Γ-CONVERGENCE AND H-CONVERGENCE OF LINEAR ELLIPTIC OPERATORS

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Abstract. We consider a sequence of linear Dirichlet problems as follows

$$\begin{cases} -\mathrm{div}(\sigma_\varepsilon \nabla u_\varepsilon) = f \text{ in } \Omega, \\ u_\varepsilon \in H_0^1(\Omega), \end{cases}$$

with (σ_ε) uniformly elliptic and possibly non-symmetric. Using *purely variational arguments* we give an alternative proof of the compactness of H-convergence, originally proved by Murat and Tartar.

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1. Introduction

The notion of H-convergence was introduced by Murat and Tartar in [10, 12] to study a wide class of homogenization problems for possibly non-symmetric elliptic equations. Let $\sigma_\varepsilon \in L^\infty(\Omega;\mathbb{R}^{n\times n})$ be a sequence of matrices satisfying uniform ellipticity and boundedness conditions on a bounded open set $\Omega \subset \mathbb{R}^n$. We say that σ_ε H-converges to matrix $\sigma_0 \in L^\infty(\Omega;\mathbb{R}^{n\times n})$ satisfying the same ellipticity and boundedness conditions if for every $f \in H^{-1}(\Omega)$ the sequence u_ε of the solutions to the problems

$$\begin{cases} -\mathrm{div}(\sigma_\varepsilon \nabla u_\varepsilon) = f \text{ in } \Omega, \\ u_\varepsilon \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

satisfy

$$u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H_0^1(\Omega) \quad \text{and} \quad \sigma_\varepsilon \nabla u_\varepsilon \rightharpoonup \sigma_0 \nabla u_0 \text{ weakly in } L^2(\Omega;\mathbb{R}^n),$$

where u_0 is the solution to

$$\begin{cases} -\mathrm{div}(\sigma_0 \nabla u_0) = f \text{ in } \Omega, \\ u_0 \in H_0^1(\Omega). \end{cases}$$

The notion of Γ-convergence was introduced by De Giorgi and Franzoni in [5, 6] to study the asymptotic behavior of the solutions of a wide class of minimization problems depending on a parameter $\varepsilon > 0$, which varies in a sequence converging to 0. Let (X, d) be a metric space and let $F_\varepsilon : X \to \overline{\mathbb{R}}$ be a sequence of functionals, we say that F_ε $\Gamma(d)$-converges to a functional $F_0 : X \to \overline{\mathbb{R}}$ if for all $x \in X$ we have

(i) (liminf inequality) for every sequence $x_\varepsilon \xrightarrow{d} x$ in X

$$F_0(x) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon);$$

(ii) (limsup inequality) there exists a sequence $\bar{x}_\varepsilon \xrightarrow{d} x$ in X such that

$$F_0(x) \geq \limsup_{\varepsilon\to 0} F_\varepsilon(\bar{x}_\varepsilon).$$

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It has been proved that when σ_ε is symmetric, the equation (1.1) has a variational structure since it can be seen as the Euler-Lagrange equation associated with

$$\mathcal{F}_\varepsilon(u) = \frac{1}{2}\int_\Omega \sigma_\varepsilon(x)\nabla u\cdot\nabla u\,dx - \int_\Omega f\,u\,dx\,,$$

or, equivalently, as the solution to the minimization problem

$$\min\{\mathcal{F}_\varepsilon(u):\ u\in H^1_0(\Omega)\}. \tag{1.2}$$

Therefore, in this case (1.2) provides a variational principle for the Dirichlet problem (1.1) and the convergence of the solutions of (1.1) can be equivalently studied by means of the Γ-convergence, with respect to the weak topology of $H^1_0(\Omega)$, of the associated functionals $\mathcal{F}_\varepsilon$ or in terms of the G-convergence of the uniformly elliptic, symmetric matrices (σ_ε) (see De Giorgi and Spagnolo [7]).

