the norm $|\cdot|_H$. The requirement that γ is a centered Gaussian measure means that for any $x^* \in X^*$, the measure $x^*_\# \gamma$ is a centered Gaussian measure on the real line $\mathbb{R}$, that is, the Fourier transform of γ is given by

$$\hat{\gamma}(x^*) = \int_X e^{-i\langle x, x^* \rangle} d\gamma(x) = \exp\left(-\frac{\langle Qx^*, x^* \rangle}{2}\right), \qquad \forall x^* \in X^*;$$

here the operator $Q \in \mathcal{L}(X^*, X)$ is the covariance operator and it is uniquely determined by formula

$$\langle Qx^*, y^* \rangle = \int_X \langle x, x^* \rangle \langle x, y^* \rangle d\gamma(x), \qquad \forall x^*, y^* \in X^*.$$

The nondegeneracy of γ implies that Q is positive definite: the boundedness of Q follows by Fernique's Theorem (see for instance [5, Theorem 2.8.5]), asserting that there exists a positive number $\beta > 0$ such that

$$\int_X e^{\beta \|x\|^2} d\gamma(x) < +\infty.$$

This implies also that the maps $x \mapsto \langle x, x^* \rangle$ belong to $L^p_\gamma(X)$ for any $x^* \in X^*$ and $p \in [1, +\infty)$, where $L^p_\gamma(X)$ denotes the space of all functions $f : X \to \mathbb{R}$ such that

$$\int_X |f(x)|^p d\gamma(x) < +\infty.$$

In particular, any element $x^* \in X^*$ can be seen as a map $x^* \in L^2_\gamma(X)$, and we denote by $R^* : X^* \to \mathcal{H}$ the identification map $R^* x^*(x) := \langle x, x^* \rangle$. The space $\mathcal{H}$ given by the closure of R^*X^* in $L^2_\gamma(X)$ is called *reproducing kernel*. By considering the map $R : \mathcal{H} \to X$ defined as

$$R\hat{h} := \int_X \hat{h}(x) x d\gamma(x),$$

we obtain that R is an injective γ–Radonifying operator, which is Hilbert–Schmidt when X is Hilbert. We also have $Q = RR^* : X^* \to X$. The space $H := R\mathcal{H}$, equipped with the inner product $[\cdot,\cdot]_H$ and norm $|\cdot|_H$ induced by $\mathcal{H}$ via R, is the Cameron-Martin space and is a dense subspace of X. The continuity of R implies that the embedding of H in X is continuous, that is, there exists $c > 0$ such that

$$\|h\|_X \le c|h|_H, \qquad \forall h \in H.$$

We have also that the measure γ is absolutely continuous with respect to translation along Cameron–Martin directions; in fact, for $h \in H$, $h = Qx^*$, the measure $\gamma_h(B) = \gamma(B - h)$ is absolutely continuous with respect to γ with density given by

$$d\gamma_h(x) = \exp\left(\langle x, x^*\rangle - \frac{1}{2}|h|^2_H\right) d\gamma(x). \tag{2.1}$$

2.2. **Cylindrical functions and differential operators.** For $j \in \mathbb{N}$ we choose $x^*_j \in X^*$ in such a way that $\hat{h}_j := R^*x^*_j$, or equivalently $h_j := R\hat{h}_j = Qx^*_j$, form an orthonormal basis of H. We order the vectors x^*_j in such a way that the numbers $\lambda_j := \|x^*_j\|^{-2}_{X^*}$ form a decreasing sequence. Given $m \in \mathbb{N}$, we also let $H_m := \langle h_1, \ldots, h_m\rangle \subseteq H$, and $\Pi_m : X \to H_m$ be the closure of the orthogonal projection from H to H_m

$$\Pi_m(x) := \sum_{j=1}^m \langle x, x^*_j\rangle h_j \qquad x \in X.$$

The map Π_m induces the decomposition $X \simeq H_m \oplus X^\perp_m$, with $X^\perp_m := \ker(\Pi_m)$, and $\gamma = \gamma_m \otimes \gamma^\perp_m$, with γ_m and $\gamma^\perp_m$ Gaussian measures on H_m and $X^\perp_m$ respectively, having H_m and $H^\perp_m$ as Cameron–Martin spaces. When no confusion is possible we identify H_m with $\mathbb{R}^m$; with this identification the measure $\gamma_m = \Pi_{m\#}\gamma$ is the standard Gaussian measure on $\mathbb{R}^m$ (see [5]). Given $x \in X$, we denote by $\underline{x}_m \in H_m$ the projection $\Pi_m(x)$, and by $\overline{x}_m \in X^\perp_m$ the infinite dimensional component of x, so that $x = \underline{x}_m + \overline{x}_m$. When we identify H_m with $\mathbb{R}^m$ we shall rather write $x = (\underline{x}_m, \overline{x}_m) \in \mathbb{R}^m \oplus X^\perp_m$.

