Geometric properties of the heat content

Luciana Angiuli[*], Umberto Massari, Michele Miranda Jr [†]

March 6, 2012

Abstract

In this paper we study the short time behavior of heat semigroup in connection with the geometry of sets with finite perimeter; we assume $C^{1,1}$ regularity and we relate the heat semigroup with curvatures of the initial datum. We also study the behavior when singularities occur; this is the case when the mean curvature is no more a function, but has to be considered as a Radon measure. This work is the natural continuation of [11] and is in the spirit of [4].

1 Introduction

The connections between the theory of semigroups and that of perimeters have been the subject of recent mathematical researches.
The interest in such results comes from the possibility to deduce geometric properties of a set $E \subset \mathbb{R}^n$ by means of the solution of suitable partial differential equations.
The pioneering paper where the first link between these theories has been established is [5] where De Giorgi noticed that by taking the heat semigroup T_t in $\mathbb{R}^n$, defined by means of the convolution with the Gauss–Weierstrass kernel,

$$T_t\chi_E(x) = g_t * \chi_E(x) = \frac{1}{(4\pi t)^{n/2}} \int_E e^{-\frac{|x-y|^2}{4t}} dy,$$

the map

$$t \mapsto \int_{\mathbb{R}^n} |\nabla T_t \chi_E| dx$$

is monotone decreasing, showing the existence of the limit

$$\lim_{t\to 0} \int_{\mathbb{R}^n} |\nabla T_t \chi_E| dx$$

defined as the *perimeter of* E and denoted by $P(E)$, for any measurable set $E \subset \mathbb{R}^n$.
A characterization of the perimeter of a set, similar to the original definition of De Giorgi can

[*]Dipartimento di Matematica "Ennio De Giorgi", Università del Salento, C.P. 193, I-73100 Lecce, Italy; e–mail: luciana.angiuli@unisalento.it

[†]Dipartimento di Matematica, via Machiavelli 35, I-44100 Ferrara, Italy; e–mail: umberto.massari@unife.it, michele.miranda@unife.it

also be found in [8], where a characterization of isoperimetric property is provided. In [8], Ledoux introduced the diffusion functional defined by

$$K_t(E,F) = \int_F T_t\chi_E(x)dx, \qquad t>0 \tag{1}$$

and he proved the following formula

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}K_t(B,B^c) = P(B) \tag{2}$$

where B is the Euclidean ball in $\mathbb{R}^n$. Recently, in [11], the authors proved that equation (2) holds for any set with finite perimeter and, moreover, an equivalent characterization of sets with finite perimeter is given, in the sense that

$$\liminf_{t\to 0}\frac{K_t(E,E^c)}{\sqrt{t}} < +\infty$$

if and only if E has finite perimeter. In that case

$$P(E) = \lim_{t\to 0}\sqrt{\frac{\pi}{t}}K_t(E,E^c)$$

Summarizing, for a set E with finite measure and perimeter, the following expansion holds

$$K_t(E,E^c) = \sqrt{\frac{t}{\pi}}P(E) + o(\sqrt{t}),$$

as t goes to 0. More generally, for any two sets E and F with finite perimeter, the following formula holds

$$\int_F T_t\chi_E(x)dx = |E\cap F| - \sqrt{\frac{t}{\pi}}\int_{\mathcal{F}E\cap\mathcal{F}F}\langle\nu_E(x),\nu_F(x)\rangle d\mathcal{H}^{n-1}(x) + o(\sqrt{t}). \tag{3}$$

where $\mathcal{F}E$, $\mathcal{F}F$ are respectively the reduced boundary of E and F.
On the other hand, in the recent paper [1], two different characterizations of $P(E,\Omega)$, the perimeter of a set in a domain are given in terms of the short-time behavior of the solution of a parabolic initial boundary value problem in Ω.
These results are similar in the spirit to that contained in [4], where it is proved that

$$Q_D(t) = |D| - \frac{2\sqrt{t}P(D)}{\sqrt{\pi}} + \frac{t}{2}\int_{\partial D}H_D(x)d\mathcal{H}^{n-1}(x) + o(t)$$

with D a bounded connected domain such that ∂D is of class C^3 and H_D is the mean curvature of ∂D; here $Q_D(t) = \int_D u(t,x)dx$ and u is the solution of the Dirichlet Laplacian on D

$$\begin{cases} \partial_t u = \Delta u & (0,+\infty)\times D \\ u(t,x) = 0 & (0,+\infty)\times\partial D \\ u(0,x) = 1 & D. \end{cases}$$

More recently, the same authors generalized the result in order to consider also Neumann boundary conditions; they also considered mixed, Dirichlet and Neumann, boundary conditions and

initial data other then χ_D (see for instance [2], [3] and the references therein).
In this paper, we are interested in studying the higher order expansion of $K_t(E,F)$ for two subsets $E, F \subset \mathbb{R}^n$ with finite perimeter; due to the presence of $\sqrt{t}$, we introduce the functions $f_{E,F}(t) = K_{t^2}(E,F)$ and $f_E(t) = K_{t^2}(E,E^c)$. A way to prove (3) simply consists in considering the limits

$$\lim_{t\to 0} f_{E,F}(t) = |E \cap F|$$

and (see [11, Theorem 3.1] for a detailed proof)

$$\lim_{t\to 0} f'_{E,F}(t) = \lim_{t\to 0} 2t \int_F \Delta T_{t^2}\chi_E(x)dx = -\frac{1}{\sqrt{\pi}} \int_{\mathcal{F}E\cap\mathcal{F}F} \langle \nu_E(x), \nu_F(x)\rangle d\mathcal{H}^{n-1}(x).$$

In [11], a more accurate description of the function $f_E(t)$ is given under additional regularity of the boundary of E; in fact, assuming $C^{1,1}$ regularity of ∂E, it is possible to prove that, for small time, the heat amount is essentially contained in a neighborhood $E_r \setminus E$, with

$$E_r = \{x \in \mathbb{R}^n : \mathrm{dist}(x,E) < r\},$$

that is $f_E(t) \sim f_{E,E_r\setminus E}(t)$. So, the fact that

$$\lim_{t\to 0} \frac{f_E(t)}{t} = \lim_{t\to 0} \frac{f_{E,E_r\setminus E}(t)}{t} = \frac{P(E)}{\sqrt{\pi}}$$

resembles the characterization of the perimeter measure by the Minkowski content

$$P(E) = \lim_{r\to 0} \frac{|E_r \setminus E|}{r},$$

justifying the terminology of *heat content* for $f_E(t)$. It is worth noticing that the heat content is much more accurate than the Minkowski content, since it gives the perimeter measure without any further condition on the regularity of the reduced boundary of E.
The study of the higher order expansion of $f_{E,F}$ as t goes to 0 is similar to the analogue computation for the Minkowski content. Indeed it has been proved that under suitable properties on E (for instance by Steiner in the case of convex sets, and by Weyl for C^2-regular sets and by Federer [7] for sets of positive reach), the quantity $|E_r|$, for r small enough, is a polynomial in r whose coefficients are geometric invariants of the set E.

However there is a crucial difference between the heat and the Minkowski content also in the class of convex sets. In fact, from the argument used, the expansion found does not reduce always to a polynomial in $\sqrt{t}$. This happens, up to a term that is infinitesimal of exponential type, when E is a polyhedral set, that is a finite intersection of halfspaces. It is easy to check that by taking the second derivative of $f_{E,F}$, the following formula

$$\begin{aligned} f''_{E,F}(t) &= -\frac{1}{(4\pi)^{n/2}t^{n+2}} \int_{\mathcal{F}F}\int_{\mathcal{F}E} \langle \nu_F(x), y-x\rangle\langle \nu_E(y), y-x\rangle e^{-\frac{|x-y|^2}{4t^2}} d\mathcal{H}^{n-1}(y)d\mathcal{H}^{n-1}(x) \\ &=: -\frac{1}{(4\pi)^{n/2}t^{n+2}} I_t(\mathcal{F}F;\mathcal{F}E) \end{aligned}$$

holds.

The behavior of the function $t \mapsto I_t(\mathcal{F}F;\mathcal{F}E)$ as $t \to 0^+$ is crucial in order to deduce an higher order expansion for $K_t(E,F)$. Indeed in this paper we make a deep analysis of the quantity $I_t(\Sigma;\Gamma)$ for two general oriented pieces of uniform $C^{1,1}$ hypersurfaces. In particular we are interested in the case when Σ and Γ are parts of the boundaries of sets of finite perimeter E, F and under mild assumption on $\partial E\cap\partial F$ we prove our main result which is stated in Theorem 1.1 below.

The proof of this theorem is a direct consequence of all the results contained in Section 3 and Section 4, Propositions 3.1, 3.3, 4.9 and 4.15 and of the occurrence of the simple formulas

$$\|T_t\chi_E\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} T_{2t}\chi_E dx = |E| - \int_{E^c} T_{2t}\chi_E dx = |E| - K_{2t}(E,E^c). \tag{4}$$

and

$$\|T_t\chi_E - \chi_E\|_{L^1(\mathbb{R}^n)} = 2\int_{E^c} T_t\chi_E(x)dx. \tag{5}$$

(see [11, Remark 3.5] for details about (5)).

To understand the statement, we refer to Section 2.1 for the definition of the mean curvature H^x_Σ and the square of the length of the second fundamental form c^2_Σ, Definition 4.1 for the notion of regular skeleton, and finally to Lemma 4.6 and Lemma 4.14 for the definitions of the quantities I^k_0 and $\delta_1 I^1_0$. Finally, we point out that this result can be easily extended to the case of two sets $E, F\subset\mathbb{R}^n$ with piecewice $C^{1,1}$–regularity, simply splitting

$$\partial E = (\partial E\cap F)\cup(\partial E\cap\bar{F}^c)\cup(\partial E\cap\partial F), \quad \partial F = (\partial F\cap E)\cup(\partial F\cap\bar{E}^c)\cup(\partial F\cap\partial E).$$

Theorem 1.1 *Let E be a set with finite perimeter such that ∂E is a finite union of $C^{1,1}$–regular surfaces; let us assume also that $\partial E\setminus\mathcal{F}E$ has regular skeleton and*

$$\partial E = \Sigma = \bigcup_{i=i}^m \Sigma_i.$$

Then, setting $A_i := \{j\neq i : S_{i,j} = \Sigma_i\cap\Sigma_j\neq\emptyset\}$, we get

$$\begin{aligned}\|T_t\chi_E\|^2_{L^2(\mathbb{R}^n)} &= |E| - \sqrt{\frac{2t}{\pi}}P(E) + t\sum_{i=1}^m\sum_{j\in A_i} I^1_0(\Sigma_i;\Sigma_j) + \\ &- \frac{\sqrt{2t^3}}{3}\left[\frac{(n-1)^2}{2\sqrt{\pi}}\int_\Sigma\left((H^x_\Sigma)^2 + \frac{2}{(n-1)^2}c^2_\Sigma\right)d\mathcal{H}^{n-1} - \sum_{i=1}^m\sum_{j\in A_i}\left(I^2_0(\Sigma_i;\Sigma_j) + \delta_1 I^1_0(\Sigma_i;\Sigma_j)\right)\right] \\ &+ o(t^{3/2}).\end{aligned}$$

and

$$\begin{aligned}\|T_t\chi_E - \chi_E\|_{L^1(\mathbb{R}^n)} &= 2\sqrt{\frac{t}{\pi}}P(E) - t\sum_{i=1}^m\sum_{j\in A_i} I^1_0(\Sigma_i;\Sigma_j) + \\ &+ \frac{\sqrt{t^3}}{3}\left[\frac{(n-1)^2}{2\sqrt{\pi}}\int_\Sigma\left((H^x_\Sigma)^2 + \frac{2}{(n-1)^2}c^2_\Sigma\right)d\mathcal{H}^{n-1} - \sum_{i=1}^m\sum_{j\in A_i}\left(I^2_0(\Sigma_i;\Sigma_j) + \delta_1 I^1_0(\Sigma_i;\Sigma_j)\right)\right] \\ &+ o(t^{3/2}).\end{aligned}$$

The paper is organized as follows; in Section 2 we fix the notations we shall use in the paper and we give the standing hypotheses on our surfaces $\mathcal{F}E$ and $\mathcal{F}F$; in Section 3 we consider the case of a single regular surface. We study the asymptotic behavior of $I_t(\partial E;\partial E)$ as t goes to 0, when ∂E is a piece of $C^{1,1}$–regular hypersurface proving that if E has finite perimeter, then

$$K_t(E,E^c)=\sqrt{\frac{t}{\pi}}P(E)-t\sqrt{t}\frac{(n-1)^2}{2\sqrt{\pi}}\int_{\partial E}\left((H^x_{\partial E})^2+\frac{2}{(n-1)^2}c^2_{\partial E}(x)\right)d\mathcal{H}^{n-1}(x),$$

where

$$H^x_{\partial E}=\frac{1}{n-1}\sum_{i=1}^{n-1}\kappa_i^{\partial E}(x),\qquad c^2_{\partial E}(x)=\sum_{i=1}^{n-1}\kappa_i^{\partial E}(x)^2.$$

and the $\kappa_i^{\partial E}(x)$'s are the principal curvatures of ∂E at x. In Section 4 we study the asymptotic behavior of $I_t(\partial E;\partial F)$ and the related expansion for $K_t(E,F)$. In this case, the coefficient of t is non trivial and depends on the presence of the singular set $\partial E\cap\partial F$. Under suitable hypothesis on the set $\partial E\cap\partial F$, we are able to go further in the expansion deducing also in this case the coefficient of $t\sqrt{t}$; we obtain this result essentially using the fact that the set $\partial E\cap\partial F$ has positive reach in both ∂E and ∂F. If ∂E and ∂F meet transversally, this is equivalent to say that $\partial E\cap\partial F$ has positive reach in $\mathbb{R}^n$, which is not too far to require that it is $C^{1,1}$–regular, that is in fact our standing hypothesis. Finally, examples of sets to which the main results of this paper apply are provided in Section 5.

We end this introduction by pointing out that a similar expansion should be true by considering convolution kernels other than the Gauss–Weierstrass one; for instance, a possibility is to take any positive symmetric regularizing kernel with some decay conditions at infinity.

2 Notations

In this section we fix the main definitions we shall use later. By $B_r(x)$ we denote the open ball centered at x and with radius $r>0$; if $x=0$, we simply write B_r. We also denote by B_r^+ the set of points $y\in B_r\subset\mathbb{R}^n$ such that $y_n>0$. Given a set $M\subset\mathbb{R}^n$, we shall use the notation $M^x_r=M\cap B_r(x)$.

