Some Basic Tools of Critical Point Theory

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Contents

Chapter I

Some basic tools of critical point theory

1 The deformation theorem

Throughout this section, we will consider a real Banach space X and a function $f : X \to \mathbb{R}$ of class C^1.

(1.1) Definition *We say that* $u \in X$ *is* a critical point *of* f, *if* $f'(u) = 0$. *We say that* $c \in \mathbb{R}$ *is* a critical value *of* f, *if there exists a critical point* u *of* f *with* $f(u) = c$. *We say that* $c \in \mathbb{R}$ *is* a regular value *of* f, *if it is not a critical value of* f.

(1.2) Definition *Let* $c \in \mathbb{R}$. *We say that* (u_h) *is* a Cerami-Palais-Smale sequence at level c *(*$(CPS)_c-$sequence, *for short) for* f, *if* $f(u_h) \to c$ *and* $(1 + \|u_h\|)f'(u_h) \to 0$.

We say that f *satisfies* the Cerami-Palais-Smale condition at level c *(*condition $(CPS)_c$, *for short), if every* $(CPS)_c-$*sequence for* f *admits a (strongly) convergent subsequence in* X.

(1.3) Remark *In the classical* Palais-Smale condition, *one considers sequences with* $f'(u_h) \to 0$ *instead of* $(1+\|u_h\|)f'(u_h) \to 0$. *This useful variant, which is clearly a weaker condition, was introduced by* Cerami *[4].*

For every $b \in \mathbb{R} \cup \{+\infty\}$ and $c \in \mathbb{R}$, we set

$$f^b := \{u \in X : f(u) \leq b\} ,$$

$$K_c := \{u \in X : f(u) = c, \, f'(u) = 0\} .$$

(1.4) Definition *Given* $u \in X$, *we say that* $v \in X$ *is* a pseudogradient vector *for* f *at* u, *if* $\|v\| \leq 2\|f'(u)\|$ *and* $\langle f'(u), v\rangle \geq \|f'(u)\|^2$.

We say that

$$V : \{u \in X : f'(u) \neq 0\} \longrightarrow X$$

is a pseudogradient vector field *for* f, *if* V *is locally Lipschitz and* $V(u)$ *is a pseudogradient vector for* f *at* u *for any* u *in the domain of* V.

(1.5) Remark *If* v *is a pseudogradient vector for* f *at* u, *we have*

$$\|f'(u)\|^2 \leq \langle f'(u), v\rangle \leq \|f'(u)\|\|v\| ,$$

hence $\|f'(u)\| \leq \|v\|$.

(1.6) Lemma *Let Y be a metric space, Z a normed space and for every $y \in Y$ let $\mathcal{F}(y)$ be a convex subset of Z. Assume that for every $y \in Y$ there exists a neighbourhood U of y such that*

$$\bigcap_{\xi \in U} \mathcal{F}(\xi) \neq \emptyset .$$

Then there exists a locally Lipschitz map $F : Y \to Z$ such that $F(y) \in \mathcal{F}(y)$ for every $y \in Y$.

Proof. For every $y \in Y$ let U_y be an open neighbourhood of y such that

$$\bigcap_{\xi \in U_y} \mathcal{F}(\xi) \neq \emptyset .$$

Since $\{U_y : y \in Y\}$ is an open cover of Y and Y is paracompact (see e.g. [8]), there exists a locally finite open cover $\{W_j : j \in J\}$ of Y refining $\{U_y : y \in Y\}$. Assume first that $W_j \neq Y$ for any $j \in J$. If we set

$$\psi_j(y) = d(y, Y \setminus W_j) , \qquad \Psi(y) = \sum_{j \in J} \psi_j(y) ,$$

then ψ_j is Lipschitz and Ψ is well defined and locally Lipschitz, as $\{W_j : j \in J\}$ is locally finite. Since $\{W_j : j \in J\}$ is an open cover, we also have $\Psi(y) \neq 0$ for every $y \in Y$. Therefore, if we set

$$\varphi_j(y) = \frac{\psi_j(y)}{\Psi(y)} ,$$

it turns out that $\{\varphi_j : j \in J\}$ is a locally Lipschitz partition of unity subordinated to $\{W_j : j \in J\}$. If there exists $j_0 \in J$ with $W_{j_0} = Y$, set $\varphi_{j_0} = 1$ and $\varphi_j = 0$ for $j \neq j_0$. Then also in this case $\{\varphi_j : j \in J\}$ is a locally Lipschitz partition of unity subordinated to $\{W_j : j \in J\}$.

Since $\{W_j : j \in J\}$ refines $\{U_y : y \in Y\}$, for every $j \in J$ we have

$$\bigcap_{y \in W_j} \mathcal{F}(y) \neq \emptyset .$$

If for every $j \in J$ we choose a $z_j \in \bigcap_{y \in W_j} \mathcal{F}(y)$, we can define a locally Lipschitz map $F : Y \to Z$ by

$$F(y) = \sum_{j \in J} \varphi_j(y)\, z_j .$$

Given $y \in Y$, there is only a finite number $W_{j_1}, \ldots, W_{j_n}$ of W_j's such that $y \in W_j$. Then

$$F(y) = \sum_{k=1}^{n} \varphi_{j_k}(y)\, z_{j_k} , \qquad \sum_{k=1}^{n} \varphi_{j_k}(y) = 1 .$$

For every $k = 1, \ldots, n$, from $y \in W_{j_k}$ it follows $z_{j_k} \in \mathcal{F}(y)$. Since $\mathcal{F}(y)$ is convex, we conclude that $F(y) \in \mathcal{F}(y)$. ■

(1.7) Theorem *There exists a pseudogradient vector field for f.*

Proof. Let

$$Y = \{u \in X : f'(u) \neq 0\} .$$

For every $u \in Y$, denote by $\mathcal{V}(u)$ the set of pseudogradient vectors for f at u. It is readily seen that $\mathcal{V}(u)$ is a convex subset of X. Moreover, for every $u \in Y$ there exists $w \in X$ such that $\|w\| \leq 1$ and $\langle f'(u), w\rangle \geq \frac{4}{5}\|f'(u)\|$. Then $v = \frac{5}{3}\|f'(u)\|\, w$ satisfies $\|v\| \leq \frac{5}{3}\|f'(u)\|$ and $\langle f'(u), v\rangle \geq \frac{4}{3}\|f'(u)\|^2$. Since f is of class C^1, there exists a neighbourhhood U of u such that $\|v\| < 2\,\|f'(\xi)\|$ and $\langle f'(\xi), v\rangle > \|f'(\xi)\|^2$ for every $\xi \in U$, so that

$$v \in \bigcap_{\xi \in U} \mathcal{V}(\xi)\,.$$

From Lemma (1.6) we deduce that there exists a locally Lipschitz map $V : Y \to X$ with $V(u) \in \mathcal{V}(u)$ and the assertion follows. ■

Now we can prove the main result of this section.

(1.8) Theorem (Deformation Theorem) *Let $c \in \mathbb{R}$ be such that f satisfies $(CPS)_c$. Then, for every $\overline{\varepsilon} > 0$, every neighbourhood U of K_c (if $K_c = \emptyset$, we allow $U = \emptyset$) and every $\lambda > 0$, there exist $\varepsilon \in]0, \overline{\varepsilon}[$ and a continuous map $\eta : X \times [0,1] \to X$ such that for every $(u,t) \in X \times [0,1]$ we have:*

(a) $\|\eta(u,t) - u\| \leq \lambda(1 + \|u\|)t$;

(b) $f(\eta(u,t)) \leq f(u)$;

(c) $\eta(u,t) \neq u \implies f(\eta(u,t)) < f(u)$;

(d) $|f(u) - c| \geq \overline{\varepsilon} \implies \eta(u,t) = u$;

(e) $\eta\left(f^{c+\varepsilon} \times \{1\}\right) \subseteq f^{c-\varepsilon} \cup U$.