We say that $u : X \to \mathbb{R}$ is a *cylindrical function* if $u(x) = v(\Pi_m(x))$ for some $m \in \mathbb{N}$ and $v : \mathbb{R}^m \to \mathbb{R}$. We denote by $\mathcal{F}C^k_b(X)$, $k \in \mathbb{N}$, the space of all C^k_b cylindrical functions, that is, functions of the form $v(\Pi_m(x))$ with $v \in C^k(\mathbb{R}^n)$, with continuous and bounded derivatives up to the order k. We denote by $\mathcal{F}C^k_b(X, H)$ the space generated by all functions of the form uh, with $u \in \mathcal{F}C^k_b(X)$ and $h \in H$.

We let

$$\nabla_\gamma u := \textstyle\sum_{j\in\mathbb{N}} \partial_j u \, h_j \qquad \text{for } u \in \mathcal{F}C^1_b(X)$$

$$\mathrm{div}_\gamma \varphi := \textstyle\sum_{j\ge 1} \partial^*_j [\varphi, h_j]_H \quad \text{for } \varphi \in \mathcal{F}C^1_b(X, H)$$

$$\Delta_\gamma u := \mathrm{div}_\gamma \nabla_\gamma u \qquad \text{for } u \in \mathcal{F}C^2_b(X)$$

where $\partial_j := \partial_{h_j}$ and $\partial_j^* := \partial_j - \hat{h}_j$ is the adjoint operator of ∂_j. With this notation, the integration by parts formula holds:

$$\int_X u \, \mathrm{div}_\gamma \varphi \, d\gamma = -\int_X [\nabla_\gamma u, \varphi]_H \, d\gamma \qquad \forall \varphi \in \mathcal{F}C_b^1(X, H). \tag{2.2}$$

In particular, thanks to (2.2), the operator ∇_γ is closable in $L^p_\gamma(X)$, and we denote by $W^{1,p}_\gamma(X)$ the domain of its closure. The Sobolev spaces $W^{k,p}_\gamma(X)$, with $k \in \mathbb{N}$ and $p \in [1, +\infty]$, can be defined analogously [5], and $\mathcal{F}C_b^k(X)$ is dense in $W^{j,p}_\gamma(X)$, for all $p < +\infty$ and $k, j \in \mathbb{N}$ with $k \geq j$.

Given a vector field $\varphi \in L^p_\gamma(X, H)$, $p \in (1, \infty]$, using (2.2) we can define $\mathrm{div}_\gamma \varphi$ in the distributional sense, taking test functions u in $W^{1,q}_\gamma(X)$ with $\frac{1}{p} + \frac{1}{q} = 1$. We say that $\mathrm{div}_\gamma \varphi \in L^p_\gamma(X)$ if this linear functional can be extended to all test functions $u \in L^q_\gamma(X)$. This is true in particular if $\varphi \in W^{1,p}_\gamma(X, H)$.

Let $u \in W^{2,2}_\gamma(X)$, $\psi \in \mathcal{F}C_b^1(X)$ and $i, j \in \mathbb{N}$. From (2.2), with $u = \partial_j u$ and $\varphi = \psi h_i$, we get

$$\int_X \partial_j u \, \partial_i \psi \, d\gamma = \int_X -\partial_j(\partial_i u) \, \psi + \partial_j u \, \psi \langle x_i^*, x \rangle d\gamma \tag{2.3}$$

Let now $\varphi \in \mathcal{F}C_b^1(X, H)$. If we apply (2.3) with $\psi = [\varphi, h_j] =: \varphi^j$, we obtain

$$\int_X \partial_j u \, \partial_i \varphi^j \, d\gamma = \int_X -\partial_j(\partial_i u) \, \varphi^j + \partial_j u \, \varphi^j \langle x_i^*, x \rangle d\gamma$$

which, summing up in j, gives

$$\int_X [\nabla_\gamma u, \partial_i \varphi] \, d\gamma = \int_X -[\nabla_\gamma(\partial_i u), \varphi] + [\nabla_\gamma u, \varphi] \langle x_i^*, x \rangle d\gamma \qquad \forall \varphi \in \mathcal{F}C_b^1(X, H). \tag{2.4}$$

The operator $\Delta_\gamma : W^{2,p}_\gamma(X) \to L^p_\gamma(X)$ is usually called the *Ornstein-Uhlenbeck operator*. Notice that, if u is a cylindrical function, that is $u(x) = v(y)$ with $y = \Pi_m(x) \in \mathbb{R}^m$ and $m \in \mathbb{N}$, then

$$\Delta_\gamma u = \sum_{j=1}^m \partial_{jj} u - \langle x_j^*, x \rangle \partial_j u = \Delta v - \langle y, \nabla v \rangle_{\mathbb{R}^m} \, . \tag{2.5}$$

We write $u \in C(X)$ if $u : X \to \mathbb{R}$ is continuous and $u \in C^1(X)$ if both $u : X \to \mathbb{R}$ and $\nabla_\gamma u : X \to H$ are continuous.

For simplicity of notation, from now on we will omit the explicit dependence on γ of operators and spaces. We also indicate by $[\cdot, \cdot]$ and $|\cdot|$ respectively the scalar product and the norm in H. When no confusion is possible, we shall also write u_i to indicate the derivative $\partial_i u$.