We shall use the notations introduced by Federer in [7]; in particular, given a set $M\subset\mathbb{R}^n$, we define the tangent cone of M at x by

$$\mathrm{Tan}(M,x)=\left\{\lambda u:u=\lim_{M\ni y\to x}\frac{y-x}{|y-x|},\lambda\geq 0\right\}.$$

If $\mathrm{Tan}(M,x)$ is a vector space, we shall denote it by T_xM; if in general $\mathrm{Tan}(M,x)$ is only a cone, we shall denote by T_xM its span, that is the smallest vector space containing it. We shall also denote by $\Pi^x_M:\mathbb{R}^n\to T_xM$ the orthogonal projection on T_xM.

We also denote the normal space to M at x by

$$\mathrm{Nor}(M,x)=\{v\in\mathbb{R}^n:\langle v,u\rangle\leq 0,\quad\forall u\in\mathrm{Tan}(M,x)\};$$

by N_xM we shall denote the orthogonal complement of T_xM, that is the linear space such that

$$\mathbb{R}^n=T_xM\otimes N_xM.$$

We define $M^x_r=M\cap B_r(x)$. We recall that a map φ is said to be L–bilipschitz, $L\geq 1$, if

$$\frac{1}{L}|y_1-y_2|\leq|\varphi(y_1)-\varphi(y_2)|\leq L|y_1-y_2|.$$

Definition 2.1 *Let M be a closed subset of $\mathbb{R}^n$ such that $\mathcal{H}^m(M \setminus i_m(M)) = 0$, $m > 0$, where*

$$i_m(M) = \{x \in M : \exists \varrho > 0 \text{ with } M^x_\varrho \text{ is } C^{1,1}\text{–diffeomorphic to an open set of } \mathbb{R}^m\}$$

is m–dimensional interior part of M. We shall say that:

- *M is* a piece of an uniform $(C^{1,1}, m, L, r)$–regular manifold, *$r > 0$ and $L \geq 1$, if for any $x \in M$ there exists a $C^{1,1}$ L bilipschitz map $\varphi^x_M : B_{Lr} \subset \mathbb{R}^m \to \mathbb{R}^n$ such that*

  1. $\varphi^x_M(0) = x$ and $\{\partial_i \varphi^x_M(0)\}_{i=1,\ldots,m}$ orthonormal;

  2. $\varphi^x_M(B_{r/L}) \subset M^x_r \subset \varphi^x_M(B_{Lr})$.

- *M is $(C^{1,1}, m, L, r)$*–regular at $x \in M$ *if*

  (i) *for $x \in i_m(M)$, in addition to requirements 1. and 2., we have that*

  $$\varphi^x_M(B_{\frac{r}{L}}) \subset M^x_r \subset \varphi^x_M(B_{Lr});$$

  (ii) *for $x \in M \setminus i_m(M)$, in addition to requirements 1. and 2., we have that*

  $$\varphi^x_M(B^+_{\frac{r}{L}}) \subset M^x_r \subset \varphi^x_M(B^+_{Lr});$$

We shall simply say a piece of and $C^{1,1}$–regular manifold if the dimension m is clear from the context and there exist $L, r > 0$ such that the manifold is a piece of or $(C^{1,1}, m, L, r)$–regular manifold. We extend this definition to the case $m = 0$ by meaning that M reduces to a single point.

Remark 2.2 In view of the previous definition, if M is $C^{1,1}$–regular and $x \in i_m(M)$, we shall use the same notation M^x_r to mean both, $M \cap B_r(x)$ or the image $\varphi^x_M(B_r)$ (or $\varphi^x_M(B_r) \cap M$ in case M is a piece of $C^{1,1}$–regular manifold). Moreover, if the part of M inside $B_r(x)$ is strictly contained in $\varphi^x_M(B_r)$, we can extend M adding a disjoint set $\widetilde{M}$ to M with $M^x_r \cup \widetilde{M} = \varphi^x_M(B_r)$. The same argument can be repeated also in the case $x \in M \setminus i_m(M)$, replacing B_r with B^+_r. We shall call this procedure the *tangential completion* of M at x.

2.1 Manifolds of codimension one

We consider now the case $M = \Sigma$ a surface of dimension $(n-1)$; with a little abuse of notation, we shall write $I_t(\Sigma; \Sigma^x_r)$ to mean the integral

$$I_t(\Sigma; \Sigma^x_r) = \int_\Sigma d\mathcal{H}^{n-1}(x) \int_{\Sigma^x_r} \langle \nu_\Sigma(x), y - x \rangle \langle \nu_\Sigma(y), y - x \rangle e^{-\frac{|y-x|^2}{4t^2}} d\mathcal{H}^{n-1}(y).$$

The second fundamental form $\vec{\mathrm{II}}^x_\Sigma : T_x\Sigma \times T_x\Sigma \to N_x\Sigma$ for an hypersurface Σ at a point x is a bilinear map and is related to the scalar second fundamental form A^x_Σ by equality

$$\vec{\mathrm{II}}^x_\Sigma(v, w) = A^x_\Sigma(v, w)\nu_\Sigma(x), \qquad \forall v, w \in T_x\Sigma.$$

The principal curvatures $\kappa^x_{\Sigma,i}$, $i = 1, \ldots, n-1$, are defined as the eigenvalues of A^x_Σ; moreover, fixed a vector $v \in T_x\Sigma$, the sectional curvature in x of Σ in direction v is defined as

$$\kappa^x_\Sigma[v] = A^x_\Sigma(v, v).$$

In this way, if $\{v_i\}_{i=1,\dots,n-1}$ are the eigenvectors of A^x_Σ associated to the $\kappa^x_{\Sigma,i}$'s, we have

$$\kappa^x_{\Sigma,i} = \kappa^x_\Sigma[v_i].$$

If no confusion may arise, we simply denote by κ_i the principal curvatures.

The mean curvature of Σ in x is defined by

$$H^x_\Sigma = \frac{1}{n-1}\sum_{i=1}^{n-1}\kappa_i;$$

in addition, the square of the length of the second fundamental form is given by

$$c^2_\Sigma(x) = \sum_{i=1}^{n-1}\kappa_i^2.$$

Remark 2.3 With the previous assumptions, we notice that if Σ is a piece of a $C^{1,1}$–regular hypersurface, then for every $x \in i_{n-1}(\Sigma)$ and r is small enough, we can parametrize Σ^x_r by

$$\varphi^x_\Sigma : A^x_r \to \Sigma^x_r, \qquad \varphi^x_\Sigma(w) = x + w + u(w)\nu_\Sigma(x), \tag{6}$$

where $A^x_r = \Pi^x_\Sigma(\Sigma^x_r)$ is an open set containing 0 and diameter less then r and $u(0) = 0$; u is a function with the same regularity in 0 as that of the surface Σ in x, then in particular u is differentiable in 0 and $\nabla u(0) = 0$. Using the Lipschitz regularity of ∇u, we also obtain that

$$|\nabla u(w)| \le L|w|, \qquad |u(w)| \le \frac{L}{2}|w|^2. \tag{7}$$

In other terms, Σ^x_r is contained in the graph of u on the tangent space $T_x\Sigma$; if x has been chosen in such a way that u is twice differentiable in 0, then the second fundamental form of Σ in x is then given by

$$\vec{\text{II}}^x_\Sigma(w,z) = \langle Hu(0)w, z\rangle \nu_\Sigma(x) \tag{8}$$

and the eigenvalues of $Hu(0)$ are the principal curvatures of Σ at x. For the normal unit field $\nu_\Sigma : \Sigma \to \mathbb{S}^{n-1}$, we have that for $y \in \Sigma^x_r$

$$|\nu_\Sigma(y) - \nu_\Sigma(x)| \le L|y - x|; \tag{9}$$

moreover, $d_x\nu_\Sigma : T_x\Sigma \to T_{\nu_x}\mathbb{S}^{n-1} = T_x\Sigma$ and, by setting $\gamma^x_z(t) = x + tz + u(tz)\nu_\Sigma(x)$, the following holds

$$d_x\nu_\Sigma[z] = \frac{d}{dt}\nu_\Sigma(\gamma^x_z(t))_{|t=0} = \lim_{t\to 0}\frac{\nu_\Sigma(x + tz + u(tz)\nu_\Sigma(x)) - \nu_\Sigma(x)}{t} = -Hu(0)z. \tag{10}$$

Finally, if $x \in \Sigma \setminus i_{n-1}(\Sigma)$, we can define as before the parametrization $\varphi^x_\Sigma : A^x_r \to \Sigma^x_r$, but in this case $0 \in \partial A^x_r$. By the $C^{1,1}$ assumption, the function u can be extended to 0 in such a way that $u(0) = \nabla u(0) = 0$, and (7)-(10) continue to hold.

2.2 Submanifolds of higher codimension

Let $M \subset \mathbb{R}^n$ be a manifold with codimension $k > 1$; then, for any $x \in M$, T_xM and N_xM have, respectively, dimension $n-k$ and k. The second fundamental form is a bilinear form $\vec{\mathrm{II}}^x_M : T_xM \times T_xM \to N_xM$, so for any $\eta \in N_xM$ we have the scalar second fundamental form in direction η

$$A^x_{M,\eta}(v,w) = \left\langle \vec{\mathrm{II}}^x_M(v,w), \eta \right\rangle, \qquad \forall v,w \in T_xM.$$

The principal curvatures of M in direction η are the eigenvalues of $A^x_{M,\eta}$, and are denoted by $(\kappa^x_{M,\eta,i})_{i=1,\dots,n-k}$. It is then defined the mean curvature of M at x in direction η by the identity

$$H^x_M[\eta] = \frac{1}{n-k} \sum_{i=1}^{n-k} \kappa^x_{M,\eta,i}.$$

Finally, if M is a parametrized manifold and $\varphi : A \to \mathbb{R}^n$, $A \subset \mathbb{R}^{n-k}$ is an open set with $\varphi(0) = x$, then the metric tensor at x is given by

$$g_{i,j}(x) = \langle \partial_i \varphi(0), \partial_j \varphi(0) \rangle;$$

using such a metric, the quadratic form $A^x_{M,\eta}$ is determined by the matrix

$$\sum_{h=1}^{n-k} g^{i,h} \langle \partial^2_{h,j} \varphi(0), \eta \rangle,$$

where g^{ih} are the coefficients of the inverse of the metric g. In particular, if M is a piece of a $C^{1,1}$–regular manifold, $x \in M$ and φ^x_M given in Definition 2.1, then the metric induced by φ^x_M is the identity in x, so that the second fundamental form is determined by the matrix

$$(A^x_{M,\eta})_{i,j} = \langle \partial^2_{i,j} \varphi(0), \eta \rangle;$$

as a consequence, the mean curvature of M at x in direction η is given by

$$H^x_M[\eta] = \frac{1}{n-k} \sum_{i=1}^{n-k} \langle \partial^2_{i,i} \varphi(0), \eta \rangle.$$

3 The functional $I_t(\Sigma)$

In this section we study $I_t(\Sigma)$ for Σ a piece of $C^{1,1}$–regular hypersurface. First of all, due to the exponential map, the part of I_t with $|y-x| \geq r$, $r > 0$ fixed, goes to 0 exponentially as $t \to 0$. With a little abuse of notation, we shall write $I_t(\Sigma, \Sigma^x_r)$ by meaning that $x \in \Sigma$ is a fixed point in the first integral of I_t and $\Sigma^x_r = \{y \in \Sigma : |y-x| < r\}$. With the change of variable $z = \frac{y-x}{t}$, we can write

$$\begin{aligned} I_t(\Sigma, \Sigma^x_r) =& t^{n+1} \int_\Sigma d\mathcal{H}^{n-1}(x) \int_{\frac{\Sigma^x_r - x}{t}} \langle \nu_\Sigma(x), z \rangle \langle \nu_\Sigma(x+tz), z \rangle e^{-\frac{|z|^2}{4}} d\mathcal{H}^{n-1}(z) \\ =& t^{n+1} \int_\Sigma d\mathcal{H}^{n-1}(x) \int \langle \nu_\Sigma(x), z \rangle \langle \nu_\Sigma(x+tz), z \rangle e^{-\frac{|z|^2}{4}} d\mu_t(z). \end{aligned}$$

We then need a time expansion of both $\nu_\Sigma(x+tz)$ and the measures $\mu_t = \mathcal{H}^{n-1} \llcorner \left(\frac{\Sigma^x_r - x}{t} \right)$.

Proposition 3.1 *Let Σ be a piece of a $C^{1,1}$–regular hypersurface; then*

$$\lim_{t\to 0}\frac{I_t(\Sigma)}{(4\pi)^{n/2}t^{n+2}}=0. \tag{11}$$

PROOF. We start with the decomposition

$$I_t(\Sigma)=I_t(\Sigma;\Sigma_r^x)+I_t(\Sigma;\Sigma\setminus\Sigma_r^x).$$

A direct computation shows that if $2t<r$

$$|I_t(\Sigma;\Sigma\setminus\Sigma_r^x)|\le r^2e^{-\frac{r^2}{4t^2}}\left(\mathcal{H}^{n-1}(\Sigma)\right)^2, \tag{12}$$

and then

$$I_t(\Sigma;\Sigma)=I_t(\Sigma;\Sigma_r^x)+o(t^k),\qquad \forall k>0, \tag{13}$$

so we can restrict our attention on $I_t(\Sigma;\Sigma_r^x)$. We can write

$$I_t(\Sigma;\Sigma_r^x)=\int_\Sigma g_t^r(x)d\mathcal{H}^{n-1}(x);$$

we fix $x\in\Sigma$ and, with the change of variable $y=x+tz$ we obtain

$$\begin{aligned}g_t^r(x)=&t^{n+1}\int_{\frac{\Sigma_r^x-x}{t}}\langle\nu_\Sigma(x),z\rangle^2e^{-\frac{|z|^2}{4}}d\mathcal{H}^{n-1}(z)+\\&+t^{n+1}\int_{\frac{\Sigma_r^x-x}{t}}\langle\nu_\Sigma(x),z\rangle\langle\nu_\Sigma(x+tz)-\nu_\Sigma(x),z\rangle e^{-\frac{|z|^2}{4}}d\mathcal{H}^{n-1}(z).\end{aligned}$$

Using the parametrization φ_Σ^x of Remark 2.3 and a change of variable, we can write

$$\begin{aligned}g_t^r(x)=t^{n+1}\int_{A_r^x/t}&e^{-\frac{|w|^2}{4}}e^{-\frac{u(tw)^2}{4t^2}}\sqrt{1+|\nabla u(tw)|^2}\left(\frac{u(tw)^2}{t^2}+\right.\\&\left.+\frac{u(tw)}{t}\left\langle\nu_\Sigma(x+tw+u(tw)\nu_\Sigma(x))-\nu_\Sigma(x),w+\frac{u(tw)}{t}\nu_\Sigma(x)\right\rangle\right)dw.\end{aligned} \tag{14}$$

As a consequence, by (7) and (9), we get

$$|g_t^r(x)|\le t^{n+3}\frac{3}{2}L^5\int_{T_x\Sigma}|w|^4(1+|w|^2)^{3/2}e^{-\frac{|w|^2}{4}}dw=ct^{n+3} \tag{15}$$

with $c=c(L,n)$ a constant depending only on L and the integral of $|w|^4(1+|w|^2)^{3/2}e^{-\frac{|w|^2}{4}}$ on $\mathbb{R}^{n-1}$. This implies (11). □

Remark 3.2 In the previous proof the $C^{1,1}$–regularity essentially shows that the dominated convergence theorem can be used; the same argument, each time $C^{1,1}$–regularity holds, can be used in the sequel, also in the case of manifolds with higher codimension, To keep the proofs a little shorter, we shall use the dominated convergence without repeating the check of it.

