NON-LOCAL GRADIENT DEPENDENT OPERATORS

C. BJORLAND, L. CAFFARELLI, AND A. FIGALLI

Abstract. In this paper we study a general class of "quasilinear non-local equations" depending on the gradient which arise from tug-of-war games. We establish a $C^\alpha/C^{1,\alpha}/C^{2,\alpha}$ regularity theory for these equations (the kind of regularity depending on the assumptions on the kernel), and we construct different non-local approximations of the p-Laplacian.

1. Introduction

For gradient dependent second order equations, the influence of the gradient on the solution can arise in different ways: On one hand there are semi-linear equations as for instance $\Delta u = g(u, \nabla u)$, with an associated idea of drift or transport. On the other hand there are quasilinear equations $a_{ij}(\nabla u)D_{ij}u = 0$ coming from calculus of variations.

When moving to a non-local setting, in the semilinear case one can simply replace the Laplacian by a fractional Laplacian (this kind of equations arise naturally, for instance, in the quasi-geostrophic equations).

On the other hand, for quasilinear equations there are two natural frameworks to incorporate fractional diffusion, corresponding to the divergence and non-divergence forms of the operators.

Consider, for example, the classical p-Laplacian. Through the calculus of variations one comes to the p-Laplacian as the Euler-Lagrange equation of the L^p norm of the gradient of a function. Using equivalent framework one may define the p-(s-Laplacian) as the Euler-Lagrange equation of the L^p norm of the s-derivative of a function:

$$\|u\|_{W^{s,p}}^p = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy.$$

This fractional version of the p-Laplacian is naturally studied through "energy" and "test function" methods (see [6]).

Alternatively the p-Laplacian may also be written in non-divergence form as

$$|\nabla u|^{p-2}(\Delta u + (p-2)u_{nn})$$

where u_{nn} denotes the second derivative in the direction of the gradient of u. This has a game-theoretical interpretation as the infinitesimal limit of a tug-of-war game with noise (see [13]). In this tug-of-war interpretation, at each turn competing players are able to impose a drift in their preferred direction to maximize/minimize the expected value of the game, but at the same time there is a random noise (an "unsteady hand") in the movement. As explained in the review paper [2, Section 4], this class of games where the players are able to impose a preferred

C. Bjorland is partially supported by NSF Grant DMS-0636586.
L. Caffarelli is partially supported by NSF Grant DMS-0654267.
A. Figalli is partially supported by NSF Grant DMS-0969962.

direction in the jump naturally leads to the general family of Isaac's equations $Iu(x) = 0$, where

$$Iu(x) := (1-s) \sup_{\xi_1 \in \mathcal{A}} \inf_{\xi_2 \in \mathcal{A}} \left(\int_{\mathbb{R}^N} \frac{[u(x+y) - u(x)]\mathcal{K}_{\xi_1}(y) + [u(x-y) - u(x)]\mathcal{K}_{\xi_2}(y)}{|y|^{N+2s}}\, dy \right), \quad (1)$$

for some family of kernels $\{\mathcal{K}_\xi\}_{\xi \in \mathcal{A}}$.

Consider for instance the following example, which we view as a non-local variant of the classical p-Laplacian Δ_p for $p \geq 2$:

$$\mathcal{A} := S^{N-1}, \qquad \mathcal{K}_\xi(y) := \frac{1}{\alpha_p} \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \xi \right). \quad (2)$$

Here, $c_p \in [0,1)$ and $\alpha_p > 0$ are suitably chosen constants, and $\mathbb{1}_{[c_p,1]}$ denotes the indicator function for the set $[c_p, 1]$. (Remark 4.5 describes the case $p \in (1,2)$.) In this model case, when $s \in (\frac{1}{2}, 1)$, both ξ_1 and ξ_2 will point in the direction of ∇u (whenever it is non-zero). Such an operator arises from the non-local tug-of-war game described in [1] if the players have *unsteady hands*: when a player picks a direction to move the token, it is instead moved in a direction randomly chosen from a cone around such direction.

The goal of this paper is to study the general class of equations (1). To give these equations some basic structure, we assume the kernel is sufficiently singular near the origin and well behaved, the precise statements of our assumptions are listed below in Subsection 2.1. Under these very general assumptions we are able to establish C^α regularity of solutions and a Harnack inequality by extending the work in [4]. Section 3 contains these regularity arguments and theorems. It should be noted that the operator (1) is not, in general, elliptic in the sense of [4, Definition 3.1], and therefore the results of [4] cannot be applied directly to (1).

In Section 4 we investigate the Dirichlet problem associated to the specific operator given by (1)-(2). We prove existence, describe the regularity of solutions, and show that they converge to solutions of the classical p-Laplacian as $s \to 1$. For operators such as the one described by (2) the questions of existence and uniqueness are complicated considerably by the interaction of local and non-local effects. We adopt a notion of solution which is stable under suitable convergence but for which we have only a partial comparison principle and non-uniqueness. This is contrasted with the approach we used in [1] which involved a stronger definition of solution allowing a full comparison principle. With the stronger definition of solution proving existence becomes very difficult, and we refer the reader to [1] for more discussion, including counterexamples, regarding this phenomenon (in particular, the non-uniqueness example given in [1, Appendix] shows the failure of a full comparison principle for the operator (1)-(2) when p is sufficiently large). All of the results from Section 4 hold also for the $p \in (1,2)$ model described in Remark 4.5.

Higher regularity of solutions associated to operators of the form (1) appears to be a very difficult question due to the delicate interaction of local and non-local effects. However, when the local effects dominate we can expect to find higher regularity of solutions if a "near by" local equation has sufficiently regular solutions. We make this idea precise in Subsection 4.4 for the kernel (2) through a perturbation argument relying on regularity results for local second order equations related to the classical p-Laplacian. In particular, we prove $C^{1,\alpha}$ regularity for solutions to (1)-(2) when s is sufficiently close to 1.

In Section 5 we examine in detail another specific example:

$$\mathcal{A} = S^{N-1}, \qquad \mathcal{K}_\xi(y) = \frac{1}{2} + (|y| \wedge 1)\psi \left(\frac{y}{|y|} \cdot \xi \right). \quad (3)$$

As in (2), we think of ψ as a cutoff function limiting the support of $\mathcal{K}_\xi$ to a cone around ξ. The decay of $|y|$ near zero indicates the non-linear term in the definition of the operator is of lower order than the fractional Laplacian. Making use of the the perturbation theorems in [5] and the special structure of our equations, we show that solutions corresponding to (3) are $C^{2,\alpha}$. Additionally we exhibit a sequence of operators of this form which approach the classical p-Laplacian Δ_p when $s \to 1$. Recalling that solutions to the p-Laplacian are at most $C^{1,\alpha}$, the $C^{2,\alpha}$ bounds cannot be uniform with respect to s (while the $C^{1,\alpha}$ bounds are). Hence, this latter example provides a non-local regularization of the classical p-Laplacian.

2. Structure of the Kernel

2.1. **Assumed Structure of the Kernel.** In this subsection we state the general assumptions we make on the set $\{\mathcal{K}_\xi\}_{\xi\in\mathcal{A}}$ to give the kernel basic structure. We keep these assumptions throughout the paper.

We require first that all kernels are non-negative and uniformly bounded:

$$\inf_{\xi\in\mathcal{A}}\inf_{y\in\mathbb{R}^N}\mathcal{K}_\xi(y)\geq 0, \qquad \sup_{\xi\in\mathcal{A}}\sup_{y\in\mathbb{R}^N}\mathcal{K}_\xi(y)<\infty.$$

Additionally we make two assumptions which control the singularity of the kernel near the origin. Assumption 2.1 ensures the kernel is sufficiently "singular" by bounding from below the measure of the set on which $\mathcal{K}_\xi$ is positive. Assumption 2.2 ensures this set is sufficiently regular.

For any $\rho > 0$ define

$$\mathbb{K}_\rho(\xi) := \{y : \mathcal{K}_\xi(y) \geq \rho\}.$$

We also use the notation $B_r = B_r(0)$ and $A_r = B_r \setminus B_{\frac{r}{2}}$ which we keep throughout.

Assumption 2.1. *We assume the existence of $\bar\rho > 0$, $\bar c > 0$, and $\bar r > 0$ such that for any $r \in (0, \bar r)$ and $\xi \in \mathcal{A}$,*

$$|\mathbb{K}_{\bar\rho}(\xi)\cap A_r| \geq \bar c|A_r|.$$

Assumption 2.2. *Let $\bar r$, $\bar\rho$ be as in Assumption 2.1. Given any $\epsilon > 0$ there exists of an integer $m = m(\epsilon) \in \mathbb{N}$ such that, for any $r \in (0, \bar r)$, $\xi \in \mathcal{A}$, and $y \in B_{\frac{r}{2^m}}$,*

$$\int_{\mathbb{R}^N} |\,\mathbb{1}_{(\mathbb{K}_{\bar\rho}(\xi)\cap A_r)/2}(z-y) - \mathbb{1}_{(\mathbb{K}_{\bar\rho}(\xi)\cap A_r)/2}(z)|\,dz \leq \epsilon|A_r|.$$

Here $\mathbb{1}_{(\mathbb{K}_{\bar\rho}(\xi)\cap A_r)/2}$ denotes the indicator function for the set $\{y : 2y \in \mathbb{K}_{\bar\rho}(\xi)\cap A_r\}$.

We observe that, compared to [4], we do not require our kernel to fill up the whole space (and not even a small ball around the origin). We believe these general assumptions may also be useful in several other problems.

3. Hölder Continuity and Harnack inequality

In this section we build upon the arguments in [4] to prove a Harnack inequality and Hölder continuity for solutions of (1). We give proofs where our assumptions change the arguments in [4] and refer to the original arguments when they are identical.

3.1. **Preliminary Definitions.** Before proceeding we introduce two function spaces which will be used throughout this paper.

Definition 3.1. *A function ϕ is said to be $C^{1,1}(x_0)$, or equivalently "$C^{1,1}$ at the point x_0" if there is a vector $p \in \mathbb{R}^N$ and numbers $M, \eta_0 > 0$ such that*

$$|\phi(x_0 + x) - \phi(x_0) - p \cdot x| \leq M|x|^2 \tag{4}$$

for $|x| < \eta_0$. We define $\nabla\phi(x_0) := p$.

It is not difficult to check that the above definition of $\nabla\phi(x_0)$ makes sense, that is, if u belongs to $C^{1,1}(x_0)$ then there exists a unique vector p for which (4) holds.

Definition 3.2. *Given $s \in (0,1)$, a function ϕ is said to be $L^1(\mathbb{R}^N; s)$ if*

$$\|\phi\|_{L^1(\mathbb{R}^N;s)} := \int_{\mathbb{R}^N} \frac{|\phi(x)|}{1 + |x|^{N+2s}}\, dx < \infty. \tag{5}$$

The space $L^1(\mathbb{R}^N; s)$ is essentially the weighted space used in [5].

Throughout this paper we use the notation

$$L(u, \xi, x) := (1 - s) \int_{\mathbb{R}^N} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{N+2s}} \mathcal{K}_\xi(y)\, dy. \tag{6}$$

This linear operator corresponds to (1) when the supremum and infimum are attained with the same choice of ξ. It is well defined whenever $u \in C^{1,1}(x) \cap L^1(\mathbb{R}^N; s)$. When u is less regular we interpret this linear operator in a viscosity sense (see [7], [4], and Definition 3.3 below). Regarding the operator (1), in this section we will actually take a general definition of solution, which includes (1) as a special case: we will just assume that u is a viscosity solution of

$$\max_{\xi \in \mathcal{A}} L(u, \xi, x) \geq 0, \qquad \min_{\xi \in \mathcal{A}} L(u, \xi, x) \leq 0.$$

Since

$$\max_{\xi \in \mathcal{A}} L(u, \xi, x) \geq Iu(x) \geq \min_{\xi \in \mathcal{A}} L(u, \xi, x),$$

this latter definition is much more general than (1) but it is sufficient to prove regularity of solutions.

Definition 3.3. *Let $L = L(\cdot, \xi, \cdot)$ be the family of operators defined in (6), let $x_0 \in \Omega$ and $C_0 \in \mathbb{R}$. An upper [resp. lower] semi continuous function $u : \mathbb{R}^N \to \mathbb{R}$ is said to satisfy $L^+u(x_0) \geq C_0$ (resp. $L^-u \leq C_0$) if*

$$\max_{\xi} L(u, \xi, x_0) \geq C_0 \qquad \text{[resp. } \min_{\xi} L(u, \xi, x_0) \geq C_0\text{]}$$

in the viscosity sense, i.e., whenever:

- *$B_r(x_0)$ is an open ball of radius r centered at x_0,*
- *$\phi \in C^{1,1}(x_0) \cap C(\bar{B}_r(x_0))$,*
- *$\phi(x_0) = u(x_0)$,*
- *$\phi(x) > u(x)$ [resp. $\phi(x) < u(x)$] for every $x \in B_r(x_0) \setminus \{x_0\}$,*

then there is a $\xi \in \mathcal{A}$ such that $L(\tilde{u}, \xi, x_0) \geq 0$ [resp. $L(\tilde{u}, \xi, x_0) \leq 0$], where

$$\tilde{u}(x) := \begin{cases} \phi(x) & \text{if } \ x \in B_r(x_0) \\ u(x) & \text{if } \ x \in \mathbb{R}^N \setminus B_r(x_0). \end{cases} \tag{7}$$

In the above definition we say the test function ϕ "touches u from above [resp. below] at x_0".

Before we begin, we recall the definition of a concave envelope which will be used throughout this section.

Definition 3.4. *Given $A \subset \mathbb{R}^N$, the concave envelope of u in A is defined by*

$$\Gamma(x) := \inf_{\ell}\{\ell(x) : \ell \geq v \text{ in } A, \ell \text{ is affine}\} \qquad \forall\, x \in A.$$

We also define the contact set $\{u = \Gamma\} := \{x \in A : u(x) = \Gamma(x)\}$.

The main goal for this section is to establish a Harnack inequality and C^α regularity when we have control of $L(u, \xi, x)$ for at least one ξ at every point $x \in \Omega$. It is important for our purposes that these estimates are uniform as $s \to 1$, so we fix $s_0 > 0$ and always take $s \in (s_0, 1)$. The constants in this section may depend on s_0 but they will never depend on s.

3.2. **ABP estimate.** First we establish an Alexandroff-Bakelman-Pucci type estimate. The following lemma is a slight generalization of [4, Lemma 8.1]. Although the argument is almost the same we provide details for completeness.

Lemma 3.5. *Let $u \leq 0$ in $\mathbb{R}^N \setminus B_1$ and Γ be the concave envelope of u^+ (the positive part of u) in B_3, setting $\Gamma(x) = 0$ for any $x \neq B_3$. Assume $L^+u(x) \geq -f(x)$ in B_1. Set $\rho_0 := 1/(8\sqrt{n})$, $r_k := \rho_0 2^{-\frac{1}{2(1-s)}-k}$, and $A_k := B_{r_k} \setminus B_{r_k/2}$. Given $M > 0$, define*

$$\tilde{A}_k(x) := A_k \cap \{y : u(x+y) < u(x) + y \cdot \nabla\Gamma(x) - Mr_k^2\}.$$

Then there is a constant $C_0 > 0$, depending only on the dimension, with the following property: For any $x \in \{u = \Gamma\}$ there are $k \in \mathbb{N}$ and $\xi \in \mathcal{A}$ such that

$$\int_{\tilde{A}_k(x)} \mathcal{K}_\xi(y)\, dy \leq C_0 \frac{f(x)}{M} |A_k|. \tag{8}$$

Proof. At any $x \in \{u = \Gamma\}$, u may be touched from above by an affine function $\ell(x)$, and so $L(u, \xi, x)$ is defined classically for any $\xi \in \mathcal{A}$ (see, for example, [4, Lemma 3.3]). Moreover, by Fatou's Lemma and dominated convergence, the map $\xi \mapsto L(u, \xi, x)$ is upper semicontinuous at these points. Hence, by the assumption $L^+u \geq -f(x)$, for any $x \in \{u = \Gamma\}$ there is $\xi \in \mathcal{A}$ satisfying pointwise $L(u, \xi, x) \geq -f(x)$.