3. Proof of Theorem 1.2

Recalling the integration by parts formula (2.2), equation (1.8) can be written in a weak form as

$$\int_X [\nabla u, \nabla \varphi] - f(u) \varphi \, d\gamma = 0 \qquad \text{for any } \varphi \in W^{1,2}(X) \tag{3.1}$$

which is meaningful for $u \in W^{1,2}(X)$. Notice that, as $\mathcal{F}C_b^1(X)$ is dense in $W^{1,2}(X)$, it is enough to require (3.1) for all $\varphi \in \mathcal{F}C_b^1(X)$.

Remark 3.1. Since $L^\infty(X) \subset L^2(X)$, by [16, Th. 4.1] we have that a bounded weak solution of (1.8) belongs to $W^{2,2}(X)$.

3.1. **The linearized equation.** We now consider the equation solved by the derivatives of the solution u.

Lemma 3.2. *Let $u \in W^{2,2}(X)$ satisfy* (1.8)*. For any $i \in \mathbb{N}$ let $u_i = \partial_i u \in W^{1,2}(X)$, then*

$$\int_X [\nabla u_i, \nabla\varphi] - f'(u)u_i\varphi + u_i\varphi\, d\gamma = 0 \qquad \text{for any } \varphi \in W^{1,2}(X). \tag{3.2}$$

Proof. Notice first that it is enough to prove (3.2) for all $\varphi \in \mathcal{F}C_b^2(X)$. Letting $\varphi \in \mathcal{F}C_b^2(X)$, we multiply (1.8) by φ_i and recall (2.4), to get

$$\begin{aligned}
0 &= \int_X [\nabla u, \nabla\varphi_i] - f(u)\varphi_i\, d\gamma \\
&= \int_X -[\nabla u_i, \nabla\varphi] + \langle x_i^*, x\rangle[\nabla u, \nabla\varphi] - f(u)\varphi_i\, d\gamma \\
&= \int_X -[\nabla u_i, \nabla\varphi] + \langle x_i^*, x\rangle[\nabla u, \nabla\varphi] + f'(u)u_i\varphi - \langle x_i^*, x\rangle x_i f(u)\varphi\, d\gamma \\
&= \int_X -[\nabla u_i, \nabla\varphi] + [\nabla u, \nabla(\langle x_i^*, x\rangle\varphi) - \varphi\nabla\langle x_i^*, x\rangle] + f'(u)u_i\varphi - \langle x_i^*, x\rangle x_i f(u)\varphi\, d\gamma \\
&= \int_X -[\nabla u_i, \nabla\varphi] - \varphi[\nabla u, \nabla\langle x_i^*, x\rangle] + f'(u)u_i\varphi\, d\gamma,
\end{aligned}$$

where the last inequality follows from (3.1), with φ replaced by $\langle x_i^*, x\rangle\varphi$. □

3.2. **A variational inequality implied by the monotonicity.** The next result shows that monotone solutions of (1.8) satisfy a variational inequality. In the Euclidean case, this fact boils down to the classical stability condition (namely, the second derivative of the energy functional being nonnegative). Differently from this, in our case, a negative eigenvalue appears in the inequality.

Lemma 3.3. *Let $u \in W^{2,2}(X)$ satisfy* (1.8) *and* (1.10)*. Then, for any $\varphi \in W^{1,2}(X)$ it holds*

$$\int_X |\nabla\varphi|^2 - f'(u)\varphi^2\, d\gamma \ge -\int_X \varphi^2\, d\gamma. \tag{3.3}$$

Proof. The proof is a variation of a classical technique (see, e.g., [1, 11]).
Without loss of generality we may assume $w = h_1$, and we let $\varphi \in W^{1,2}(X)$ be such that $\varphi^2/u_1 \in W^{1,2}(X)$. Notice that, thanks to (1.10), the space of such functions is dense in

$W^{1,2}(X)$. We use (3.2), with $i=1$ and test function φ^2/u_1, and we obtain

$$\begin{aligned}
&\int_X f'(u)\varphi^2 - \varphi^2\, d\gamma \\
&= \int_X [\nabla u_1, \nabla(\varphi^2/u_1)]\, d\gamma \\
&= \int_X 2(\varphi/u_1)[\nabla u_1, \nabla\varphi] - (\varphi/u_1)^2|\nabla u_1|^2\, d\gamma \\
&= \int_X |\nabla\varphi|^2 - \Big|(\varphi/u_1)\nabla u_1 - \nabla\varphi\Big|^2\, d\gamma \\
&\le \int_X |\nabla\varphi|^2\, d\gamma.
\end{aligned}$$

□

3.3. **A geometric Poincaré inequality.** We show that a sort of geometric Poincaré inequality stems from solutions of (1.8) satisfying (3.3). In the Euclidean case, it boils down to the inequality discovered in [18, 19].