We can go further in the expansion, in order to obtain the following result.

Theorem 3.3 *Let Σ be a piece of a $C^{1,1}$–regular hypersurface; then*

$$
\begin{aligned}
\lim_{t\to 0}\frac{I_t(\Sigma)}{(4\pi)^{n/2}t^{n+3}} &= -\frac{1}{4(4\pi)^{n/2}}\int_\Sigma d\mathcal{H}^{n-1}(x)\int_{T_x\Sigma} A^x_\Sigma(z,z)^2 e^{-\frac{|z|^2}{4}}\,dz \\
&= -\frac{(n-1)^2}{2\sqrt{\pi}}\int_\Sigma \left((H^x_\Sigma)^2+\frac{2}{(n-1)^2}c^2_\Sigma(x)\right)d\mathcal{H}^{n-1}(x).
\end{aligned}
\tag{16}
$$

Proof. Thanks to estimate (15), we can use the dominated convergence and compute the pointwise limit

$$
\lim_{t\to 0}\frac{g^r_t(x)}{(4\pi)^{n/2}t^{n+3}}.
$$

So we fix $x\in\Sigma$ and denote simply by ν the vector $\nu_\Sigma(x)$; we also use the parametrization φ^x_Σ of Remark 2.3. We can also assume that the point x has been chosen in such a way that $u\in C^{1,1}$ and u twice differentiable in 0. By (14), we can write

$$
\begin{aligned}
g^r_t(x) =& t^{n+3}\int_{A^x_r/t} e^{-\frac{|w|^2}{4}-\frac{|u(tw)|^2}{4t^2}}\sqrt{1+|\nabla u(tw)|^2}\Big(\frac{u(tw)^2}{t^4}+ \\
&+\frac{u(tw)}{t^2}\left\langle\frac{\nu_\Sigma(x+tw+u(tw)\nu)-\nu}{t}, w+\frac{u(tw)}{t}\nu\right\rangle\Big)dw
\end{aligned}
$$

where in the last integral we have performed the change of variable $z=tw$. By the dominated convergence, it suffices to consider the pointwise limits as $t\to 0$ of $u(tw)/t$, $\nabla u(tw)$ and $u(tw)/t^2$. We have that

$$
\lim_{t\to 0}\frac{u(tw)}{t}=0,\qquad \lim_{t\to 0}\nabla u(tw)=0,\qquad \lim_{t\to 0}\frac{u(tw)}{t^2}=\frac{1}{2}\langle Hu(0)w,w\rangle.
$$

We also have that

$$
\lim_{t\to 0}\left\langle\frac{\nu_\Sigma(x+tw+u(tw)\nu)-\nu}{t}, w+\frac{u(tw)}{t}\nu\right\rangle = -\langle Hu(0)w,w\rangle,
$$

as can be easily proved by assuming $\nu=e_n$, the last element of the canonical base of $\mathbb{R}^n$ and by writing

$$
\nu_\Sigma(x+tw+u(tw)\nu)=\frac{(-\nabla u(tw),1)}{\sqrt{1+|\nabla u(tw)|^2}}.
$$

Summarizing, we can conclude that

$$
\lim_{t\to 0}\frac{g^r_t(x)}{(4\pi)^{n/2}t^{n+3}} = -\frac{1}{4(4\pi)^{n/2}}\int_{T_x\Sigma}\langle Hu(0)w,w\rangle^2 e^{-\frac{|w|^2}{4}}\,dw := I(x).
$$

We take then an orthonormal basis in $T_x\Sigma$ of eigenvector $(\tau_i)_{i=1,\dots,n-1}$ with $Hu(0)\tau_i = \kappa_i\tau_i$, κ_i the principal curvatures of Σ at x; so we can write $w = \sum_{i=1}^{n-1} w_i\tau_i$ so that we have

$$\begin{aligned}
I(x) &= -\frac{1}{4(4\pi)^{n/2}}\int_{T_x\Sigma} \langle Hu(0)w, w\rangle^2 e^{-\frac{|w|^2}{4}}\,dw \\
&= -\frac{1}{4(4\pi)^{n/2}}\int_{\mathbb{R}^{n-1}} \Big(\sum_{i=1}^{n-1}\kappa_i w_i^2\Big)^2 e^{-\frac{w_1^2+\ldots+w_{n-1}^2}{4}}\,dw_1\ldots dw_{n-1} \\
&= -\frac{1}{4(4\pi)^{n/2}}\sum_{i,j=1}^{n-1}\kappa_i\kappa_j\int_{\mathbb{R}^{n-1}} w_i^2 w_j^2 e^{-\frac{|w|^2}{4}}\,dw \\
&= -\frac{1}{2\sqrt{\pi}}\sum_{i,j=1}^{n-1}\kappa_i\kappa_j - \frac{1}{\sqrt{\pi}}\sum_{i=1}^{n-1}\kappa_i^2 = -\frac{(n-1)^2}{2\sqrt{\pi}}(H_\Sigma^x)^2 - \frac{1}{\sqrt{\pi}}c_\Sigma^2(x).
\end{aligned}$$

□

4 The functional $I_t(\Sigma;\Gamma)$

In this section we study the quantity $I_t(\Sigma;\Gamma)$ for two oriented pieces of uniform $C^{1,1}$ hypersurfaces. In this computation, a crucial rôle is played by the intersection $S = \Sigma\cap\Gamma$ of the two manifolds. Since we are interested in the case when Σ and Γ are parts of the boundaries of sets of finite perimeter, we can restrict to the case when for $\mathcal{H}^{n-2}$–almost every $x_0 \in S$, $\mathrm{Tan}(\Sigma, x_0)$ and $\mathrm{Tan}(\Gamma, x_0)$ are both half hyperplanes. On S we shall always assume the following regularity.

Definition 4.1 (Regular skeleton) *We shall say that a set S with $\mathcal{H}^{n-2}(S) < +\infty$ has regular skeleton if, denoting by $S_0 = S$, the sets $S_k = S_{k-1}\setminus i_{n-k}(S_{k-1})$ $k = 1,\dots,n-1$ are finite union of closed pieces of $C^{1,1}$–regular $(n-1-k)$–dimensional manifolds. Notice that with this definition we have that $S_1 = S_0$.*

Given Σ and Γ, we define the sets

$$\Sigma_r = \{x\in\Sigma : \mathrm{dist}(x,S)\le r\}, \qquad \Gamma_r = \{y\in\Gamma : \mathrm{dist}(y,S)\le r\};$$

we point out that if $y\in\Gamma\setminus\Gamma_r$, then

$$\mathrm{dist}(y,\Sigma)\ge cr \tag{17}$$

for some positive constant $c > 0$. This in particular implies that

$$I_t(\Sigma;\Gamma) = I_t(\Sigma_r;\Gamma_r) + o(t^m), \qquad \forall m\in\mathbb{N}. \tag{18}$$

In fact, we can write

$$I_t(\Sigma;\Gamma) = I_t(\Sigma_r;\Gamma_r) + I_t(\Sigma\setminus\Sigma_r;\Gamma_r) + I_t(\Sigma;\Gamma\setminus\Gamma_r),$$

and, as in (12), there holds

$$|I_t(\Sigma;\Gamma\setminus\Gamma_r)| \le r^2 e^{-\frac{r^2}{4t^2}}\mathcal{H}^{n-1}(\Sigma)\mathcal{H}^{n-1}(\Gamma). \tag{19}$$

Same estimate holds for $I_t(\Sigma \setminus \Sigma_r; \Gamma_r)$.

From now on we consider Σ_r; the same considerations and constructions can be done also for Γ_r. Most of this considerations and notations are based on the paper of Federer [7] and Zähle [19] and a fundamental tool is the Federer coarea formula (see also Thäle [16]); our setting is a little bit simpler, since we are assuming regular skeleton. If r is small enough, it is well defined the projection $\pi : \Sigma_r \to S$; for any $k = 0, \dots, n-1$, we define the sets

$$S^k = \{x_0 \in S : \dim \pi^{-1}(x_0) = k\}, \qquad \Sigma_r^{k,x_0} = \pi^{-1}(x_0), x_0 \in S^k.$$

By the fact that S has regular skeleton, we deduce that

$$i_{n-1-k}(S_k) \subset S^k \subset S_k, \qquad k = 1, \dots n-1 \tag{20}$$

and, since $\mathcal{H}^{n-1-k}(S_{k+1}) = 0$, $\mathcal{H}^{n-1-k}$ a.e. point of S_k belongs to S^k. The sets $\Sigma_r^k = \pi^{-1}(S^k)$ are $(n-1)$–dimensional manifolds for any $k \geq 1$, while $\mathcal{H}^{n-1}(\Sigma_r^0) = 0$; we then have the following disjoint decomposition

$$\Sigma_r = \bigcup_{k=0}^{n-1} \Sigma_r^k. \tag{21}$$

Moreover, by (20), for $\mathcal{H}^{n-1-k}$ a.e. $x_0 \in S^k$, we have the decomposition

$$\mathrm{Tan}(\Sigma_r^k, x_0) = T_{x_0} S^k \oplus \mathrm{Tan}(\Sigma_r^{k,x_0}, x_0). \tag{22}$$

In the same way we can define the sets Γ_r^k and Γ_r^{k,x_0}. For $\mathcal{H}^{n-1-k}$ a.e. $x_0 \in S^k$ the cones $\mathrm{Tan}(\Sigma_r^{k,x_0}, x_0)$ and $\mathrm{Tan}(\Gamma_r^{k,x_0}, x_0)$ are k–dimensional and are generated by k vectors that we shall denote by $\{\sigma_j^k(x_0)\}_{j=1,\dots,k}$ and $\{\gamma_j^k(x_0)\}_{j=1,\dots,k}$. Eventually dividing Σ and Γ in two parts in case that they pass through each other, we can assume that $\mathrm{Tan}(\Sigma_r^{k,x_0}, x_0)$ and $\mathrm{Tan}(\Gamma_r^{k,x_0}, x_0)$ are positive cones, meaning by positive cone generated by the vectors $v_1, \dots, v_m$ any combination of the type

$$\lambda_1 v_1 + \dots + \lambda_m v_m, \qquad \lambda_i \geq 0, \ i = 1, \dots, m.$$

We give the following definition.

Definition 4.2 *We say that Σ meets Γ transversally if there exists $c > 0$ such that for almost every $x_0 \in S$, $-1 + c \leq \langle \sigma_1^1(x_0), \gamma_1^1(x_0) \rangle \leq 1 - c$, or equivalently $|\sigma_1^1(x_0) \wedge \gamma_1^1(x_0)| \geq c$.*

Remark 4.3 In what follows, the transversality can also be replaced by a weaker condition, requiring the existence of a function $\omega \in L^1(S, \mathcal{H}^{n-2})$ such that

$$\frac{1}{|\sigma_1^1(x_0) \wedge \gamma_1^1(x_0)|} \leq \omega(x_0), \qquad \mathcal{H}^{n-2} - \text{a.e. } x_0 \in S.$$

We also define, for $\mathcal{H}^{n-1-k}$ a.e. $x_0 \in S^k$, the cone

$$V_{x_0}^k = \mathrm{Tan}(\Gamma_r^{k,x_0}, x_0) - \mathrm{Tan}(\Sigma_r^{k,x_0}, x_0);$$

which can be parametrized by using the maps $Q\Sigma_{x_0}^k : \mathbb{R}_+^k \to \mathrm{Tan}(\Sigma_r^{k,x_0}, x_0)$, $Q\Gamma_{x_0}^k : \mathbb{R}_+^k \to \mathrm{Tan}(\Gamma_r^{k,x_0}, x_0)$

$$Q\Sigma_{x_0}^k(\alpha) = \sum_{i=1}^k \alpha_i \sigma_i^k(x_0), \qquad Q\Gamma_{x_0}^k(\beta) = \sum_{i=1}^k \beta_i \gamma_i^k(x_0);$$

such a parametrization is given by $Q^k_{x_0} : \mathbb{R}^{2k}_+ \to V^k_{x_0}$,

$$Q^k_{x_0}(\alpha, \beta) = Q\Gamma^k_{x_0}(\beta) - Q\Sigma^k_{x_0}(\alpha)$$

and is determined by the matrix

$$\begin{pmatrix} -\sigma^k_1(x_0) & \dots & -\sigma^k_k(x_0) & \gamma^k_1(x_0) & \dots & \gamma^k_k(x_0) \end{pmatrix}. \tag{23}$$

The dimension of $V^k_{x_0}$ is given by the rank of (23), that is, due to the transversality condition on Σ and Γ, always equals to $k+1$. We shall then denote by $J_k Q^k_{x_0}$ the factor

$$D_k Q^k_{x_0} = \begin{cases} |\sigma^1_1(x_0) \wedge \gamma^1_1(x_0)| & \text{if } k = 1 \\ \\ \dfrac{J_k dQ\Sigma^k_{x_0} J_k dQ\Gamma^k_{x_0}}{C_k dQ^k_{x_0}} & \text{if } k > 1, \end{cases} \tag{24}$$

where by $J_k dQ\Sigma^k_{x_0}$ and $J_k dQ\Gamma^k_{x_0}$ we denote the area factor of the maps $Q\Sigma^k_{x_0}$ and $Q\Gamma^k_{x_0}$, and by $C_k dQ^k_{x_0}$ the coarea factor associated to $Q^k_{x_0}$.