We claim that, for any $x \in \{u = \Gamma\}$, $\delta(y) := u(y+x) + u(y-x) - 2u(x) \leq 0$. Indeed, at any $x \in \{u = \Gamma\}$ it must be that $u(x) \geq 0$. So, $\delta(y) \leq 0$ whenever $u(x+y), u(x-y) \leq 0$. On the other hand, if $u(x+y) > 0$ or $u(x-y) > 0$ then $x+y, x-y \in B_3$, and the inequality $\delta(y) \leq 0$ follows from the fact that u is dominated by the concave function Γ inside B_3.

Now, the negativity of $\delta(y)$ implies that $f(x) \geq 0$ and

$$\begin{aligned} -f(x) \leq L(u, \xi, x) &= (1-s) \int_{\mathbb{R}^N} \frac{\delta(y)}{|y|^{N+2s}} \mathcal{K}_\xi(y)\, dy \\ &\leq (1-s) \int_{B_{r_0}} \frac{\delta(y)}{|y|^{N+2s}} \mathcal{K}_\xi(y)\, dy. \end{aligned} \tag{9}$$

By way of contradiction, assume that for every $C_0 > 0$ there is an $x \in \{u = \Gamma\}$ and an $M > 0$ such that, for every $k > 0$,

$$\int_{\tilde{A}_k(x)} \mathcal{K}_\xi(y)\, dy \geq 2C_0 \frac{f(x)}{M} |B_1 \setminus B_{1/2}| r_k^N. \tag{10}$$

Since $x \in \{u = \Gamma\}$ we have

$$\tilde{A}_k \subset \{y : -\delta(y) > 2Mr_k^2\} \cap A_k,$$

so that, using (10),

$$\begin{aligned}\int_{A_k(x)} \frac{-\delta(y)}{|y|^{N+2s}} \mathcal{K}_\xi(y)\, dx &\geq Mr_k^2 \int_{\tilde{A}_k(x)} \frac{\mathcal{K}_\xi(y)}{|y|^{N+2s}}\, dy \\ &\geq c_N C_0 f(x) r_k^{2(1-s)},\end{aligned}$$

where $c_N > 0$ is a dimensional constant. Combined with (9) we find

$$\begin{aligned} f(x) &\geq c_N C_0 f(x)(1-s) \sum_{r=0}^{\infty} r_k^{2(1-s)} \\ &= c_N C_0 f(x) \rho_0^{(1-s)} \frac{2(1-s)}{1 - 2^{-2(1-s)}}. \end{aligned}$$

Since $\rho_0^{(1-s)} \frac{(1-s)}{1-2^{-2(1-s)}}$ is bounded away from zero, independently of $s \in (0,1)$, this is a contradiction for C_0 sufficiently large. $\square$

Next we generalize [4, Lemma 8.4]. Here and in the sequel, we say that a constant is *universal* if it depends only on the dimension and on the quantities appearing in Assumptions 2.1 and 2.2.

Lemma 3.6. *Let $r \in (0, \bar{r})$ with $\bar{r}$ as in Assumption 2.1, and Γ a concave function in B_r. Given $h > 0$ set*

$$\tilde{A}(x) := A_r \cap \{y : \Gamma(x+y) < \Gamma(x) + y \cdot \nabla\Gamma(x) - h\}.$$

There exist universal constants $\epsilon > 0$ and $m \in \mathbb{N}$ with the following property:

If, for some $\xi \in \mathcal{A}$,

$$\int_{\tilde{A}(x)} \mathcal{K}_\xi(y)\, dy \leq \epsilon |A_r|, \tag{11}$$

then $\Gamma(x+y) \geq \Gamma(x) + y \cdot \nabla\Gamma(x) - h$ for $y \in B_{r/2^m}$.

Proof. Given any ξ, let $\bar{\rho}$ and $\mathbb{K}_{\bar{\rho}}(\xi)$ be as in Assumption 2.1. If (11) holds then

$$|\tilde{A} \cap \mathbb{K}_{\bar{\rho}}(\xi)| \leq \frac{1}{\bar{\rho}} \int_{\tilde{A}} \mathcal{K}_\xi(y)\, dy \leq \frac{\epsilon}{\bar{\rho}} |A_r|.$$

We are interested in the set

$$K := (\mathbb{K}_{\bar{\rho}}(\xi) \setminus \tilde{A}) \cap A_r.$$

Combining the above estimate on $A \cap \mathbb{K}_{\bar{\rho}}(\xi)$ with Assumption 2.1, for ϵ small we have

$$|K| \geq \frac{\bar{c}}{2} |A_r|, \tag{12}$$

with $\bar{c}$ as in Assumption 2.1.

To finish the proof we will show that for all y in some neighborhood of zero there are two points $z_1, z_2 \in K$ such that $\frac{z_1+z_2}{2} = y$. The concavity of Γ will then imply the result. We deduce the existence of such a neighborhood from the convolution

$$g(y) := \int_{\mathbb{R}^N} \mathbb{1}_{K/2}(z)\, \mathbb{1}_{-K/2}(y-z)\, dz,$$

where $K/2 := \{z : 2z \in K\}$ and $-K/2 := \{z : -2z \in K\}$. Indeed, by (12),

$$g(0) = |K/2| = \frac{1}{2^N}|K| \geq \frac{\bar{c}}{2^N}|A_r|.$$

Moreover, by Assumption 2.2, there exists $m > 0$ such that for any $y \in B_{\frac{r}{2^m}}$,

$$\max |g(\cdot - y) - g(\cdot)| \leq \int_{\mathbb{R}^N} |\, \mathbb{1}_{K/2}(z-y) - \mathbb{1}_{K/2}(z)|\, dz \leq \frac{\bar{c}}{2^{N+1}}|A_r|.$$

Thus, g is strictly positive in $B_{r/2^m}$.

To conclude, we observe that for any y in this ball there must be a $z \in K/2$ such that $y - z \in -K/2$ (as otherwise we would have $g(y) = 0$). Setting $z_1 := 2z$ and $z_2 := 2y - 2z$, we have $z_1, z_2 \in K$ and $y = (z_1 + z_2)/2$, as desired. □

Corollary 3.7. *For any $\epsilon > 0$ and $\rho_0 > 0$ there are universal constants $C_1 > 0$ and $m \in \mathbb{N}$ with the following property. Let u and Γ be as in Lemma 3.5, and let $x \in \{u = \Gamma\}$. Then there is $r \in (0, \rho_0 2^{-\frac{1}{2(1-s)}})$ such that*

$$\tilde{A} := A_r \cap \{y : u(x+y) < u(x) + y \cdot \nabla\Gamma(x) - C_1 f(x) r^2\}$$

satisfies

$$\int_{\tilde{A}} \mathcal{K}_\xi(y)\, dy \leq \epsilon |A_r|,$$

$$|\nabla\Gamma(B_{\frac{r}{2^{m+1}}}(x))| \leq \big(2C_1 f(x)\big)^n |B_{\frac{r}{2^{m+1}}}|.$$

Proof. To prove the result, let C_0 and ϵ be given by Lemmas 3.5 and 3.6, and set $C_1 := C_0|B_1 \setminus B_{1/2}|/\epsilon$. Then, by Lemma 3.6 with $h = C_1 f(x) r^2$ we deduce that

$$\Gamma(x+y) \geq \Gamma(x) + y \cdot \nabla\Gamma(x) - C_1 f(x) r^2 \qquad \forall\, y \in B_{r/2^m}(x).$$

By concavity, this implies

$$|\nabla\Gamma(x+y) - \nabla\Gamma(x)| \leq 2C_1 f(x) r \qquad \forall\, y \in B_{r/2^{m+1}}(x),$$

and the result follows. □

From here the proof of the ABP estimate follows exactly as in [4] and we have the following version of [4, Theorem 8.7].

Lemma 3.8. *Let u and Γ be as in Lemma 3.5. There exist a universal constant C and open cubes Q_j with diameters d_j such that:*

(a) $Q_i \cap Q_j = \emptyset$ *if* $i \neq j$.
(b) $\{u = \Gamma\} \subset \cup \bar{Q}_j$.
(c) $Q_i \cap \{u = \Gamma\} = \emptyset$ *for all* i.
(d) $d_j \leq \rho_0 2^{-1/(2-2s)}$, *where* $\rho_0 = 1/(8\sqrt{n})$.
(e) $|\nabla\Gamma(\bar{Q}_j)| \leq (2C \max_{\bar{Q}_j} f)^n |Q_j|$.
(f) $|\{y \in 4\sqrt{N} Q_j : u(y) > \Gamma(y) - C(\max_{\bar{Q}_j} f) d_j^2\}| \geq \mu |Q_j|$.

Proof. This lemma is deduced directly from Corollary 3.7 exactly as in [4, Theorem 8.7]. □

3.3. **Barrier Function.** Analogous to [4, Section 8] we now demonstrate the existence of a barrier function which is a subsolution in a suitable annulus.

Before we proceed we mention a short lemma which is a consequence of Assumption 2.1:

Lemma 3.9. *Let $\bar{r}$ and $\bar{\rho}$ be as in Assumption 2.1. There exists a universal constant $c_1 > 0$ such that, for any $r \in (0, \bar{r})$ and $\xi \in \mathcal{A}$,*

$$\int_{A_r} y_1^2 \mathcal{K}_\xi(y)\, dy \geq c_1 \bar{\rho} |A_r|^{1+\frac{2}{N}}.$$

Proof. For any $\xi \in \mathcal{A}$ and $\mathbb{K}_{\bar{\rho}}(\xi)$ as in Assumption 2.1,

$$\int_{A_r} y_1^2 \mathcal{K}_\xi(y)\, dy \geq \int_{A_r \cap \mathbb{K}_{\bar{\rho}}(\xi)} y_1^2 \mathcal{K}_\xi(y)\, dy \geq \bar{\rho} \int_{A_r \cap \mathbb{K}_{\bar{\rho}}(\xi)} y_1^2\, dy.$$

Choose $\eta > 0$ sufficiently small (depending only on the dimension) so that

$$|\{|y_1| < \eta r\} \cap A_r| \leq \frac{\bar{c}}{2} |A_r|.$$

Then, by Assumption 2.1, the set

$$E := \{|y_1| > \eta r\} \cap A_r \cap \mathbb{K}_{\bar{\rho}}(\xi)$$

has measure at least $\bar{c}|A_r|/2$, and we get

$$\int_{A_r \cap \mathbb{K}_{\bar{\rho}}(\xi)} y_1^2\, dy \geq \int_E y_1^2\, dy \geq \eta^2 r^2 \frac{\bar{c}}{2} |A_r|.$$

Since $|A_r|$ is proportional to r^N, the proof is complete. □

Lemma 3.10. *Given $\hat{\rho}_0 > 1$ there is $p > 0$ such that*

$$f(x) = \min(2^p, |x|^{-p})$$

satisfies

$$L(f, \xi, x) \geq 0$$

for every $s \in (s_0, 1)$, $\xi \in \mathcal{A}$, and $|x| \in [1, \hat{\rho}_0)$.

Proof. Modifying the arguments in [4, Section 8], we will establish $L(f, \xi, x) \geq 0$ for any $\xi \in \mathcal{A}$ and $x = \rho e_1$, $1 \leq \rho \leq \hat{\rho}_0$. The remaining cases are recovered through rotation. The argument relies on the following estimates which hold for $a > b > 0$ and $q > 0$:

$$(a+b)^{-q} \geq a^{-q}\left(1 - q\frac{b}{a}\right),$$

$$(a+b)^{-q} + (a-b)^{-q} \geq 2a^{-q} + \frac{1}{2} q(q+1) b^2 a^{-q-2}.$$

Since $f(\rho x) \geq \rho^{-p} f(x)$ and $f(\rho e_1) = \rho^{-p} f(e_1)$ for all $\rho \geq 1$, using a change of variables we may scale out ρ:

$$\begin{aligned} L(f, \xi, x) &= \rho^{-2s}(1-s) \int_{\mathbb{R}^N} \frac{f(\rho(e_1+z)) + f(\rho(e_1-z)) - 2f(\rho e_1)}{|z|^{N+2s}} \mathcal{K}_\xi(\rho z)\, dz \\ &\geq \rho^{-2s-p}(1-s) \int_{\mathbb{R}^N} \frac{f(e_1+z) + f(e_1-z) - 2f(e_1)}{|z|^{N+2s}} \mathcal{K}_\xi(\rho z)\, dz. \end{aligned}$$

If $|z| \le 1/2$ we have

$$\begin{aligned}\delta := &\, f(e_1+z)+f(e_1-z)-2f(e_1)\\ &\ge 2(1+|z|^2)^{-p/2}+\frac{p}{2}(p+2)z_1^2(1+|z|^2)^{-p/2-2}-2\\ &\ge \frac{p}{2}(p+2)z_1^2-p|z|^2-\frac{p}{4}(p+2)(p+4)z_1^2|z|^2.\end{aligned}$$

Then, for $r \le 1/2$ and $s \in (0,1)$,

$$\begin{aligned}\rho^{2s+p}L(f,\xi,x) &= (1-s)\int_{B_r}\frac{\delta}{|z|^{N+2s}}\mathcal{K}_\xi(\rho z)\,dz+(1-s)\int_{\mathbb{R}^N\setminus B_r}\frac{\delta}{|z|^{N+2s}}\mathcal{K}_\xi(\rho z)\,dz\\ &\ge \frac{p}{2}(p+2)(1-s)\int_{B_r}\frac{z_1^2}{|z|^{N+2s}}\mathcal{K}_\xi(\rho z)\,dz-Cp\|\mathcal{K}_\xi\|_\infty|\partial B_1|r^{2(1-s)}\\ &\quad -C\|\mathcal{K}_\xi\|_\infty(1-s)\frac{p}{4}(p+2)(p+4)r^{4-2s}-C\|\mathcal{K}_\xi\|_\infty\frac{1-s}{s}r^{-2s},\end{aligned}$$

where moving to the last line we used $\delta \ge -1$ for all z. Let $\bar r$ and c_1 be as in Lemma 3.9. If $r < \bar r/\hat\rho_0$ then

$$\begin{aligned}\int_{B_r}\frac{z_1^2}{|z|^{N+2s}}\mathcal{K}_\xi(\rho z)\,dz &= \rho^{-2(1-s)}\int_{B_{\rho r}}\frac{z_1^2}{|z|^{N+2s}}\mathcal{K}_\xi(z)\,dz\\ &= \rho^{-2(1-s)}\sum_{k=0}^{\infty}\int_{A_{\frac{\rho r}{2^k}}}\frac{z_1^2}{|z|^{N+2s}}\mathcal{K}_\xi(z)\,dz\\ &\ge \rho^{-2(1-s)}(\rho r)^{2(1-s)}c_1\sum_{k=0}^{\infty}|A_{2^{-k}}|^{1+\frac{2}{N}}\\ &= C_N r^{2(1-s)}c_1.\end{aligned}$$

This implies

$$\begin{aligned}\rho^{2s+p}L(f,\xi,e_1) \ge &\, C_N\frac{p}{2}(p+2)c_1r^{2-2s}-Cp\|\mathcal{K}_\xi\|_\infty|\partial B_1|r^{2-2s}\\ &-C\|\mathcal{K}_\xi\|_\infty(1-s)\frac{p}{4}(p+2)(p+4)r^{4-2s}-C\|\mathcal{K}_\xi\|_\infty\frac{1-s}{s}r^{-2s}. \qquad (13)\end{aligned}$$

Ignoring the common factor r^{-2s}, the positive term on the right hand side of (13) is $\sim p^2r^2$ while the negative terms are $\sim pr^2$, $\sim p^3r^4$, and ~ 1 respectively. Hence, choosing $r = p^{-3/4}$, for p sufficiently large we get that $r < \bar r/\hat\rho_0$ and the right hand side of (13) is non-negative, as desired. □

Corollary 3.11. *There is a function* Φ *such that:*

- Φ *is continuous in* $\mathbb{R}^N$.
- $\Phi(x) = 0$ *on* $\mathbb{R}^N \setminus B_{2\sqrt{N}}$.
- $\Phi(x) > 2$ *on* Q_3.
- $L^-\Phi(x) > -\psi(x)$ *in* $\mathbb{R}^N$ *for some* $\psi(x) \ge 0$ *supported in* $\bar B_{1/4}$.

for every $s \in (s_0,1)$.