Proof. From condition $(CPS)_c$ it easily follows that K_c is compact. Therefore there exists $\varrho > 0$ such that $\mathrm{B}_{3\varrho}(K_c) \subseteq U$.

We claim there exist $\hat{\varepsilon} \in \left]0, \frac{1}{2}\overline{\varepsilon}\right[$ and $\sigma > 0$ such that

$$c - 2\hat{\varepsilon} \leq f(u) \leq c + 2\hat{\varepsilon},\ u \notin \mathrm{B}_{\varrho}(K_c) \implies (1 + \|u\|)\|f'(u)\| \geq \sigma\,. \tag{1.9}$$

Actually, assume for a contradiction that (u_h) is a sequence in X with $f(u_h) \to c$, $u_h \notin \mathrm{B}_{\varrho}(K_c)$ and $(1 + \|u_h\|)\|f'(u_h)\| \to 0$. Then, up to a subsequence, (u_h) is convergent to some u with $f(u) = c$, $u \notin \mathrm{B}_{\varrho}(K_c)$ and $f'(u) = 0$, which is clearly impossible.

Let $\chi : X \to [0,1]$ be a locally Lipschitz function such that

$$\left(|f(u) - c| \geq 2\hat{\varepsilon} \text{ or } u \in \overline{\mathrm{B}_{\varrho}(K_c)}\right) \implies \chi(u) = 0\,,$$

$$\left(|f(u) - c| \leq \hat{\varepsilon} \text{ and } u \notin \mathrm{B}_{2\varrho}(K_c)\right) \implies \chi(u) = 1\,,$$

let $\mu > 0$ with

$$\exp\mu - 1 \leq \lambda$$

and let

$$W(u) = \begin{cases} \sigma\mu\chi(u)\,\dfrac{V(u)}{\|V(u)\|^2} & \text{if } |f(u) - c| \leq 2\hat{\varepsilon} \text{ and } u \notin \mathrm{B}_{\varrho}(K_c)\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where V is a pseudogradient vector field for f. Then $W : X \to X$ is locally Lipschitz. Moreover, if $|f(u) - c| \leq 2\hat{\varepsilon}$ and $u \notin \mathrm{B}_\varrho(K_c)$, we deduce from (1.9) and the definition of pseudogradient vector that

$$\|W(u)\| \leq \sigma\mu \frac{1}{\|V(u)\|} \leq \sigma\mu \frac{1}{\|f'(u)\|} \leq \mu(1 + \|u\|),$$

$$\langle f'(u), W(u)\rangle = -\sigma\mu\chi(u) \frac{\langle f'(u), V(u)\rangle}{\|V(u)\|^2} \leq -\sigma\mu\chi(u) \frac{\|f'(u)\|^2}{\|V(u)\|^2} \leq -\frac{1}{4}\sigma\mu\chi(u).$$

It follows

$$\forall u \in X : \|W(u)\| \leq \mu(1 + \|u\|), \tag{1.10}$$

$$\forall u \in X : \langle f'(u), W(u)\rangle \leq -\frac{1}{4}\sigma\mu\chi(u). \tag{1.11}$$

Therefore the Cauchy problem

$$\begin{cases} \dfrac{\partial\eta}{\partial t}(u,t) = W(\eta(u,t)) \\ \eta(u,0) = u \end{cases}$$

defines a continuous map $\eta : X \times \mathbb{R} \to X$ such that $\eta(u,t) = u$ whenever $|f(u) - c| \geq 2\hat{\varepsilon}$, whence assertion (d). From (1.11) also (b) and (c) easily follow.

By (1.10) we have

$$\begin{aligned} \|\eta(u,t) - u\| &\leq \int_0^t \|W(\eta(u,\tau))\| \, d\tau \leq \\ &\leq \mu \int_0^t (1 + \|\eta(u,\tau)\|) \, d\tau \leq \\ &\leq \mu \int_0^t \|\eta(u,\tau) - u\| \, d\tau + \mu(1 + \|u\|)t, \end{aligned}$$

hence

$$\int_0^t \|\eta(u,\tau) - u\| \, d\tau \leq \frac{1 + \|u\|}{\mu} (\exp(\mu t) - 1) - (1 + \|u\|)t.$$

If $0 \leq t \leq 1$, it follows

$$\|\eta(u,t) - u\| \leq (1 + \|u\|)\,(\exp(\mu t) - 1) \leq (1 + \|u\|)\,(\exp\mu - 1)\,t \leq (1 + \|u\|)\lambda t,$$

whence assertion (a). Since $\eta(u,t_2) = \eta(\eta(u,t_1), t_2 - t_1)$, we also have

$$0 \leq t_1 \leq t_2 \leq 1 \implies \|\eta(u,t_2) - \eta(u,t_1)\| \leq \lambda(1 + \|\eta(u,t_1)\|)(t_2 - t_1).$$

Finally, to prove assertion (e), consider $R > 0$ such that $\overline{\mathrm{B}_{2\varrho}(K_c)} \subseteq \mathrm{B}_R(0)$ and $\varepsilon \in]0, \hat{\varepsilon}]$ such that

$$8\varepsilon \leq \sigma\mu, \qquad 8\lambda(1 + R)\varepsilon \leq \sigma\mu\varrho.$$

Let $u \in f^{c+\varepsilon}$ and assume, for a contradiction, that $f(\eta(u,1)) > c - \varepsilon$ and $\eta(u,1) \notin U$. First of all, we have $c - \varepsilon < f(\eta(u,t)) \leq c + \varepsilon$ for every $t \in [0,1]$. Moreover, it is not possible to have $\eta(\{u\} \times [0,1]) \cap \mathrm{B}_{2\varrho}(K_c) = \emptyset$, for otherwise from (1.11) it would follow

$$2\varepsilon > f(u) - f(\eta(u,1)) \geq \frac{1}{4}\sigma\mu.$$

Therefore there exist $0 \leq t_1 < t_2 \leq 1$ such that

$$d(\eta(u,t_1),K_c) = 2\varrho\,, \qquad d(\eta(u,t_2),K_c) = 3\varrho\,,$$

$$\forall t \in]t_1,t_2[:\ 2\varrho < d(\eta(u,t),K_c) < 3\varrho\,.$$

We have

$$2\varepsilon > f(\eta(u,t_1)) - f(\eta(u,t_2)) \geq \frac{1}{4}\,\sigma\mu(t_2-t_1)\,,$$

hence

$$\varrho \leq \|\eta(u,t_2)-\eta(u,t_1)\| \leq \lambda(1+\|\eta(u,t_1)\|)(t_2-t_1) < \lambda(1+R)\frac{8\varepsilon}{\sigma\mu}$$

and a contradiction follows. ■

We end this section by providing a useful criterion for the verification of condition $(CPS)_c$.

(1.12) Definition *Let Y, Z be two normed spaces. A map $F : Y \to Z$ is said to be* completely continuous*, if*

(*a*) *F is continuous;*

(*b*) *for every bounded sequence (u_h) in Y, $(F(u_h))$ admits a (strongly) convergent subsequence in Z.*

(1.13) Theorem *Assume that*

$$f'(u) = Lu - F(u)$$

where $L : X \to X^*$ is linear, continuous, with closed range and finite dimensional null space and $F : X \to X^*$ is completely continuous.