In this paper we consider the equivalence between H-convergence and Γ-convergence in the possibly non-symmetric case. To every elliptic matrix $\sigma\in L^\infty(\Omega;\mathbb{R}^{n\times n})$ we associate a suitable quadratic integral functional $F: L^2(\Omega;\mathbb{R}^n)\times H^1_0(\Omega)\to[0,+\infty)$ (see (2.12)) and we consider the Γ-convergence with respect to the distance d defined by

$$d((\alpha,\varphi),(\beta,\psi)) = \|\alpha-\beta\|_{H^{-1}(\Omega;\mathbb{R}^n)} + \|\mathrm{div}(\alpha-\beta)\|_{H^{-1}(\Omega)} + \|\varphi-\psi\|_{L^2(\Omega)}\,.$$

We prove (Theorem 3.2) that the H-convergence of σ_ε to σ_0 is equivalent to the $\Gamma(d)$-convergence of the functionals F_ε corresponding to σ_ε to the functional F_0 corresponding to σ_0. In [2] this result was proved using compactness properties of H-convergence [10, 12], while in the present paper the equivalence is obtained as a consequence of a general compactness theorem for integral functionals with respect to $\Gamma(d)$-convergence [1]. Moreover, as a consequence of the results proved in [1], we also give an independent proof (Theorem 3.1) of the compactness of H-convergence based only on Γ-convergence arguments.

2. Notation and preliminaries

In this section we introduce a few notation and we recall some preliminary results we employ in the sequel. For any $A\in\mathbb{R}^{n\times n}$ we denote by A^s and A^a the symmetric and the anti-symmetric part of A, respectively; *i.e.,*

$$A^s := \frac{A+A^T}{2},\quad A^a := \frac{A-A^T}{2},$$

where A^T is the transpose matrix of A. We use bold capital letters to denote matrices in $\mathbb{R}^{2n\times 2n}$. The scalar product of two vectors ξ and η is denoted by $\xi\cdot\eta$.

Let Ω be an open bounded subset of $\mathbb{R}^n$. For $0<c_0\le c_1<+\infty$, $\mathcal{M}(c_0,c_1,\Omega)$ denotes the set of matrix-valued functions $\sigma\in L^\infty(\Omega;\mathbb{R}^{n\times n})$ satisfying

$$\sigma(x)\xi\cdot\xi\ge c_0|\xi|^2,\quad \sigma^{-1}(x)\xi\cdot\xi\ge c_1^{-1}|\xi|^2,\quad \text{for every } \xi\in\mathbb{R}^n,\ \text{for a.e. } x\in\Omega, \tag{2.1}$$

or, equivalently, satisfying

$$\sigma(x)\xi\cdot\xi\ge c_0|\xi|^2,\quad \sigma(x)\xi\cdot\xi\ge c_1^{-1}|\sigma(x)\xi|^2,\quad \text{for every } \xi\in\mathbb{R}^n,\ \text{for a.e. } x\in\Omega. \tag{2.2}$$

Note that (2.1) (or (2.2)) implies that

$$|\sigma(x)|\le c_1\quad \text{for a.e. } x\in\Omega,$$

and that necessarily $c_0\le c_1$. To not overburden notation, in all that follows we always write σ in place of $\sigma(x)$.

Given $\sigma\in\mathcal{M}(c_0,c_1,\Omega)$ we consider the $(2n\times 2n)$-matrix-valued function $\mathbf{\Sigma}\in L^\infty(\Omega;\mathbb{R}^{2n\times 2n})$ having the following block structure

$$\mathbf{\Sigma} := \begin{pmatrix} (\sigma^s)^{-1} & -(\sigma^s)^{-1}\sigma^a \\ \sigma^a(\sigma^s)^{-1} & \sigma^s-\sigma^a(\sigma^s)^{-1}\sigma^a \end{pmatrix}. \tag{2.3}$$

Notice that $\mathbf{\Sigma}$ is symmetric. Moreover, the assumption $\sigma \in \mathcal{M}(c_0, c_1, \Omega)$ easily implies that $\mathbf{\Sigma}$ is uniformly coercive (see [2, Section 3.1.1] for the details); specifically, there exists a constant $C(c_0, c_1) > 0$, depending only on c_0 and c_1, such that

$$\mathbf{\Sigma}\mathrm{w} \cdot \mathrm{w} \geq C(c_0, c_1)|\mathrm{w}|^2, \tag{2.4}$$

for every $\mathrm{w} \in \mathbb{R}^{2n}$, and a.e. in Ω.