Proof. Let p and f be given by Lemma 3.10 with $\hat\rho_0 = 8\sqrt{N}$, so that after scaling $L(f(\cdot/4),\xi,x) \ge 0$ for all $\xi \in \mathcal{A}$ and x such that $\frac{1}{4} \le |x| \le 2\sqrt{N}$.

Then we set

$$\Phi = c \begin{cases} 0 & \text{if} \quad x \in \mathbb{R}^N \setminus B_{2\sqrt{N}}, \\ 4^p(|x|^{-p} - (2\sqrt{N})^{-p}) & \text{if} \quad x \in B_{2\sqrt{N}} \setminus B_{1/4}, \\ q & \text{if} \quad x \in B_{1/4}. \end{cases}$$

where q is a quadratic polynomial chosen so that Φ is $C^{1,1}$ across $\partial B_{1/4}$ and c is chosen so that $\Phi > 2$ in Q_3. It is easy to check that Φ satisfies all the desired properties. □

3.4. **Point Estimates.** We recall that I denotes the operator defined in (1).

Lemma 3.12. *Let $s \in (s_0, 1)$. There exist $\epsilon_0 > 0$, $0 < \mu < 1$, and $M > 1$ such that if*

- *$u \geq 0$ in $\mathbb{R}^N$,*
- *$\inf_{Q_3} u \leq 1$,*
- *$L^- u(x) \leq \epsilon_0$ inside $Q_{4\sqrt{N}}$,*

then $|\{u \leq M\} \cap Q_1| > \mu$. These constants are independent of $s \in (s_0, 1)$.

Proof. Consider $v := \Phi - u$ where Φ is as in Corollary 3.11. Then, for every $x \in Q_{4\sqrt{N}}$ there exists $\xi \in \mathcal{A}$ such that $L(v, \xi, x) \geq -\psi(x) - 2\epsilon_0$. The proof now follows the one of [4, Lemma 10.1]. □

The following theorem is a direct consequence (using a covering argument and scaling) of Lemma 3.12. The Harnack inequality and Hölder estimates can be deduced from it.

Theorem 3.13. *Let $u \geq 0$ in $\mathbb{R}^N$, $u(0) \leq 1$, and assume $L^- u(x) \leq C_0$ for any $x \in B_{2r}$. Then,*

$$|\{u > t\} \cap B_r| \leq C r^N (u(0) + C_0 r^{2s})^\epsilon t^{-\epsilon}$$

for every $t > 0$ and $s \in (s_0, 1)$. Here, the constants C and ϵ depend only on N, s, and the constants in Assumptions 2.1-2.2.

3.5. **Harnack Inequality.**

Theorem 3.14. *Let $u \geq 0$ in $\mathbb{R}^N$, and assume*

$$L^+ u \geq -C_0', \quad L^- u \leq C_0' \qquad \text{in } B_1.$$

Then $u(x) \leq C'(u(0) + C_0')$ for every $x \in B_{1/2}$ and $s \in (s_0, 1)$. Here C' depends only on N, s, and the constants in Assumptions 2.1-2.2.

Proof. This is proved arguing exactly as in [4, Section 11], using Theorem 3.13 above. □

3.6. **Hölder Estimates.**

Theorem 3.15. *Let $s \in (0, s_0)$ and u a bounded function in $\mathbb{R}^N$. Furthermore, assume there is a constant $C_0' > 0$ such that*

$$L^+ u \geq -C_0', \quad L^- u \leq C_0' \qquad \text{in } B_1.$$

Then there are constants C'' and $\alpha > 0$, depending only on N, s, and the constants in Assumptions 2.1-2.2, such that $u \in C^\alpha(B_{1/2})$ and

$$\|u\|_{C^\alpha(B_{1/2})} \leq C''(\sup_{\mathbb{R}^N} |u| + C_0').$$

Proof. This is proved arguing exactly as in [4, Section 12], using Theorem 3.13 above. □

4. Non-local p-Laplacian

Throughout this section we use Δ_p^s to denote the operator (1) with (2). We restrict ourselves to the case $s \in (\frac{1}{2}, 1)$ and $p \in [2, \infty)$ but all of the results hold for the $p \in (1, 2)$ case described in Remark 4.5. Our main goals are to demonstrate how $\Delta_p^s \to \Delta_p$, the classical p-Laplacian, as $s \to 1$, and describe the existence and regularity properties for solutions.

4.1. **Preliminary Definitions.**

4.1.1. *Local Definitions.* Recall the classical p-Laplacian:

$$\Delta_p u := \Delta u + (p-2)|\nabla u|^{-2} D^2 u : \nabla u \otimes \nabla u.$$

This is the Euler-Lagrange formula (after canceling $|\nabla u|^{p-2}$) associated with the functional $\int_\Omega |\nabla u|^p \, dx$.

The p-Laplacian is not obviously defined when $\nabla u = 0$ and we adopt the following convention (see also [9]).

- If $\nabla u(x) \neq 0$ then
$$\Delta_{p,+} u(x) \equiv \Delta_{p,-} u(x) := \Delta_p u(x).$$
- If $\nabla u(x) = 0$ then
$$\Delta_{p,+} u(x) = \Delta u(x) + (p-2) \sup_{\xi \in S^{N-1}} D^2 u(x) : \xi \otimes \xi,$$
$$\Delta_{p,-} u(x) = \Delta u(x) + (p-2) \inf_{\xi \in S^{N-1}} D^2 u(x) : \xi \otimes \xi.$$

We interpret solutions of the classical p-Laplacian in the following viscosity sense (the definition of viscosity solution can be given equivalently either with quadratic polynomials or with C^2 test functions, but for convenience we prefer to adopt the first one):

Definition 4.1. *An upper [resp. lower] semi continuous function $u : \bar{\Omega} \to \mathbb{R}$ is said to be a* subsolution *[resp.* supersolution*] at $x_0 \in \Omega$, and we write $\Delta_p u(x_0) \geq 0$ [resp. $\Delta_p u(x_0) \leq 0$], if every time a quadratic polynomial ϕ touches u from above [resp. below] at x_0 we have $\Delta_{p,+}\phi(x_0) \geq 0$ [resp. $\Delta_{p,-}\phi(x_0) \leq 0$]. If a function is both a subsolution and a supersolution, we say it is a solution.*

We recall that the viscosity definition of $\Delta_{p,+}$ coincides with the variational one, see [11]. We record now some well known facts that will be useful throughout this section.

Theorem 4.2. *Given a continuous $f : \partial\Omega \to \mathbb{R}$ the Dirichlet problem*

$$\begin{cases} \Delta_p v(x) &= 0 \quad \text{if} \quad x \in \Omega, \\ v(x) &= f(x) \quad \text{if} \quad x \in \partial\Omega. \end{cases} \tag{14}$$

has a unique solution in the sense of Definition 4.1.

The regularity associated with the classical p-Laplacian is also well known.

Theorem 4.3. *Let $u : B_1 \to \mathbb{R}$ solve (4.1) with $\Omega = B_1$. Then*

$$\|u\|_{C^{1,\bar{\alpha}}(B_{1/2})} \leq C(N,p)\|f\|_{L^\infty(\partial B_1)}$$

for some $\bar{\alpha} = \bar{\alpha}(N,p) > 0$.

Proof. See, for example, [8], [10], [12], [14], [15], and [16]. ☐

4.1.2. *Non-local Definitions.* Reflecting the definition of solutions for the classical p-Laplacian, throughout this section we adopt the following definition of solutions corresponding to Δ_p^s, $s \in (\frac{1}{2}, 1)$. This definition makes use of the following observation: for the kernel (2), when $u \in C^{1,1}(x)$ and $\nabla u(x) \neq 0$, the infimum and supremum in definition (1) are obtained in the direction of of $\nabla u(x)$ (see also [1]).

Define $\Delta_{p,+}^s$ and $\Delta_{p,-}^s$ in the following way: (See (15) and the discussion below for the precise definition of α_p, c_p. At the moment, think of them just as two positive constants.)

- If $\nabla u(x) \neq 0$ then
$$\Delta_{p,+}^s u(x) \equiv \Delta_{p,-}^s u(x) := \frac{(1-s)}{\alpha_p} \int_{\mathbb{R}^N} \frac{[u(x+y) + u(x-y) - 2u(x)]\, \mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|} \cdot \xi\right)}{|y|^{N+2s}}\, dy,$$
with $\xi = \frac{\nabla u(x)}{|\nabla u(x)|}$.
- If $\nabla u(x) = 0$ then
$$\Delta_{p,+}^s u(x) = \frac{(1-s)}{\alpha_p} \sup_{\xi \in S^{N-1}} \int_{\mathbb{R}^N} \frac{[u(x+y) + u(x-y) - 2u(x)]\, \mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|} \cdot \xi\right)}{|y|^{N+2s}}\, dy,$$
$$\Delta_{p,-}^s u(x) = \frac{(1-s)}{\alpha_p} \inf_{\xi \in S^{N-1}} \int_{\mathbb{R}^N} \frac{[u(x+y) + u(x-y) - 2u(x)]\, \mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|} \cdot \xi\right)}{|y|^{N+2s}}\, dy.$$

Definition 4.4. *An upper [resp. lower] semi continuous function $u : \mathbb{R}^N \to \mathbb{R}$ is said to be a* subsolution *[resp.* supersolution*] at $x_0 \in \Omega$, and we write $\Delta_p^s u(x_0) \geq 0$ [resp. $\Delta_p^s u(x_0) \leq 0$], if every time a test function $\phi \in C^{1,1}(x_0)$ touches u from above [resp. below] at x_0 we have $\Delta_{p,+}^s \tilde{u}(x_0) \geq 0$ [resp. $\Delta_{p,-}^s \tilde{u}(x_0) \leq 0$] where $\tilde{u}$ is as in (7). If a function is both a subsolution and a supersolution, we say it is a solution.*

We also use the notation $\Delta_p^1 \equiv \Delta_p$, $\Delta_{p,+}^1 \equiv \Delta_{p,+}$, and $\Delta_{p,-}^1 \equiv \Delta_{p,-}$.

4.2. **Connection with the p-Laplacian.** In this subsection we demonstrate how $\Delta_p^s u(x) \to \Delta_p u(x)$ as $s \to 1$ when u is smooth. To begin we make precise the choices of α_p and c_p in (2). We define

$$\alpha_p := \frac{1}{2} \int_{\partial B_1} (\omega \cdot e_2)^2\, \mathbb{1}_{[c_p,1]}(\omega \cdot e_1)\, d\sigma(\omega), \tag{15}$$

$$\beta_p := \frac{1}{2} \int_{\partial B_1} (\omega \cdot e_1)^2\, \mathbb{1}_{[c_p,1]}(\omega \cdot e_1)\, d\sigma(\omega) - \alpha_p. \tag{16}$$

For $p \geq 2$ we choose $c_p \in [0,1]$ such that $\beta_p/\alpha_p = p - 2$. Such a choice is possible for any $p \in [2, \infty)$. (The case $p = 2$ corresponds to $c_p = 0$.)

Remark 4.5. *To handle the cases $p \in (1,2)$ one should instead use the kernel*
$$\mathcal{A} := S^{N-1}, \qquad \mathcal{K}_\xi(y) := \frac{1}{\alpha_p}\, \mathbb{1}_{[0,c_p]}\left(\frac{y}{|y|} \cdot \xi\right).$$
The difference between this kernel and (2) is the cone which defines the kernel: here we use the indicator for the set $[0, c_p]$ while (2) uses $[c_p, 1]$. The ideas and theorems in this section may be applied to this model in a straightforward way.

For the remainder of this subsection let $u \in C^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N; s_0)$ for some $s_0 \in (0,1)$ and assume always that $s > s_0$.

4.2.1. *Limit in the case* $\nabla u(x) \neq 0$. First we handle the case $\nabla u(x) \neq 0$. Set $\epsilon_s = (1-s)^{\frac{1}{2(N+2s)}}$. Then, for s sufficiently close to 1:

$$\begin{aligned}
\Delta_p^s u(x) &= \frac{(1-s)}{\alpha_p} \int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\, \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \frac{\nabla u}{|\nabla u|} \right) dy \\
&= \frac{(1-s)}{\alpha_p} D^2 u(x) : \int_{B_{\epsilon_s}} \frac{y \otimes y}{|y|^{N+2s}}\, \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \frac{\nabla u}{|\nabla u|} \right) dy \\
&\quad + \frac{(1-s)}{\alpha_p} \int_{\mathbb{R}^N \setminus B_{\epsilon_s}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\, \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \frac{\nabla u}{|\nabla u|} \right) dy \\
&\quad + \frac{(1-s)}{\alpha_p} \int_{B_{\epsilon_s}} \frac{o(|y|^2)}{|y|^{N+2s}}\, \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \frac{\nabla u}{|\nabla u|} \right) dy. \\
&=: A_1 + A_2 + A_3.
\end{aligned}$$

With a change of variables,

$$\begin{aligned}
A_1 &= \epsilon_s^{2(1-s)} \frac{(1-s)}{\alpha_p} D^2 u(x) : \int_{B_1} \frac{y \otimes y}{|y|^{N+2s}}\, \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \frac{\nabla u}{|\nabla u|} \right) dy, \\
&= \epsilon_s^{2(1-s)} \frac{(1-s)}{2\alpha_p} D^2 u(x) : \int_{\partial B_1} \omega \otimes \omega\, \mathbb{1}_{[c_p,1]} \left(\omega \cdot \frac{\nabla u}{|\nabla u|} \right) d\sigma(\omega) \\
&= \epsilon_s^{2(1-s)} \left(\Delta u(x) + \frac{\beta_p}{\alpha_p} |\nabla u|^{-2} D^2 u : \nabla u \otimes \nabla u \right).
\end{aligned}$$

Also, since

$$|y|^{-N-2s} \leq \frac{2\epsilon_s^{-N-2s}}{1+|y|^{N+2s_0}} \qquad \text{for } |y| \geq \epsilon_s \text{ and } s \geq s_0,$$

we get

$$A_2 \leq 4\epsilon_s^{-N-2s} \frac{(1-s)}{\alpha_p} \int_{\mathbb{R}^N} \frac{|u(x+y)-u(x)|}{1+|y|^{N+2s_0}}\, dy$$

and we see $A_2 \to 0$ as $s \to 1$ (observe that $(1-s)\epsilon_s^{-N-2s} = \sqrt{1-s}$).