Lemma 3.4. *Let $u \in W^{2,2}(X)$ satisfy* (1.8) *and* (3.3)*. For any $\varphi \in W^{1,\infty}(X)$ we have*

$$\int_X \big(|\nabla^2 u|^2 - \big|\nabla|\nabla u|\big|^2\big)\varphi^2\, d\gamma \le \int_X |\nabla u|^2|\nabla\varphi|^2\, d\gamma \tag{3.4}$$

where

$$|\nabla^2 u|^2 := \sum_{i,j} u_{ij}^2\,.$$

Proof. We use (3.3) with test function $|\nabla u|\,\varphi$, and we see that

$$\begin{aligned}
&\int_X \big(f'(u)-1\big)|\nabla u|^2\varphi^2\, d\gamma \\
&\le \int_X \big|\nabla(|\nabla u|\varphi)\big|^2\, d\gamma \\
&= \int_X \varphi^2\big|\nabla|\nabla u|\big|^2 + |\nabla u|^2|\nabla\varphi|^2 + 2[\nabla|\nabla u|, \nabla\varphi]\,|\nabla u|\varphi\, d\gamma \\
&= \int_X \varphi^2\big|\nabla|\nabla u|\big|^2 + |\nabla u|^2|\nabla\varphi|^2 + \frac{1}{2}[\nabla|\nabla u|^2, \nabla\varphi^2]\, d\gamma.
\end{aligned}\tag{3.5}$$

We now exploit (3.2) with test function $u_i\varphi^2$ and we get

$$\begin{aligned}
&\int_X \big(f'(u)-1\big)u_i^2\varphi^2\, d\gamma \\
&= \int_X [\nabla u_i, \nabla(u_i\varphi^2)]\, d\gamma \\
&= \int_X |\nabla u_i|^2\varphi^2 + u_i[\nabla u_i, \nabla\varphi^2]\, d\gamma \\
&= \int_X |\nabla u_i|^2\varphi^2 + \frac{1}{2}[\nabla u_i^2, \nabla\varphi^2]\, d\gamma.
\end{aligned}$$

Summing over $i \in \mathbb{N}$, we conclude that

$$\int_X \big(f'(u)-1\big)|\nabla u|^2\varphi^2\,d\gamma = \int_X |\nabla^2 u|^2\varphi^2 + \frac{1}{2}[\nabla|\nabla u|^2, \nabla\varphi^2]\,d\gamma. \tag{3.6}$$

From (3.5) and (3.6), we conclude that

$$\int_X |\nabla^2 u|^2\varphi^2 + \frac{1}{2}[\nabla|\nabla u|^2, \nabla\varphi^2]\,d\gamma \le \int_X \varphi^2\big|\nabla|\nabla u|\big|^2 + |\nabla u|^2|\nabla\varphi|^2 + \frac{1}{2}[\nabla|\nabla u|^2, \nabla\varphi^2]\,d\gamma$$

which gives (3.4). □

Let $u \in C^1(X)\cap L^\infty(X)$ satisfying (1.9), let $N \in \mathbb{N}$ and $\overline{x}_N \in X_N^\perp$. We consider the map $\psi_{N,\overline{x}_N} : \mathbb{R}^N \to \mathbb{R}$ defined as $\psi_{N,\overline{x}_N}(\underline{x}_N) := u(\underline{x}_N, \overline{x}_N)$, and let

$$\begin{aligned}\mathcal{N}_N(\overline{x}_N) &:= \big\{\underline{x}_N \in \mathbb{R}^N \,:\, \nabla\psi_{N,\overline{x}_N}(\underline{x}_N) \neq 0\big\}\\ &= \big\{\underline{x}_N \in \mathbb{R}^N \,:\, \exists\, i \in \{1,\ldots,N\} \text{ such that } u_i(\underline{x}_N, \overline{x}_N) \neq 0\big\}\end{aligned}$$

be its noncritical set. By the Implicit Function Theorem, the level set of $\psi_{N,\overline{x}_N}$ in $\mathcal{N}_N(\overline{x}_N)$ are $(N-1)$-dimensional hypersurfaces of class C^2. Thus we can consider the principal curvatures of these hypersurfaces, that we denote by $\kappa_{1,N},\ldots,\kappa_{N-1,N}$, and the tangential gradient of $\psi_{N,\overline{x}_N}$[1], that we denote by $\nabla_{T,N}$. We also set

$$\nabla_N u := \Pi_N \nabla u = \nabla\psi_{N,\overline{x}_N} \qquad \nabla_N^2 u := \nabla_N\big(\nabla_N u\big) = \nabla^2\psi_{N,\overline{x}_N} \qquad \mathcal{K}_N := \sqrt{\sum_{i=1}^{N-1}\kappa_{i,N}^2}$$

and

$$\mathcal{N}_N := \Big\{x = (\underline{x}_N, \overline{x}_N) \in X \,:\, \underline{x}_N \in \mathcal{N}_N(\overline{x}_N)\Big\} = \Big\{x \in X \,:\, \nabla_N u(x) \neq 0\Big\}.$$

With this notation, we have the following

Lemma 3.5. *Let $u \in C^1(X)\cap L^\infty(X)$ satisfy* (1.8), (1.9) *and* (3.3), *and fix $N \in \mathbb{N}$. For any $\varphi \in W^{1,\infty}(X)$ we have*

$$\begin{aligned}\int_{\mathcal{N}_N} \Big(|\nabla_N u|^2\mathcal{K}_N^2 + \big|\nabla_{T,N}|\nabla_N u|\big|^2\Big)\varphi^2\,d\gamma &\le \int_X \Big(|\nabla^2 u|^2 - \big|\nabla|\nabla u|\big|^2\Big)\varphi^2\,d\gamma \\ &\le \int_X |\nabla u|^2|\nabla\varphi|^2\,d\gamma.\end{aligned} \tag{3.7}$$