Then for every $c \in \mathbb{R}$ the following assertions are equivalent:

(*a*) *f satisfies condition $(CPS)_c$;*

(*b*) *every $(CPS)_c-$sequence for f is bounded in X.*

Proof.
$(a) \implies (b)$ If (u_h) is an unbounded $(CPS)_c-$sequence for f, there exists a subsequence (u_{h_k}) with $\|u_{h_k}\| \to \infty$. Then (u_{h_k}) is a $(CPS)_c-$sequence which cannot admit any convergent subsequence.
$(b) \implies (a)$ Let (u_h) be a $(CPS)_c-$sequence for f. In particular, we have $f'(u_h) \to 0$ in X^*. Since (u_h) is bounded in X, up to a subsequence $(F(u_h))$ is convergent in X^*. Consequently, also (Lu_h) is convergent in X^*. Let Y be a closed subspace of X with $X = \mathcal{N}(L) \oplus Y$ and let $P_0 : X \to \mathcal{N}(L)$, $P_1 : X \to Y$ be the projections associated with the direct decomposition. Of course, we have $LP_1u_h = Lu_h$. Since $L : Y \to \mathcal{R}(L)$ is bijective and $\mathcal{R}(L)$ is closed, from the Open Mapping Theorem we deduce that (P_1u_h) is convergent in Y, hence in X. On the other hand, up to a subsequence also (P_0u_h) is convergent, as $\mathcal{N}(L)$ is finite dimensional. Then the assertion follows. ■

2 Mountain pass theorems

Throughout this section, we will consider again a real Banach space X and a function $f : X \to \mathbb{R}$ of class C^1.

(2.1) Definition *Let $A, B \subseteq X$. We say that B* links *A, if $B \cap A = \emptyset$ and B is not contractible in $X \setminus A$.*

(2.2) Remark *Of course any $B \subseteq X$ is contractible in X.*

The next result is a general mountain pass theorem which will be specialized in some corollaries later. Our kind of approach is taken from [5, 11]. We want also to recall that the possibility to consider also the large inequality in the $\sup - \inf -$estimate involving B and A is due to [7].

(2.3) Theorem *Let A be a nonempty closed subset of X, B a nonempty subset of X and let $\mathcal{C}_B$ be the family of all contractions of B in X. Assume that B links A, that*

$$\sup_B f \leq \inf_A f\,,$$

$$c := \inf_{\mathcal{H}\in\mathcal{C}_B} \sup_{B\times[0,1]} f \circ \mathcal{H} < +\infty$$

and that f satisfies $(CPS)_c$.

Then $c \geq \inf\limits_A f$ and c is a critical value of f. Moreover, if $c = \inf\limits_A f$, there exists a critical point u of f with $f(u) = c$ and $u \in A$.

Proof. Since B links A, we have $\mathcal{H}(B \times [0,1]) \cap A \neq \emptyset$ for every $\mathcal{H} \in \mathcal{C}_B$. It follows $c \geq \inf\limits_A f$.

Now, consider first the case $c = \inf\limits_A f$ and assume, for a contradiction, that $K_c \cap A = \emptyset$. Let U be a neighbourhood of K_c with $U \cap A = \emptyset$ and let $\varepsilon > 0$ and $\eta : X \times [0,1] \to X$ be as in the Deformation Theorem. Let also $\mathcal{H} \in \mathcal{C}_B$ be such that $f(\mathcal{H}(u,t)) \leq c + \varepsilon$ for every $(u,t) \in B \times [0,1]$. If we define $\mathcal{K} : B \times [0,1] \to X$ by

$$\mathcal{K}(u,t) = \begin{cases} \eta(u,2t) & \text{if } 0 \leq t \leq \frac{1}{2}\,, \\ \eta(\mathcal{H}(u,2t-1),1) & \text{if } \frac{1}{2} \leq t \leq 1\,, \end{cases}$$

it is readily seen that $\mathcal{K} \in \mathcal{C}_B$. For every $u \in B$ we have either $\eta(u,2t) = u$ or $f(\eta(u,2t)) < f(u) \leq \inf\limits_A f$. In both cases it follows $\eta(u,2t) \notin A$. On the other hand

$$\eta(\mathcal{H}(u,2t-1),1) \subseteq f^{c-\varepsilon} \cup U$$

and $(f^{c-\varepsilon} \cup U) \cap A = \emptyset$. Therefore $\mathcal{K}$ is a contraction of B in $X \setminus A$ and this contradicts the assumption that B links A.

Finally, consider the case $c > \inf\limits_A f$ and assume, for a contradiction, that $K_c = \emptyset$. Let $U = \emptyset$ and let $\varepsilon > 0$ and $\eta : X \times [0,1] \to X$ be as in the Deformation Theorem. Let also $\mathcal{H} \in \mathcal{C}_B$ be such that $f(\mathcal{H}(u,t)) \leq c + \varepsilon$ for every $(u,t) \in B \times [0,1]$. If we define $\mathcal{K} : B \times [0,1] \to X$ by

$$\mathcal{K}(u,t) = \begin{cases} \eta(u,2t) & \text{if } 0 \leq t \leq \frac{1}{2}\,, \\ \eta(\mathcal{H}(u,2t-1),1) & \text{if } \frac{1}{2} \leq t \leq 1\,, \end{cases}$$

we have again $\mathcal{K} \in \mathcal{C}_B$. On the other hand, for every $u \in B$ we have

$$f(\eta(u,2t)) \leq f(u) \leq \sup_B f\,,$$

$$f(\eta(\mathcal{H}(u, 2t-1), 1)) \leq c - \varepsilon \,.$$

Since $\sup\limits_B f < c$, this contradicts the definition of c. ∎

(2.4) Corollary *Let A be a nonempty closed subset of X, B a nonempty subset of X and let $\mathcal{C}_B$ be the family of all contractions of B in X. Assume that B links A, that*

$$\sup_B f \leq \inf_A f$$

and that

$$c := \inf_{\mathcal{H}\in\mathcal{C}_B} \sup_{B\times[0,1]} f\circ\mathcal{H} < +\infty \,.$$

Then $c \geq \inf\limits_A f$ and there exists a $(CPS)_c-$sequence (u_h) for f.

Proof. As before, one easily verifies that $c \geq \inf\limits_A f$. Now assume, for a contradiction, that there are no $(CPS)_c-$sequences for f. Then there exists $\sigma > 0$ such that

$$c - \sigma \leq f(u) \leq c + \sigma \implies (1 + \|u\|)\|f'(u)\| \geq \sigma,. \tag{2.5}$$

Therefore condition $(CPS)_c$ holds and from Theorem (2.3) we deduce that c is a critical value of f. This contradicts (2.5). ∎

The first particular case we consider is the classical mountain pass theorem of Ambrosetti-Rabinowitz (see [1, 10]).

(2.6) Corollary (Mountain Pass Theorem) *Assume there exist $u_1 \in X$ and $r > 0$ such that $\|u_1\| > r$ and*

$$\max\{f(0), f(u_1)\} \leq \inf\{f(u) : \|u\| = r\} \,.$$

Set

$$\Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = 0, \gamma(1) = u_1\} \,,$$

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$$

and suppose that f satisfies $(CPS)_c$.

Then $c \geq \inf\{f(u) : \|u\| = r\}$ and c is a critical value of f. Moreover, if $c = \inf\{f(u) : \|u\| = r\}$, there exists a critical point u of f with $f(u) = c$ and $\|u\| = r$.

Proof. Set $A = \{u \in X : \|u\| = r\}$ and $B = \{0, u_1\}$. It is evident that B links A and that $c < +\infty$. If $\gamma \in \Gamma$, then

$$\mathcal{H}(u,t) = \begin{cases} \gamma(t) & \text{if } u = 0 \\ u_1 & \text{if } u = u_1 \end{cases}$$

is clearly a contraction of B in X. Therefore

$$c \geq \inf_{\mathcal{H}\in\mathcal{C}_B} \sup_{B\times[0,1]} f\circ\mathcal{H} \,.$$

Conversely, if $\mathcal{H}$ is a contraction of B in X, then

$$\gamma(t) = \begin{cases} \mathcal{H}(0, 2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \mathcal{H}(u_1, 2-2t) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

belongs to Γ, whence

$$c \le \inf_{\mathcal{H} \in \mathcal{C}_B} \sup_{B \times [0,1]} f \circ \mathcal{H} \,.$$

From Theorem (2.3) the assertion follows. ∎

(2.7) Lemma *Let Y be a finite dimensional normed space, U a bounded open subset of Y and $y_0 \in U$. Then ∂U is not contractible in $Y \setminus \{y_0\}$.*

Proof. Assume, for a contradiction, that $\mathcal{H} : \partial U \times [0,1] \to Y \setminus \{y_0\}$ is a contraction of ∂U in $Y \setminus \{y_0\}$ to some point y_1. If $F : \overline{U} \to Y$ is the map with constant value y_1, by well known properties of Brouwer's degree (see e.g. [6, 12]), we have

$$1 = \deg\left(\mathrm{Id}, U, y_0\right) = \deg\left(F, U, y_0\right) = 0 \,,$$

which is clearly absurd. ∎

Now we come to the saddle theorem of Rabinowitz (see [10]).