Moreover, since $\epsilon_s^{2(1-s)} \to 1$ as $s \to 1$, we see that $A_1 \to \Delta_p u(x)$ and $A_3 \to 0$ (recall that $\beta_p/\alpha_p = (p-2)$).

Hence

$$\lim_{s \to 1} \Delta_p^s u(x) = \Delta_p u(x),$$

as desired.

4.2.2. *Limit in the case* $\nabla u(x) = 0$. For the same choice of ϵ_s we have

$$\begin{aligned}
\Delta_{p,+}^s u(x) \leq{}& \frac{(1-s)}{\alpha_p} \sup_{\xi \in S^{N-1}} D^2 u(x) : \int_{B_{\epsilon_s}} \frac{y \otimes y}{|y|^{N+2s}}\, \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \xi \right) dy \\
&+ 2\frac{(1-s)}{\alpha_p} \sup_{\xi \in S^{N-1}} \int_{\mathbb{R}^N \setminus B_{\epsilon_s}} \frac{|u(x+y)-u(x)|}{|y|^{N+2s}}\, \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \xi \right) dy \\
&+ \frac{(1-s)}{\alpha_p} \int_{B_{\epsilon_s}} \frac{o(|y|^2)}{|y|^{N+2s}}\, \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \xi \right) dy.
\end{aligned}$$

Arguing similar to the case $\nabla u(x) \neq 0$ above, one can show the second and third terms on the right hand side tend to zero as $s \to 1$, so that

$$0 \leq \lim_{s\to 1} \Delta^s_{p,+} u(x) \leq \Delta u(x) + (p-2) \sup_{\xi \in S^{N-1}} \partial^2_\xi u(x) = \Delta_{p,+} u(x).$$

Likewise,

$$0 \geq \lim_{s\to 1} \Delta^s_{p,-} u(x) \geq \Delta u(x) + (p-2) \inf_{\xi \in S^{N-1}} \partial^2_\xi u(x) = \Delta_{p,-} u(x).$$

4.3. **Basic Properties of Δ^s_p.** For the remainder of this section we investigate the following Dirichlet problem.

Given a domain Ω and data $f : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ we are interested in solutions of

$$\begin{cases} \Delta^s_p u(x) & = 0 \quad \text{if} \quad x \in \Omega, \\ u(x) & = f(x) \quad \text{if} \quad x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{17}$$

In order to construct barriers at the boundary, we will assume Ω is a bounded subset of $\mathbb{R}^N$ which satisfies an exterior sphere condition, that is, we assume there is a fixed radius $R_0 > 0$ so that any point on $\partial\Omega$ can be touched from the outside by a sphere of radius R_0. Furthermore, we also assume that $s \in [s_0, 1)$ for some $s_0 \in (\frac{1}{2}, 1)$ we have fixed beforehand.

4.3.1. *Stability.* We will now show Δ^s_p is stable under suitable convergence of solutions. The definition of half-relaxed limit is borrowed from [7, Section 6].

Theorem 4.6. *Let $u_n : \mathbb{R}^N \to \mathbb{R}$ be a family of upper semi continuous functions, and $\{s_n\} \subset (\frac{1}{2}, 1)$ be a sequence converging to $s \in (\frac{1}{2}, 1]$ as $n \to \infty$. Let $\Omega \subset \mathbb{R}^N$ and assume that*

- *there is a function $g \in L^1(\mathbb{R}^N)$ such that $\frac{|u_n(x)|}{1+|x|^{N+2s_n}} \leq g(x)$ for all n and $x \in \mathbb{R}^N$,*
- *$\Delta^{s_n}_{p,+} u_n \geq 0$ in Ω.*

Let u denote the "half-relaxed limit" of u_n, i.e.,

$$u^+(x) := \limsup_{j\to\infty} \{u_n(z) : z \in B_{1/j}(x) \cap \Omega,\ n \geq j\}$$

Then $\Delta^s_{p,+} u^+ \geq 0$ in Ω.

Notice that if $u_n \to u$ locally uniformly, then $u^+ = u$ and so u is a subsolution.

Proof. First of all, using the definition of half-relaxed limit it is easy to check that u^+ is upper semi continuous.

Let us first assume $s < 1$, so that $\{s_n\} \leq \bar{s} < 1$ for n large. As shown in [4, Lemma 4.3], it suffices to test the subsolution condition with C^2 test functions.

Let $\phi \in C^2(B_r(x_0))$ touch u^+ from above at $x_0 \in \Omega$, and let x_n be a sequence as in the assumption above. Since ϕ touches u strictly at x_0, using the definition of half-relaxed limit we see that for n sufficiently large there exists a small constant $\delta_n \in \mathbb{R}$ such that $\phi + \delta_n$ touches u_n above at a point $x_n \in B_r(x_0)$. Define $r_n = r - |x_n - x_0|$ (observe that $r_n \to r$ as $n \to \infty$), and define

$$\tilde{u}_n = \begin{cases} \phi(x) + \delta_n & \text{if} \quad |x - x_n| < r_n, \\ u_n(x) & \text{if} \quad |x - x_n| \geq r_n. \end{cases}$$

By assumption, $\Delta^{s_n}_{p,+} \tilde{u}_n(x_n) \geq 0$. We will show $\Delta^s_{p,+} \tilde{u}(x_0) \geq 0$.

Case I: If $\nabla\phi(x_0) \neq 0$, taking n large enough we can ensure $\nabla\phi(x_n) \neq 0$. Let $\xi_n \in S^{N-1}$ denote the direction of $\nabla\phi(x_n)$ and ξ_0 denote the direction of $\nabla\phi(x_0)$. We have

$$\begin{aligned} 0 \leq (1-s_n) \int_{B_{r_n}} & \frac{[\phi(x_n+y)+\phi(x_n-y)-2\phi(x_n)]\, \mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi_n\right)}{|y|^{N+2s_n}}\, dy \\ &+ (1-s_n)\int_{\mathbb{R}^N\setminus B_{r_n}} \frac{[u_n(x_n+y)+u_n(x_n-y)-2u_n(x_n)]\, \mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi_n\right)}{|y|^{N+2s_n}}\, dy. \end{aligned} \tag{18}$$

The first integral on the right hand side is bounded by the integrable function $M|y|^{2(1-\bar{s})-N}$. By assumption, the integrand in the second integral is also bounded by the integrable function g. Since $\xi_n \to \xi_0$ and $r_n \to r_0$ as $x_n \to x_0$, the dominated convergence theorem implies

$$\begin{aligned} 0 \leq (1-s) \int_{B_r} & \frac{[\phi(x_0+y)+\phi(x_0-y)-2\phi(x_0)]\, \mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi_0\right)}{|y|^{N+2s}}\, dy \\ &+ (1-s)\int_{\mathbb{R}^N\setminus B_r} \frac{[u_0(x_0+y)+u_0(x_0-y)-2u_0(x_0)]\, \mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi_0\right)}{|y|^{N+2s}}\, dy. \end{aligned}$$

This is exactly $\Delta^x_{p,+}\tilde{u}(x_0) \geq 0$.

Case II: If $\nabla\phi(x_0) = 0$, for any $\epsilon > 0$ there is a sequence $\xi_n \in S^{N-1}$ so that

$$\begin{aligned} -\epsilon \leq (1-s_n) \int_{B_{r_n}} & \frac{[\phi(x_n+y)+\phi(x_n-y)-2\phi(x_n)]\, \mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi_n\right)}{|y|^{N+2s_n}}\, dy \\ &+ (1-s_n)\int_{\mathbb{R}^N\setminus B_{r_n}} \frac{[u_n(x_n+y)+u_n(x_n-y)-2u_n(x_n)]\, \mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi_n\right)}{|y|^{N+2s_n}}\, dy. \end{aligned} \tag{19}$$

Since S^{N-1} is compact, there is a $\xi_0 \in S^{N-1}$ and subsequence (which we do not relabel) so that $\xi_n \to \xi_0$. Arguing as in the $\nabla\phi(x_0) \neq 0$ case we find

$$\begin{aligned} -\epsilon \leq \int_{B_r} & \frac{[q(x_0+y)+q(x_0-y)-2q(x_0)]\, \mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi_0\right)}{|y|^{N+2s}}\, dy \\ &+ \int_{\mathbb{R}^N\setminus B_r} \frac{[u_0(x_0+y)+u_0(x_0-y)-2u_0(x_0)]\, \mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi_0\right)}{|y|^{N+2s}}\, dy \\ \leq& \sup_{\xi\in S^{N-1}} \int_{\mathbb{R}^N} \frac{[\tilde{u}_0(x_0+y)+\tilde{u}_0(x_0-y)-2\tilde{u}_0(x_0)]\, \mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi\right)}{|y|^{N+2s}}\, dy, \end{aligned}$$

where $\tilde{u}_0$ is as in (7). As $\epsilon > 0$ is arbitrary we conclude the right hand side is positive which implies $\Delta^s_{p,+}\tilde{u}_0(x) \geq 0$.

To finish we need to handle the case $s_n \to 1$.

Let $\phi \equiv q$ be a quadratic polynomial (see Definition 4.1). We follow a path similar to Subsection 4.2: again assume $\nabla q(x_0) \neq 0$ and rewrite the first integral on the right hand side of (18):

$$(1-s_n)\int_{B_{r_n}} \frac{[q(x_n+y)+q(x_n-y)-2q(x_n)]\,\mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi_n\right)}{|y|^{N+2s_n}}\,dy = r_n^{2(1-s_n)}\Delta_{p,+}q(x_n).$$

The second integral on the right hand side of (18) is bounded by

$$C(1-s_n)\left[\|g\|_{L^1(\mathbb{R}^N)} + \frac{|u_n(x_n)|}{r_n^{2s}}\right],$$

which tends to zero as $n\to\infty$. Thus,

$$0 \le \lim_{n\to\infty}\Delta_{p,+}^{s_n}\tilde{u}_n(x_n) = \Delta_{p,+}q(x_0).$$

If $\nabla q(x_0)=0$ we start instead with (19) and find $\Delta_{p,+}q(x_0)\ge 0$. □

Remark 4.7. *A similar statement can be made about supersolutions u_n by applying this lemma to $-u_n$.*

Remark 4.8. *If, in the statement for Theorem 4.6, we replace the assumption "$\Delta_{p,+}^{s_n}u_n \ge 0$ in Ω" with:*

For every $x\in\Omega$ there is $\xi\in S^{N-1}$ such that

$$\int_{\mathbb{R}^N} \frac{[u_n(x+y)+u_n(x-y)-2u_n(x)]\,\mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi\right)}{|y|^{N+2s_n}}\,dy \ge 0 \quad (\textit{resp. } \le 0)$$

interpreted in the viscosity sense,

we have instead the conclusion:

$$\int_{\mathbb{R}^N} \frac{[u^+(x+y)+u^+(x-y)-2u^+(x)]\,\mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi\right)}{|y|^{N+2s}}\,dy \ge 0 \quad (\textit{resp. } \le 0) \quad \textit{if} \quad s<1,$$

$$\Delta u^+(x) + (p-2)\partial_\xi^2 u^+(x) \ge 0 \quad (\textit{resp. } \le 0) \quad \textit{if} \quad s=1.$$

The following corollary will be used to construct solutions using Perron's method.

Corollary 4.9. *Let $\mathcal{F}$ be a set of subsolutions such that $\frac{|u(x)|}{1+|x|^{N+2s}} \le g(x)$ for all $u\in\mathcal{F}$, where $g\in L^1(\mathbb{R}^N)$. Set*

$$w(x) := \sup_{u\in\mathcal{F}} u(x) < \infty \qquad \forall\, x\in\Omega,$$

and let w^* be its upper semi continuous envelope, i.e.,

$$w^*(x) := \limsup_{j\to\infty}\{w(z) : z\in B_{1/j}(x)\cap\Omega\}.$$

Then w^* is a subsolution.

Proof. Since the maximum of two subsolution is a subsolution, there is a sequence of subsolution w_n whose half-relaxed limit is given by w^*. The result then follows from Theorem 4.6. □

4.3.2. *Partial Comparison.* Subsolutions and supersolutions for Δ_p^s satisfy a weak comparison principle given by the following lemma. This comparison principle is not strong enough for a full uniqueness theory but we will use it to prove growth estimates away from the boundary.

To prove it, we will rely on the well known inf/sup-convolution approximation (see, for example, [3, Section 5.1]):

Definition 4.10. *Given a continuous function u, the "sup-convolution approximation" u^ϵ is given by*

$$u^\epsilon(x_0) = \sup_{x\in\mathbb{R}^N} \left\{ u(x) + \epsilon - \frac{|x-x_0|^2}{\epsilon} \right\}.$$

Given a continuous function v, the "inf-convolution approximation" v_ϵ is given by

$$v_\epsilon(x_0) = \inf_{x\in\mathbb{R}^N} \left\{ v(x) - \epsilon + \frac{|x-x_0|^2}{\epsilon} \right\}.$$

We state the following lemma without proof, as it is standard in the theory of viscosity solutions (see, for instance, [3, Section 5.1]).

Lemma 4.11. *Assume that $u, w : \mathbb{R}^N \to \mathbb{R}$ are two continuous functions which grow at most as $|x|^2$ at infinity. The following properties hold:*

- *$u^\epsilon \downarrow u$ [resp. $w_\epsilon \uparrow w$] uniformly on compact sets as $\epsilon \to 0$. Moreover, if u [resp. w] is uniformly continuous on the whole $\mathbb{R}^N$, then the convergence is uniform on $\mathbb{R}^N$.*
- *At every point there is a concave [resp. convex] paraboloid of opening $2/\epsilon$ touching u^ϵ [resp. w_ϵ] from below [resp. from above]. (We informally refer to this property by saying that u^ϵ [resp. w_ϵ] is $C^{1,1}$ from below [resp. above].)*
- *If $\Delta_p^s u \geq 0$ [resp. $\Delta_p^s w \leq 0$] in Ω in the viscosity sense, then $\Delta_p^s u^\epsilon \geq 0$ [resp. $\Delta_p^s w^\epsilon \leq 0$] in Ω.*

Theorem 4.12. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Also, let $s \in (\frac{1}{2}, 1)$ and $u, v \in L^1(\mathbb{R}^N, s)$ satisfy*

$$\sup_{\xi\in S^{N-1}} \int_{\mathbb{R}^N} \frac{[u(x+y)+u(x-y)-2u(x)]\, \mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi\right)}{|y|^{N+2s}}\, dy \geq 0, \tag{20}$$

$$\sup_{\xi\in S^{N-1}} \int_{\mathbb{R}^N} \frac{[v(x+y)+v(x-y)-2v(x)]\, \mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi\right)}{|y|^{N+2s}}\, dy \leq 0, \tag{21}$$

inside Ω. (Here u is a subsolution in the sense of Definition 3.3, while we are assuming stronger control on v.) If $u(x) \leq v(x)$ for all $x \in \mathbb{R}^N \setminus \Omega$ then $u \leq v$ in Ω.

Proof. Assume by contradiction that there is a point $x_0 \in \Omega$ such that $u(x_0) > v(x_0)$. Replacing u and v by u^ϵ and v_ϵ, we have $u^\epsilon(x_0) - v_\epsilon(x_0) \geq c > 0$ for ϵ sufficiently small, and $(u^\epsilon - v_\epsilon) \vee 0 \to 0$ as $\epsilon \to 0$ locally uniformly outside Ω. Thanks to these properties, the continuous function $u^\epsilon - v_\epsilon$ attains its maximum over $\overline{\Omega}$ at some interior point $\bar{x} \in \Omega$. Set $\delta = u^\epsilon(\bar{x}) - v_\epsilon(\bar{x}) \geq c > 0$. Since u^ϵ is $C^{1,1}$ from below, $v_\epsilon + \delta$ is $C^{1,1}$ from above, and $v_\epsilon + \delta$ touches u^ϵ from above at $\bar{x}$, it is easily seen that both u^ϵ and $v_\epsilon + \delta$ are $C^{1,1}(\bar{x})$. Thus, we can proceed directly without appealing to test functions.