For the restrictions $\pi_k : \Sigma^k_r \to S^k$ of π, we can consider, for any measurable function g, the coarea formula

$$\int_{\Sigma^k_r} g(x) C_k d_x \pi_k d\mathcal{H}^{n-1}(x) = \int_{S^k} d\mathcal{H}^{n-1-k}(x_0) \int_{\Sigma^{k,x_0}_r} g(x) d\mathcal{H}^k(x),$$

where $C_k d_x \pi_k$ is the coarea factor associated to π_k. Equivalently, since $C_k d_x \pi_k \neq 0$ on Σ^k_r,

$$\int_{\Sigma^k_r} g(x) d\mathcal{H}^{n-1}(x) = \int_{S^k} d\mathcal{H}^{n-1-k}(x_0) \int_{\Sigma^{k,x_0}_r} \frac{g(x)}{C_k d_x \pi_k} d\mathcal{H}^k(x).$$

Remark 4.4 We can write the coarea factor using local parametrizations; we do it explicitly since we shall use it in Lemma 4.5. Using a partition of unity argument, we can assume that Σ^k_r is a parametrized surface in a neighbourhood of a fixed point $x_0 \in S^k$. We shall then assume to have a map $\psi : A^k_{x_0} \times I^k_{x_0} \to \mathbb{R}^n$ with the following properties:

1. $A^k_{x_0}$ is an open subset of $\mathbb{R}^{n-1-k}$ and $I^k_{x_0} \subset \mathbb{R}^k$, both sets containing 0;
2. $\psi(0,0) = x_0$;
3. the map $a \mapsto \phi(a) = \psi(a, 0)$ is a parametrization for S^k;
4. for any $a \in A^k_{x_0}$, the map $a' \mapsto \psi(a, a')$ is a parametrization of $\Sigma^k_r \cap \mathrm{Nor}(S^k, \phi(a))$;
5. the set $\{\partial_i \psi(0,0)\}_{i=1,\dots,n-1-k}$ is an orthonormal basis of $T_{x_0} S^k$ and

$$\partial_{n-1-k+j} \psi(0,0) = \sigma^k_j(x_0), \qquad j = 1, \dots, k.$$

We shall also write ψ^{-1} and ϕ^{-1} to denote the inverses of ψ and ϕ defined on Σ^k_r and S^k respectively. With the previous assumptions, it is clear that

$$\pi_k(\psi(a, a')) = \phi(a).$$

Moreover, the parametrization ψ induces then metrics g_{n-1}, g_{n-1-k} and g_k on Σ^k_r, S^k and $\Sigma^{k,y}_r$, $y \in S^k$ so that the coarea factor tunrs out to be given by

$$C_k d_x \pi_k = \sqrt{\frac{\det[g_{n-1-k}(\phi^{-1}(\pi_k(x)), 0)] \det[g_k(\psi^{-1}(x))]}{\det[g_{n-1}(\psi^{-1}(x))]}}. \tag{25}$$

The following lemma contains the main properties of the coarea factor $C_k d_{x_0}\pi_k$ we use in the sequel; we omit the proof since it uses standard arguments.

Lemma 4.5 *Let $C_k d\pi_k$ be the k–th coarea factor associated to π_k; then, if $x_0 \in S^k$,*

$$C_k d_{x_0}\pi_k = 1, \tag{26}$$

while for $z \in \mathrm{Tan}(\Sigma_r^{k,x_0}, x_0)$,

$$d_{x_0} C_k d_{x_0}\pi_k[z] = (n-1-k) H^{x_0}_{S^k}[z], \tag{27}$$

where $H^{x_0}_{S^k}[z]$ is the mean curvature of S^k at x_0 in direction $z \in N_{x_0}S^k$.

Using the decomposition (21), we can write

$$I_t(\Sigma_r;\Gamma) = \sum_{k=1}^{n-1} I_t(\Sigma_r^k;\Gamma); \tag{28}$$

the following result holds.

Lemma 4.6 *The quantity $I_t(\Sigma_r^k;\Gamma)$ is asymptotic to t^{n+k+1} and there holds*

$$I_0^k(\Sigma;\Gamma) := \lim_{t\to 0} \frac{I_t(\Sigma_r^k;\Gamma)}{(4\pi)^{n/2} t^{n+1+k}} = \int_{S^k} \Theta_k(x_0) d\mathcal{H}^{n-1-k}(x_0),$$

where, if $k=1$,

$$\Theta_1(x_0) = \frac{1}{4\pi|\sigma^1(x_0)\wedge\gamma_1^1(x_0))|} \int_{V^1_{x_0}} \langle \nu_\Sigma(x_0), v\rangle \langle \nu_\Gamma(x_0), v\rangle \exp\left(-\frac{|v|^2}{4}\right) dv$$

and if $k>1$,

$$\Theta_k(x_0) = \frac{D_k Q^k_{x_0}}{(4\pi)^{\frac{k+1}{2}}} \int_{V^k_{x_0}} \langle \nu_\Sigma(x_0), v\rangle \langle \nu_\Gamma(x_0), v\rangle e^{-\frac{|v|^2}{4}} \mathcal{H}^{k-1}((Q^k_{x_0})^{-1}(v)) dv$$

with $D_k Q^k_{x_0}$ given by (24).

Proof. First of all we note that, using the changes of variables $x = x_0 + tz$ and $y = x_0 + tw$,

$$\begin{aligned} I_t(\Sigma_r^k;\Gamma) &= \int_{\Sigma_r^k} d\mathcal{H}^{n-1}(x) \int_\Gamma \langle \nu_\Sigma(x), y-x\rangle \langle \nu_\Gamma(x), y-x\rangle e^{-\frac{|y-x|^2}{4t^2}} d\mathcal{H}^{n-1}(y) \\ &= \int_{S^k} d\mathcal{H}^{n-1-k}(x_0) \int_{\Sigma_r^{k,x_0}} d\mathcal{H}^k(x) \int_\Gamma \frac{\langle \nu_\Sigma(x), y-x\rangle \langle \nu_\Gamma(x), y-x\rangle}{C_k d_x \pi_k} e^{-\frac{|y-x|^2}{4t^2}} d\mathcal{H}^{n-1}(y) \\ &= t^{n+1+k} \int_{S^k} d\mathcal{H}^{n-1-k}(x_0) \int_{\frac{\Sigma_r^{k,x_0}-x_0}{t}} d\mathcal{H}^k(z) \int_{\frac{\Gamma - x_0}{t}} F^k_{x_0}(t,z,w) d\mathcal{H}^{n-1}(w), \end{aligned}$$

with

$$F^k_{x_0}(t,z,w) = \frac{\langle \nu_\Sigma(x_0+tz), w-z\rangle \langle \nu_\Gamma(x_0+tw), w-z\rangle}{C_k d_{x_0+tz}\pi_k} e^{-\frac{|w-z|^2}{4}}.$$

Using the convergences

$$\lim_{t\to 0} F^k_{x_0}(t,z,w) = \langle \nu_\Sigma(x_0), w-z\rangle\langle \nu_\Gamma(x_0), w-z\rangle e^{-\frac{|w-z|^2}{4}},$$

$$\mathcal{H}^k \llcorner \left(\frac{\Sigma^{k,x_0}_r - x_0}{t}\right) \rightharpoonup \mathcal{L}^k \llcorner \operatorname{Tan}(\Sigma^{k,x_0}_r, x_0), \qquad \mathcal{H}^{n-1} \llcorner \left(\frac{\Gamma - x_0}{t}\right) \rightharpoonup \mathcal{L}^{n-1} \llcorner \operatorname{Tan}(\Gamma, x_0)$$

and the decomposition $\operatorname{Tan}(\Gamma, x_0) = T_{x_0}S^k \oplus \operatorname{Tan}(\Gamma^{k,x_0}_r, x_0)$, we get

$$\lim_{t\to 0} \frac{I_t(\Sigma^k_r;\Gamma)}{(4\pi)^{n/2} t^{n+1+k}} = \int_{S^k} \Theta_k(x_0)\, d\mathcal{H}^{n-1-k}(x_0),$$

with

$$\Theta_k(x_0) = \frac{1}{(4\pi)^{\frac{k+1}{2}}} \int_{\operatorname{Tan}(\Sigma^{k,x_0}_r, x_0)} dz \int_{\operatorname{Tan}(\Gamma^{k,x_0}_r, x_0)} \langle \nu_\Sigma(x_0), w-z\rangle\langle \nu_\Gamma(x_0), w-z\rangle e^{-\frac{|w-z|^2}{4}} dw.$$

To compute the last integral we distinguish the cases $k=1$ and $k>1$; in the first case we parametrize $\operatorname{Tan}(\Sigma^{1,x_0}, x_0)$ and $\operatorname{Tan}(\Gamma^{1,x_0}, x_0)$ using the maps

$$Q\Sigma^1_{x_0}(\alpha) = \alpha\sigma^1_1(x_0), \qquad Q\Gamma^1_{x_0}(\beta) = \beta\gamma^1_1(x_0),$$

whose area factor is 1, so that we obtained

$$\begin{aligned}\Theta_1(x_0) =& \frac{1}{4\pi}\int_{\mathbb{R}^2_+} \big\langle \nu_\Sigma(x_0), Q^1_{x_0}(\alpha,\beta)\big\rangle\big\langle \nu_\Gamma(x_0), Q^1_{x_0}(\alpha,\beta)\big\rangle e^{-\frac{|Q^1_{x_0}(\alpha,\beta)|^2}{4}} d(\alpha,\beta)\\ =& \frac{1}{4\pi|\sigma^1_1(x_0)\wedge\gamma^1_1(x_0)|}\int_{V^1_{x_0}} \langle \nu_\Sigma(x_0), v\rangle\langle \nu_\Gamma(x_0), v\rangle e^{-\frac{|v|^2}{4}} dv\end{aligned}$$

The case $k>1$ is similar, but since $Q^k_{x_0}$ maps $\mathbb{R}^{2k}$ onto the $(k+1)$–dimensional cone $V^k_{x_0}$, we have to use the coarea formula, that is

$$\begin{aligned}\Theta_k(x_0) =& \frac{J_k dQ\Sigma^k_{x_0} J_k dQ\Gamma^k_{x_0}}{(4\pi)^{\frac{k+1}{2}}}\int_{\mathbb{R}^{2k}_+} \big\langle \nu_\Sigma(x_0), Q^k_{x_0}(\alpha,\beta)\big\rangle\big\langle \nu_\Gamma(x_0), Q^k_{x_0}(\alpha,\beta)\big\rangle e^{-\frac{|Q^k_{x_0}(\alpha,\beta)|^2}{4}} d(\alpha,\beta)\\ =& \frac{D_k Q^k_{x_0}}{(4\pi)^{\frac{k+1}{2}}}\int_{V^k_{x_0}} \langle \nu_\Sigma(x_0), v\rangle\langle \nu_\Gamma(x_0), v\rangle e^{-\frac{|v|^2}{4}} \mathcal{H}^{k-1}((Q^k_{x_0})^{-1}(v)) dv.\end{aligned}$$

□

Remark 4.7 We point out some consequence and possible generalization of the previous result; it states that the quantity $I_t(\Sigma;\Gamma)$ is infinitesimal of order strictly related to the dimension of the intersection $\Sigma\cap\Gamma$. In general, if M_1 and M_2 are two oriented manifolds meeting transversally of dimensions m_1 and m_2 respectively, $S_{12} = M_1\cap M_2$ has dimension $m < \min\{m_1, m_2\}$ and ξ_1, ξ_2 are two vector fields in $\mathbb{R}^n$, then the quantity

$$I_t(M_1,\xi_1;M_2,\xi_2) := \int_{M_1} d\mathcal{H}^{m_1}(x)\int_{M_2} \langle \xi_1(x), y-x\rangle\langle \xi_2(y), y-x\rangle e^{-\frac{|y-x|^2}{4t^2}} d\mathcal{H}^{m_2}(y)$$

is infinitesimal of order $t^{m_1+m_2-m+2}$. The proof of this essentially uses the projection on the intersection S_{12}; this fact has been exploited in the previous lemma by the quantity $I_t(\Sigma^k;\Gamma)$, since in this case the intersection between Σ^k and Γ, having dimension $n-1$, is contained in S^k, which has dimension $n-1-k$.

Since we are interested in the cases $k = 1$, we compute Θ_1 explicitly. In order to do this, we define the angle ϑ_0 as the unique $\vartheta_0 \in [0, \pi)$ such that

$$\big\langle \sigma_1^1(x_0), \gamma_1^1(x_0) \big\rangle = \cos \vartheta_0.$$

With this choice, we also have that

$$\big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0) \big\rangle = \pm \sin \vartheta_0, \qquad \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0) \big\rangle = \pm \sin \vartheta_0$$

where the signs depend on the orientations of Σ and Γ.

Lemma 4.8 *Let us fix $x_0 \in S^1$; for $k = 1$, we have that*

$$\Theta_1(x_0) = -\frac{\big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0) \big\rangle \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0) \big\rangle}{\pi \sin^2 \vartheta_0} \Big(1 + (\pi - \vartheta_0) \mathrm{ctg} \vartheta_0 \Big).$$

Proof. For $\eta \in V_{x_0}^1$, we write $\eta = \eta_1 v_1 + \eta_2 v_2$ where $\{v_1, v_2\}$ is the orthogonal system determined by

$$v_1 = \sigma_1^1(x_0), \qquad v_2 = \frac{1}{\sqrt{1 - \langle \sigma_1^1(x_0), \gamma_1^1(x_0) \rangle^2}} (\langle \sigma_1^1(x_0), \gamma_1^1(x_0) \rangle \sigma_1^1(x_0) - \gamma_1^1(x_0)).$$

With this choice, we obtain

$$\begin{aligned}
4\pi \Big(1 - \big\langle \sigma_1^1(x_0), \gamma_1^1(x_0) \big\rangle^2 \Big) \Theta_1(x_0) = & - \big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0) \big\rangle \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0) \big\rangle \int_{V_{x_0}^1} \eta_1 \eta_2 e^{-\frac{|\eta|^2}{4}} d\eta + \\
& - \frac{\big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0) \big\rangle \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0) \big\rangle \big\langle \sigma_1^1(x_0), \gamma_1^1(x_0) \big\rangle}{\sqrt{1 - \langle \sigma_1^1(x_0), \gamma_1^1(x_0) \rangle^2}} \int_{V_{x_0}^1} \eta_2^2 e^{-\frac{|\eta|^2}{4}} d\eta \\
= & - \big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0) \big\rangle \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0) \big\rangle \int_0^{+\infty} \varrho^3 e^{-\frac{\varrho^2}{4}} d\varrho \int_0^\alpha \sin \vartheta \cos \vartheta d\vartheta + \\
& - \frac{\big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0) \big\rangle \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0) \big\rangle \big\langle \sigma_1^1(x_0), \gamma_1^1(x_0) \big\rangle}{\sqrt{1 - \langle \sigma_1^1(x_0), \gamma_1^1(x_0) \rangle^2}} \int_0^{+\infty} \varrho^3 e^{-\frac{\varrho^2}{4}} d\varrho \int_0^\alpha \sin^2 \vartheta d\vartheta,
\end{aligned}$$

with $\alpha = \pi - \vartheta_0$ since $V_{x_0}^1$ is the positive cone generated by $\sigma_1^1(x_0)$ and $-\gamma_1^1(x_0)$. □

Proposition 4.9 *Let Σ and Γ two pieces of $C^{1,1}$–regular surfaces such that $S = \Sigma \cap \Gamma$ has regular skeleton; then*

$$\lim_{t \to 0} \frac{I_t(\Sigma; \Gamma)}{(4\pi)^{n/2} t^{n+2}} = I_0^1(\Sigma; \Gamma) = \int_{S^1} \Theta_1(x_0) d\mathcal{H}^{n-2}(x_0).$$

Proof. The proposition follows by using the decomposition (28), the estimate (19) and applying Lemma 4.6. □

Remark 4.10 In the proof of Lemma 4.6, we have essentially used the weak convergence of

$$\mu_t^{\Sigma_r^{k,x_0}} = \mathcal{H}^k \llcorner \left(\frac{\Sigma_r^{k,x_0} - x_0}{t} \right)$$

to the measure

$$\mu_0^{\Sigma_r^{k,x_0}} = \mathcal{L}^k \llcorner \mathrm{Tan}(\Sigma_r^{k,x_0}, x_0)$$

and the weak convergence of

$$\mu_t^{\Gamma_r} = \mathcal{H}^{n-1} \llcorner \left(\frac{\Gamma_r - x_0}{t}\right)$$

to the measure

$$\mu_0^{\Gamma_r} = \mathcal{L}^{n-1} \llcorner \mathrm{Tan}(\Gamma, x_0).$$

In the next lemma we shall investigate the behavior of the distributions defined by

$$\delta_1 \mu_0^{\Sigma_r^{1,x_0}} = \lim_{t\to 0} \frac{\mu_t^{\Sigma_r^{1,x_0}} - \mu_0^{\Sigma_r^{1,x_0}}}{t}, \qquad \delta_1 \mu_0^{\Gamma_r^{x_0}} = \lim_{t\to 0} \frac{\mu_t^{\Gamma_r^{x_0}} - \mu_0^{\Gamma_r^{x_0}}}{t}. \tag{29}$$

in the case that Σ_r^{1,x_0} and $\Gamma_r^{x_0}$ are $C^{1,1}$–regular at x_0 (see Definition 2.1).