[1]the tangential gradient of a function g along a hypersurface with normal ν is $\nabla g - (\nabla g\cdot\nu)\nu$, that is, the tangential component of the full gradient

Proof. Let

$$\begin{aligned}\mathcal{D}_N &:= |\nabla_N^2 u|^2 - \big|\nabla_N|\nabla_N u|\big|^2 \\ &= \sum_{1\le i,j\le N} u_{ij}^2 - \sum_{1\le i\le N}\left[\frac{\nabla_N u}{|\nabla_N u|}, \nabla_N u_i\right]^2 \\ &= \sum_{1\le i,j\le N}\left(u_{ij}^2 - \left(\frac{u_j u_{ij}}{|\nabla_N u|}\right)^2\right).\end{aligned}$$

Since $|\nabla_{N-1}u| \le |\nabla_N u|$ and

$$\left|\frac{u_j u_{ij}}{|\nabla_N u|}\right| \le \frac{|u_j|}{|\nabla_N u|}\,|u_{ij}| \le |u_{ij}|$$

for any i, $j \le N$, it follows that

$$\mathcal{D}_N - \mathcal{D}_{N-1} \ge \sum_{\substack{N-1\le i,j\le N \\ \max(i,j)=N}}\left(u_{ij}^2 - \left(\frac{u_j u_{ij}}{|\nabla_N u|}\right)^2\right) \ge 0$$

so that $\mathcal{D}_N$ is nondecreasing in N. Accordingly,

$$|\nabla^2 u|^2 - \big|\nabla|\nabla u|\big|^2 = \lim_{M\to+\infty}\mathcal{D}_M \ge \mathcal{D}_N \tag{3.8}$$

for any $N \in \mathbb{N}$. Moreover, by Stampacchia's Theorem we have that $\nabla_N|\nabla_N u| = 0$ for almost any $\underline{x}_N \in \mathbb{R}^N \setminus \mathcal{N}_N(\overline{x}_N)$, and similarly $u_{ij} = 0$ for almost any $\underline{x}_N \in \mathbb{R}^N \setminus \mathcal{N}_N(\overline{x}_N)$. Therefore

$$\mathcal{D}_N = |\nabla_N^2 u|^2 - \big|\nabla_N|\nabla_N u|\big|^2 = 0 \text{ for almost any } \underline{x}_N \in \mathbb{R}^N \setminus \mathcal{N}_N(\overline{x}_N). \tag{3.9}$$

On the other hand, by [19, Formula (2.1)],

$$\mathcal{D}_N = |\nabla_N u|^2\mathcal{K}_N^2 + \big|\nabla_{T,N}|\nabla_N u|\big|^2 \quad \text{when } \underline{x}_N \in \mathcal{N}_N(\overline{x}_N).$$

From this, (3.8) and (3.9), we obtain

$$\begin{aligned}&\int_X \left(|\nabla^2 u|^2 - \big|\nabla|\nabla u|\big|^2\right)\varphi^2\,d\gamma \\ &\quad\ge \int_X \mathcal{D}_N\varphi^2\,d\gamma \\ &\quad= \int_{\mathcal{N}_N}\mathcal{D}_N\varphi^2\,d\gamma \\ &\quad= \int_{\mathcal{N}_N}\left(|\nabla_N u|^2\mathcal{K}_N^2 + \big|\nabla_{T,N}|\nabla_N u|\big|^2\right)\varphi^2\,d\gamma\,,\end{aligned}$$

which, recalling (3.4), implies (3.7). □

3.4. **A symmetry result.** We now use the previous material to obtain a one-dimensional symmetry result for the N-dimensional projection of the solution. The idea of using geometric Poincaré inequalities as the ones in [18, 19] in order to obtain symmetry properties goes back to [10] and it was widely used in [11] in the finite dimensional Euclidean setting. The result we present here is the following:

Proposition 3.6. *Fix $N \in \mathbb{N}$ and $\overline{x}_N \in X_N^\perp$. Let $u \in C^1(X) \cap L^\infty(X)$ satisfy* (1.8), (1.9) *and* (3.3). *Then, the map $\psi_{N,\overline{x}_N}$ is one-dimensional, i.e. there exists $U_{N,\overline{x}_N} : \mathbb{R} \to \mathbb{R}$ and $\omega_{N,\overline{x}_N} \in \mathbb{R}^N$, with $|\omega_{N,\overline{x}_N}| = 1$, such that*

$$u(\underline{x}_N, \overline{x}_N) = U_{N,\overline{x}_N}\big(\langle \omega_{N,\overline{x}_N}, \underline{x}_N\rangle\big) \tag{3.10}$$

for any $\underline{x}_N \in \mathbb{R}^N$.