To begin we note (20)-(21) and a slight variation of Lemma 4.11 imply

$$\sup_{\xi\in S^{N-1}} \int_{\mathbb{R}^N} \frac{[u^\epsilon(\bar x+y)+u^\epsilon(\bar x-y)-2u^\epsilon(\bar x)]\,\mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi\right)}{|y|^{N+2s}}\,dy \geq 0,$$

$$\sup_{\xi\in S^{N-1}} \int_{\mathbb{R}^N} \frac{[v_\epsilon(\bar x+y)+v_\epsilon(\bar x-y)-2v_\epsilon(\bar x)]\,\mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi\right)}{|y|^{N+2s}}\,dy \leq 0.$$

Let $\xi_0 \in S^{N-1}$ be such that

$$0 \leq (1-s)\int_{\mathbb{R}^N} \frac{u^\epsilon(\bar x+y)+u^\epsilon(\bar x-y)-2u^\epsilon(\bar x)}{|y|^{N+2s}}\,\mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi_0\right). \tag{22}$$

(We are always able to find such a ξ_0 because S^{N-1} is a compact set and u^ϵ is $C^{1,1}(\bar x)\cap L^1(\mathbb{R}^N;s)$, so $L(u^\epsilon,\xi,\bar x)$ is continuous in ξ.) Then,

$$0 \leq (1-s)\int_{\mathbb{R}^N} \frac{w(\bar x+y)+w(\bar x-y)}{|y|^{N+2s}}\,\mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi_0\right)dy$$

where

$$w(\bar x+y) := u^\epsilon(\bar x+y)-v^\epsilon(\bar x+y)-\delta$$

Notice $w\leq 0$ so that $w(\bar x+y)\equiv 0$ for all $y\in\mathbb{R}^N\setminus B_r$ such that $\mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi_0\right)>0$. This contradicts $u^\epsilon\leq v_\epsilon$ in $\mathbb{R}^N\setminus\Omega$ completing the proof. □

Before proceeding we record another partial comparison theorem relating to subsolutions in the sense of Definition 4.4 that will also be useful.

Theorem 4.13. *Let $s\in(\frac{1}{2},1)$ and $u,v\in L^1(\mathbb{R}^N;s)$ satisfy*

$$\Delta^s_{p,+}u(x)\geq 0,$$
$$\Delta^s_{p,+}v(x)\leq 0,$$

for every x in the bounded set Ω. If $u(x)\leq v(x)$ for all $x\in\mathbb{R}^N\setminus\Omega$ then $u\leq v$ in Ω.

Proof. The proof of this theorem is argued the same as the proof of Theorem 4.12. In fact, if u^ϵ and $v^\epsilon+\delta$ touch at $\bar x$ and $\nabla u^\epsilon(\bar x)=\nabla v_\epsilon(\bar x)=0$ the exact same argument applies. If $\nabla u^\epsilon(\bar x)=\nabla v_\epsilon(\bar x)\neq 0$ then one uses this common direction as the choice of ξ along which to compare. □

Notice that Theorems 4.12 and 4.13 are not symmetric in u and v, as we are asking more on v and less on u. In the sequel we will use Theorem 4.12 when we are comparing solutions with barrier functions, and 4.13 when we are comparing solutions with each other.

4.3.3. *Growth From the Boundary.* In this subsection we examine how solutions of (17) attain their boundary values. The argument below uses a barrier function and the partial comparison theorems to bound growth from the boundary.

Lemma 4.14. *There is continuous function ϕ such that:*

- *$\phi=0$ in B_1.*
- *$\phi\geq 0$ in $\mathbb{R}^N$.*
- *$\phi=1$ in $\mathbb{R}^N\setminus B_2$.*

- *For any $s \in (s_0, 1)$ and $x \in \mathbb{R}^N \setminus B_1$, ϕ satisfies*

$$\sup_{\xi \in S^{N-1}} \int_{\mathbb{R}^N} \frac{[\phi(x+y) + \phi(x-y) - 2\phi(x)] \, \mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|} \cdot \xi\right)}{|y|^{N+2s}} \, dy \leq 0. \tag{23}$$

Proof. We claim there are $\alpha, r > 0$ so that the function $u(x) = ((|x|-1)^+)^\alpha$ satisfies $\Delta^s_{p,+} u(x) \leq 0$ in $B_{1+r} \setminus B_1$ for any $s \in (s_0, 1]$. Indeed, it can be computed directly that $\Delta \log(|x|-1) + (p-2)\partial^2_\xi \log(|x|-1) \to -\infty$ as $|x| \to 1$, uniformly in ξ. So the proof of this claim is argued exactly as in the proof of [5, Lemma 3.1], using Remark 4.8 to address stability for fixed ξ.

Set $\phi(x) = \min(1, r^{-\alpha}((|x|-1)^+)^\alpha)$. It is immediate that (23) holds for $x \in B_{1+r} \setminus B_1$. Since ϕ attains its global maximum at any point in $\mathbb{R}^N \setminus B_{1+r}$, (23) follows for $x \in \mathbb{R}^N \setminus B_{1+r}$. □

We now demonstrate how bounded subsolutions grow from the boundary. By applying this lemma to $-u$ we find a similar estimate for supersolutions.

Lemma 4.15. *Let $\Omega \subset \mathbb{R}^N$ be a bounded subset satisfying the exterior sphere condition. Fix $z \in \partial\Omega$ and let $s \in (s_0, 1)$. Assume there is a modulus of continuity ρ and a bounded function $u : \mathbb{R}^N \to \mathbb{R}$ satisfying*

- $$\sup_{\xi \in S^{N-1}} \int_{\mathbb{R}^N} \frac{[u(x+y) + u(x-y) - 2u(x)] \, \mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|} \cdot \xi\right)}{|y|^{N+2s}} \, dy \geq 0 \qquad \forall\, x \in \Omega,$$
- $$u(x) - u(z) \leq \rho(|x-z|) \qquad \forall\, z \in \partial\Omega,\ x \in \mathbb{R}^N \setminus \Omega.$$

Then there is another modulus of continuity $\tilde{\rho}$, independent of $s \in (s_0, 1)$, such that

$$u(x) - u(z) \leq \tilde{\rho}(|x-z|) \qquad \forall\, z \in \partial\Omega,\ x \in \mathbb{R}^N.$$

Proof. This proof follows that of [5, Lemma 3.5]. Let $\nu \in S^{N-1}$ denote the outward unit normal at $z \in \partial\Omega$. For each $R > 0$ small, the barrier function

$$b(x) := u(z) + \rho(3R) + \|u\|_{L^\infty(\mathbb{R}^N)} \phi\left(\frac{x-z}{R} - \nu\right)$$

satisfies

$$\begin{cases} b(x) & \geq u(x) \quad \text{if} \quad x \in \mathbb{R}^N \setminus (\Omega \cap B_{3R}(z)), \\ \Delta^s_{p,+} b(x) & \leq 0 \qquad \text{if} \quad x \in \Omega \cap B_{3R}(z). \end{cases}$$

Lemma 4.14 combined with Theorem 4.12 then implies $u(x) \leq b(x)$ on $\mathbb{R}^N$, so that

$$u(x) - u(z) \leq \tilde{\rho}(|x-z|) := \inf_{R_0 > R > 0} \left(\rho(3R) + \|u\|_{L^\infty(\mathbb{R}^N)} \phi\left(\frac{x-z}{R} - \nu\right)\right) \qquad \forall\, x \in \mathbb{R}^N,$$

where $R_0 > 0$ is the radius given by the exterior sphere condition. □

The following corollary gives control on the boundary growth for solutions which are unbounded but controlled in how they approach infinity.

Corollary 4.16. *Let $\Omega \subset \mathbb{R}^N$ be a bounded subset satisfying the exterior sphere condition. Fix $z \in \partial\Omega$ and let $s \in (s_0, 1)$. Assume there is a modulus of continuity ρ and a function $u : \mathbb{R}^N \to \mathbb{R}$ satisfying*

- $$\sup_{\xi\in S^{N-1}}\int_{\mathbb{R}^N}\frac{[u(x+y)+u(x-y)-2u(x)]\,\mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi\right)}{|y|^{N+2s}}\,dy\geq 0\qquad\forall\,x\in\Omega,$$
- $$u(x)-u(z)\leq\rho(|x-z|)\qquad\forall\,z\in\partial\Omega,\,x\in\mathbb{R}^N\setminus\Omega,$$
- $$|u(x)|\leq M(1+|x|)^\alpha\qquad\textit{for some }\alpha\in(0,2s_0).$$

Then there is another modulus of continuity $\tilde\rho$, independent of $s\in(s_0,1)$, such that

$$u(x)-u(z)\leq\tilde\rho(|x-z|)\qquad\forall\,z\in\partial\Omega,\,x\in\mathbb{R}^N.$$

Proof. Fix $R_0>0$ and let $R>0$ be large enough that $dist(\Omega,\mathbb{R}^N\setminus B_R)>R_0$ and truncate u at $M(1+R)^\alpha$:

$$w(x):=\min\Big\{-M(1+R)^\alpha,\max\big\{u(x),M(1+R)^\alpha\big\}\Big\}.$$

Then, it is easy to check that, for every $x\in\Omega$,

$$\Delta^s_{p,+}w(x)\geq -C_0(M,R_0,s_0,\alpha).$$

Moreover, there is a constant $c_0=c_0(M,R_0,s_0,\alpha)>0$ such that $p(x)=\max(0,1-|x|^2/R^2)$ satisfies

$$\sup_{\xi\in S^{N-1}}\int_{\mathbb{R}^N}\frac{[p(x+y)+p(x-y)-2p(x)]\,\mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi\right)}{|y|^{N+2s}}\,dy\leq -c_0$$

inside Ω. Then $w-\frac{C_0}{c_0}p$ is bounded and satisfies the assumptions of Lemma 4.15. Since p is uniformly continuous, we have proven this corollary. □

Remark 4.17. *Assume that u solves* (17) *and satisfies the uniform bound*

$$|u(z)-u(x)|\leq\tilde\rho(|x-z|)\qquad\forall\,z\in\partial\Omega,\,x\in\mathbb{R}^N.$$

Then, by the same argument as in the proof of [5, Lemma 3.6] *one can use the Hölder estimates from Theorem 3.15 (suitably rescaled) to find a modulus of continuity $\hat\rho$ such that*

$$|u(z)-u(x)|\leq\hat\rho(|x-z|)\qquad\forall\,z\in\Omega,\,x\in\mathbb{R}^N.$$

4.3.4. *Existence.* To prove existence of solutions to (17), we use Perron's method. Hence, we need the following standard "bump" construction:

Lemma 4.18. *Let $u:\mathbb{R}^N\to\mathbb{R}$ be a subsolution in Ω, that is $\Delta^s_p u(x)\geq 0$ for all $x\in\Omega$. If u is not a supersolution in Ω then there is another function w such that $w(x_0)>u(x_0)$, $w=u$ in $\mathbb{R}^N\setminus\Omega$, and w is a subsolution in Ω.*

Proof. We recall that, by [4, Lemma 4.3], one can use C^2 functions as test functions.

If u is not a supersolution then there are two constants $a,r>0$, a point $x_0\in\Omega$, and a function $\phi\in C^2(B_r(x_0))$ touching u from below at x_0, such that $\Delta^s_{p,-}\hat u(x_0)>a$ where

$$\hat u(x):=\begin{cases}\phi(x) & \text{if}\quad x\in B_r(x_0),\\ u(x) & \text{if}\quad x\in\mathbb{R}^N\setminus B_r(x_0).\end{cases}$$

Notice that if $x\in B_r(x_0)$ we may evaluate $\Delta^s_{p,-}\hat u(x)$ classically.

Claim: There exists $r_0 < r$ such that $\Delta^s_{p,-}\hat{u}(x) \geq a$ inside $B_{r_0}(x_0)$.

In the case when $\nabla\phi(x_0) \neq 0$, $\Delta^s_{p,-}\hat{u}(x)$ is actually continuous. Indeed, there is a small neighborhood of x_0 so that $\nabla\phi(x) \neq 0$ in this neighborhood. The continuity of $\Delta^s_{p,-}\hat{u}(x)$ in this neighborhood is a now a consequence of the dependence of the operator on $\nabla\phi$, and the claim follows.

In the case $\nabla\phi(x_0) = 0$ the claim will follow immediately once we show

$$\inf_{x\in B_{r_0}(x_0)} \inf_{\xi\in S^{N-1}} L(\hat{u},\xi,x) \geq a$$

for some $r_0 \in (0,r)$ (recall the definition of L in (6)). Assume for the sake of contradiction that there are sequences $x_n \to x_0$ and $\xi_n \in S^{N-1}$ such that

$$\lim_{n\to\infty} L(\hat{u},\xi_n,x_n) < a.$$

Relying on the compactness of S^{N-1} there is an $\xi_0 \in S^{N-1}$ and a subsequence (which we do not relabel) such that $\xi_n \to \xi_0$. Arguing as in the proof of the Lemma 4.6 it is easy to show $L(\hat{u},\xi_0,x_0) \leq a$ contradicting $\Delta^s_{p,-}\hat{u}(x_0) > a$, and the claim is proven also in this case.