(2.8) Corollary (Saddle Theorem) *Assume that*

(a) $X = X_- \oplus X_+$, *where* $\dim X_- < \infty$ *and* X_+ *is closed in* X;

(b) *there exists* $R > 0$ *such that*

$$\max\left\{f(u) : \, u \in X_-, \, \|u\| = R\right\} \le \inf\left\{f(u) : \, u \in X_+\right\} \,;$$

(c) f *satisfies* $(CPS)_c$, *where*

$$c = \inf_{\varphi \in \Phi} \max_{u \in D} f(\varphi(u)) \,,$$

$$D = \left\{u \in X_- : \, \|u\| \le R\right\} \,,$$

$$\Phi = \left\{\varphi \in C(D; X) : \, \varphi(u) = u \textit{ whenever } \|u\| = R\right\} \,.$$

Then $c \ge \inf\limits_{X_+} f$ and c is a critical value of f. Moreover, if $c = \inf\limits_{X_+} f$, there exists a critical point u of f with $f(u) = c$ and $u \in X_+$.

Proof. Set $A = X_+$ and

$$B = \left\{u \in X_- : \, \|u\| = R\right\} \,.$$

Since D is compact, it is evident that $c < +\infty$. Moreover, if $\mathcal{H}$ is a contraction of B in $X \setminus X_+$ and $P_- : X \to X_-$ is the projection associated with the direct decomposition, then

$$\mathcal{K}(u,t) = P_- \mathcal{H}(u,t)$$

is a contraction of B in $X_- \setminus \{0\}$. Since this contradicts Lemma (2.7), it follows that B links A.

If $\varphi \in \Phi$, then

$$\mathcal{H}(u,t) = \varphi((1-t)u)$$

is a contraction of B in X. Therefore

$$c \geq \inf_{\mathcal{H}\in\mathcal{C}_B} \sup_{B\times[0,1]} f\circ\mathcal{H}\,.$$

Conversely, if $\mathcal{H}$ is a contraction of B in X to some point u_1, we can define a continuous map

$$\psi : (B\times[0,1]) \cup (D\times\{1\}) \to X$$

by

$$\psi(u,t) = \begin{cases} \mathcal{H}(u,t) & \text{if } (u,t)\in B\times[0,1]\,, \\ u_1 & \text{if } (u,t)\in D\times\{1\}\,. \end{cases}$$

There exists a homeomorphism

$$F : D \to (B\times[0,1]) \cup (D\times\{1\})$$

with $F(B) = B\times\{0\}$. Then we have that $\psi\circ F\in\Phi$, whence

$$c \leq \inf_{\mathcal{H}\in\mathcal{C}_B} \sup_{B\times[0,1]} f\circ\mathcal{H}\,.$$

From Theorem (2.3) the assertion follows. ■

Finally, we derive the linking theorem of Benci-Rabinowitz (see [10] and [2] for the corresponding version in the strongly indefinite case).

(2.9) Corollary (Linking Theorem) *Assume that*

(*a*) $X = X_-\oplus X_+$, *where* $\dim X_- < \infty$ *and* X_+ *is closed in* X*;*

(*b*) *there exist* $0<r<R$ *and* $v\in X_+$ *with* $\|v\|=1$ *such that*

$$\max\{f(u):\, u\in B\} \leq \inf\{f(u):\, u\in S\}\,,$$

where B *is the boundary of*

$$D := \{u+tv :\, u\in X_-,\, t\geq 0,\, \|u+tv\|\leq R\}$$

in $X_-\oplus\mathbb{R}v$ *and*

$$S = \{u\in X_+ :\, \|u\| = r\}\,;$$

(*c*) f *satisfies* $(CPS)_c$, *where*

$$c = \inf_{\varphi\in\Phi}\max_{u\in D} f(\varphi(u))\,,$$

$$\Phi = \{\varphi\in C(D;X) :\, \varphi(u) = u \text{ whenever } u\in B\}\,.$$

Then $c\geq\inf_S f$ *and* c *is a critical value of* f*. Moreover, if* $c=\inf_S f$*, there exists a critical point* u *of* f *with* $f(u)=c$ *and* $u\in S$*.*

Proof. Since D is compact, it is evident that $c < +\infty$. If $\mathcal{H}$ is a contraction of B in $X \setminus S$, consider the projections $P_\pm : X \to X_\pm$ associated with the direct decomposition. Then

$$\mathcal{K}(u,t) = P_- \mathcal{H}(u,t) + \|P_+ \mathcal{H}(u,t)\| v$$

is a contraction of B in $(X_- \oplus \mathbb{R}v) \setminus \{rv\}$. Since this contradicts Lemma (2.7), it follows that B links A.

Now, the same argument used in the proof of the Saddle Theorem shows that

$$c = \inf_{\mathcal{H} \in \mathcal{C}_B} \sup_{B \times [0,1]} f \circ \mathcal{H}\,.$$

From Theorem (2.3) the assertion follows. ■

3 Nemytskij operator

Throughout this section, E will denote a measurable subset of $\mathbb{R}^n$ and $\| \|_p$ the usual norm of L^p $(1 \leq p \leq \infty)$.

(3.1) Definition *We say that $g : E \times \mathbb{R}^N \to \mathbb{R}^k$ is a* Carathéodory function, *if*

(*a*) *for every $s \in \mathbb{R}^N$ the function $\{x \longmapsto g(x,s)\}$ is measurable on E;*

(*b*) *for a.e. $x \in E$ the function $\{s \longmapsto g(x,s)\}$ is continuous on $\mathbb{R}^N$.*

If $u : E \to \mathbb{R}^N$ is a function, we denote by $g(x,u)$ the function

$$\begin{array}{ccc} E & \longrightarrow & \mathbb{R}^k \\ x & \longmapsto & g(x,u(x)) \end{array}.$$

(3.2) Theorem *Let $g : E \times \mathbb{R}^N \to \mathbb{R}^k$ be a Carathéodory function.*

Then for every measurable function $u : E \to \mathbb{R}^N$ we have that $g(x,u) : E \to \mathbb{R}^k$ is measurable. Moreover, if u, v agree a.e. in E, then also $g(x,u)$ and $g(x,v)$ agree a.e. in E.

Proof. Let $u : E \to \mathbb{R}^N$ be a simple function, namely a measurable function with a finite number of values. If $u(E) = \{s_1, \ldots, s_m\}$, set $E_h = u^{-1}(s_h)$. Then $\{E_1, \ldots, E_m\}$ is a measurable partition of E and we have

$$\forall x \in E : \; g(x,u(x)) = \sum_{h=1}^{m} \chi_{E_h}(x)\, g\left(x, s_h\right)\,.$$

Therefore $g(x,u)$ is measurable.

Let now $u : E \to \mathbb{R}^N$ be a measurable function. It is well known that there exists a sequence (u_h) of simple functions pointwise convergent to u. Then we have

$$\lim_h g(x,u_h) = g(x,u) \qquad \text{a.e. in } E,$$

whence the measurability of $g(x,u)$.