Proof. We fix $R > 1$, to be taken arbitrarily large in what follows, and let $\Lambda = \max_i \lambda_i$. Let $\Phi \in C^\infty(\mathbb{R})$ be such that $\Phi(t) = 1$ if $t \le R$, $\Phi(t) = 0$ if $t \ge R + 1$ and $|\Phi'(t)| \le 3$ for any $t \in [R, R + 1]$. We take $\varphi(x) := \Phi(|x|)$. Then $|\nabla\varphi(x)| \le \sqrt{\Lambda}\, |\Phi'(|x|)| \le 3\sqrt{\Lambda}$, and (3.7) yields

$$\int_{\mathcal{N}_N \cap \{|x| \le R\}} |\nabla_N u|^2 \mathcal{K}_N^2 + \big|\nabla_{T,N} |\nabla_N u|\big|^2 \, d\gamma \;\le\; 9\Lambda \int_{\{R \le |x| \le R+1\}} |\nabla u|^2 \, d\gamma. \tag{3.11}$$

Also, due to our assumptions on u,

$$\int_X |\nabla u|^2 \, d\gamma < +\infty.$$

Therefore, by sending $R \to +\infty$ in (3.11), we conclude that

$$|\nabla_N u|^2 \mathcal{K}_N^2 + \big|\nabla_{T,N} |\nabla_N u|\big|^2 = 0$$

for any $x \in \mathcal{N}_N$. From this and [11, Lemma 2.11] we get (3.10). □

From the finite dimensional symmetry result of Proposition 3.6, one can take the limit as $N \to +\infty$ and obtain:

Corollary 3.7. *Let $u \in C^1(X) \cap L^\infty(X)$ satisfy* (1.8), (1.9) *and* (3.3). *Then, u is necessarily one-dimensional, i.e. there exists $U : \mathbb{R} \to \mathbb{R}$ and $\omega \in X^*$ such that*

$$u(x) = U(\langle \omega, x\rangle)$$

for any $x \in X$.

Proof. We first show that there exists $h \in H$ such that

$$\frac{\nabla u}{|\nabla u|} = h \qquad \text{in } \mathcal{N} := \big\{x \in X : \nabla u(x) \ne 0\big\} = \bigcup_{N \in \mathbb{N}} \mathcal{N}_N\,. \tag{3.12}$$

Let $V \subset X$ be defined as $V = \cup_N H_N$. Since V is a dense subset of X, it is enough to show that (3.12) holds in $\mathcal{N} \cap V = \cup_N V_N$, where $V_N := \mathcal{N}_N \cap H_N$.
However, from Proposition 3.6 we know that

$$\frac{\nabla_N u}{|\nabla_N u|} = \omega_{N,0} \qquad \text{in } V_N, \tag{3.13}$$

which implies that

$$\frac{\nabla u}{|\nabla u|} = \lim_{N\to\infty} \frac{\nabla_N u}{|\nabla_N u|} = \lim_{N\to\infty} \omega_{N,0} =: h \qquad \text{in } V.$$

From (3.12) it follows that there exists a function $U : \mathbb{R} \to \mathbb{R}$ such that $U(t) = u(th)$ for all $t \in \mathbb{R}$, and

$$u(x) = U(\hat{h}(x)) \qquad x \in X. \tag{3.14}$$

Moreover, U is a bounded nondecreasing solution to the ODE

$$U'' - t\,U' + f(U) = 0 \qquad t \in \mathbb{R}.$$

Being u continuous, if U is nonconstant (otherwise the thesis follows immediately) then the function $\hat{h}$ is also continuous, so that $h \in QX^*$ and $\hat{h}(x) = \langle \omega, x\rangle$ for some $\omega \in X^*$, which implies the thesis. □

3.5. **Proof of Theorem 1.2.** The proof of Theorem 1.2 follows directly from Lemma 3.3 and Corollary 3.7. □

Remark 3.8. We observe that, in the infinite dimensional case, there may exist weak solutions to (1.8), satisfying (1.10), which are not continuous. Indeed, given $U : \mathbb{R} \to \mathbb{R}$ satisfying (4.1) and (4.2) below, the function $u(x) = U(\hat{h}(x))$ in (3.14) is a solution to (1.8), monotone in the direction given by h, for any $h \in H$. However, such a solution is continuous only if $h \in QX^*$. As a possible generalization of Theorem 1.2, one could ask if any bounded weak solution to (1.8), satisfying $[\nabla u, w] > 0$ for some $w \in H$, is of this form.

4. Heteroclinic solutions

The results in Theorems 1.1 and 1.2 may be seen either as classification results (when one knows explicitly the solutions of the associated one-dimensional problem) or as nonexistence result (when the associated one-dimensional problem does not admit any solution). For this, we now give some simple conditions on the nonlinearity f ensuring existence or nonexistence of bounded solutions to the ODE

$$U'' - t\,U' + f(U) = 0 \qquad t \in \mathbb{R} \tag{4.1}$$

satisfying

$$U' > 0 \qquad t \in \mathbb{R}. \tag{4.2}$$

Notice that, from (4.2) it follows that there exist $U^{\pm} \in \mathbb{R}$, with $U^- < U^+$, such that

$$\lim_{t\to\pm\infty} U(t) = U^{\pm}. \tag{4.3}$$

Moreover, passing to the limit in (4.1) we also get

$$f(U^-) = f(U^+) = 0. \tag{4.4}$$

We start with a nonexistence result.