Lemma 4.11 *If $\phi \in C_c^1(\mathbb{R}^n)$; if $x_0 \in S$ and Σ_r^{1,x_0} is $C^{1,1}$–regular at x_0, then there holds*

$$\left\langle \phi, \delta_1 \mu_0^{\Sigma_r^{1,x_0}} \right\rangle = \frac{1}{2} \int_{\mathrm{Tan}(\Sigma_r^{1,x_0}, x_0)} \langle \nabla\phi(z), \nu_\Sigma(x_0)\rangle \kappa_\Sigma^{x_0}[z] dz. \tag{30}$$

Moreover, if $x_0 \in S$ and Γ is $C^{1,1}$–regular at x_0,

$$\left\langle \phi, \delta_1 \mu_0^{\Gamma_r^{x_0}} \right\rangle = \frac{1}{2} \int_{\mathrm{Tan}(\Gamma, x_0)} \langle \nabla\phi(w), \nu_\Gamma(x_0)\rangle A_\Gamma^{x_0}(w,w) dw - \frac{1}{2} \int_{T_{x_0} S} \phi(s) A^{x_0}_{S,\gamma_1^1(x_0)}(s,s) ds. \tag{31}$$

Proof. We start by proving (31); we use the parametrization of $\frac{\Gamma_r^{x_0} - x_0}{t}$ given in Remark 2.3; this parametrization is given by $\varphi_t : B^+_{r/t} \to \mathbb{R}^n$

$$\begin{aligned}\varphi_t(w,b) =& \frac{\varphi(tw,tb) - x_0}{t} = \sum_{h=1}^{n-2} \partial_h \varphi(0) w_h + \partial_{n-1}\varphi(0) b + \\ &+ \frac{t}{2}\left(\sum_{h,k=1}^{n-2} \partial^2_{h,k}\varphi(0) w_h w_k + 2\sum_{h=1}^{n-2} \partial^2_{h,n-1}\varphi(0) w_h b + \partial^2_{n-1,n-1}\varphi(0) b^2\right) + o(t).\end{aligned}$$

We may also assume that $\partial_i \varphi(0)$ coincide with the elements e_i of the standard basis for $i = 1, \ldots, n-1$. Moreover, for the metric we have

$$\begin{aligned}\frac{d}{dt}\sqrt{\det g(tw,tb)}_{t=0} =& \sum_{h,k=1}^{n-2} \partial^2_{h,k}\varphi^k(0) w_h + \sum_{h=1}^{n-2} \partial^2_{n-1,h}\varphi^h(0) b + \\ &+ \sum_{h=1}^{n-2} \partial^2_{h,n-1}\varphi^{n-1}(0) w_h + \partial^2_{n-1,n-1}\varphi^{n-1}(0) b.\end{aligned}$$

We now use the fact that for $\phi \in C_c^1(\mathbb{R}^n)$ and t small enough, we have that

$$\mathrm{spt}(\phi) \cap \frac{\Gamma_r - x_0}{t} \subset \varphi_t(B^+_{r/t}).$$

We also have that for $i \in \{1, \dots, n-2\}$ there hold

$$\int_{\mathbb{R}^{n-1}_+} \partial_i \phi(w, b, 0) w_h w_k dw db = -\int_{\mathbb{R}^{n-1}_+} \phi(w, b, 0)(\delta_{ih} w_k + w_h \delta_{ik}) dw db,$$

$$\int_{\mathbb{R}^{n-1}_+} \partial_i \phi(w, b, 0) w_h b dw db = -\int_{\mathbb{R}^{n-1}_+} \phi(w, b, 0) \delta_{ih} b dw db,$$

and

$$\int_{\mathbb{R}^{n-1}_+} \partial_i \phi(w, b, 0) b^2 dw db = 0,$$

while

$$\int_{\mathbb{R}^{n-1}_+} \partial_{n-1} \phi(w, b, 0) w_h w_k dw db = -\int_{\mathbb{R}^{n-2}} \phi(w, 0, 0) w_h w_k dw,$$

$$\int_{\mathbb{R}^{n-1}_+} \partial_{n-1} \phi(w, b, 0) w_h b dw db = -\int_{\mathbb{R}^{n-1}_+} \phi(w, b, 0) w_h dw db,$$

and

$$\int_{\mathbb{R}^{n-1}_+} \partial_{n-1} \phi(w, b, 0) b^2 dw db = -2\int_{\mathbb{R}^{n-1}_+} \phi(w, b, 0) b dw db.$$

In this way we obtain that

$$\begin{aligned}
\int \phi d\mu_t^{\Gamma_r^{x_0}} &= \int_{\mathbb{R}^{n-1}_+} \phi(w, b, 0) dw db + \frac{t}{2}\Big(\sum_{h,k=1}^{n-2} \partial^2_{h,k} \varphi^n(0) \int_{\mathbb{R}^{n-1}_+} \partial_n \phi(w, b, 0) w_h w_k dw db + \\
&+ 2 \sum_{h=1}^{n-2} \partial^2_{h,n-1} \varphi^n(0) \int_{\mathbb{R}^{n-1}_+} \partial_n \phi(w, b, 0) w_h b dw db + \partial^2_{n-1,n-1} \varphi^n(0) \int_{\mathbb{R}^{n-1}_+} \partial_n \phi(w, b, 0) b^2 dw db\Big) + \\
&- \frac{t}{2} \sum_{h,k=1}^{n-2} \partial^2_{h,k} \varphi^{n-1}(0) \int_{\mathbb{R}^{n-2}} \phi(w, 0, 0) w_h w_k dw + o(t) \\
&= \int \phi d\mu_0^{\Gamma_r^{x_0}} + \frac{t}{2} \int_{\mathrm{Tan}(\Gamma, x_0)} \langle \nabla \phi(w), \nu_\Gamma(x_0) \rangle A_\Gamma^{x_0}(w, w) dw - \frac{t}{2} \int_{T_{x_0} S} \phi(s) A^{x_0}_{S, \gamma_1^1(x_0)}(s, s) ds + o(t)
\end{aligned}$$

and this proves (31). The proof of (30) is similar. □

Remark 4.12 Lemma 4.11 applies also to function in the Schwarz space and in particular, as in our case, to a polynomial times the Gaussian.

Remark 4.13 In the proof of next result we need to compute integrals of the type

$$F_{hk} = \int_0^{+\infty} d\alpha \int_0^{+\infty} d\beta \alpha^h \beta^k \exp\left(-\frac{\alpha^2 + \beta^2 - 2\alpha\beta \cos \vartheta_0}{4} \right).$$

The integrals we are interested in are F_{10}, F_{21}, F_{30} and F_{32}. With standard computation, we get

$$F_{10} = \frac{2\sqrt{\pi}}{1 - \cos \vartheta_0}, \quad F_{30} = \frac{4\sqrt{\pi}(2 - \cos \vartheta_0)}{(1 - \cos \vartheta_0)^2}, \quad F_{21} = \frac{4\sqrt{\pi}}{(1 - \cos \vartheta_0)^2}, \qquad F_{32} = \frac{16\sqrt{\pi}}{(1 - \cos \vartheta_0)^3}.$$

In order to go further in the expansion of $I_t(\Sigma;\Gamma)$ we have to compute the derivative of the function $t \mapsto \frac{I_t(\Sigma^1_r;\Gamma)}{(4\pi)^{n/2}t^{n+2}}$. In such derivative, the distributions $\delta_1\mu_0^{\Sigma^{1,x_0}_r}$ and $\delta_1\mu_0^{\Gamma^{x_0}_r}$ defined in (29) are involved. By Lemma 4.11 we know how these distribution act on smooth functions when Σ^{1,x_0}_r and $\Gamma^{x_0}_r$ are $C^{1,1}$-regular at x_0. Since, in general, this is not the case, we have to complete our surfaces Σ and Γ in such a way that the assumptions of Lemma 4.11 are satisfied.

In the next lemma we will consider points belonging to $S_2 = S \setminus i_{n-2}(S)$; for $\mathcal{H}^{n-3}$–almost every $x_0 \in S^2 \subset S_2$, we have the $(n-3)$–dimensional tangent space $T_{x_0}S^2$ and a vector $s^1_1(x_0)$, pointing inside S, in such a way that the $(n-2)$–dimensional cone $\mathrm{Tan}(S,x_0)$ is given by

$$\mathrm{Tan}(S,x_0) = T_{x_0}S^2 \otimes \mathbb{R}_+\langle s^1_1(x_0)\rangle.$$

Moreover, for such $x_0 \in S^2$, $\mathrm{Tan}(\Sigma,x_0)$ is the product of $T_{x_0}S^2$ and a positive cone generated by two vectors, $s^1_1(x_0)$ and a second vector $\widetilde{\sigma}^1_1(x_0)$, belonging to the plane generated by $\sigma^1_1(x_0)$ (a vector orthogonal to $s^1_1(x_0)$) and $s^1_1(x_0)$. If $\big\langle \widetilde{\sigma}^1_1(x_0), s^1_1(x_0)\big\rangle > 0$, then $\Sigma^{2,x_0}_r = \emptyset$ and then Σ has a defect of orthogonality around x_0, that is for points $x \in S^1 \cap B_r(x_0)$ close to x_0, $\Sigma^{1,x}_r$ is not $C^{1,1}$–regular at x. Then we can complete $\Sigma^{1,x}_r$ using the sets $\widetilde{\Sigma}^{1,x}_r$ of Remark 2.2; we set

$$\widetilde{\Sigma}_r = \bigcup_{x_0\in S_2}\ \bigcup_{x\in S\cap B_r(x_0)} \widetilde{\Sigma}^{1,x}_r.$$

In case $\big\langle \widetilde{\sigma}^1_1(x_0), s^1_1(x_0)\big\rangle < 0$, then Σ has an excess of orthogonality and then $\Sigma^{2,x_0}_r \neq \emptyset$ and as sets of generators of $\mathrm{Tan}(\Sigma^{2,x_0}_r, x_0)$ we can choose $\sigma^2_1(x_0) = \sigma^1_1(x_0)$, $\sigma^2_2(x_0) = \widetilde{\sigma}^1_1(x_0)$. In case $\big\langle \widetilde{\sigma}^1_1(x_0), s^1_1(x_0)\big\rangle = 0$, then both Σ^{2,x_0}_r and $\widetilde{\Sigma}_r \cap B_r(x_0)$ are empty. In the same way, for Γ, we have the vectors $\gamma^1_1(x_0)$ and $\widetilde{\gamma}^1_1(x_0)$ such that $\mathrm{Tan}(\Gamma,x_0)$ is given by the product of $T_{x_0}S^2$ and the cone generated by $s^1_1(x_0)$ and $\widetilde{\gamma}^1_1(x_0)$. The fact that Γ can be not $C^{1,1}$–regular at $x_0 \in S^2$ means that Γ has a defect of tangentiality at x_0; in this case, $\mathrm{Tan}(\Gamma^{2,x_0}_r, x_0)$ is the positive cone generated by $s^1_1(x_0)$ and $\widetilde{\gamma}^1_1(x_0)$, that is we can set $\gamma^2_1(x_0) = s^1_1(x_0)$ and $\gamma^2_2(x_0) = \widetilde{\gamma}^1_1(x_0)$. We also use the definition of

$$\widetilde{\Gamma}_r = \bigcup_{x_0\in S_2}\ \bigcup_{x\in B_r(x_0)} \widetilde{\Gamma}^x_r,$$

given by Remark 2.2 is such a way that $\Gamma \cup \widetilde{\Gamma}_r$ is $C^{1,1}$–regular at any point of S; finally, for almost any point $x_0 \in S^2$, we notice that $\mathrm{Tan}(\widetilde{\Gamma}_r, x_0)$ is given by $T_{x_0}S^2$ and the positive cone generated by $\widetilde{\gamma}^2_1(x_0) = -s^1_1(x_0)$ and $\widetilde{\gamma}^2_2(x_0) = \widetilde{\gamma}^1_1(x_0)$.