If we consider the matrix-valued functions $A, B, C \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ defined as

$$A = (\sigma^s)^{-1}, \quad B = -(\sigma^s)^{-1}\sigma^a, \quad C = \sigma^s - \sigma^a(\sigma^s)^{-1}\sigma^a, \tag{2.5}$$

the matrix $\mathbf{\Sigma}$ can be rewritten as

$$\mathbf{\Sigma} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}. \tag{2.6}$$

We notice that, for a.e. $x \in \Omega$, the matrix $\mathbf{\Sigma}$ belongs to the indefinite special orthogonal group $SO(n, n)$; *i.e.,*

$$\mathbf{\Sigma}\mathbf{J}\mathbf{\Sigma} = \mathbf{J} \quad \text{a.e. in } \Omega, \quad \text{with } \mathbf{J} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{2.7}$$

where $I \in \mathbb{R}^{n\times n}$ is the identity matrix (see [2, Section 3.1.1]). Moreover, taking into account the symmetry of $\mathbf{\Sigma}$, it is immediate to show that (2.7) is equivalent to the following system of identities for the block decomposition (2.6):

$$\begin{cases} AB^T + BA = 0 \\ AC + B^2 = I, & \text{a.e. in } \Omega. \\ CB + B^T C = 0 \end{cases} \tag{2.8}$$

Conversely, one can prove that, if $\mathbf{M} \in L^\infty(\Omega; \mathbb{R}^{2n\times 2n})$ is symmetric and has the block decomposition

$$\mathbf{M} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \tag{2.9}$$

with $A, B, C \in L^\infty(\Omega; \mathbb{R}^{n\times n})$, A and C symmetric, and $\det A \neq 0$, then the first two equations in (2.8) imply the third one, and (2.8) implies that $\mathbf{M}$ is equal to the matrix $\mathbf{\Sigma}$ defined in (2.3) with $\sigma = A^{-1} - A^{-1}B$ (see [2, Proposition 3.1]).

Throughout the paper the parameter ε varies in a strictly decreasing sequence of positive real numbers converging to zero. Let (σ_ε) be a sequence in $\mathcal{M}(c_0, c_1, \Omega)$ and consider the sequence $(\mathbf{\Sigma}_\varepsilon) \subset L^\infty(\Omega; \mathbb{R}^{2n\times 2n})$ defined by (2.3) with $\sigma = \sigma_\varepsilon$. Let $Q_\varepsilon \colon L^2(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^n) \to [0, +\infty)$ be the quadratic forms associated with $\mathbf{\Sigma}_\varepsilon$; *i.e.,*

$$Q_\varepsilon(a, b) := \int_\Omega \mathbf{\Sigma}_\varepsilon \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} dx. \tag{2.10}$$

Their gradients $\operatorname{grad} Q_\varepsilon \colon L^2(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^n) \to L^2(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^n)$ are given by

$$\operatorname{grad} Q_\varepsilon(a, b) = (A_\varepsilon a + B_\varepsilon b, B_\varepsilon^T a + C_\varepsilon b), \tag{2.11}$$

where $A_\varepsilon, B_\varepsilon$, and C_ε are as in (2.5) with $\sigma = \sigma_\varepsilon$. We also consider the quadratic forms $F_\varepsilon : L^2(\Omega; \mathbb{R}^n) \times H_0^1(\Omega) \to [0, +\infty)$ defined by

$$F_\varepsilon(\alpha, \psi) := Q_\varepsilon(\alpha, \nabla\psi) \tag{2.12}$$

For every $\lambda, \mu \in H^{-1}(\Omega)$; we consider the sequence of constrained functionals $F_\varepsilon^{\lambda,\mu} \colon L^2(\Omega; \mathbb{R}^n) \times H_0^1(\Omega) \to [0, +\infty]$ defined as follows