Proposition 4.1. *Assume that there exists* $U_0 \in (U^-, U^+)$ *such that*

$$f \geq 0 \text{ in } [U^-, U_0] \qquad \text{or} \qquad f \leq 0 \text{ in } [U_0, U^+].$$

Then, there are no solutions to (4.1) *satisfying* (4.2).

Proof. Let us assume that $f \leq 0$ in $[U_0, U^+]$, since the argument is analogous in the other case, and assume by contradiction that we are given a solution U of (4.1), (4.2).

Letting $t_0 > 0$ be such that $u(t_0) \in [U_0, U^+]$, we have that U satisfies the differential inequality

$$U'' \geq t\, U' \qquad \text{for all } t \in [t_0, +\infty),$$

which implies, by direct integration,

$$U'(t) \geq U'(t_0) e^{(t^2 - t_0^2)/2} \geq U'(t_0) > 0 \qquad \text{for all } t \in [t_0, +\infty),$$

contradicting (4.3). □

We consider the potential $F : \mathbb{R} \to \mathbb{R}$, defined as

$$F(t) \, = \, -\int_0^t f(s)\, ds + k\,.$$

where $k \in \mathbb{R}$. Notice that, if F is convex or concave, from (4.4) if follows that $f \equiv 0$ in $[U^-, U^+]$, so that by Proposition 4.1 there are no solutions to (4.1) satisfying (4.2).

Given $U : (0, +\infty) \to \mathbb{R}$, we let

$$G(U) := \int_0^{+\infty} \left(\frac{(U'(t))^2}{2} + F(U(t)) \right) d\gamma(t) \tag{4.5}$$

where $d\gamma(t) = e^{-t^2/2} dt$. Notice that (4.1) is the Euler-Lagrange equation of G.

As a counterpart of the nonexistence result in Proposition 4.1, we now give an existence result for monotone solutions to (4.1).

Proposition 4.2. *Assume that* F *satisfies the following properties:*

$$\begin{array}{ll} F(c) = F(-c) = 0 & \text{for some } c > 0 \\ F(r) > 0 & \text{for any } r \notin \{c, -c\} \\ F(r) = F(-r) & \text{for any } r \in [0, +\infty). \\ f(r) = 0 & \text{iff } r \in \{c, -c, 0\}. \end{array} \tag{4.6}$$

Assume also that there exists $U \in W^{1,2}_\gamma((0, +\infty))$ *such that* $U(0) = 0$ *and*

$$G(U) < G(0) = \sqrt{\frac{\pi}{2}}\, F(0). \tag{4.7}$$

Then, there exists a monotone solution to (4.1)*, connecting* $-c$ *to* c.

Proof. Let $\overline{U}$ be a solution to the minimum problem

$$\min \left\{ G(U) : \ U \in W^{1,2}_\gamma((0, +\infty)), \ U(0) = 0 \right\}. \tag{4.8}$$

Note that (4.6) implies

$$G\left(\min\left(|\overline{U}|, c\right)\right) \leq G\left(\overline{U}\right),$$

so that we may assume $\overline{U}(t) \in [0, c]$ for all $t \in (0, +\infty)$.

Let now $\overline{U}^\star$ be the Ehrhard rearrangement of $\overline{U}$ [9], which is defined in such a way that $\overline{U}^\star$ is nondecreasing on $(0,+\infty)$, and

$$\gamma\Big(\Big\{t:\overline{U}^\star(t)>r\Big\}\Big)=\gamma\Big(\Big\{t:\overline{U}(t)>r\Big\}\Big)\qquad\text{for all } r\in(0,c).$$

Notice that $\overline{U}^\star(0)=0$ and $\overline{U}(t)\in[0,c]$ for all $t\in(0,+\infty)$. By [9] (see also [13, Prop. 3.12]), we have $\overline{U}^\star\in W^{1,2}_\gamma((0,+\infty))$ and

$$\begin{aligned}\int_0^{+\infty}\frac{(U^{\star\prime}(t))^2}{2}d\gamma(t) &\le \int_0^{+\infty}\frac{(U'(t))^2}{2}d\gamma(t)\\ \int_0^{+\infty}F(U^\star(t))d\gamma(t) &= \int_0^{+\infty}F(U(t))d\gamma(t),\end{aligned}$$

so that

$$G(\overline{U}^\star)\le G(\overline{U}).$$

In particular, we may assume that $\overline{U}=\overline{U}^\star$, i.e. that $\overline{U}$ is nondecreasing on $(0,+\infty)$.