Since the following decomposition

$$I_t(\Sigma_r;\Gamma_r) = I_t(\Sigma_r \cup \widetilde{\Sigma}_r; \Gamma_r \cup \widetilde{\Gamma}_r) - I_t(\widetilde{\Sigma}_r; \Gamma_r \cup \widetilde{\Gamma}_r) - I_t(\Sigma_r; \widetilde{\Gamma}_r),$$

holds, we shall deal with the quantities $I_t(\tilde{\Sigma}_r; \Gamma_r \cup \tilde{\Gamma}_r)$ and $I_t(\tilde{\Gamma}_r; \Sigma_r)$ which (as pointed out in Remark 4.7) are infinitesimal with order $n-3$. We need to consider the dimensional decompositions of $\tilde{\Sigma}_r$, $\tilde{\Gamma}_r$ and $\Gamma_r \cup \tilde{\Gamma}_r$ induced by the projections on S_2 and with an abuse of notation, we continue to denote by $\tilde{\Sigma}^{k,x_0}_r$, $(\tilde{\Gamma}_r)^{k,x_0}$ and $(\Gamma_r \cup \tilde{\Gamma}_r)^{k,x_0}$ such slicings. We shall use in particular the spaces

$$\widetilde{V}^2_{x_0} = \mathrm{Tan}((\Gamma \cup \widetilde{\Gamma}_r)^{2,x_0}_r, x_0) - \mathrm{Tan}(\widetilde{\Sigma}^{2,x_0}_r, x_0), \qquad \widetilde{W}^2_{x_0} = \mathrm{Tan}(\widetilde{\Gamma}^{2,x_0}_r, x_0) - \mathrm{Tan}(\Sigma^{2,x_0}_r, x_0)$$

and the linear maps $\widetilde{Q}^2_{x_0} : \mathbb{R}^3_+ \times \mathbb{R} \to \widetilde{V}^2_{x_0}$ and $\widehat{Q}^2_{x_0} : \mathbb{R}^4_+ \to \widetilde{W}^2_{x_0}$, defined by

$$\widetilde{Q}^2_{x_0}(\alpha,\beta) = -\alpha_1\sigma^1_1(x_0) - \alpha_2\widetilde{\sigma}^1_1(x_0) + \beta_1\gamma^1_1(x_0) + \beta_2 s^1_1(x_0).$$

and

$$\widehat{Q}^2_{x_0}(\alpha,\beta) = -\alpha_1\tilde{\sigma}^1_1(x_0) - \alpha_2 s^1_1(x_0) + \beta_1\tilde{\gamma}^1_1(x_0) - \beta_2 s^1_1(x_0).$$

Lemma 4.14 *Let Σ and Γ as before; then*

$$\delta_1 I_0^1(\Sigma;\Gamma) := \lim_{t\to 0}\frac{1}{t}\left(\frac{I_t(\Sigma_r^1;\Gamma)}{(4\pi)^{n/2}t^{n+2}} - I_0^1(\Sigma;\Gamma)\right) = \int_S T_2(x_0)d\mathcal{H}^{n-2} - \int_{S^2}\widetilde{\Theta}_2(x_0)d\mathcal{H}^{n-3}(x_0),$$

where

$$\begin{aligned}T_2(x_0) =&\frac{(\cos\vartheta_0-2)}{\sqrt{\pi}(1-\cos\vartheta_0)^2}\Big[\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle\kappa_\Sigma^{x_0}[\sigma_1^1(x_0)] + \big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle\kappa_\Gamma^{x_0}[\gamma_1^1(x_0)]\\ &+\frac{(n-2)\cos\vartheta_0}{(\cos\vartheta_0-2)}\Big(\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle H_S^{x_0}[\nu_\Gamma(x_0)] + \big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle H_S^{x_0}[\nu_\Sigma(x_0)]\Big)\Big] \quad (32)\end{aligned}$$

and

$$\begin{aligned}\widetilde{\Theta}_2(x_0) =&\frac{D_2\widetilde{Q}^2_{x_0}}{(4\pi)^{3/2}}\int_{\widetilde{V}^2_{x_0}}\langle\nu_\Sigma(x_0),v\rangle\langle\nu_\Gamma(x_0),v\rangle e^{-\frac{|v|^2}{4}}\mathcal{H}^1((\widetilde{Q}^2_{x_0})^{-1}(v))dv+\\ &+\frac{D_2\widehat{Q}^2_{x_0}}{(4\pi)^{3/2}}\int_{\widetilde{W}^2_{x_0}}\langle\nu_\Sigma(x_0),v\rangle\langle\nu_\Gamma(x_0),v\rangle e^{-\frac{|v|^2}{4}}\mathcal{H}^1((\widehat{Q}^2_{x_0})^{-1}(v))dv.\end{aligned}$$

Proof. By (19), we have to compute the derivative of

$$\frac{I_t(\Sigma_r^1;\Gamma_r)}{(4\pi)^{n/2}t^{n+2}} = \int_S d\mathcal{H}^{n-2}(x_0)\int d\mu_t^{\Sigma_r^{1,x_0}}(z)\int F_{x_0}(t,z,w)d\mu_t^{\Gamma_r}(w)$$

where

$$F_{x_0}(t,z,w) = \frac{1}{(4\pi)^{n/2}}\frac{\langle\nu_\Sigma(x_0+tz),w-z\rangle\langle\nu_\Gamma(x_0+tw),w-z\rangle}{J\pi_1(x_0+tz)}e^{-\frac{|w-z|^2}{4}}.$$

First of all, we have that

$$\begin{aligned}\partial_t F_{x_0}(0,z,w) =& \frac{e^{-\frac{|w-z|^2}{4}}}{(4\pi)^{n/2}}\Big(\langle d_{x_0}\nu_\Sigma[z],w-z\rangle\langle\nu_\Gamma(x_0),w-z\rangle+\\ &+\langle\nu_\Sigma(x_0),w-z\rangle\langle d_{x_0}\nu_\Gamma[w],w-z\rangle - (n-2)\langle\nu_\Sigma(x_0),w-z\rangle\langle\nu_\Gamma(x_0),w-z\rangle H_S^{x_0}[z]\Big)\\ =&\frac{e^{-\frac{|w-z|^2}{4}}}{(4\pi)^{n/2}}\Big(\langle\nu_\Gamma(x_0),z\rangle A_\Sigma^{x_0}(z,\Pi_\Sigma^{x_0}(w)-z)+\\ &-\langle\nu_\Sigma(x_0),w\rangle A_\Gamma^{x_0}(w,w-\Pi_\Gamma^{x_0}(z)) + (n-2)\langle\nu_\Sigma(x_0),w\rangle\langle\nu_\Gamma(x_0),z\rangle H_S^{x_0}[z]\Big), \quad (33)\end{aligned}$$

where in the last line we have used the fact that $z\in\mathrm{Tan}(\Sigma_r^{1,x_0},x_0)$ and $w\in\mathrm{Tan}(\Gamma,x_0)$. If we write $w=w_\tau+\zeta$, with $w_\tau\in T_{x_0}S$ and $\zeta\in\mathrm{Tan}(\Gamma_r^{1,x_0},x_0)$ and fix an orthonormal basis $\{e_1,\dots,e_{n-2}\}$ of $T_{x_0}S$. Using the fact that $T_{x_0}S$ is a vector space, we can discard the summand of (33) that are odd in the variable w_τ; in addition, we use the fact that $\langle\nu_\Sigma(x_0),w\rangle = \langle\nu_\Sigma(x_0),\zeta\rangle$ and the decompositions $w_\tau=\sum_{i=1}^{n-2}w_\tau^i e_i$ and

$$\zeta = \big\langle\zeta,\sigma_1^1(x_0)\big\rangle\sigma_1^1(x_0) + \langle\zeta,\nu_\Sigma(x_0)\rangle\nu_\Sigma(x_0), \qquad z = \big\langle z,\gamma_1^1(x_0)\big\rangle\gamma_1^1(x_0) + \langle z,\nu_\Gamma(x_0)\rangle\nu_\Gamma(x_0),$$

whence the fact that

$$\Pi_\Sigma^{x_0}(\zeta) = \big\langle\zeta,\sigma_1^1(x_0)\big\rangle\sigma_1^1(x_0), \qquad \Pi_\Gamma^{x_0}(z) = \big\langle z,\gamma_1^1(x_0)\big\rangle\gamma_1^1(x_0).$$

In this way we obtain

$$
\begin{aligned}
&\int_{\mathrm{Tan}(\Sigma_r^{1,x_0},x_0)} dz \int_{\mathrm{Tan}(\Gamma,x_0)} \partial_t F_{x_0}(0,z,w)dw = \\
=&\int_{T_{x_0}S} dw_\tau \frac{e^{-\frac{|w_\tau|^2}{4}}}{(4\pi)^{n/2}} \int_{\mathrm{Tan}(\Sigma_r^{1,x_0},x_0)} dz \int_{\mathrm{Tan}(\Gamma_r^{1,x_0},x_0)} \Big(\langle \nu_\Gamma(x_0), z\rangle \big\langle \zeta, \sigma_1^1(x_0)\big\rangle A_\Sigma^{x_0}(z,\sigma_1^1(x_0)) + \\
&- \langle \nu_\Gamma(x_0), z\rangle A_\Sigma^{x_0}(z,z) - \langle \nu_\Sigma(x_0), \zeta\rangle A_\Gamma^{x_0}(w_\tau,w_\tau) - \langle \nu_\Sigma(x_0), \zeta\rangle A_\Gamma^{x_0}(\zeta,\zeta) + \\
&+ \langle \nu_\Sigma(x_0), \zeta\rangle \big\langle z, \gamma_1^1(x_0)\big\rangle A_\Gamma^{x_0}(\zeta,\gamma_1^1(x_0)) + (n-2)\langle \nu_\Sigma(x_0), \zeta\rangle \langle \nu_\Gamma(x_0), z\rangle H_S^{x_0}[z]\Big) e^{-\frac{|\zeta-z|^2}{4}} d\zeta.
\end{aligned}
$$

We also need the fact that

$$
\int_{T_{x_0}S} A_\Gamma^{x_0}(w_\tau,w_\tau) e^{-\frac{|w_\tau|^2}{4}} dw_\tau = 2(4\pi)^{\frac{n-2}{2}} \sum_{i=1}^{n-2} A_\Gamma^{x_0}(e_i,e_i) = 2(4\pi)^{\frac{n-2}{2}}(n-2) H_S^{x_0}[\nu_\Gamma(x_0)],
$$

so we get

$$
\begin{aligned}
&\int_{\mathrm{Tan}(\Sigma_r^{1,x_0},x_0)} dz \int_{\mathrm{Tan}(\Gamma,x_0)} \partial_t F_{x_0}(0,z,w)dw = \\
=&\frac{1}{4\pi}\Big[\kappa_\Sigma^{x_0}[\sigma_1^1(x_0)]\left(\big\langle \gamma_1^1(x_0), \sigma_1^1(x_0)\big\rangle \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle F_{21} - \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle F_{30}\right) + \\
&+ \kappa_\Gamma^{x_0}[\gamma_1^1(x_0)]\left(\big\langle \gamma_1^1(x_0), \sigma_1^1(x_0)\big\rangle \big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle F_{12} - \big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle F_{03}\right) + \\
&+ (n-2)\big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle H_S^{x_0}[\sigma_1^1(x_0)] F_{21} + \\
&- 2(n-2)\big\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\big\rangle H_S^{x_0}[\nu_\Gamma(x_0)] F_{01}\Big]; \qquad (34)
\end{aligned}
$$

here the F_{hk} are the coefficients defined in Remark 4.13. We now have to consider the derivatives of the measures $\mu_t^{\Sigma_r^{1,x_0}}$ and $\mu_t^{\Gamma_r}$; in order to apply Lemma 4.11, we need that for any $x_0 \in S^1$, Σ_r^{1,x_0} and Γ have to be $C^{1,1}$–regular at x_0. So, in case, we can consider the completions $\widetilde{\Sigma}_r$ of Σ_r and $\widetilde{\Gamma}_r$ of Γ. We also notice that

$$
I_t(\Sigma_r;\Gamma_r) = I_t(\Sigma_r \cup \widetilde{\Sigma}_r; \Gamma_r \cup \widetilde{\Gamma}_r) - I_t(\widetilde{\Sigma}_r; \Gamma_r \cup \widetilde{\Gamma}_r) - I_t(\Sigma_r; \widetilde{\Gamma}_r).
$$

To deal with $I_t(\widetilde{\Sigma}_r; \Gamma_r \cup \widetilde{\Gamma}_r)$, we consider the projection $\widetilde{\pi}_\Sigma : \widetilde{\Sigma}_r \to S_2$ and define the sets

$$
(S_2)^k = \{x_0 \in S_2 : \dim \widetilde{\pi}^{-1}(x_0) = k\}, \quad \text{and} \quad \widetilde{\Sigma}_r^k = \widetilde{\pi}^{-1}((S_2)^k).
$$

Since S_2 is an $(n-3)$–dimensional set, we deduce that $\mathcal{H}^{n-1}(\widetilde{\Sigma}_r^0) = \mathcal{H}^{n-1}(\widetilde{\Sigma}_r^1) = 0$, so that

$$
I_t(\widetilde{\Sigma}_r; \Gamma_r \cup \widetilde{\Gamma}_r) = \sum_{k=2}^{n-1} I_t(\widetilde{\Sigma}_r^k; \Gamma_r \cup \widetilde{\Gamma}_r)
$$

and, arguing as in Lemma 4.6, the term $I_t(\widetilde{\Sigma}_r^k; \Gamma_r \cup \widetilde{\Gamma}_r)$ is asymptotic to t^{n+1+k}. In the same way, by considering the projection $\widetilde{\pi}_\Gamma : \widetilde{\Gamma}_r \to S_2$, we can write

$$
I_t(\widetilde{\Gamma}_r; \Sigma_r) = \sum_{k=2}^{n-1} I_t(\widetilde{\Gamma}_r^k; \Sigma_r).
$$

We shall still denote by $\widetilde{\Sigma}_r^{2,x_0}$ and by $(\Gamma\cup\widetilde{\Gamma_r})_r^{2,x_0}$ the sections induced by the projection's $\widetilde{\pi}$.