$$F_\varepsilon^{\lambda,\mu}(\alpha, \psi) := \begin{cases} F_\varepsilon(\alpha, \psi) - \langle \mu, \psi \rangle & \text{if } \quad -\mathrm{div}\alpha = \lambda, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.13}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual paring between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Given a symmetric matrix $\mathbf{M} \in L^\infty(\Omega;\mathbb{R}^{2n\times 2n})$, we consider the quadratic functionals $Q_{\mathbf{M}} : L^2(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{R}^n) \to [0,+\infty)$ and $F_{\mathbf{M}} : L^2(\Omega;\mathbb{R}^n) \times H^1_0(\Omega) \to [0,+\infty)$ defined by

$$Q_{\mathbf{M}}(a,b) := \int_\Omega \mathbf{M}\begin{pmatrix} a \\ b \end{pmatrix}\cdot\begin{pmatrix} a \\ b \end{pmatrix} dx \qquad \text{and} \qquad F_{\mathbf{M}}(\alpha,\psi) := Q_{\mathbf{M}}(\alpha,\nabla\psi)\,. \tag{2.14}$$

Considering the block decomposition (2.9), the gradient of $Q_{\mathbf{M}}$ is given by

$$\operatorname{grad} Q_{\mathbf{M}}(a,b) = (Aa + Bb, B^T a + Cb)\,. \tag{2.15}$$

Finally, for every $\lambda,\mu \in H^{-1}(\Omega)$; we consider the constrained functional $F^{\lambda,\mu}_{\mathbf{M}}\colon L^2(\Omega;\mathbb{R}^n) \times H^1_0(\Omega) \to [0,+\infty]$ defined as follows

$$F^{\lambda,\mu}_{\mathbf{M}}(\alpha,\psi) := \begin{cases} F_{\mathbf{M}}(\alpha,\psi) - \langle\mu,\psi\rangle & \text{if} \quad -\operatorname{div}\alpha = \lambda, \\ +\infty & \text{otherwise.} \end{cases}$$

Let w be the weak topology of $L^2(\Omega;\mathbb{R}^n) \times H^1_0(\Omega)$ and let d be the distance in $L^2(\Omega;\mathbb{R}^n) \times H^1_0(\Omega)$ defined by

$$d((\alpha,\varphi),(\beta,\psi)) = \|\alpha-\beta\|_{H^{-1}(\Omega;\mathbb{R}^n)} + \|\operatorname{div}(\alpha-\beta)\|_{H^{-1}(\Omega)} + \|\varphi-\psi\|_{L^2(\Omega)}\,.$$

The following result is proved in [1, Corollary 2.9].

Theorem 2.1. *Let (σ_ε) be a sequence in $\mathcal{M}(c_0,c_1,\Omega)$. There exist a subsequences of ε, not relabeled, and a symmetric matrix $\mathbf{M} \in L^\infty(\Omega;\mathbb{R}^{2n\times 2n})$, such that the functionals F_ε defined by (2.12) $\Gamma(d)$-converge to the functional $F_{\mathbf{M}}$ defined in (2.14). Moreover, $\mathbf{M}$ is positive definite and satisfies the coercivity condition* (2.4).

The following result is a consequence of [1, Theorem 3.3] and of the stability of Γ-convergence under continuous perturbations.

Theorem 2.2. *Let (σ_ε) be a sequence in $\mathcal{M}(c_0,c_1,\Omega)$ and let $\mathbf{M} \in L^\infty(\Omega;\mathbb{R}^{2n\times 2n})$ be a symmetric, positive define matrix satisfying* (2.4). *Assume that the functionals F_ε defined by (2.12) $\Gamma(d)$-converge to the functional $F_{\mathbf{M}}$ defined in (2.14). Then, for every $\lambda,\mu \in H^{-1}(\Omega)$, the functionals $(F^{\lambda,\mu}_\varepsilon)$ defined by (2.13) $\Gamma(w)$-converges to the functional $F^{\lambda,\mu}$ defined by*

$$F^{\lambda,\mu}(\alpha,\psi) := \begin{cases} F_{\mathbf{M}}(\alpha,\psi) - \langle\mu,\psi\rangle & \textit{if} \quad -\operatorname{div}\alpha = \lambda, \\ +\infty & \textit{otherwise.} \end{cases}$$