As $U=c$ and $U=0$ are solutions to (4.1), which is the Euler-Lagrange equation of G, we get that either $\overline{U}=0$ or $\overline{U}=c$ or

$$\overline{U}(t)\in(0,c)\qquad\text{for all } t\in(0,+\infty).\tag{4.9}$$

On the other hand, thanks to (4.7) and the fact that $\overline{U}(0)=0$, we can exclude the first two possibilities, so that (4.9) holds. Moreover, since $\overline{U}$ is nondecreasing and $f(r)\neq 0$ for all $r\in(0,c)$, it follows that $\overline{U}'(t)>0$ for all $t\in(0,+\infty)$ and

$$\lim_{t\to+\infty}\overline{U}(t)=c.$$

Since by (4.6) the function $t\to-\overline{U}(-t)$ is a monotone solution to (4.1) on $(-\infty,0)$, we get that the odd extension of $\overline{U}$ on $\mathbb{R}$ is a solution to (4.1) on the whole of $\mathbb{R}$ which satisfies (4.2) and connects $-c$ to c. □

Remark 4.3. Notice that for all $U\in W^{1,2}_\gamma((0,+\infty))$ we have

$$G(U)<\widetilde{G}(U):=\int_0^{+\infty}\left(\frac{(U'(t))^2}{2}+F(U(t))\right)dt.$$

If we let $\overline{U}$ be the unique solution to

$$\begin{aligned}U''(t)+f(U(t)) &= 0 \qquad t\in(0,+\infty)\\ U(0) &= 0\\ \lim_{t\to+\infty}U(t) &= c,\end{aligned}$$

we have

$$G(\overline{U})<\widetilde{G}(\overline{U})=\int_0^c\sqrt{2F(r)}\,dr\,.$$

In particular, condition (4.7) is verified whenever

$$\int_0^c\sqrt{2F(r)}\,dr\le\sqrt{\frac{\pi}{2}}\,F(0)$$

which is satisfied, for instance, by the standard double-well potential $F(t)=(1-t^2)^2/4$.

References

[1] G. Alberti, L. Ambrosio, X. Cabré. On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property. *Acta Appl. Math.*, 65(1-3):9–33, 2001.

[2] L. Ambrosio, X. Cabré. Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi. *J. Amer. Math. Soc.*, 13:725–739, 2000.

[3] L. Ambrosio, S. Maniglia, M. Miranda, D. Pallara. BV functions in abstract Wiener spaces. *J. Funct. Anal.*, 258(3):785–813, 2010.

[4] H. Berestycki, L. Caffarelli, L. Nirenberg. Further qualitative properties for elliptic equations in unbounded domains. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 25(1-2), 69–94, 1997/98.

[5] V. I. Bogachev. *Gaussian measures.* Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998.

[6] E. De Giorgi. Convergence problems for functionals and operators. In *Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978)*, pp. 131–188, Pitagora, Bologna, 1979.

[7] M. del Pino, M. Kowalczyk, J. Wei. On a conjecture by De Giorgi in dimensions 9 and higher. *Symmetry for elliptic PDEs, 115–137, Contemp. Math.*, 528, Amer. Math. Soc., Providence, RI, 2010.

[8] K. Ecker and G. Huisken. Mean curvature evolution of entire graphs. *Ann. of Math. (2)* 130(3):453–471, 1998.

[9] A. Ehrhard. Symmetrization in Gaussian spaces. *Math. Scand.* 53(2):281–301, 1983.

[10] A. Farina. Propriétés qualitatives de solutions d'équations et systèmes d'équations non-linéaires. *Habilitation à diriger des recherches, Paris VI*, 2002.

[11] A. Farina, B. Sciunzi, E. Valdinoci. Bernstein and De Giorgi type problems: new results via a geometric approach. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 7(4):741–791, 2008.

[12] A. Farina, E. Valdinoci. The state of the art for a conjecture of De Giorgi and related problems. In *Recent progress on reaction-diffusion systems and viscosity solutions*, pp. 74–96, World Sci. Publ., Hackensack, NJ, 2009.

[13] M. Goldman, M. Novaga. Approximation and relaxation of perimeter in the Wiener space. *Annales IHP - Analyse Nonlinéaire*, to appear.

[14] N. Ghoussoub, C. Gui. On a conjecture of De Giorgi and some related problems. *Math. Ann.*, 311:481–491, 1998.

[15] M. Ledoux. A short proof of the Gaussian isoperimetric inequality. High dimensional probability (Oberwolfach, 1996). *Progr. Probab.*, Birkhäuser, 43:229-232, 1998.

[16] A. Lunardi. On the Ornstein-Uhlenbeck operator in L^2 spaces with respect to invariant measures. *Trans. Amer. Math. Soc.*, 349:155–169, 1997.

[17] O. Savin. Regularity of flat level sets in phase transitions. *Ann. of Math. (2)* 169(1):41–78, 2009.

[18] P. Sternberg, K. Zumbrun. A Poincaré inequality with applications to volume-constrained area-minimizing surfaces. *J. Reine Angew. Math.*, 503:63–85, 1998.

[19] P. Sternberg, K. Zumbrun. Connectivity of phase boundaries in strictly convex domains. *Arch. Rational Mech. Anal.*, 141(4):375–400, 1998.

[20] L. Wang. A Bernstein type theorem for self-similar shrinkers. *Geom. Dedicata*, 151:297–303, 2011.

Annalisa Cesaroni and Matteo Novaga: Dipartimento di Matematica, Università di Padova, Via Trieste 63, 35121 Padova (Italy), e-mail: acesar@math.unipd.it, novaga@math.unipd.it

Enrico Valdinoci: Dipartimento di Matematica, Università di Milano, Via Cesare Saldini 50, 20133 Milano (Italy), e-mail: enrico@math.utexas.edu