We are interested in the term asymptotic to t^{n+3}; we have that

$$
\begin{aligned}
&\lim_{t\to 0}\frac{I_t(\widetilde{\Sigma}_r^2;\Gamma_r\cup\widetilde{\Gamma}_r)}{(4\pi)^{n/2}t^{n+3}} := \widetilde{I}_0(\Sigma;\Gamma) \qquad (35)\\
&=\int_{S_2} d\mathcal{H}^{n-3}(x_0)\int_{\mathrm{Tan}(\widetilde{\Sigma}_r^{2,x_0},x_0)} dz\int_{\mathrm{Tan}(\Gamma\cup\widetilde{\Gamma}_r,x_0)}\frac{\langle\nu_\Sigma(x_0),w-z\rangle\langle\nu_\Gamma(x_0),w-z\rangle}{(4\pi)^{n/2}}e^{-\frac{|w-z|^2}{4}}dw\\
&=\int_{S_2} d\mathcal{H}^{n-3}(x_0)\int_{\mathrm{Tan}(\widetilde{\Sigma}_r^{2,x_0},x_0)} dz\int_{\mathrm{Tan}((\Gamma\cup\widetilde{\Gamma}_r^{2,x_0},x_0)}\frac{\langle\nu_\Sigma(x_0),w-z\rangle\langle\nu_\Gamma(x_0),w-z\rangle}{(4\pi)^{3/2}}e^{-\frac{|w-z|^2}{4}}dw\\
&=\int_{S_2} d\mathcal{H}^{n-3}(x_0)\frac{D_2\widetilde{Q}^2_{x_0}}{(4\pi)^{3/2}}\int_{\widetilde{V}^2_{x_0}}\langle\nu_\Sigma(x_0),v\rangle\langle\nu_\Gamma(x_0),v\rangle e^{-\frac{|v|^2}{4}}\mathcal{H}^1((\widetilde{Q}^2_{x_0})^{-1}(v))dv;
\end{aligned}
$$

moreover we get that

$$
\begin{aligned}
&\lim_{t\to 0}\frac{I_t(\widetilde{\Gamma}_r^2;\Sigma_r)}{(4\pi)^{n/2}t^{n+3}} := \widetilde{I}_1(\Sigma;\Gamma) \qquad (36)\\
&=\frac{D_2\widehat{Q}^2_{x_0}}{(4\pi)^{3/2}}\int_{S_2} d\mathcal{H}^{n-3}(x_0)\int_{\widetilde{W}^2_{x_0}}\langle\nu_\Sigma(x_0),v\rangle\langle\nu_\Gamma(x_0),v\rangle e^{-\frac{|v|^2}{4}}\mathcal{H}^1((\widehat{Q}^2_{x_0})^{-1}(v))dv,
\end{aligned}
$$

For the term $I_t(\Sigma_r\cup\widetilde{\Sigma}_r;\Gamma\cup\widetilde{\Gamma}_r)$ we can apply Lemma 4.11 and the fact that $\mathrm{Tan}(\Sigma_r^{1,x_0},x_0)=\mathrm{Tan}((\Sigma\cup\widetilde{\Sigma}_r)_r^{1,x_0},x_0)$, so that

$$
\begin{aligned}
\delta_1 I_0^1(\Sigma;\Gamma) =& \int_S d\mathcal{H}^{n-2}(x_0)\int_{\mathrm{Tan}(\Sigma_r^{1,x_0},x_0)} dz\int_{\mathrm{Tan}(\Gamma,x_0)}\partial_t F(0,z,w)dw+\\
&+\int_S\left(\left\langle\phi_\Gamma,\delta_1\mu_0^{(\Sigma\cup\widetilde{\Sigma}_r)_r^{1,x_0}}\right\rangle+\left\langle\phi_\Sigma,\delta_1\mu_0^{\Gamma\cup\widetilde{\Gamma}_r}\right\rangle\right)d\mathcal{H}^{n-2}(x_0)-\widetilde{I}_0(\Sigma;\Gamma)-\widetilde{I}_1(\Sigma;\Gamma),
\end{aligned}
$$

where

$$
\phi_\Gamma(z)=\int_{\mathrm{Tan}(\Gamma,x_0)}F_{x_0}(0,z,w)dw,\qquad \phi_\Sigma(w)=\int_{\mathrm{Tan}(\Sigma_r^{1,x_0},x_0)}F_{x_0}(0,z,w)dz.
$$

We start with ϕ_Γ; using the decomposition $\mathrm{Tan}(\Gamma,x_0)=T_{x_0}S\oplus\mathrm{Tan}(\Gamma_r^{1,x_0},x_0)$, we deduce that

$$
\phi_\Gamma(z)=-\frac{\langle\nu_\Gamma(x_0),z\rangle}{4\pi}\int_{\mathrm{Tan}(\Gamma_r^{1,x_0},x_0)}\langle\nu_\Sigma(x_0),w-z\rangle e^{-\frac{|w-z|^2}{4}}dw,
$$

so that

$$
\begin{aligned}
\nabla\phi_\Gamma(z)=&-\frac{1}{4\pi}\int_{\mathrm{Tan}(\Gamma_r^{1,x_0},x_0)}\Big(\langle\nu_\Sigma(x_0),w-z\rangle\nu_\Gamma(x_0)-\langle\nu_\Gamma(x_0),z\rangle\nu_\Sigma(x_0)+\\
&+\frac{1}{2}\langle\nu_\Sigma(x_0),w-z\rangle\langle\nu_\Gamma(x_0),z\rangle(w-z)\Big)e^{-\frac{|w-z|^2}{4}}dw.
\end{aligned}
$$

We then obtain that

$$\left\langle \phi_\Gamma, \delta_1\mu_0^{\Sigma_r^{1,x_0}}\right\rangle = -\frac{1}{8\pi}\int_{\mathrm{Tan}(\Sigma_r^{1,x_0},x_0)} dz \int_{\mathrm{Tan}(\Gamma_r^{1,x_0},x_0)} \Big(\langle\nu_\Sigma(x_0),w\rangle\langle\nu_\Sigma(x_0),\nu_\Gamma(x_0)\rangle +$$

$$-\langle\nu_\Gamma(x_0),z\rangle + \frac{1}{2}\langle\nu_\Sigma(x_0),w\rangle^2\langle\nu_\Gamma(x_0),z\rangle\Big)\kappa_\Sigma^{x_0}[z]e^{-\frac{|w-z|^2}{4}}dw$$

$$= -\frac{1}{8\pi}\kappa_\Sigma^{x_0}[\sigma_1^1(x_0)]\Big(\langle\nu_\Sigma(x_0),\nu_\Gamma(x_0)\rangle\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle F_{21}+$$

$$-\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle F_{30} + \frac{1}{2}\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle^2\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle F_{32}\Big). \qquad (37)$$

Concerning $\nabla\phi_\Sigma$, we have

$$\nabla\phi_\Sigma(w) = \frac{1}{(4\pi)^{\frac{n}{2}}}\int_{\mathrm{Tan}(\Sigma_r^{1,x_0},x_0)} (\langle\nu_\Gamma(x_0),w-z\rangle\nu_\Sigma(x_0) + \langle\nu_\Sigma(x_0),w\rangle\nu_\Gamma(x_0)+$$

$$-\frac{1}{2}\langle\nu_\Sigma(x_0),w\rangle\langle\nu_\Gamma(x_0),w-z\rangle(w-z)\Big)\, e^{-\frac{|w-z|^2}{4}}dz;$$

we also get

$$\left\langle \phi_\Sigma, \delta_1\mu_0^{\Gamma_r^{x_0}}\right\rangle = \frac{1}{8\pi}\kappa_\Gamma^{x_0}[\gamma_1^1(x_0)]\Big(-\langle\nu_\Sigma(x_0),\nu_\Gamma(x_0)\rangle\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle F_{12}+$$

$$+\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle F_{03} - \frac{1}{2}\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle^2 F_{23}\Big)+$$

$$+\frac{(n-2)}{4\pi}H_S^{x_0}[\nu_\Gamma(x_0)]\Big(-\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle\langle\nu_\Sigma(x_0),\nu_\Gamma(x_0)\rangle F_{10}+$$

$$+\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle F_{01} - \frac{1}{2}\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle^2 F_{21}\Big) \qquad (38)$$

Summing the three terms (34), (37) and (38), we obtain the term

$$T_2(x_0) = \frac{1}{4\pi}\Big\{\kappa_\Sigma^{x_0}[\sigma_1^1(x_0)]\Big(-\frac{1}{2}\langle\nu_\Sigma(x_0),\nu_\Gamma(x_0)\rangle\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle F_{21} - \frac{1}{2}\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle F_{30}+$$

$$-\frac{1}{4}\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle^2\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle F_{32} + \big\langle\gamma_1^1(x_0),\sigma_1^1(x_0)\big\rangle\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle F_{21}\Big)+$$

$$+\kappa_\Gamma^{x_0}[\gamma_1^1(x_0)]\Big(-\frac{1}{2}\langle\nu_\Sigma(x_0),\nu_\Gamma(x_0)\rangle\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle F_{12} - \frac{1}{2}\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle F_{03}+$$

$$-\frac{1}{4}\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle^2\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle F_{23} + \big\langle\gamma_1^1(x_0),\sigma_1^1(x_0)\big\rangle\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle F_{12}\Big)+$$

$$+(n-2)H_S^{x_0}[\nu_\Gamma]\Big(-\langle\nu_\Sigma(x_0),\nu_\Gamma(x_0)\rangle\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle F_{10} - \big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle F_{01}+$$

$$-\frac{1}{2}\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle^2 F_{21}\Big)+$$

$$+(n-2)H_S^{x_0}[\sigma_1^1(x_0)]\big\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\big\rangle\big\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\big\rangle F_{21}\Big\}.$$

We write $\sigma_1^1(x_0)$ as a combination of $\nu_\Sigma(x_0)$ and ν_Γ

$$\sigma_1^1(x_0) = \alpha\nu_\Sigma(x_0) + \beta\nu_\Gamma(x_0),$$

where, since $\big\langle\nu_\Sigma(x_0),\sigma_1^1(x_0)\big\rangle = 0$ and $1 = \big\langle\sigma_1^1(x_0),\sigma_1^1(x_0)\big\rangle$,

$$\alpha = \frac{\cos\vartheta_0}{\langle\nu_\Sigma(x_0),\gamma_1^1(x_0)\rangle}, \qquad \beta = \frac{1}{\langle\nu_\Gamma(x_0),\sigma_1^1(x_0)\rangle}.$$

By the fact that $\cos\vartheta_0 = \big\langle \sigma_1^1(x_0), \gamma_1^1(x_0)\big\rangle$, we also obtain that

$$\langle \nu_\Sigma(x_0), \nu_\Gamma(x_0)\rangle = -\frac{\cos\vartheta_0 \big\langle \nu_\Gamma(x_0), \sigma_1^1(x_0)\big\rangle}{\langle \nu_\Sigma(x_0), \gamma_1^1(x_0)\rangle}.$$

whence equation (32). $\square$

The results obtained so far immediately prove the following.

Proposition 4.15 *Let Σ and Γ as before, then there holds*

$$\frac{I_t(\Sigma;\Gamma)}{(4\pi)^{n/2} t^{n+2}} = I_0^1(\Sigma;\Gamma) + t\left(I_0^2(\Sigma;\Gamma) + \delta_1 I_0^1(\Sigma;\Gamma)\right) + o(t)$$

Remark 4.16 It is clear that the previous expansion can also continue for higher powers of t; this expansion will contains higher derivatives of the objects involved so far, but also the higher codimensional part of the skeleton of S.

5 Examples

5.1 Two-dimensional region

Example 5.1 Let $\gamma : [0, L] \to \mathbb{R}^2$ be a $C^{1,1}$ planar simple curve parametrized by arclength; then

$$\lim_{t\to 0} \frac{I_t(\gamma)}{4\pi t^5} = -\frac{3}{2\sqrt{\pi}} \int_0^L \kappa_\gamma^2(\alpha)d\alpha,$$

with κ_γ the curvature of γ. In particular, if E is a bounded set with ∂E parametrized by γ, then

$$\|T_t\chi_E\|^2_{L^2(\mathbb{R}^2)} = |E| - \sqrt{\frac{2t}{\pi}} L - \sqrt{\frac{t^3}{2\pi}} \int_0^L \kappa_\gamma^2(\alpha)d\alpha + o(t^{3/2}).$$

A first example is given by the circle $E = B_r(0)$ in the plane; in this case ∂E is parametrized by $\gamma(\alpha) = (r\cos\frac{\alpha}{r}, r\sin\frac{\alpha}{r})$, and $\kappa_\gamma(\alpha) = r^{-1}$ for every $\alpha \in [0, 2\pi r)$. Then

$$\|T_t\chi_E\|^2_{L^2(\mathbb{R}^2)} = \pi r^2 - 2r\sqrt{2\pi t} - \frac{\sqrt{2\pi t^3}}{r} + o(t^{3/2}).$$

The following example shows how the second order expansion of the heat content of a set E with finite perimeter takes into account the behavior of the boundary ∂E along the 0-singular set of ∂E.

Example 5.2 Let E be a simple oriented polygonal region in the plane with angles $\alpha_i \in (0, \pi), i = 1, \ldots, m$, then in the expansion of its heat content the coefficients of t is not zero and depends on the not $C^{1,1}$ contact of pair of consecutive segments; indeed if Σ and Γ are two segments that have a common endpoint x_0 and generate an angle $\alpha \in (0, \pi)$, by Lemma 4.8 we get that

$$I_0^1(\Sigma;\Gamma) = \Theta_1(x_0) = -\frac{1}{\pi}\big(1 + (\pi - \alpha)\operatorname{ctg}\alpha\big).$$

In the case of E as before we have to consider m segments and for each of them two contacts with the adjacent segments; moreover, since the curvature of a line is zero, the heat content of E is, up to an exponential infinitesimal term, a quadratic polynomial in $\sqrt{t}$

$$\|T_t\chi_E\|^2_{L^2(\mathbb{R}^2)} = |E| - \sqrt{\frac{2t}{\pi}}P(E) - 2t\Big(\frac{m}{\pi} + \sum_{i=1}^{m}\frac{\operatorname{ctg}\alpha_i}{\pi}(\pi-\alpha_i)\Big) + o(t^h) \quad \forall\, h>1.$$

In the case of the square $E=[0,1]^2\subset\mathbb{R}^2$, ∂E is the union of four orthogonal segments and

$$\lim_{t\to 0}\frac{\|T_t\chi_E\|^2_{L^2(\mathbb{R}^2)} - |E| + \sqrt{\frac{2t}{\pi}}P(E)}{t} = -\frac{8}{\pi}$$

being $\alpha_i=\frac{\pi}{2}$, $i=1,..,4$. Hence,

$$\|T_t\chi_E\|^2_{L^2(\mathbb{R}^2)} = 1 - 4\sqrt{\frac{2t}{\pi}} - \frac{8}{\pi}t + o(t^h) \quad \forall\, h>1.$$

5.2 Three-dimensional region

Example 5.3 Let $E=B_r(0)\subset\mathbb{R}^3$, then ∂E is $C^{1,1}$- regular and being $\kappa^x_{\partial E,i}=r^{-1}$ for $i=1,2$ and for every $x\in\partial E$, as immediate consequence of Theorem 1.1 we get that

$$\|T_t\chi_{B_r(0)}\|^2_{L^2(\mathbb{R}^3)} = \frac{4}{3}\pi r^3 - 4\sqrt{2\pi t}r^2 - \frac{16}{3}\sqrt{2\pi t^3} + o(t^{3/2})$$

Example 5.4 We consider now the set $E=B_r^+(0)=B_r(0)\cap\{(x,y,z): z>0\}$; we divide $\partial E=\Sigma_1\cup\Sigma_2$, where $\Sigma_1=\{(x,y,z): x^2+y^2<r^2, z=0\}$ and $\Sigma_2=\{(x,y,z): x^2+y^2+z^2=r^2, z>0\}$; both Σ_1 and Σ_2 can be parametrized by uniformly Lipschitz functions. The presence of the 1-singular set $S=\{x^2+y^2=r^2, z=0\}$ gives a nontrivial coefficient of t in the asymptotic expansion of the heat content of E. We have that

$$-(4\pi)^{3/2}t^5 f''_E(t) = I_t(\partial E) = \sum_{i,j=1}^{2} I_t(\Sigma_i;\Sigma_j)$$

By Theorem 3.3, it holds that

$$I_t(\Sigma_i) = I_t(\Sigma_i;\Sigma_i) = -16\pi t^6\int_{\Sigma_i}\Big((H^x_{\Sigma_i})^2 + \frac{1}{2}c^2_{\Sigma_i}(x))\Big)d\mathcal{H}^2(x) + o(t^6) \quad i=1,2.$$