For the reader's sake, here we briefly recall a fundamental tool we employ in what follows, the Cherkaev-Gibiansky variational principle [3] (see also Fannjiang-Papanicolaou [8] and Milton [9]) , which will be presented in the notational setting which is suitable for our purposes. Loosely speaking, this variational principle amounts to associate to the two following Dirichlet problems

$$\begin{cases} -\operatorname{div}(\sigma_\varepsilon\nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon \in H^1_0(\Omega), \end{cases} \qquad\qquad \begin{cases} -\operatorname{div}(\sigma_\varepsilon^T\nabla v_\varepsilon) = g & \text{in } \Omega, \\ v_\varepsilon \in H^1_0(\Omega). \end{cases} \tag{2.16}$$

with $f,g \in H^{-1}(\Omega)$, a quadratic functional whose Euler-Lagrange equation is solved by a suitable combination of solutions to (2.16) and of their momenta. We set

$$a_\varepsilon := \sigma_\varepsilon\nabla u_\varepsilon \qquad \text{and} \qquad b_\varepsilon := \sigma_\varepsilon^T\nabla v_\varepsilon\,. \tag{2.17}$$

For every $\varepsilon > 0$, $\lambda, \mu \in H^{-1}(\Omega)$ the unique minimizer $(\alpha_\varepsilon, \psi_\varepsilon)$ of $F_\varepsilon^{\lambda,\mu}$ satisfies the constraint $-\mathrm{div}\alpha_\varepsilon = \lambda$ and the following system of Euler-Lagrange equations:

$$\begin{cases} \displaystyle\int_\Omega (A_\varepsilon \alpha_\varepsilon + B_\varepsilon \nabla\psi_\varepsilon)\cdot\beta\,dx = 0, \\ \displaystyle\int_\Omega (B_\varepsilon^T \alpha_\varepsilon + C_\varepsilon \nabla\psi_\varepsilon)\cdot\nabla\varphi\,dx = \langle\mu,\varphi\rangle, \end{cases} \tag{2.18}$$

for every $\beta \in L^2(\Omega;\mathbb{R}^n)$ with $\mathrm{div}\beta = 0$ and for every $\varphi \in H_0^1(\Omega)$.

If $u_\varepsilon, v_\varepsilon$ satisfy (2.16) then we can prove (see [2, Section 3.2] for details) that the pair

$$(a_\varepsilon + b_\varepsilon, u_\varepsilon - v_\varepsilon) \tag{2.19}$$

solves (2.18), with $\lambda = f+g$, $\mu = f-g$, and thus minimizes $F_\varepsilon^{f+g,f-g}$.

In the same way, it can be seen that the pair

$$(a_\varepsilon - b_\varepsilon, u_\varepsilon + v_\varepsilon) \tag{2.20}$$

minimizes $F_\varepsilon^{f-g,f+g}$.

3. The main result

In this section we state and prove the main result of this paper: an alternative and purely variational proof of the sequential compactness of $\mathcal{M}(c_0,c_1,\Omega)$ with respect to H-convergence, originally proved by Murat and Tartar [10, 12].

Theorem 3.1 (Compactness of H-convergence)**.** *Let (σ_ε) be a sequence in $\mathcal{M}(c_0,c_1,\Omega)$. Then there exist a subsequence (not relabeled) and a matrix $\sigma_0 \in \mathcal{M}(c_0,c_1,\Omega)$ such that (σ_ε) H-converges to σ_0 and (σ_ε^T) H-converges to σ_0^T.*

Proof. By Theorem 2.1 there exist a subsequence of F_ε, not relabeled, and a symmetric, positive definite matrix $\mathbf{M} \in L^\infty(\Omega;\mathbb{R}^{2n\times 2n})$, with the block decomposition (2.9), such that F_ε $\Gamma(d)$-converges to $F_{\mathbf{M}}$. In the rest of this proof we show that (σ_ε) H-converges to σ_0 and (σ_ε^T) H-converges to σ_0^T, where $\sigma_0 := A^{-1} - A^{-1}B$.