It is evident that, if u, v agree a.e. in E, then also $g(x,u)$ and $g(x,v)$ agree a.e. in E. ■

(3.3) Theorem *Let $g : E \times \mathbb{R}^N \to \mathbb{R}^k$ be a Carathéodory function and let $p, q \in [1, \infty[$. Assume there exist $a \in L^q(E)$ and $b \in \mathbb{R}$ such that*

$$|g(x, s)| \leq a(x) + b|s|^{\frac{p}{q}}$$

for a.e. $x \in E$ and every $s \in \mathbb{R}^N$.

Then for every $u \in L^p(E; \mathbb{R}^N)$ we have $g(x, u) \in L^q(E; \mathbb{R}^k)$ and the map

$$\begin{array}{ccc} L^p(E; \mathbb{R}^N) & \longrightarrow & L^q(E; \mathbb{R}^k) \\ u & \longmapsto & g(x, u) \end{array}$$

is continuous.

Proof. For any $u \in L^p(E; \mathbb{R}^N)$ we have

$$|g(x, u)|^q \leq \left(a(x) + b|u|^{\frac{p}{q}}\right)^q \leq 2^{q-1} \left(a(x)^q + b^q|u|^p\right) .$$

Combining this fact with Theorem (3.2), we deduce that $g(x, u) \in L^q(E; \mathbb{R}^k)$.

Now, let (u_h) be a sequence convergent to some u in $L^p(E; \mathbb{R}^N)$. Up to a subsequence, (u_h) is convergent a.e. to u and there exists $w \in L^p(E)$ such that

$$|u_h| \leq w \qquad \text{a.e. in } E$$

(see e.g. [3, Theorem IV.9]). Therefore we have

$$\lim_h g(x, u_h) = g(x, u) \qquad \text{a.e. in } E,$$

$$\begin{aligned} |g(x, u_h) - g(x, u)|^q &\leq 2^{q-1}(|g(x, u_h)|^q + |g(x, u)|^q) \leq \\ &\leq 4^{q-1} \left(2a(x)^q + b^q|u_h|^p + b^q|u|^p\right) \leq \\ &\leq 4^{q-1} \left(2a(x)^q + b^q w^p + b^q|u|^p\right) \qquad \text{a.e. in } E. \end{aligned}$$

From Lebesgue's Theorem we deduce that $(g(x, u_h))$ is convergent to $g(x, u)$ in $L^q(E; \mathbb{R}^k)$. ■

(3.4) Theorem *Let $g : E \times \mathbb{R}^N \to \mathbb{R}^k$ be a Carathéodory function and let $q \in [1, \infty[$. Assume that for every $M > 0$ there exists $a_M \in L^q(E)$ such that*

$$|g(x, s)| \leq a_M(x)$$

for a.e. $x \in E$ and every $s \in \mathbb{R}^N$ with $|s| \leq M$.

Then for every $u \in L^\infty(E; \mathbb{R}^N)$ we have $g(x, u) \in L^q(E; \mathbb{R}^k)$ and the map

$$\begin{array}{ccc} L^\infty(E; \mathbb{R}^N) & \longrightarrow & L^q(E; \mathbb{R}^k) \\ u & \longmapsto & g(x, u) \end{array}$$

is continuous.

Proof. If $u \in L^\infty(E; \mathbb{R}^N)$ and $M = \|u\|_\infty$, we have $|g(x, u)| \leq a_M(x)$ a.e. Taking into account Theorem (3.2), we easily deduce that $g(x, u) \in L^q(E; \mathbb{R}^k)$.

If (u_h) is a sequence convergent to some u in $L^\infty(E;\mathbb{R}^N)$, there exists $M>0$ such that $\|u_h\|_\infty \leq M$ for every h. It follows that $(g(x,u_h))$ is convergent to $g(x,u)$ a.e. and

$$|g(x,u_h)-g(x,u)|^q \leq 2^q a_M(x)^q\,.$$

From Lebesgue's Theorem we deduce that $(g(x,u_h))$ is convergent to $g(x,u)$ in $L^q(E;\mathbb{R}^k)$. ■

(3.5) Definition *The map*

$$\begin{array}{ccc} L^p(E;\mathbb{R}^N) & \longrightarrow & L^q(E;\mathbb{R}^k) \\ u & \longmapsto & g(x,u) \end{array}$$

is called Nemytskij operator *or* superposition operator *associated with g.*

(3.6) Definition *We say that $G:E\times\mathbb{R}^N\to\mathbb{R}^k$ is a* C^1-Carathéodory function, *if*

(a) *for every $s\in\mathbb{R}^N$ the function $\{x\longmapsto G(x,s)\}$ is measurable on E;*

(b) *for a.e. $x\in E$ the function $\{s\longmapsto G(x,s)\}$ is of class C^1 on $\mathbb{R}^N$.*

(3.7) Proposition *Let $g:E\times\mathbb{R}\to\mathbb{R}^k$ be a Carathéodory function and set $G(x,s)=\int_0^s g(x,t)\,dt$.*
Then $G:E\times\mathbb{R}\to\mathbb{R}^k$ is a C^1-Carathéodory function with $G(x,0)=0$.

Proof. It is evident that $\{s\longmapsto G(x,s)\}$ is of class C^1 for a.e. $x\in E$. Moreover, for every $s\in\mathbb{R}$ we have

$$G(x,s)=\lim_k\left(\sum_{h=1}^k \frac{s}{k}\,g\left(x,h\frac{s}{k}\right)\right)\qquad \text{a.e. in } E.$$

Therefore $\{x\longmapsto G(x,s)\}$ is measurable for every $s\in\mathbb{R}$. ■

(3.8) Theorem *Let $G:E\times\mathbb{R}^N\to\mathbb{R}^k$ be a C^1-Carathéodory function, let $1\leq q<p<\infty$, let $r>1$ be such that*

$$\frac{1}{r}+\frac{1}{p}=\frac{1}{q}$$

and set $g(x,s)=D_sG(x,s)$. Assume that $G(x,0)\in L^q(E)$ and that there exist $a\in L^r(E)$ and $b\in\mathbb{R}$ such that

$$|g(x,s)|\leq a(x)+b|s|^{\frac{p}{q}-1} \tag{3.9}$$

for a.e. $x\in E$ and every $s\in\mathbb{R}^N$.

Then $g:E\times\mathbb{R}^N\to\mathbb{R}^{Nk}$ is a Carathéodory function, we have $G(x,u)\in L^q(E;\mathbb{R}^k)$ for every $u\in L^p(E;\mathbb{R}^N)$ and the Nemytskij operator

$$\begin{array}{cccc} \mathcal{G}: & L^p(E;\mathbb{R}^N) & \longrightarrow & L^q(E;\mathbb{R}^k) \\ & u & \longmapsto & G(x,u) \end{array}$$

associated with G is of class C^1. Moreover we have

$$\forall u,v\in L^p(E;\mathbb{R}^N):\ \mathcal{G}'(u)v=g(x,u)v\,.$$

Proof. It is evident that $\{s \longmapsto g(x,s)\}$ is continuous for a.e. $x \in E$. Moreover, for every $s, \sigma \in \mathbb{R}^N$ we have

$$g(x,s)\sigma = \lim_k k\left(G\left(x, s + \frac{1}{k}\sigma\right) - G(x,s)\right) \qquad \text{a.e. in } E.$$

Therefore $\{x \longmapsto g(x,s)\}$ is measurable for every $s \in \mathbb{R}$.