Since $\kappa^x_{\Sigma_1,i}=0$ for $i=1,2$, $x\in\Sigma_1$ and $\kappa^x_{\Sigma_2,i}=\frac{1}{r}$ for $i=1,2$, $x\in\Sigma_2$ then $I_t(\Sigma_1)=0$ and

$$\lim_{t\to 0}\frac{I_t(\Sigma_2)}{(4\pi)^{3/2}t^6} = -8\sqrt{\pi}$$

Whereas for $i,j=1,2$ and $i\neq j$ we have that

$$I_t(\Sigma_i;\Sigma_j) = I_t(\Sigma_j;\Sigma_i) = (4\pi)^{3/2}t^5\int_S \Theta_1(x)d\mathcal{H}^1(x) + o(t^5)$$

and

$$I_0^1(\Sigma_1;\Sigma_2) = I_0^1(\Sigma_2;\Sigma_1) = \lim_{t\to 0}\frac{I_t(\Sigma_2;\Sigma_1)}{(4\pi)^{3/2}t^5} = \int_S \Theta_1(x)d\mathcal{H}^1(x) = -2r$$

being $\theta_0 = \frac{\pi}{2}$ and $\Theta_1(x_0) = -\frac{1}{\pi}$ for every $x_0 \in S$. In addition, since $\kappa^{x_0}_{\Sigma_2}[e_3] = \frac{1}{r}$ for any $x_0 \in S$, we also deduce that $T_2(x_0) = -\frac{2}{r\sqrt{\pi}}$ for any $x_0 \in S$. Since $S^2 = \emptyset$, we can conclude that

$$\|T_t\chi_{B_r^+(0)}\|^2_{L^2(\mathbb{R}^3)} = \frac{2}{3}\pi r^3 - 3\sqrt{2\pi t}r^2 - 4rt - \frac{16}{3}\sqrt{2\pi t^3} + o(t^{3/2}).$$

Example 5.5 The example of the square in $\mathbb{R}^2$ can be easily extended to the case $E = [0,1]^3 \subset \mathbb{R}^3$; in this case

$$\partial E = \sum_{1=1}^{6} \Sigma_i,$$

where the Σ_i's are 1 side-squares and $|A_i| = |\{j \neq i : S_{i,j} = \Sigma_i \cap \Sigma_j \neq \emptyset\}| = 4$ for every $i = 1, \dots, 6$. Moreover, there holds that

$$I^1_0(\Sigma_i;\Sigma_j) = \int_{S_{i,j}} \Theta_1(x) d\mathcal{H}^1(x) = -\frac{1}{\pi}, \quad i = 1,\dots,6,\ j \in A_i.$$

Since $\kappa^x_{\Sigma_i} = 0$ for every $x \in \Sigma_i$ and $S^2_{i,j} = \emptyset$ the coefficient of $\sqrt{t^3}$ in the asymptotic expansion of the heat content of E reduces to $\sum_{i=1}^6 \sum_{j \in A_i} \delta_1 I^1_0(\Sigma_i;\Sigma_j)$. In order to describe a generic term of the form $\delta_1 I^1_0(\Sigma_i;\Sigma_j)$ we consider an orthogonal coordinate system, we fix $\Sigma = \Sigma_i$ and $\Gamma = \Sigma_j$ with $S = \Sigma \cap \Gamma \neq \emptyset$ and, without loss of generality, we assume that $\Sigma = \{0\} \times [0,1]^2$ and $\Gamma = [0,1]^2 \times \{0\}$, then the origin of the axis belongs to S_2. Using the notation introduced in Section 4 we have that

$$\nu_\Sigma(0) = e_1,\ \nu_\Gamma(0) = e_3,\ \sigma^1_1(0) = e_3,\ \gamma^1_1(0) = e_1,\ s^1_1(0) = e_2,$$

where $\{e_i\}_{i=1,2,3}$ denote the canonical basis in $\mathbb{R}^3$. It is obvious that, being $\kappa^x_{\Sigma_i} = 0$, $\kappa^x_{S_{i,j}} = 0$ respectively for every $x \in \Sigma_i$ and $x \in S_{i,j}$ and $\tilde{\Sigma}^{2,0}_r = \emptyset$, there holds that $\delta_1 I^1_0(\Sigma;\Gamma)$ reduces to $-\tilde{I}_1(\Sigma;\Gamma)$ which depends on the defect of tangentiality of Γ in 0; in this case we have to consider $\tilde{\Gamma}^{2,0}_r$ in such a way that $\Gamma \cup \tilde{\Gamma}^{2,0}_r$ is $C^{1,1}$-regular at 0, then $\mathrm{Tan}(\tilde{\Gamma}^{2,0}_r, 0)$ is generated by $-e_2$ and e_1. We have to consider $\bar{\pi} : \Sigma_r \to (\Gamma \cup \tilde{\Gamma}^{2,0}_r) \cap \Sigma_r$ and $\Sigma^{2,0}_{r,\bar{\pi}} = \bar{\pi}^{-1}(0) = \Sigma_r$; in this case $\mathrm{Tan}(\Sigma^{2,0}_{r,\bar{\pi}}, 0)$ is generated by e_2 and e_3. In this case we get that

$$\widetilde{W}^2_0 = \mathrm{Tan}(\tilde{\Gamma}^{2,0}_r, 0) - \mathrm{Tan}(\Sigma^{2,0}_{r,\bar{\pi}}, 0) = (0,+\infty) \times (-\infty, 0)^2$$

and

$$\widehat{Q}^2_0 : \mathbb{R}^4_+ \to \widetilde{W}^2_0,\ (\alpha,\beta) \mapsto (\beta_1, -\alpha_2 - \beta_2, -\alpha_1),$$

then $D_2\widehat{Q}^2_0 = \frac{1}{\sqrt{2}}$

$$\tilde{I}_1(\Sigma;\Gamma) = \frac{1}{8\pi\sqrt{2\pi}} \int_{\widetilde{W}^2_0} v_2 \langle \nu_\Sigma(0), v\rangle \langle \nu_\Gamma(0), v\rangle e^{-\frac{|v|^2}{4}} \mathcal{H}^1((\hat{Q}^2_0)^{-1}(v)) dv = -\frac{1}{\sqrt{\pi^3}}.$$

Hence, we get

$$\|T_t\chi_E\|^2_{L^2(\mathbb{R}^3)} = 1 - 6\sqrt{\frac{2t}{\pi}} - \frac{24}{\pi}t - 8\sqrt{\frac{2t^3}{\pi^3}} + o(t^h) \quad \forall\, h > 3/2.$$

Example 5.6 Let E be an oriented regular tetrahedron, that is a polyhedron whose four faces are triangles equilateral T_i (with side equal to a), three of which meet at each vertex. In this case $|E| = \frac{\sqrt{2}}{12}a^3$, $P(E) = \sqrt{3}a^2$; moreover $\partial E = \sum_{i=1}^4 T_i$ and $|A_i| = |\{j \neq i : T_{i,j} = T_i \cap T_j \neq \emptyset\}| = 3$ for every $i = 1,..,4$. We notice that $\vartheta_0 = \frac{\pi}{3}$ and for every $x_0 \in T_{i,j}$, $\Theta_1(x_0) = -(\frac{1}{\pi} + \frac{2}{\sqrt{3}})$, hence

$$I_0^1(T_i;T_j) = -a\left(\frac{1}{\pi} + \frac{2}{\sqrt{3}}\right), \quad i = 1,\ldots,4,\ j \in A_i.$$

It is easy to see that in the expansion of the heat content of E all the terms which depend on the curvatures of T_i and $T_{i,j}$ will be zero. However, in order to go further in the expansion we first observe that, since $(T_i)_r^2 = \emptyset$ then $I_0^2(T_i;T_j) = 0$ for every $i = 1,..,4$ and $j \in A_i$; moreover, fixed T_i, T_j (with $i = 1,\ldots,4$, $j \in A_i$), $\delta_1 I_0^1(T_i;T_j)$ reduces to $\widetilde{I}_0(T_i;T_j) + \widetilde{I}_1(T_i;T_j) = 2\widetilde{\Theta}_2(x_0)$ with $x_0 \in \partial T_{i,j}$. Without loss of generality, we fix an orthogonal coordinate system, we assume that x_0 coincides with the origin of the axis and that Γ and Σ are two faces of the tetrahedron whose common side coincides with the segment of endpoints the origin O and $A(a,0,0)$. Assume that Γ belongs to $z = 0$ whereas Σ is contained in the plane $z - \sqrt{3}y = 0$. In order to compute $\widetilde{\Theta}_2(0)$ we have to complete $\Sigma_r^{1,0}$ which has a defect of orthogonality around 0 using the set $\widetilde{\Sigma}_r^{1,0}$, and also Γ, which has a defect of tangentiality around 0 using the set $\widetilde{\Gamma}_r$ both introduced in Remark 4.13. In this case $\nu_\Gamma = (0,0,1)$, $\gamma = (0,1,0)$, $\tilde{\gamma} = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$, $\nu_\Sigma = (0, \frac{\sqrt{3}}{2}, -\frac{1}{2})$, $\sigma = (0, \frac{1}{2}, \frac{\sqrt{3}}{2})$, $\tilde{\sigma} = \left(\frac{1}{2}, \frac{\sqrt{3}}{4}, \frac{3}{4}\right)$. By (35), we get that

$$\widetilde{I}_0 = \widetilde{I}_0(\Sigma;\Gamma) = \frac{2D_2\widetilde{Q}_0^2}{(4\pi)^{3/2}} \int_{\widetilde{V}_0^2} \langle \nu_\Sigma, v\rangle\langle \nu_\Gamma, v\rangle e^{-\frac{|v|^2}{4}} \mathcal{H}^1(\widetilde{Q}_0^2)^{-1}(v)dv$$

where

$$\widetilde{Q}_0^2 : \mathbb{R}_+^3 \times \mathbb{R} \to \widetilde{V}_0^2,\ (\alpha,\beta) \mapsto \left(-\frac{1}{2}\alpha_2 + \beta_1, -\frac{1}{2}\alpha_1 - \frac{\sqrt{3}}{4}\alpha_2 + \beta_2, -\frac{\sqrt{3}}{2}\alpha_1 - \frac{3}{4}\alpha_2\right),$$

and $\widetilde{V}_0^2 = \{v_3 \le 0\} \cap \{v_3 - \sqrt{3}v_2 \le 0\}$. Analogously, by (36), we get

$$\widetilde{I}_1 = \widetilde{I}_1(\Sigma;\Gamma) = \frac{2D_2\widehat{Q}_0^2}{(4\pi)^{3/2}} \int_{\widetilde{W}_0^2} \langle \nu_\Sigma, v\rangle\langle \nu_\Gamma, v\rangle e^{-\frac{|v|^2}{4}} \mathcal{H}^1(\widehat{Q}_0^2)^{-1}(v)dv,$$

where

$$\widehat{Q}_0^2 : \mathbb{R}_+^4 \to \widetilde{W}_0^2,\ (\alpha,\beta) \mapsto \left(-\alpha_1 - \frac{1}{2}\alpha_2 - \beta_1 + \frac{1}{2}\beta_2, -\frac{\sqrt{3}}{4}\alpha_2 + \frac{\sqrt{3}}{2}\beta_2, -\frac{3}{4}\alpha_2\right),$$

and $\widetilde{W}_0^2 = \{v_2 \ge \max\{0, v_1\sqrt{3}\}, 3v_1 - v_2\sqrt{3} \le v_3 \le 0\} \cup \{v_1\frac{\sqrt{3}}{2} \le v_2 \le 0, 3v_1 - v_2\sqrt{3} \le v_3 \le \sqrt{3}v_2\}$. Finally, summing all the terms obtained, we get

$$\|T_t\chi_E\|^2_{L^2(\mathbb{R}^3)} = \frac{\sqrt{2}}{12}a^3 - a^2\sqrt{\frac{6t}{\pi}} - 12a\left(\frac{1}{\pi} + \frac{2}{\sqrt{3}}\right)t + 4\sqrt{2t^3}(\widetilde{I}_0 + \widetilde{I}_1) + o(t^h) \quad \forall\, h > 3/2.$$

Acknowledgments We would like to thank Diego Pallara and Fabio Paronetto; the collaboration and useful discussions with them was the starting point of this paper.

References

[1] L. Angiuli, M. Miranda Jr., D. Pallara, F. Paronetto. BV functions and parabolic initial boundary value problems on domains *Ann. Mat. Pura et Applicata*, 188(2):297–331, 2009.

[2] M. van den Berg, P. Gilkey, and R. Seeley. Heat content asymptotics with singular initial temperature distributions. *J. Funct. Anal.*, 254(12):3093–3122, 2008.

[3] M. van den Berg, P. Gilkey, K. Kirsten, and V. A. Kozlov. Heat content asymptotics for Riemannian manifolds with Zaremba boundary conditions. *Potential Anal.*, 26(3):225–254, 2007.

[4] M. van den Berg and J.-F. Le Gall. Mean curvature and the heat equation. *Math. Z.*, 215(3):437–464, 1994.

[5] E. De Giorgi. Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni. *Ann. Mat. Pura Appl. (4)*, 36:191–213, 1954.

[6] E. De Giorgi. Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad r dimensioni. *Ricerche Mat.*, 4:95–113, 1955.

[7] H. Federer. Curvature measures *Trans. Amer. Math. Soc.*, 93:418–491, 1959.

[8] M. Ledoux. Semigroup proofs of the isoperimetric inequality in Euclidean and Gauss space. *Bull. Sci. Math.*, 118(6):485–510, 1994.

[9] U. Menne. Second order rectifiability of integral varifolds of locally bounded first variation *arXiv:0808.3665v1*, 0808.3665v1, 27 Aug 2008.

[10] U. Menne. Some applications of the isoperimetric inequality for integral varifolds. *Adv. Calc. Var.*, 2(3):247–269, 2009.

[11] M. Miranda, Jr., D. Pallara, F. Paronetto, and M. Preunkert. Short-time heat flow and functions of bounded variation in R^N. *Ann. Fac. Sci. Toulouse Math. (6)*, 16(1):125–145, 2007.

[12] J. Rataj and M. Zähle. Curvatures and Currents for Unions of Sets with Positive Reach II. *Ann. Glob. An. and Geom.*, (20):1–21, 2001.

[13] J. Rataj and M. Zähle. Normal cycles of Lipschitz manifolds by approximation with parallel sets. *Differential Geom. Appl.*, 19(1):113–126, 2003.

[14] J. Rataj and S. Winter. A note on measures of parallel sets. Preprint., arXiv:0905.327v1

[15] R. Schätzle. Quadratic tilt-excess decay and strong maximum principle for varifolds. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 3(1):171–231, 2004.

[16] C. Thäle. 50 years sets with positive reach—a survey. *Surv. Math. Appl.*, 3:123–165, 2008.

[17] F.-Y. Wang. Second fundamental form and gradient of Neumann semigroups. *J. Funct. Anal.*, 256(10):3461–3469, 2009.

[18] H. Weyl. On the Volume of Tubes. *Amer. J. Math.*, 61(2):461–472, 1939.

[19] M. Zähle. Integral and current representation of Federer's curvature measures. *Arch. Math.*, 46(6):557–567, 1986.