Let $f, g \in H^{-1}(\Omega)$, let $u_\varepsilon, v_\varepsilon$ be as in (2.16), and let $a_\varepsilon, b_\varepsilon$ be as in (2.17). By standard variational estimates we have that (u_ε) and (v_ε) are bounded in $H_0^1(\Omega)$ while (a_ε) and (b_ε) are bounded in $L^2(\Omega;\mathbb{R}^n)$. Therefore, up to subsequences (not relabeled),

$$u_\varepsilon \rightharpoonup u_0, \quad v_\varepsilon \rightharpoonup v_0 \quad \text{weakly in } H_0^1(\Omega) \quad \text{and} \quad a_\varepsilon \rightharpoonup a_0, \quad b_\varepsilon \rightharpoonup b_0 \quad \text{weakly in } L^2(\Omega;\mathbb{R}^n), \tag{3.1}$$

for some $u_0, v_0 \in H_0^1(\Omega)$ and $a_0, b_0 \in L^2(\Omega;\mathbb{R}^n)$.

Since $(a_\varepsilon + b_\varepsilon, u_\varepsilon - v_\varepsilon)$ are minimizers of $F_\varepsilon^{f+g,f-g}$ and these functionals Γ-converge to $F_{\mathbf{M}}^{f+g,f-g}$ by Theorem 2.2, appealing to the fundamental property of Γ-convergence we find that

$$\lim_{\varepsilon\to 0} F_\varepsilon^{f+g,f-g}(a_\varepsilon + b_\varepsilon, u_\varepsilon - v_\varepsilon) = F_{\mathbf{M}}^{f+g,f-g}(a_0+b_0, u_0-v_0) = \min F_{\mathbf{M}}^{f+g,f-g}\,. \tag{3.2}$$

Similarly, since $(a_\varepsilon - b_\varepsilon, u_\varepsilon + v_\varepsilon)$ minimizes $F_\varepsilon^{f-g,f+g}$, we have also

$$\lim_{\varepsilon\to 0} F_\varepsilon^{f-g,f+g}(a_\varepsilon - b_\varepsilon, u_\varepsilon + v_\varepsilon) = F_{\mathbf{M}}^{f-g,f+g}(a_0-b_0, u_0+v_0) = \min F_{\mathbf{M}}^{f-g,f+g}\,. \tag{3.3}$$

Thanks to Theorem 2.2, (3.2), (3.3), and in view of [1, Proposition 2.10] we are now in a position to invoke the result about the convergence of momenta proved in [1, Corollary 4.6], hence we obtain

$$\mathrm{grad}\, Q_\varepsilon(a_\varepsilon + b_\varepsilon, \nabla u_\varepsilon - \nabla v_\varepsilon) \rightharpoonup \mathrm{grad}\, Q_{\mathbf{M}}(a+b, \nabla u - \nabla v)\,, \tag{3.4}$$

$$\mathrm{grad}\, Q_\varepsilon(a_\varepsilon - b_\varepsilon, \nabla u_\varepsilon + \nabla v_\varepsilon) \rightharpoonup \mathrm{grad}\, Q_{\mathbf{M}}(a_0 - b_0, \nabla u_0 + \nabla v_0) \tag{3.5}$$

weakly in $L^2(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{R}^n)$. By (2.11) and (2.15), considering only the first component, we get

$$A_\varepsilon(a_\varepsilon+b_\varepsilon)+B_\varepsilon(\nabla u_\varepsilon-\nabla v_\varepsilon)\rightharpoonup A(a_0+b_0)+B(\nabla u_0-\nabla v_0), \tag{3.6}$$
$$A_\varepsilon(a_\varepsilon-b_\varepsilon)+B_\varepsilon(\nabla u_\varepsilon+\nabla v_\varepsilon)\rightharpoonup A(a_0-b_0)+B(\nabla u_0+\nabla v_0) \tag{3.7}$$

weakly in $L^2(\Omega;\mathbb{R}^n)$. Since by (2.5) $A_\varepsilon(a_\varepsilon+b_\varepsilon)+B_\varepsilon(\nabla u_\varepsilon-\nabla v_\varepsilon)=\nabla u_\varepsilon+\nabla v_\varepsilon$ and $A_\varepsilon(a_\varepsilon-b_\varepsilon)+B_\varepsilon(\nabla u_\varepsilon+\nabla v_\varepsilon)=\nabla u_\varepsilon-\nabla v_\varepsilon$, from (3.6) and (3.7) we deduce that