Using the claim we now prove the lemma. Possibly shrinking the ball we may assume $r_0 < r/2$ and $B_{r_0} \subset \Omega$. Next choose $\delta_0 > 0$ small enough that $\phi(x) + \delta_0 < u(x)$ for any $x \in \partial B_{r_0}(x_0)$ and

$$\frac{(1-s)\delta_0|\partial B_1|}{\alpha_p s r_0^{2s}} \leq a.$$

It is left to prove that $w := u \wedge (\phi+\delta)$ is a subsolution for any $\delta < \delta_0$. For any $\bar{x} \in \Omega$ let $\varphi \in C^{1,1}(\bar{x})$ touch w from above at $\bar{x}$. If φ touches u from above, we use that $w \geq u$ and that u is a subsolution to see $\Delta^s_{p,+}\tilde{w}(\bar{x}) \geq \Delta^s_{p,+}\tilde{u}(\bar{x}) \geq 0$ (where $\tilde{w}$ and $\tilde{u}$ are as in (7)). If φ does not touch u, then it must touch $\phi+\delta$ from above. Hence, since $r - r_0 \geq r_0$ (as $r_0 < r/2$ by assumption), we get

$$\Delta^s_{p,+}\tilde{w}(\bar{x}) \geq \Delta^s_{p,+}\hat{u}(\bar{x}) - \frac{2(1-s)\delta}{\alpha_p}\int_{\mathbb{R}^N\setminus B_{r-r_0}} \frac{1}{|y|^{N+2s}}\,dy \geq a - \frac{(1-s)\delta_0|\partial B_1|}{\alpha_p s(r-r_0)^{2s}} \geq 0,$$

concluding the proof. □

Theorem 4.19. *Let $\Omega \subset \mathbb{R}^N$ be a bounded subset satisfying the exterior sphere condition and $f : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ a uniformly continuous and bounded function. Furthermore assume $f \in L^1(\mathbb{R}^N \setminus \Omega; s)$. Then there exists a solution for the Dirichlet problem (17). Moreover any solution is uniformly continuous up to the boundary of Ω, with a modulus of continuity independent of s.*

Proof. Similar to the proof of Lemma 4.15, for each $x \in \partial\Omega$ let ν_x be the unit outward normal and let ρ_x be the modulus of continuity for f. Then, for any $R \in (0,R_0)$ (R_0 being the radius from the exterior ball condition) we define the barrier functions

$$b^-_{x,R}(y) := f(x) - \rho(3R) - \|f\|_{L^\infty(\mathbb{R}^N\setminus\Omega)}\phi\left(\frac{y-x}{R} - \nu_x\right),$$

$$b^+_{x,R}(y) := f(x) + \rho(3R) + \|f\|_{L^\infty(\mathbb{R}^N\setminus\Omega)}\phi\left(\frac{y-x}{R} - \nu_x\right).$$

In this definition ϕ is the function given by Lemma 4.14. Set

$$\mathcal{F} := \{u : \Delta^s_{p,+}u(y) \geq 0 \ \ \forall y \in \Omega,\, u(y) = f(y) \ \ \forall y \in \mathbb{R}^N \setminus \Omega\}.$$

As in Lemma 4.15 the functions $b^-_{x,R}$ are subsolutions, so Lemma 4.9 implies

$$w_-(y) = \begin{cases} \sup_{x\in\partial\Omega}\sup_{R\in(0,R_0)} b^-_{x,R}(y) & \text{if} \quad y \in \Omega, \\ f(y) & \text{if} \quad y \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

is a subsolution, and the set $\mathcal{F}$ is nonempty. Moreover, by Theorem 4.12 applied with $v = b^+_{x,R}$ (see Lemma 4.14), each $u \in \mathcal{F}$ is pointwise bounded by $b^+_{x,R}$ for any $x \in \partial\Omega$. So, if we define

$$w_+(y) = \begin{cases} \inf_{x\in\partial\Omega}\inf_{R\in(0,R_0)} b^+_{x,R}(y) & \text{if} \quad y \in \Omega, \\ f(y) & \text{if} \quad y \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

we find

$$w(x) := \sup_{u\in\mathcal{F}} u(x)$$

satisfies $w^- \leq w \leq w^+$. In particular, $w = f$ on $\mathbb{R}^N \setminus \Omega$ and attains continuously its boundary value.

To conclude the proof one would like to use Corollary 4.9 to deduce that w is a subsolution, and Lemma 4.18 to get that it is a supersolution. The only point where one needs to pay some attention is that a priori w is not continuous, while the definition of viscosity solution requires continuity (see Definition 3.3). This can be addressed however in a rather standard way: since we have already a uniform growth near the boundary, one observe that w may be also defined as

$$w(x) := \sup_{u\in\mathcal{F},\, w^-\leq u\leq w^+} R_u(x),$$

where

$$R_u(x) := \sup_{z\in\mathbb{R}^N} u(x+z) - \tilde\rho(z).$$

Here $\tilde\rho$ is a modulus of continuity which is weak enough so that $R_u = f$ outside Ω, and moreover so that the sup in the definition of R_u in attained at $x + z \in \Omega$ when $x \in \Omega$ (this can be done since, by assumption, $w^- \leq u \leq w^+$).

In this way, since $u(\cdot+z)$ is a subsolution whenever $x+z \in \Omega$, R_u is a subsolution by Corollary 4.9. Moreover, all functions $\{R_u\}_{u\in\mathcal{F},\, w^-\leq u\leq w^+}$ have $\tilde\rho$ as a uniform modulus of continuity. This implies that w is (uniformly) continuous, and so it is a viscosity solution.

Finally, to show that *any* viscosity solution of (17) is uniformly continuous up to the boundary of Ω, it suffices to apply Lemma 4.15 and Remark 4.17. $\square$

4.3.5. *Regularity up to the boundary, compactness and stability.* By a simple approximation argument and using Corollary 4.16, Theorem 4.19 can be extended to boundary data which are not necessarily bounded (we leave the details of the proof to the interested reader):

Proposition 4.20. *Let $s \in (s_0, 1)$ and let us solve (17) with $f : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ a uniformly continuous function satisfying the growth estimate $|f(x)| \leq M(1 + |x|)^\alpha$ for some $\alpha \in (0, 2s_0)$. Then there is a modulus of continuity $\tilde\rho$, independent of s, such that*

$$|u(z) - u(x)| \leq \tilde\rho(|x - z|) \qquad \forall\, z \in \Omega,\, x \in \mathbb{R}^N, .$$

We have the following corollary:

Corollary 4.21. *Let $\Omega \subset \mathbb{R}^N$ be a bounded subset satisfying the exterior sphere condition and $f : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ a uniformly continuous function satisfying the growth estimate $|f(x)| \leq M(1 + |x|)^\alpha$ for some $\alpha \in (0, 2s_0)$. Let $\{s_n\}_{n\in\mathcal{N}} \subset (s_0, 1)$, and u_n be solutions of (17) with $s = s_n$. Then $\{u_n\}_{n\in\mathcal{N}}$ is uniformly bounded and equicontinuous on Ω.*

In particular, the uniqueness of solutions for the classical p-Laplacian gives the following stability result as $s \to 1$:

Proposition 4.22. *Let $\Omega \subset \mathbb{R}^N$ be a bounded subset satisfying the exterior sphere condition and $f : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ a uniformly continuous function satisfying the growth estimate $|f(x)| \leq M(1+|x|)^\alpha$ for some $\alpha \in (0, 2s_0)$. Let $\{s_n\}_{n\in\mathcal{N}} \subset (s_0, 1)$ be a sequence such that $s_n \to 1$ and u_n be solutions of (17) with $s = s_n$. Then u_n converges uniformly to u_0, the unique solution of (17) for $s = 1$.*

Proof. Combining Corollary 4.21 with Arzelà-Ascoli Theorem gives, for any subsequence u_{n_j}, a limit u_0 so that $u_{n_j} \to u_0$ uniformly on Ω. Theorem 4.6 implies u_0 solves (17) with $s = 1$, so it is unique and we conclude the whole sequence u_n converges to u_0. □

Since the moduli of continuity of the functions u_n inside $\overline{\Omega}$ depend only on the modulus of continuity of f, M, and α (see Proposition 4.20), the above result can be restated as follows:

Proposition 4.23. *Let $\Omega \subset \mathbb{R}^N$ be a bounded subset satisfying the exterior sphere condition and $f : \mathbb{R}^N \to \mathbb{R}$ a uniformly continuous function satisfying the growth estimate $|f(x)| \leq M(1+|x|)^\alpha$ for some $\alpha \in (0, 2s_0)$. Let v be the corresponding solution of (17) with $s = 1$. Given $\epsilon > 0$ there is $s_1 \in (s_0, 1)$ such that if $s \in (s_1, 1)$ and u solves (17) with s, then*

$$\sup_{\Omega} |u - v| < \epsilon.$$

The constant s_1 depends only on the modulus of continuity of f, M, α and ϵ.

4.4. $C^{1,\alpha}$ **regularity of the non-local p-Laplacian.** In Proposition 4.22 we showed that solutions of our non-local p-Laplacian converge to a solution of the classical p-Laplacian when $s \to 1$. Moreover, by Theorem 3.15 we also have C^α bounds which are independent of s. Finally, we recall that solutions of the classical p-Laplacian are $C^{1,\alpha}$. We will now demonstrate that these ingredients imply that, for s sufficiently close to 1, solutions of our non-local p-Laplacian are $C^{1,\alpha}$ as well

4.4.1. *Subtracting Linear functions from solutions.* To prove the $C^{1,\alpha}$ regularity we will follow the argument established in [5]. We will measure the difference between a solution of (17) for s close to 1 and a sequence of affine functions. The affine functions will be the linear part of solutions of $\Delta_p h = 0$, with $s = 1$, at different scales. One obstacle to this approach is that the operators Δ_p^s depend on the gradient, so the difference of a solution and an affine function is not a solution. To handle this problem we introduce modified versions of Δ_p^s which also depend upon a vector $b \in \mathbb{R}^N$.

We define

$$I_p^s(u, b, x) := \Delta_p^s(u - b \cdot x).$$

Solutions are interpreted analogously to Definition 4.1.

The work in this section can be extended to I^s. Specifically we will use analogues of Proposition 4.23 and Theorem 4.2 which we record here:

Theorem 4.24. *Given a continuous function $f : \partial B_1 \to \mathbb{R}$ and $b \in \mathbb{R}^N$, the Dirichlet problem*

$$\begin{cases} I_p^1(u, b, x) &= 0 \quad \text{if} \quad x \in B_1, \\ \quad u(x) &= f(x) \quad \text{if} \quad x \in \partial B_1. \end{cases} \tag{24}$$

has a unique solution in the sense of Definition 4.1. Moreover, $u \in C^{1,\bar{\alpha}}(B_{1/2})$, with

$$\|u\|_{C^{1,\bar{\alpha}}(B_{1/2})} \leq C(N,p)(1 + \|f\|_{L^\infty(\partial B_1)},$$

and $\bar\alpha = \bar\alpha(N,p) > 0$ as in Theorem 4.2.

Proof. The non-trivial fact in the above result is that the bound on the $C^{1,\bar\alpha}$ norm of u is independent of b.

To show this we observe that if b belongs to a bounded sets (say $|b| \le R$ for some uniform constant $R = R(N,p) > 0$), then the uniform $C^{1,\alpha}$ estimate follows from Theorem 4.3 since solutions of (24) will also satisfy $\Delta_p u = 0$ in B_1 and $u = f + b\cdot x$ on ∂B_1. On the other hand, if $|b| \ge R(N,p)$ and $R(N,p)$ is sufficiently large, then the operator I^1_p becomes uniformly close to the second order constant coefficient operator $\triangle u + (p-2)\partial^2_{\hat b} u$ ($\hat b := b/|b|$), for which uniform (with respect to b) interior C^2-estimates hold. Arguing as in [16] (see in particular Lemmata 2.2, 2.3 and 3.2), a compactness argument completes the proof. □

We now get an analogous of Proposition 4.23:

Lemma 4.25. *Let $\Omega \subset \mathbb{R}^N$ be a bounded subset satisfying the exterior sphere condition and $f : \mathbb{R}^N\setminus\Omega \to \mathbb{R}$ a uniformly continuous function satisfying the growth estimate $|f(x)| \le M(1+|x|)^\alpha$ for some $\alpha \in (0, 2s_0)$. Let v solve (24) for some $b \in \mathbb{R}^N$. Given $\epsilon > 0$ there is an $s_1 \in (s_0, 1)$ such that if $s \in (s_1, 1)$ and u solves*

$$\begin{cases} I^s_p(u,b,x) &= 0 \quad \text{if} \quad x \in B_1, \\ \quad u(x) &= f(x) \quad \text{if} \quad x \in \partial B_1, \end{cases} \tag{25}$$

then

$$\sup_\Omega |u - v| < \epsilon.$$

The constant s_1 depends only on the modulus of continuity of f, M, α and ϵ.

Proof. With respect to Proposition 4.23, we need to check that s_1 does not depend on b. This follows from the fact that the family of operators $I^s_p(\cdot, b, \cdot)$, $b \in \mathbb{R}^N$, satisfies the assumption of the previous sections, uniformly with respect to b. Indeed the precence of b only modifies the domain of integration in the definition of $\Delta^s_{p,\pm}$, not the "size" of the operator: for instance, if $\nabla u(x) + b \ne 0$ then

$$I^s_p(u,b,x) = \frac{(1-s)}{\alpha_p}\int_{\mathbb{R}^N} \frac{[u(x+y)+u(x-y)-2u(x)]\,\mathbb{1}_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi\right)}{|y|^{N+2s}}\,dy,$$

with $\xi = \frac{\nabla u(x)+b}{|\nabla u(x)+b|}$. Thus, by a covering argument the Hölder estimate from Theorem 3.15 (suitably rescaled) implies uniform continuity of u in the interior of Ω, independently of b and $s \in (s_0, 1)$. Then the same argument leading to Proposition 4.23 gives the desired result. □

We are now ready to prove $C^{1,\alpha}$ regularity when s is close to 1.

Theorem 4.26. *Let $\bar\alpha$ be the exponent in Lemma 4.24. Let $\Omega \subset \mathbb{R}^N$ be a bounded subset satisfying the exterior sphere condition and $f : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ a uniformly continuous function satisfying the growth estimate $|f(x)| \le M(1+|x|)^\alpha$ for some $\alpha \in (0, 2s_0)$. There is an $s_1 \in (s_0, 1)$ such that if $s \in (s_1, 1)$ and u is a solution of (17), then $u \in C^{1,\beta}_{loc}(\Omega)$ for any $\beta < \min\{\bar\alpha, 2s_1 - 1\}$.*

Proof. Since the argument is standard, we only sketch the proof, referring the reader to [5, Theorem 5.2] for more details.

Without loss of generality we can assume that $0 \in \Omega$, and we prove that u is $C^{1,\alpha}$ at the origin. Then the result follows by standard arguments.

The proof of this theorem follows the inductive argument used to prove [5, Theorem 5.2]. Set $l_0 = 0$ and fix $\lambda > 0$ which we will be picked small to finish the inductive argument. Fix $0 < \beta < \beta_1 < \min(\bar{\alpha}, 2s_1 - 1)$, and assume we are given $l_k(x) = a_k + b_k \cdot x$ satisfying

$$\sup_{B_{\lambda^k}} |u - l_k| \leq \lambda^{k(1+\beta)}, \tag{26}$$

$$|a_{k+1} - a_k| \leq \lambda^{k(1+\beta)}, \tag{27}$$

$$|b_{k+1} - b_k| \leq C_2 \lambda^{k\beta}, \tag{28}$$

$$|u(x) - l_k(x)| \leq |x|^{1+\beta_1} \qquad \forall\, x \in \mathbb{R}^N \setminus B_{\lambda^k}. \tag{29}$$

After scaling the original equation we can assume these assumptions hold for l_0. Define

$$w_k(x) = \frac{[u - l_k](\lambda^k x)}{\lambda^{k(1+\beta)}}.$$

Consider the Dirichlet problem

$$\begin{cases} I^{\bar{s}}(u, b_k \lambda^{-k\beta}, x) & = 0 \qquad \text{if} \quad x \in B_1, \\ u(x) & = w_k(x) \quad \text{if} \quad x \in \mathbb{R}^N \setminus B_1. \end{cases} \tag{30}$$

The function w_k is a solution of (30) with $\bar{s} = s$. Let h be the solution of (30) for $\bar{s} = 1$. Lemma 4.25 implies there is a s_1 such that if $s \in (s_1, 1)$ then $|w_k - h| < \lambda^{1+\beta}$. This choice of s_1 depends only on λ but not on k. Theorem 4.24 implies h is $C^{1,\bar{\alpha}}(B_{1/2})$ with a uniform a priori estimate (since $|w_k| \leq 1$ on ∂B_1). The argument now follows directly the proof of [5, Theorem 5.2]: if $\bar{l} = \nabla h(0)$ then

$$l_{k+1}(x) = l_k(x) + \lambda^{k(1+\beta)} \bar{l} \left(\frac{x}{\lambda^k} \right)$$

satisfies (26)-(29) for a small enough λ. This implies u is $C^{1,\beta}$ at the origin, completing the proof. □

5. Kernels with lower order gradient dependence.

In this section we consider the following non-local operator:

$$Iu(x) := -(-\Delta)^s u(x) + \sup_{\xi_1 \in S^{N-1}} \left(\int_{\mathbb{R}^N} \frac{u(x+y) - u(x)}{|y|^{N+2s}} \mathcal{K}_{\xi_1}(y)\, dy \right)$$
$$+ \inf_{\xi_2 \in S^{N-1}} \left(\int_{\mathbb{R}^N} \frac{u(x-y) - u(x)}{|y|^{N+2s}} \mathcal{K}_{\xi_2}(y)\, dy \right), \tag{31}$$

$$\mathcal{K}_\xi(y) := (|y| \wedge 1) \psi \left(\frac{y}{|y|} \cdot \xi \right). \tag{32}$$

Here $\psi : [-1, 1] \to \mathbb{R}$ is some bounded non-negative function, and the operator $(-\Delta)^s$ is defined by

$$-(-\Delta)^s u(x) = 2N(1-s) \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}}\, ds, \tag{33}$$

(the constant $2N(1-s)$ ensures that $-(-\Delta)^s \to \Delta$ as $s \to 1$).