Since $\frac{p}{p-q} = \frac{r}{q}$, from (3.9) and Young's inequality we deduce that

$$\begin{aligned} |G(x,s)| &\leq |G(x,0)| + a(x)|s| + \frac{bq}{p}|s|^{\frac{p}{q}} \leq |G(x,0)| + \frac{p-q}{p}\,a(x)^{\frac{p}{p-q}} + \frac{q}{p}\,|s|^{\frac{p}{q}} + \frac{bq}{p}|s|^{\frac{p}{q}} = \\ &= |G(x,0)| + \frac{p-q}{p}\,a(x)^{\frac{r}{q}} + (1+b)\,\frac{q}{p}\,|s|^{\frac{p}{q}}\,. \end{aligned}$$

Since $a^{\frac{r}{q}} \in L^q(E)$, from Theorem (3.3) it follows that the Nemytskij operator

$$\begin{array}{rccc} \mathcal{G}: & L^p(E;\mathbb{R}^N) & \longrightarrow & L^q(E;\mathbb{R}^k) \\ & u & \longmapsto & G(x,u) \end{array}$$

is well defined and continuous. Since $\frac{p}{q} - 1 = \frac{p}{r}$, also the Nemytskij operator

$$\begin{array}{ccc} L^p(E;\mathbb{R}^N) & \longrightarrow & L^r(E;\mathbb{R}^{Nk}) \\ u & \longmapsto & g(x,u) \end{array}$$

is well defined and continuous.

Now let $u \in L^p(E;\mathbb{R}^N)$. By Hölder's inequality it is readily seen that the map

$$\begin{array}{ccc} L^p(E;\mathbb{R}^N) & \longrightarrow & L^q(E;\mathbb{R}^k) \\ v & \longmapsto & g(x,u)v \end{array}$$

is well defined, linear and continuous. Let (v_h) be a sequence convergent to 0 in $L^p(E;\mathbb{R}^N)$. Up to a subsequence, (v_h) is convergent to 0 also a.e. and there exists $w \in L^p(E)$ such that $|v_h| \leq w$ a.e. Set $z_h = \frac{v_h}{\|v_h\|_p}$ and define

$$\alpha_h = \begin{cases} \dfrac{G(x,u+v_h) - G(x,u) - g(x,u)v_h}{|v_h|} & \text{where } v_h(x) \neq 0, \\ 0 & \text{where } v_h(x) = 0. \end{cases}$$

Then (α_h) is convergent to 0 a.e. and, by Lagrange's Inequality, we have

$$\begin{aligned} |\alpha_h|^r &\leq |g(x,u+\vartheta_h v_h) - g(x,u)|^r \leq \\ &\leq \left(a + b|u+\vartheta_h v_h|^{\frac{p}{r}} + a + b|u|^{\frac{p}{r}}\right)^r \leq \\ &\leq \left(a + b(|u|+|w|)^{\frac{p}{r}} + a + b|u|^{\frac{p}{r}}\right)^r, \end{aligned}$$

where $0 < \vartheta_h < 1$. Therefore (α_h) is convergent to 0 also in $L^r(E;\mathbb{R}^k)$. From Hölder's inequality it follows

$$\begin{aligned} \int_E \left|\frac{G(x,u+v_h) - G(x,u) - g(x,u)v_h}{\|v_h\|_p}\right|^q dx &= \int_E |\alpha_h|^q |z_h|^q\,dx \leq \\ &\leq \|\alpha_h\|_r^q \|z_h\|_p^q = \|\alpha_h\|_r^q\,, \end{aligned}$$

hence

$$\lim_h \int_E \left|\frac{G(x,u+v_h) - G(x,u) - g(x,u)v_h}{\|v_h\|_p}\right|^q dx = 0\,.$$

Therefore $\mathcal{G}$ is Fréchet differentiable at u and $\mathcal{G}'(u)v = g(x,u)v$.

Finally, for every $u_1, u_2, v \in L^p(E;\mathbb{R}^N)$ we have

$$\|\mathcal{G}'(u_1)v - \mathcal{G}'(u_2)v\|_q = \|g(x,u_1)v - g(x,u_2)v\|_q \le \|g(x,u_1) - g(x,u_2)\|_r \|v\|_p\,,$$

hence

$$\|\mathcal{G}'(u_1) - \mathcal{G}'(u_2)\|_{\mathcal{L}(L^p;L^q)} \le \|g(x,u_1) - g(x,u_2)\|_r\,.$$

Therefore $\mathcal{G}$ is of class C^1. ∎

(3.10) Theorem *Let $G : E \times \mathbb{R}^N \to \mathbb{R}^k$ be a C^1-Carathéodory function, let $1 \le q < \infty$ and set $g(x,s) = D_s G(x,s)$. Assume that $G(x,0) \in L^q(E;\mathbb{R}^k)$ and that for every $M > 0$ there exists $a_M \in L^q(E)$ such that*

$$|g(x,s)| \le a_M(x)$$

for a.e. $x \in E$ and every $s \in \mathbb{R}^N$ with $|s| \le M$.

Then $g : E \times \mathbb{R}^N \to \mathbb{R}^{Nk}$ is a Carathéodory function, we have $G(x,u) \in L^q(E;\mathbb{R}^k)$ for every $u \in L^\infty(E;\mathbb{R}^N)$ and the Nemytskij operator

$$\begin{array}{rccc} \mathcal{G}: & L^\infty(E;\mathbb{R}^N) & \longrightarrow & L^q(E;\mathbb{R}^k) \\ & u & \longmapsto & G(x,u) \end{array}$$

associated with G is of class C^1. Moreover we have

$$\forall u, v \in L^\infty(E;\mathbb{R}^N) : \mathcal{G}'(u)v = g(x,u)v\,.$$

Proof. As before, we have that g is a Carathéodory function. Moreover,

$$|G(x,s)| \le |G(x,0)| + a_M(x)|s| \le |G(x,0)| + M a_M(x)$$

for a.e. $x \in E$ and every $s \in \mathbb{R}^N$ with $|s| \le M$. From Theorem (3.4) it follows that the Nemytskij operators

$$\begin{array}{rccc} \mathcal{G}: & L^\infty(E;\mathbb{R}^N) & \longrightarrow & L^q(E;\mathbb{R}^k) \\ & u & \longmapsto & G(x,u) \end{array}$$

and

$$\begin{array}{ccc} L^\infty(E;\mathbb{R}^N) & \longrightarrow & L^q(E;\mathbb{R}^{Nk}) \\ u & \longmapsto & g(x,u) \end{array}$$

are well defined and continuous. Then it is possible to argue, with minor variants, as in the proof of Theorem (3.8). ∎

Let now Ω be an open subset of $\mathbb{R}^n$.

(3.11) Theorem *The following facts hold:*

(a) *if $1 \le p < n$, then we have $W_0^{1,p}(\Omega;\mathbb{R}^N) \subseteq L^{\frac{np}{n-p}}(\Omega;\mathbb{R}^N)$ and there exists $c(n,p) > 0$ such that*

$$\forall u \in W_0^{1,p}(\Omega;\mathbb{R}^N) : \|u\|_{\frac{np}{n-p}} \le c(n,p)\,\|\nabla u\|_p\,;$$

(b) *if $n < p < \infty$, then we have $W_0^{1,p}(\Omega;\mathbb{R}^N) \subseteq L^\infty(\Omega;\mathbb{R}^N)$ and there exists $c(n,p) > 0$ such that*

$$\forall u \in W_0^{1,p}(\Omega;\mathbb{R}^N) : \|u\|_\infty \leq c(n,p) \left(\|\nabla u\|_p^p + \|u\|_p^p\right)^{\frac{1}{p}} .$$

(c) *if $a, b \in \mathbb{R}$ and $1 \leq p \leq \infty$, then we have $W^{1,p}(]a,b[;\mathbb{R}^N) \subseteq L^\infty(]a,b[;\mathbb{R}^N)$ and there exists $c(a,b) > 0$ such that*

$$\forall u \in W^{1,p}(]a,b[;\mathbb{R}^N) : \|u\|_\infty \leq c(a,b) \left(\|u'\|_p^p + \|u\|_p^p\right)^{\frac{1}{p}} .$$

Proof. See for instance [3, Theorems IX.9, IX.12 and VIII.7]. ∎

(3.12) Theorem *Let $1 \leq p \leq \infty$. Then every bounded sequence (u_h) in $W^{1,p}(\Omega;\mathbb{R}^N)$ admits a subsequence convergent a.e. to some $u \in L^p(\Omega;\mathbb{R}^N)$.*

Proof. If Ω is an open ball, the Rellich-Kondrachov Theorem (see e.g. [3, Theorem IX.16]) implies that there exists a subsequence (u_{h_k}) strongly convergent in $L^p(\Omega;\mathbb{R}^N)$ to some u. Then a further subsequence is convergent to u a.e.