The operator (32) is exactly the one described by (3) and satisfies Assumptions 2.1-2.2. Throughout we assume for convenience that $|\psi| \leq 1$, but this assumption does not impact the following arguments in any substantial way.

5.1. $C^{1,\alpha}$ **Regularity.** We can deduce $C^{1,\alpha}$ regularity for solutions related to the operator (3) using the regularity results established in [4] and [5]. These results rely on the following notion of uniform ellipticity for non-local operators.

Definition 5.1. *Given a family of kernels $\{K_{\xi_1,\xi_2}\}$, a nonlocal operator $\tilde{I}$ defined by*

$$\tilde{I}u(x) := \inf_{\xi_1} \sup_{\xi_2} \int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x)) K_{\xi_1,\xi_2}(y)\, dy$$

is said to be uniformly elliptic if there exist $\lambda, \Lambda > 0$ such that

$$(1-s)\frac{\lambda}{|y|^{N+2s}} \le K_{\xi_1,\xi_2}(y) \le (1-s)\frac{\Lambda}{|y|^{N+2s}}.$$

Definition 5.1 is a consequence of [4, Definition 3.1] applied to the class of operators in [4, Equation (1.4)]. Indeed, as shown in [4, Lemmas 3.2 and 4.2], under the above assumption $\tilde{I}u(x)$ is well defined for $u \in C^{1,1}(x) \cap L^1(\mathbb{R}; s)$, and moreover $\tilde{I}u(x)$ is continuous in Ω (as a function of x) whenever $u \in C^2(\Omega) \cap L^1(\mathbb{R}; s)$.

For $\epsilon \in (0,1)$ define

$$I^{(1,\epsilon)}u(x) := -(-\Delta)^s u(x) + \sup_{\xi \in S^{N-1}} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} \mathcal{K}_\xi(y) \rho_\epsilon(y)\, dy, \tag{34}$$

$$I^{(2,\epsilon)}u(x) := -(-\Delta)^s u(x) + \inf_{\xi \in S^{N-1}} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} \mathcal{K}_\xi(y) \rho_\epsilon(y)\, dy. \tag{35}$$

Here, $\rho_\epsilon(y) := \rho_0(y/\epsilon)$, with ρ_0 a smooth cutoff function equal to 1 when $|y| < 1/2$ and supported in B_1, and $\mathcal{K}_\xi$ is as in (32). In Lemma 5.7 we will see the relation between the operator I defined in (31) and suitable rescaling on the above operators.

The operators $I^{(1,\epsilon)}$ and $I^{(2,\epsilon)}$ should be thought of as uniformly elliptic (in the sense of Definition 5.1) perturbations of $(-\Delta)^s$ and we will show they are close to the s-fractional Laplacian on every scale. This is measured using the following norms.

Definition 5.2. *Given a nonlocal operator $\tilde{I}$ as in Definition 5.1, we define its norm $\|\tilde{I}\|$ inside a domain Ω by*

$$\begin{aligned}\|\tilde{I}\| := \sup\{|\tilde{I}u(x)|/(1+M) : x \in \Omega,\ & \\ & u \in C^{1,1}(x), \\ & |u(y) - u(x) - (y-x)\cdot \nabla u(x)| \le M|x-y|^2 \quad \forall\, y \in B_1(x), \qquad (36) \\ & \|u\|_{L^1(\mathbb{R}^N;s)} \le M.\} \qquad (37)\end{aligned}$$

Definition 5.3. *Given $s \in (0,1)$ and an operator $\tilde{I}$, we define the rescaled operator*

$$\tilde{I}_\lambda w(x) := \lambda^{2s} \tilde{I} w_\lambda(\lambda x), \qquad w_\lambda(x) := w(x/\lambda).$$

The associated norm reflecting this scaling is defined

$$\|\tilde{I}^{(1)} - \tilde{I}^{(2)}\|_{sc} := \sup_{\lambda < 1} \|\tilde{I}^{(1)}_\lambda - \tilde{I}^{(2)}_\lambda\|.$$

We make use of the following regularity by perturbation from [5, Theorem 5.2].

Theorem 5.4. *(Caffarelli-Silvestre) Fix $s_0 \in (\frac{1}{2}, 1)$ and let $s \in (s_0, 1)$. Let $I^{(1)}$ and $I^{(2)}$ be two nonlocal operators as in Definition 5.1, satisfying*

$$\| - (-\Delta)^s - I^{(i)}\|_{sc} \le \eta$$

for some $\eta > 0$ *and* $i = 1, 2$.

Moreover, let u *be a bounded function satisfying*

$$\begin{cases} I^{(1)}u(x) & \geq -\eta \quad in \quad B_1, \\ I^{(2)}u(x) & \leq \eta \quad \ \ in \quad B_1. \end{cases}$$

Then there exists $\eta_0 > 0$, *independent of* $I^{(i)}$ *and* u, *such that if* $\eta \in (0, \eta_0]$ *then* $u \in C^{1,\alpha}(B_{1/2})$ *for any* $\alpha < 2s - 1$, *and*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C(\|u\|_{L^\infty(\mathbb{R}^N)} + \eta).$$

The constants depend on s_0, λ, Λ, *and* α, *but not on* s.

Remark 5.5. *The statement in* [5, Theorem 5.2] *is more general, as* $(-\Delta)^s$ *can be replaced with any translation invariant operator for which* $C^{1,\alpha}$ *estimates are known.*

Lemma 5.6. *Given any* $\eta > 0$ *there is an* $\epsilon_0 \in (0, 1)$ *such that for every* $\epsilon \in (0, \epsilon_0]$ *the operators (34)-(35) satisfy*

$$\|(-\Delta)^s - I^{(i,\epsilon)}\|_{sc} \leq \eta$$

for $i = 1, 2$.

Proof. Starting with a change of variables we find

$$\begin{aligned} -(-\Delta)^s_\lambda w(x) &- I^{(1,\epsilon)}_\lambda w(x) \\ &= \lambda^{2s} \sup_{\xi \in S^{N-1}} \int_{\mathbb{R}^N} \frac{w(x + y/\lambda) + w(x - y/\lambda) - 2w(x)}{|y|^{N+2s}} \mathcal{K}_\xi(y)\rho_\epsilon(y)\, dy \\ &= \sup_{\xi \in S^{N-1}} \int_{\mathbb{R}^N} \frac{w(x + y) + w(x - y) - 2w(x)}{|y|^{N+2s}} \mathcal{K}_\xi(\lambda y)\rho_\epsilon(\lambda y)\, dy. \end{aligned}$$

Then $\mathcal{K}_\xi(\lambda y)\rho_\epsilon(\lambda y) \leq |\lambda y|$ implies

$$| - (-\Delta)^s_\lambda w(x) - I^{(1,\epsilon)}_\lambda w(x)| \leq \lambda \int_{B_{\epsilon/\lambda}} \frac{|w(x + y) + w(x - y) - 2w(x)|}{|y|^{N+2s-1}}\, dy.$$

We proceed in two cases: $\lambda < \epsilon$ and $\lambda \geq \epsilon$. In the case $\lambda \geq \epsilon$ we apply (36) to see

$$\int_{B_{\epsilon/\lambda}} \frac{|w(x + y) + w(x - y) - 2w(x)|}{|y|^{N+2s-1}}\, dy \leq \frac{M}{3 - 2s} \left(\frac{\epsilon}{\lambda}\right)^{1+2(1-s)}.$$

If $\lambda < \epsilon$, we observe that (36)-(37) imply the bound $|w| < CM$ inside the bounded set Ω. (In this estimate C is independent of s but does depend on the size of Ω.)

Now, inside B_1 we apply (36), while in $B_{\epsilon/\lambda} \setminus B_1$ we use the inequality

$$\frac{1}{|y|^{N+2s-1}} \leq \left(\frac{\epsilon}{\lambda}\right) \frac{2}{1 + |y|^{N+2s-1}}$$

together with (37) and the bound $|w(x)| \leq CM$. Thus,

$$\begin{aligned}\int_{B_{\epsilon/\lambda}} \frac{|w(x+y)+w(x-y)-2w(x)|}{|y|^{N+2s-1}}\,dy &\leq \int_{B_1} \frac{|w(x+y)+w(x-y)-2w(x)|}{|y|^{N+2s-1}}\,dy \\ &\quad + \int_{B_{\epsilon/\lambda}\setminus B_1} \frac{|w(x+y)+w(x-y)-2w(x)|}{|y|^{N+2s-1}}\,dy \\ &\leq \frac{M}{3-2s} + CM\left(\frac{\epsilon}{\lambda}\right) \\ &\leq C\frac{M}{3-2s}\left(\frac{\epsilon}{\lambda}\right).\end{aligned}$$

All together we have

$$\left|-(-\Delta)^s_\lambda w(x) - I^{(1,\epsilon)}_\lambda w(x)\right| \leq C\frac{M}{3-2s}\epsilon \leq CM\epsilon,$$

which concludes the proof by taking ϵ sufficiently small. □

We now relate the operator I in (31) to the operators $I^{(1,\epsilon)}$ and $I^{(2,\epsilon)}$ defined in (34) and (35):

Lemma 5.7. *Given $r > 0$, let u be a bounded function satisfying $\|u\|_{L^1(\mathbb{R}^N;s)} < M$ and*

$$Iu = f(x) \qquad in B_r$$

for some bounded functions f Then, given any $\eta > 0$ and $\epsilon \in (0,1)$, there exists $\rho_0 \in (0,1)$ (depending only on M, $\|f\|_{L^\infty(B_1)}$, η, and ϵ) such that if $\rho \in (0,\rho_0]$, then $w(x) := u(\rho x)$ satisfies

$$\begin{cases} I^{(1,\epsilon)}_\rho w & \geq -\eta \quad in \quad B_{r/\rho}, \\ I^{(2,\epsilon)}_\rho w & \leq \eta \quad\ \ in \quad B_{r/\rho}, \end{cases}$$

Proof. We first observe that $Iu = f$ implies that

$$-(-\Delta)^s u(x) + \sup_{\xi\in S^{N-1}} \int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}} \mathcal{K}_\xi(y)\,dy \geq f(x)$$

in the viscosity sense. Recalling the definition of $I^{(1,\epsilon)}$ (see (34)), we get

$$\begin{aligned} I^{(1,\epsilon)}u &\geq f(x) - \sup_{\xi\in S^{N-1}} \int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}} \mathcal{K}_\xi(y)[1-\rho_\epsilon(y)]\,dy && (38) \\ &\geq f(x) - C(M,\epsilon), && (39)\end{aligned}$$

where in the last inequality we used that $1-\rho_\epsilon(y) = 0$ inside $B_{\epsilon/2}$. The result now follows from the observation that

$$I^{(1,\epsilon)}_\rho w(x) = \rho^{2s}\big(I^{(1,\epsilon)}u\big)(\rho x) \geq -\rho^{2s}\big(\|f\|_\infty + C(M,\epsilon)\big),$$

by choosing $\rho > 0$ small enough. (The case of $I^{(2,\epsilon)}$ is completely analogous.) □

Theorem 5.8. *Let u satisfy the same hypothesis as Lemma 5.7. Then $u \in C^{1,\alpha}(B_{r/2})$ for any $\alpha < 2s-1$.*

Proof. Let us observe that if $I^{(i)}$ are nonlocal operators satisfying the assumptions of Theorem 5.4, then also the kernels $I^{(i)}_\rho$ satisfy the same assumptions for any $\rho \leq 1$, uniformly with respect to ρ: indeed

$$\|(-\Delta)^s - I^{(i)}_\rho\|_{sc} \leq \|(-\Delta)^s - I^{(i)}\|_{sc} \qquad \forall\, \rho \leq 1,$$

and $I_\rho^{(i)}$ satisfy the assumptions from Definition 5.1 with the same constants λ, Λ.

Hence we first choose η_0 as in Theorem 5.4 with $\lambda = 1$ and $\Lambda = 2$ (recall that $|\psi| \le 1$), then we apply Lemma 5.6 to find ϵ_0 so that

$$\|(-\Delta)^s - I^{(1,\epsilon_0)}\|_{sc} \le \eta_0, \qquad \|(-\Delta)^s - I^{(2,\epsilon_0)}\|_{sc} \le \eta_0,$$

and we finally use Lemma 5.7 to find $\rho_0 \in (0,1)$ so that $w(x) := u(\rho_0 x)$ satisfies

$$\begin{cases} I_{\rho_0}^{(1,\epsilon_0)} w & \ge -\eta_0 \quad \text{in} \quad B_{r/\rho_0}, \\ I_{\rho_0}^{(2,\epsilon_0)} w & \le \eta_0 \quad \text{in} \quad B_{r/\rho_0}. \end{cases}$$

We then conclude from Theorem 5.4. □

5.2. $C^{2,\alpha}$ **Regularity.** Here we prove $C^{2,\alpha}$ regularity for solutions of $Iu = 0$, with I as in (31)-(32) (the same result would be true for solutions of $Iu = f$ with f Lipschitz).

To establish this higher regularity, we examine the equation solved by the difference quotient

$$w_h(x) := \frac{u(x+h) - u(x)}{|h|^\beta}.$$

We observe that, since u solves $Iu(x) = 0$, then

$$\begin{aligned}(1-s) \inf_{\xi \in S^{N-1}} \int_{\mathbb{R}^N} & \frac{w_h(x+y) - w_h(x)}{|y|^{N+2s}} \mathcal{K}_\xi(y)\, dy \\ & \le -(-\Delta)^s w_h(x) \\ & \le (1-s) \sup_{\xi \in S^{N-1}} \int_{\mathbb{R}^N} \frac{w_h(x+y) - w_h(x)}{|y|^{N+2s}} \mathcal{K}_\xi(y)\, dy. \qquad (40)\end{aligned}$$

The idea for proving $C^{2,\alpha}$ is that the decay of $\mathcal{K}_\xi$ near the origin implies that both integrals are finite, giving control over $(-\Delta)^s w^h$. More precisely, let us recall that by Theorem 5.8 we already know that solutions are $C^{1,\alpha}$ for any $\alpha < 2s - 1$.

Now, our goal is to show that, if $u \in C^{1,\beta}$ for some $\beta > 0$, then the control above yields $C^{1,\alpha+\beta}$ regularity for u. Iterating this result finitely many times, this will imply $C^{2,\alpha}$ regularity for u.

To begin we localize our considerations. Assume $C^{1,\beta}$ regularity for u in some ball B_r. Take $\delta \ll r$, and let $\rho_\delta(y)$ be a smooth cutoff function equal to 1 in $B_{r-\frac{\delta}{4}}$ and equal to zero outside of B_r. Define

$$\begin{aligned} u^1(x) &= \frac{u(x+h)\rho_\delta(x+h) - u(x)\rho_\delta(x)}{|h|^\beta}, \\ u^2(x) &= \frac{u(x+h)(1-\rho_\delta(x+h)) - u(x)(1-\rho_\delta(x))}{|h|^\beta}. \end{aligned}$$

We will now argue $C^{1,\alpha}$ regularity for u^1.

A consequence of the $C^{1,\beta}$ regularity for u is that u^1 is uniformly Lipschitz. Also, $u^2 = 0$ in $B_{r-\frac{\delta}{4}}$. Hence, using the equation $Iu(x) = 0$,

$$\begin{aligned}\left|-(-\Delta)^s u^1(x)\right| \le (1-s) \int_{\mathbb{R}^N} \frac{|u^1(x+y) - u^1(x)|}{|y|^{N+2s-1}}\, dy \\ + (1-s) \sup_{\xi \in S^{N-1}} \left| \int_{\mathbb{R}^N} \frac{w^2(x+y) - w^2(x)}{|y|^{N+2s}} \big(2N + \mathcal{K}_\xi(y)\big)\, dy \right|. \\ =: A_1 + A_2.\end{aligned}$$

Relying on the uniform Lipschitz bound for u^1, A_1 is bounded.

For A_2, we notice that $u^2(x) = 0$ for $x \in B_{r-\delta/2}$ and h small enough. Moreover, $1-\rho_\delta(x+y) = 0$ for $y \in B_{\delta/4}$. Hence, denoting $K_\xi(y) := \frac{2N+\mathcal{K}_\xi(y)}{|y|^{N+2s}}$, and changing variables:

$$\begin{aligned}\left| \int_{\mathbb{R}^N} [u^2(x+y) - u^2(x)] K_\xi(y)\, dy \right| &= \left| \int_{\mathbb{R}^N} u^2(x+y) K_\xi(y)\, dy \right| \\ &= |h|^{-\beta} \left| \int_{\mathbb{R}^N} u(x+y)(1-\rho_\delta(x+y))[K_\xi(y) - K_\xi(y-h)]\, dy \right| \\ &\le \|u\|_\infty |h|^{1-\beta} \left| \int_{\mathbb{R}^N \setminus B_{\delta/4}} \frac{K_\xi(y) - K_\xi(y-h)}{h}\, dy \right|\end{aligned}$$

So, if we assume that

$$\sup_{\xi \in S^{n-1}} |\nabla_y \mathcal{K}_\xi(y)| \le \frac{C}{|y|}, \tag{41}$$

then

$$\sup_{h>0} \sup_{\xi \in S^{N-1}} \left| \frac{1}{|h|} \int_{\mathbb{R}^N \setminus B_{\delta/4}} \left(\frac{2N + \mathcal{K}_\xi(y)}{|y|^{N+2s}} - \frac{2N + \mathcal{K}_\xi(y-h)}{|y-h|^{N+2s}} \right) dy \right| < \infty.$$

and we deduce that $\left|(-\Delta)^s u^1\right|$ is bounded inside $x \in B_{r-\delta/2}$.

This implies that u^1 is $C^{1,\alpha}$, so $u \in C^{1,\alpha+\beta}$. Iterating this finitely many times, we obtain:

Theorem 5.9. *Let I be given by (31)-(32), and assume that (41) holds. Let $u \in L^\infty(\mathbb{R}^N)$ solve $Iu = 0$ in some ball B_r. Then $u \in C^{2,\alpha}(B_{r/2})$ for any $\alpha < 2s-1$.*

5.3. **Non-local p-Laplacian with lower order gradient dependence.** The goal of this subsection is to construct a family of operators which belong to the class (31)-(32), and which approaches the classical p-Laplacian as $s \to 1$. To accomplish this we use a coefficient which tends to infinity as $s \to 1$ with the lower order term.

$$\begin{aligned}I^s u(x) := -\frac{3}{4} (-\Delta)^s u(x) + \sup_{\xi_1 \in S^{N-1}} \left(\int_{\mathbb{R}^N} \frac{u(x+y) - u(x)}{|y|^{N+2s}} \mathcal{K}_{\xi_1}(y)\, dy \right) \\ + \inf_{\xi_2 \in S^{N-1}} \left(\int_{\mathbb{R}^N} \frac{u(x-y) - u(x)}{|y|^{N+2s}} \mathcal{K}_{\xi_2}(y)\, dy \right), \\ \mathcal{K}_\xi(y) := \frac{1-s}{2\alpha_p} \left(\frac{|y|}{\delta_s} \wedge 1 \right) \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \xi \right).\end{aligned}$$

In this subsection we define α_p and β_p as in (15)-(16), but choose c_p so that $\beta_p/(4\alpha_p) = p-2$. Such a choice is possible for any $p \in [2,\infty)$. Also, we choose $\delta_s := e^{-\frac{1}{(1-s)^2}}$ so that it pushes the lower order term into a second order term in the limit $s \to 1$. (Other choices of δ_s will work but it is important that $\delta_s^{(1-s)} \to 0$ as $s \to 1$.)

We remark that Theorem 3.15 applies to this class of operators: indeed, if we consider the expression

$$\sup_{\xi_1 \in S^{N-1}} \left(\int_{\mathbb{R}^N} \frac{u(x+y)-u(x)}{|y|^{N+2s}} \mathcal{K}_{\xi_1}(y)\, dy \right) + \inf_{\xi_2 \in S^{N-1}} \left(\int_{\mathbb{R}^N} \frac{u(x-y)-u(x)}{|y|^{N+2s}} \mathcal{K}_{\xi_2}(y)\, dy \right),$$

then since for any ξ_1 in the maximization problem one can choose $\xi_2 = \xi_1$, it follows that the above expression is bounded from above by

$$\sup_{\xi_1 \in S^{N-1}} \left(\int_{\mathbb{R}^N} \frac{u(x+y)-u(x)}{|y|^{N+2s}} \mathcal{K}_{\xi_1}(y)\, dy + \int_{\mathbb{R}^N} \frac{u(x-y)-u(x)}{|y|^{N+2s}} \mathcal{K}_{\xi_1}(y)\, dy \right).$$

Analogously, it is bounded from below by

$$\inf_{\xi_2 \in S^{N-1}} \left(\int_{\mathbb{R}^N} \frac{u(x+y)-u(x)}{|y|^{N+2s}} \mathcal{K}_{\xi_2}(y)\, dy + \int_{\mathbb{R}^N} \frac{u(x-y)-u(x)}{|y|^{N+2s}} \mathcal{K}_{\xi_2}(y)\, dy \right).$$

Hence, given $s_0 \in (0,1)$, Theorem 3.15 holds uniformly for $s \in [s_0, 1]$.

Moreover, Theorems 5.8 and 5.9 also all apply to this class of operators: more precisely, the $C^{1,\alpha}$ regularity is uniform for $s \in (1/2, 1]$ (as a consequence of Theorem 5.8, and a variant of Theorem 4.26 to deal with the case $s \to 1$ once the convergence result to Δ_p will be proved), but the $C^{2,\alpha}$ regularity degenerates as $s \to 1$.

The main goal of this subsection is to show how this operator approaches the classical p-Laplacian as $s \to 1$. Combining this convergence with the uniform $C^{1,\alpha}$ estimate and the uniqueness of solutions for the classical p-Laplacian, we may conclude that any sequence of solutions u_s of $I^s u_s = 0$ will converge in $C^{1,\alpha}$ to the solution of $\Delta_p u = 0$ as $s \to 1$. Hence, we regard I^s as a non-local regularization of the classical p-Laplacian.

Assume $u \in C^2 \cap L^1(\mathbb{R}^N; s)$ and set, as in Subsection 4.2, $\epsilon_s = (1-s)^{\frac{1}{2(N+2s)}}$. Observe that $\delta_s < \epsilon_s$ for s close to 1. Then

$$\begin{aligned}
&\frac{1-s}{2\alpha_p} \int_{\mathbb{R}^N} \frac{u(x+y)-u(x)}{|y|^{N+2s}} \left(\frac{|y|}{\delta_s} \wedge 1 \right) \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \xi \right) dy \\
&\quad = \frac{1-s}{2\alpha_p} \nabla u(x) \cdot \int_{B_{\epsilon_s}} \frac{y}{|y|^{N+2s}} \left(\frac{|y|}{\delta_s} \wedge 1 \right) \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \xi \right) dy \\
&\qquad + \frac{1-s}{2\alpha_p} \frac{1}{2} D^2 u(x) : \int_{B_{\epsilon_s}} \frac{y \otimes y}{|y|^{N+2s}} \left(\frac{|y|}{\delta_s} \wedge 1 \right) \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \xi \right) dy \\
&\qquad\quad + \frac{1-s}{2\alpha_p} \int_{\mathbb{R}^N \setminus B_{\epsilon_s}} \frac{u(x+y)-u(x)}{|y|^{N+2s}} \left(\frac{|y|}{\delta_s} \wedge 1 \right) \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \xi \right) dy \\
&\qquad\qquad + \frac{1-s}{2\alpha_p} \int_{B_{\epsilon_s}} \frac{o(|y|^2)}{|y|^{N+2s}} \left(\frac{|y|}{\delta_s} \wedge 1 \right) \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \xi \right) dy \\
&\quad = A_1 + A_2 + A_3 + A_4.
\end{aligned}$$

To start,

$$
\begin{aligned}
A_1 &= \frac{\gamma_p}{2\alpha_p} \nabla u(x) \cdot \xi \left(\frac{1}{2} \delta_s^{1-2s} + \frac{1-s}{1-2s} \left(\epsilon_s^{1-2s} - \delta_s^{1-2s} \right) \right), \qquad (42) \\
\gamma_p &:= \int_{\partial B_1} \omega \cdot e_1 \, \mathbb{1}_{[c_p,1]} \left(\omega \cdot e_1 \right) \, d\sigma(\omega).
\end{aligned}
$$

Arguing as in Subsection 4.2 one also finds:

$$
\begin{aligned}
A_2 &= \frac{1}{2} \left(\Delta u + \frac{\beta_p}{\alpha_p} \partial_\xi^2 u \right) \left(\frac{1-s}{3-2s} \delta_s^{2(1-s)} + \frac{1}{2} \left(\epsilon_s^{2(1-s)} - \delta_s^{2(1-s)} \right) \right), \\
|A_3| &\leq \frac{1-s}{\alpha_p} \frac{1}{\epsilon_s^{N+2s}} \int_{\mathbb{R}^N} \frac{|u(x+y) - u(x)|}{1 + |y|^{N+2s}} \, dy = o(1), \\
A_4 &= o(1).
\end{aligned}
$$

5.3.1. *Limit in the case* $\nabla u(x) \neq 0$. Notice A_2, A_3 and A_4 are uniformly bounded as $s \to 1$. Hence, since $\nabla u(x)/|\nabla u(x)|$ is an admissible choice in the supremum

$$
\sup_{\xi_1 \in S^{N-1}} \int_{\mathbb{R}^N} \frac{u(x+y) - u(x)}{|y|^{N+2s}} \left(\frac{|y|}{\delta_s} \wedge 1 \right) \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \xi_1 \right) dy,
$$

by (42) we have $\nabla u(x) \cdot \xi_1 \geq |\nabla u(x)| + O(\delta_s^{2s-1})$ where ξ_1 is the value which attains the supremum. Likewise, the ξ_2 which attains the infimum in

$$
\inf_{\xi_2 \in S^{N-1}} \int_{\mathbb{R}^N} \frac{u(x-y) - u(x)}{|y|^{N+2s}} \left(\frac{|y|}{\delta_s} \wedge 1 \right) \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \xi_2 \right) dy
$$

will satisfy $\nabla u(x) \cdot \xi_2 \geq |\nabla u(x)| + O(\delta_s^{2s-1})$. This implies $\xi_1, \xi_2 \to \frac{\nabla u(x)}{|\nabla u(x)|}$ as $s \to 0$. Since ξ_1 is an admissible choice in the infimum, for s close to 1 we have

$$
Iu(x) \leq -\frac{3}{4} (-\Delta)^s u(x) + \frac{1}{4} \left(\Delta u(x) + \frac{\beta_p}{\alpha_p} \partial_{\xi_1}^2 u(x) \right) \epsilon_s^{2(1-s)} + A_3 + A_4,
$$

so

$$
\lim_{s \to 1} Iu(x) \leq \Delta_p u(x).
$$

Similarly one can establish the opposite inequality and conclude

$$
\lim_{s \to 1} Iu(x) = \Delta_p u(x).
$$

5.3.2. *Limit in the case* $\nabla u(x) = 0$. In the $\nabla u(x) = 0$ case first note that the ξ_1 which attains the supremum in

$$
\sup_{\xi_1 \in S^{N-1}} \int_{\mathbb{R}^N} \frac{u(x+y) - u(x)}{|y|^{N+2s}} \left(\frac{|y|}{\delta_s} \wedge 1 \right) \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \xi_1 \right) dy
$$

is an admissible choice in the infimum of

$$
\inf_{\xi_2 \in S^{N-1}} \int_{\mathbb{R}^N} \frac{u(x-y) - u(x)}{|y|^{N+2s}} \left(\frac{|y|}{\delta_s} \wedge 1 \right) \mathbb{1}_{[c_p,1]} \left(\frac{y}{|y|} \cdot \xi_2 \right) dy
$$

so that

$$
\lim_{s \to 1} Iu(x) \leq \Delta u(x) + \frac{\beta_p}{4\alpha_p} \sup_{\xi_1 \in S^{N-1}} \partial_{\xi_1}^2 u(x) = \Delta_p u(x).
$$

Likewise

$$\lim_{s \to 1} Iu(x) \geq \Delta_p u(x),$$

concluding the proof of the convergence result.

References

[1] C. Bjorland, L. Caffarelli, and A. Figalli. Non-local tug-of-war and the infinity fractional Laplacian. *Comm. Pure Appl. Math.*, To appear.

[2] L. Caffarelli. Non local diffusions, drifts and games. Preprint.

[3] L. A. Caffarelli and X. Cabré. *Fully nonlinear elliptic equations*, volume 43 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1995.

[4] L. A. Caffarelli and L. Silvestre. Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math.*, 62(5):597–638, 2009.

[5] L. A. Caffarelli and L. Silvestre. Regularity results for nonlocal equations by approximation. *Arch. Rational Mech. Anal.*, 200(1):59, 2011.

[6] A. Chambolle, E. Lindgren, and Monneau R. The Hölder infinite Laplacian and Hölder extensions. *Preprint.*, 2010.

[7] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[8] E. DiBenedetto. $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. *Nonlinear Anal.*, 7(8):827–850, 1983.

[9] L. C. Evans and J. Spruck. Motion of level sets by mean curvature. I. *J. Differential Geom.*, 33(3):635–681, 1991.

[10] Lawrence C. Evans. A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e. *J. Differential Equations*, 45(3):356–373, 1982.

[11] Petri Juutinen, Peter Lindqvist, and Juan J. Manfredi. On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation. *SIAM J. Math. Anal.*, 33(3):699–717 (electronic), 2001.

[12] J. L. Lewis. Regularity of the derivatives of solutions to certain degenerate elliptic equations. *Indiana Univ. Math. J.*, 32(6):849–858, 1983.

[13] Y. Peres and S. Sheffield. Tug-of-war with noise: a game-theoretic view of the p-Laplacian. *Duke Math. J.*, 145(1):91–120, 2008.

[14] K. Uhlenbeck. Regularity for a class of non-linear elliptic systems. *Acta Math.*, 138(3-4):219–240, 1977.

[15] N. N. Ural′ceva. Degenerate quasilinear elliptic systems. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 7:184–222, 1968.

[16] L. Wang. Compactness methods for certain degenerate elliptic equations. *J. Differential Equations*, 107(2):341–350, 1994.