Since any open subset of $\mathbb{R}^n$ is a countable union of open balls, also in the general case we may find a subsequence convergent a.e. to some u. From Fatou's Lemma it is easy to deduce that $u \in L^p(\Omega;\mathbb{R}^N)$. ∎

(3.13) Theorem *Let $1 \leq p < n$, let X be a subspace of $W^{1,p}(\Omega;\mathbb{R}^N)$ continuously imbedded in $L^{\frac{np}{n-p}}(\Omega;\mathbb{R}^N)$, let $G : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a C^1-Carathéodory function and set $g(x,s) = \nabla_s G(x,s)$. Assume that $G(x,0) \in L^1(\Omega)$ and that there exist $a \in L^{\frac{np}{n(p-1)+p}}(\Omega)$ and $b \in \mathbb{R}$ such that*

$$|g(x,s)| \leq a(x) + b|s|^{\frac{np}{n-p}-1}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}^N$.

Then for every $u \in X$ we have $G(x,u) \in L^1(\Omega)$ and the functional

$$\begin{array}{rccc} f: & X & \longrightarrow & \mathbb{R} \\ & u & \longmapsto & \int_\Omega G(x,u)\,dx \end{array}$$

is of class C^1. Moreover we have

$$\forall u, v \in X : f'(u)v = \int_\Omega g(x,u) \cdot v \, dx .$$

Proof. Since

$$\frac{n(p-1)+p}{np} + \frac{n-p}{np} = 1 ,$$

it follows from Theorem (3.8) that the Nemytskij operator

$$\begin{array}{rccc} \mathcal{G}: & L^{\frac{np}{n-p}}(\Omega;\mathbb{R}^N) & \longrightarrow & L^1(\Omega) \\ & u & \longmapsto & G(x,u) \end{array}$$

is of class C^1 with $\mathcal{G}'(u)v = g(x,u) \cdot v$.

On the other hand X is continuously included in $L^{\frac{np}{n-p}}(\Omega;\mathbb{R}^N)$ and $\left\{w \mapsto \int_\Omega w\,dx\right\}$ is a continuous and linear functional on $L^1(\Omega)$. Then the assertion easily follows. ■

(3.14) Definition *Let $1 \le p < n$. We say that a Carathéodory function $g : \Omega \times \mathbb{R}^N \to \mathbb{R}^k$ has* subcritical growth *with respect to $W^{1,p}(\Omega;\mathbb{R}^N)$, if for every $\varepsilon > 0$ there exists $a_\varepsilon \in L^{\frac{np}{n(p-1)+p}}(\Omega)$ such that*

$$|g(x,s)| \le a_\varepsilon(x) + \varepsilon |s|^{\frac{np}{n-p}-1}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}^N$.

(3.15) Remark *If Ω has finite measure and*

$$|g(x,s)| \le a(x) + b|s|^q$$

with $a \in L^{\frac{np}{n(p-1)+p}}(\Omega)$, $b \in \mathbb{R}$ and $0 < q < \frac{np}{n-p} - 1$, then g has subcritical growth with respect to $W^{1,p}(\Omega;\mathbb{R}^N)$.

Proof. Let $rq = \frac{np}{n-p} - 1$. From Young's inequality we deduce that

$$|g(x,s)| \le a(x) + \frac{1}{r'}\left(\frac{b}{\delta}\right)^{r'} + \frac{1}{r}\,\delta^r |s|^{\frac{np}{n-p}-1}$$

for every $\delta > 0$. Since the constant $\frac{1}{r'}\left(\frac{b}{\delta}\right)^{r'}$ belongs to $L^{\frac{np}{n(p-1)+p}}(\Omega)$ and δ^r can be made arbitrarily small, the assertion follows. ■

(3.16) Theorem *Let $1 \le p < n$, let X be a subspace of $W^{1,p}(\Omega;\mathbb{R}^N)$ continuously imbedded in $L^{\frac{np}{n-p}}(\Omega;\mathbb{R}^N)$, let $G : \Omega\times\mathbb{R}^N \to \mathbb{R}$ be a C^1-Carathéodory function and set $g(x,s) = \nabla_s G(x,s)$. Assume that $G(x,0) \in L^1(\Omega)$ and that g has subcritical growth with respect to $W^{1,p}(\Omega;\mathbb{R}^N)$.*

Then the functional

$$\begin{array}{rccc} f: & X & \longrightarrow & \mathbb{R} \\ & u & \longmapsto & \int_\Omega G(x,u)\,dx \end{array}$$

is of class C^1 with

$$\forall u, v \in X : \; f'(u)v = \int_\Omega g(x,u)\cdot v\,dx$$

and the derivative $f' : X \to X^*$ is completely continuous.

Proof. Since g has subcritical growth, it follows from Theorem (3.13) that f is well defined and of class C^1 with

$$\forall u, v \in X : \; f'(u)v = \int_\Omega g(x,u)\cdot v\,dx\,.$$

For every $u_1, u_2, v \in X$, Hölder's inequality yields,

$$\begin{aligned} |f'(u_1)v - f'(u_2)v| &= \left|\int_\Omega (g(x,u_1) - g(x,u_2))\cdot v\,dx\right| \le \\ &\le \|g(x,u_1) - g(x,u_2)\|_{\frac{np}{n(p-1)+p}} \|v\|_{\frac{np}{n-p}} \le \\ &\le c\|g(x,u_1) - g(x,u_2)\|_{\frac{np}{n(p-1)+p}} \left(\|\nabla v\|_p^p + \|v\|_p^p\right)^{\frac{1}{p}}\,. \end{aligned}$$

Therefore it is sufficient to prove assertion (b) of Definition (1.12) for the map

$$\begin{array}{ccc} X & \longrightarrow & L^{\frac{np}{n(p-1)+p}}(\Omega;\mathbb{R}^N) \\ u & \longmapsto & g(x,u) \end{array} .$$

Without loss of generality, we can suppose $N = 1$. Let us treat first of all the case in which

$$|g(x,s)| \leq a(x)$$

with $a \in L^{\frac{np}{n(p-1)+p}}(\Omega)$. Let (u_h) be a bounded sequence in X. By Theorem (3.12), up to a subsequence (u_h) is convergent a.e. to some $u \in L^p(\Omega)$. Since

$$|g(x,u_h) - g(x,u)| \leq 2a(x)\,,$$

from Lebesgue's Theorem we deduce that $(g(x,u_h))$ is strongly convergent to $g(x,u)$ in $L^{\frac{np}{n(p-1)+p}}(\Omega)$.

In the general case, set for any $\varepsilon > 0$

$$g_\varepsilon(x,s) = \min\left\{\max\left\{g(x,s), -a_\varepsilon(x)\right\}, a_\varepsilon(x)\right\}\,.$$

Since $|g_\varepsilon(x,s)| \leq a_\varepsilon(x)$, the map

$$\begin{array}{ccc} X & \longrightarrow & L^{\frac{np}{n(p-1)+p}}(\Omega) \\ u & \longmapsto & g_\varepsilon(x,u) \end{array}$$

satisfies condition (b) of Definition (1.12) by the previous step. On the other hand, we have

$$|g_\varepsilon(x,s) - g(x,s)| \leq \varepsilon |s|^{\frac{np}{n-p}-1}\,,$$

hence, for every $u \in X$,

$$\|g_\varepsilon(x,u) - g(x,u)\|_{\frac{np}{n(p-1)+p}} \leq \varepsilon \|u\|_{\frac{np}{n-p}}^{\frac{np}{n-p}-1} \leq \varepsilon c^{\frac{np}{n-p}-1}\left(\|\nabla u\|_p^p + \|u\|_p^p\right)^{\frac{n}{n-p}-\frac{1}{p}}\,.$$

We deduce that

$$\lim_{\varepsilon\to 0} \|g_\varepsilon(x,u) - g(x,u)\|_{\frac{np}{n(p-1)+p}} = 0$$

uniformly on bounded subsets of X and the assertion follows from well known properties of completely continuous operators (see e.g. [9, Proposition III.5.4]). ■

(3.17) Theorem *Let $n < p \leq \infty$, let X be a subspace of $W^{1,p}(\Omega;\mathbb{R}^N)$ continuously imbedded in $L^\infty(\Omega;\mathbb{R}^N)$, let $G : \Omega\times\mathbb{R}^N \to \mathbb{R}$ be a C^1-Carathéodory function and set $g(x,s) = \nabla_s G(x,s)$. Assume that $G(x,0) \in L^1(\Omega)$ and that for every $M > 0$ there exists $a_M \in L^1(\Omega)$ such that*

$$|g(x,s)| \leq a_M(x)$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}^N$ with $|s| \leq M$.

Then for every $u \in X$ we have $G(x,u) \in L^1(\Omega)$ and the functional

$$\begin{array}{cccc} f: & X & \longrightarrow & \mathbb{R} \\ & u & \longmapsto & \int_\Omega G(x,u)\,dx \end{array}$$

is of class C^1. Moreover we have

$$\forall u, v \in X : f'(u)v = \int_\Omega g(x,u) \cdot v \, dx$$

and the derivative $f' : X \to X^*$ is completely continuous.

Proof. From Theorem (3.10) we deduce that the Nemytskij operator

$$\begin{array}{rccc} \mathcal{G}: & L^\infty(\Omega;\mathbb{R}^N) & \longrightarrow & L^1(\Omega) \\ & u & \longmapsto & G(x,u) \end{array}$$

is of class C^1 with $\mathcal{G}'(u)v = g(x,u) \cdot v$.

On the other hand, X is continuously included in $L^\infty(\Omega;\mathbb{R}^N)$ and $\{w \mapsto \int_\Omega w \, dx\}$ is a continuous and linear functional on $L^1(\Omega)$. Then it is easy to show that f is of class C^1 with $f'(u)v = \int_\Omega g(x,u) \cdot v \, dx$.

For every $u_1, u_2, v \in X$, Hölder's inequality yields,

$$\begin{aligned} |f'(u_1)v - f'(u_2)v| &= \left| \int_\Omega (g(x,u_1) - g(x,u_2)) \cdot v \, dx \right| \leq \\ &\leq \|g(x,u_1) - g(x,u_2)\|_1 \|v\|_\infty \leq \\ &\leq c\|g(x,u_1) - g(x,u_2)\|_1 \left(\|\nabla v\|_p^p + \|v\|_p^p\right)^{\frac{1}{p}} . \end{aligned}$$

Therefore it is sufficient to show that the map

$$\begin{array}{ccc} X & \longrightarrow & L^1(\Omega;\mathbb{R}^N) \\ u & \longmapsto & g(x,u) \end{array}$$

satisfies condition (b) of Definition (1.12).

Let (u_h) be a bounded sequence in X and let $M > 0$ be such that $\|u_h\|_\infty \leq M$. By Theorem (3.12), up to a subsequence (u_h) is convergent a.e. to some $u \in L^p(\Omega;\mathbb{R}^N)$. Since

$$|g(x,u_h) - g(x,u)| \leq 2a_M(x) ,$$

from Lebesgue's Theorem we deduce that $(g(x,u_h))$ is strongly convergent to $g(x,u)$ in $L^1(\Omega;R^N)$. ■

(3.18) Corollary *Let Ω be bounded, let $G : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a C^1-Carathéodory function with $G(x,0) \in L^1(\Omega)$ and assume that $g := \nabla_s G(x,s)$ has subcritical growth with respect to $W^{1,2}(\Omega;\mathbb{R}^N)$.*

Then the functional $f : W_0^{1,2}(\Omega) \to \mathbb{R}$ defined by

$$f(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega G(x,u) \, dx$$

is of class C^1 with

$$\forall u, v \in W_0^{1,2}(\Omega) : f'(u)v = \int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega g(x,u) \cdot v \, dx$$

and the derivative $f' : W_0^{1,2}(\Omega;\mathbb{R}^N) \to W^{-1,2}(\Omega;\mathbb{R}^N)$ has the form required in Theorem (1.13).

Proof. Define $f_1 : W_0^{1,2}(\Omega;\mathbb{R}^N) \to \mathbb{R}$ by

$$f_1(u) = -\int_\Omega \widetilde{G}(x,u) \, dx ,$$

where $\widetilde{G}(x,s) = G(x,s) + \frac{1}{2}|s|^2$. Taking into account Remark (3.15), it is easy to see that also $\tilde{g} := \nabla_s \widetilde{G}$ has subcritical growth with respect to $W^{1,2}(\Omega;\mathbb{R}^N)$. From Theorem (3.16) it follows that f_1 is well defined, of class C^1 with $f_1{}'$ completely continuous.

Since

$$f(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |u|^2\right) dx - f_1(u)\,,$$

we have $f'(u) = Lu - f_1{}'(u)$, where $L : W_0^{1,2}(\Omega;\mathbb{R}^N) \to W^{-1,2}(\Omega;\mathbb{R}^N)$ is an isomorphism, and the assertion follows ■

(3.19) Corollary *Let*

$$X = \left\{u \in W^{1,2}(]-\pi,\pi[;\mathbb{R}^N) : \; u(-\pi) = u(\pi)\right\}$$

let $G :]-\pi,\pi[\times\mathbb{R}^N \to \mathbb{R}$ be a C^1-Carathéodory function and set $g(x,s) = \nabla_s G(x,s)$. Assume that $G(x,0) \in L^1(]-\pi,\pi[)$ and that for every $M > 0$ there exists $a_M \in L^1(]-\pi,\pi[)$ such that

$$|g(x,s)| \leq a_M(x)$$

for a.e. $x \in]-\pi,\pi[$ and every $s \in \mathbb{R}^N$ with $|s| \leq M$.

Then X is a closed subspace of $W^{1,2}(]-\pi,\pi[;\mathbb{R}^N)$, the functional $f : X \to \mathbb{R}$ defined by

$$f(u) = \frac{1}{2}\int_{-\pi}^{\pi} |u'|^2\,dx - \int_{-\pi}^{\pi} G(x,u)\,dx$$

is of class C^1 with

$$\forall u, v \in X : \; f'(u)v = \int_{-\pi}^{\pi} u' \cdot v'\,dx - \int_{-\pi}^{\pi} g(x,u)\cdot v\,dx$$

and the derivative $f' : X \to X^*$ has the form required in Theorem (1.13).

Proof. It is well known that $W^{1,2}(]-\pi,\pi[;\mathbb{R}^N)$ is continuously imbedded in $C([-\pi,\pi];\mathbb{R}^N)$ (see e.g. [3]). Therefore X is well defined and is in fact a closed linear subspace of $W^{1,2}(]-\pi,\pi[;\mathbb{R}^N)$.

If we define again $f_1 : X \to \mathbb{R}$ by

$$f_1(u) = -\int_{-\pi}^{\pi} \widetilde{G}(x,u)\,dx\,,$$

where $\widetilde{G}(x,s) = G(x,s) + \frac{1}{2}|s|^2$, we deduce now from Theorem (3.17) that $(f_1)' : X \to X^*$ is completely continuous. Then the assertion easily follows. ■

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