$$\nabla u_\varepsilon+\nabla v_\varepsilon\rightharpoonup A(a_0+b_0)+B(\nabla u_0-\nabla v_0), \tag{3.8}$$
$$\nabla u_\varepsilon-\nabla v_\varepsilon\rightharpoonup A(a_0-b_0)+B(\nabla u_0+\nabla v_0) \tag{3.9}$$

weakly in $L^2(\Omega;\mathbb{R}^n)$. Hence, adding up (3.8) and (3.9) entails $\nabla u_\varepsilon\rightharpoonup Aa_0+B\nabla u_0$ in $L^2(\Omega;\mathbb{R}^n)$, which gives $\nabla u_0=Aa_0+B\nabla u_0$ by (3.1). This implies

$$a_0=\sigma_0\,\nabla u_0, \tag{3.10}$$

with

$$\sigma_0:=A^{-1}-A^{-1}B. \tag{3.11}$$

Since $-\mathrm{div}a_\varepsilon=f$, by (2.16) and (2.17) we get that $-\mathrm{div}a_0=f$. Hence, (3.10) implies that u_0 is the solution to

$$\begin{cases}-\mathrm{div}(\sigma_0\nabla u_0)=f & \text{in }\Omega,\\ u_0\in H^1_0(\Omega).\end{cases} \tag{3.12}$$

So far we have proved that for every $f\in H^{-1}(\Omega)$ the solutions u_ε of (2.16) converge weakly in $H^1_0(\Omega)$ to the solution u_0 of (3.12) and their momenta $\sigma_\varepsilon\nabla u_\varepsilon$ converge weakly in $L^2(\Omega;\mathbb{R}^n)$ to $\sigma_0\nabla u_0$. Thus, to conclude the proof of the H-convergence of (σ_ε) to σ_0 it remains to show that σ_0 belongs to $\mathcal{M}(c_0,c_1,\Omega)$. To this end, let $u\in H^1_0(\Omega)$ and choose

$$f:=-\mathrm{div}(\sigma_0\nabla u); \tag{3.13}$$

in this way the solution u_0 of the equation (3.12) coincides with u.

Let $\varphi\in C^\infty_c(\Omega)$. Using φu_ε as a test function in the equation $-\mathrm{div}(\sigma_\varepsilon\nabla u_\varepsilon)=f$ and then passing to the limit on ε we get

$$\int_\Omega f\varphi u_0\,dx=\lim_{\varepsilon\to0}\int_\Omega f\varphi u_\varepsilon\,dx=\lim_{\varepsilon\to0}\Big(\int_\Omega(\sigma_\varepsilon\nabla u_\varepsilon\cdot\nabla u_\varepsilon)\varphi\,dx\Big)+\int_\Omega\sigma_0\nabla u_0\cdot u_0\nabla\varphi\,dx, \tag{3.14}$$

where to compute the limit of the last term in (3.14) we appealed to the strong $L^2(\Omega)$ convergence of u_ε to u_0. On the other hand, since by (3.12)

$$\int_\Omega f\varphi u_0\,dx=\int_\Omega(\sigma_0\nabla u_0\cdot\nabla u_0)\varphi\,dx+\int_\Omega\sigma_0\nabla u_0\cdot u_0\nabla\varphi\,dx$$

from (3.14) we deduce that

$$\lim_{\varepsilon\to0}\int_\Omega(\sigma_\varepsilon\nabla u_\varepsilon\cdot\nabla u_\varepsilon)\varphi\,dx=\int_\Omega(\sigma_0\nabla u_0\cdot\nabla u_0)\varphi\,dx, \tag{3.15}$$

for every $\varphi\in C^\infty_c(\Omega)$. Hence, choosing $\varphi\geq0$, combining (3.15), the first condition in (2.1), and the equality $u=u_0$, we have

$$\int_\Omega(\sigma_0\nabla u\cdot\nabla u)\varphi\,dx\geq c_0\liminf_{\varepsilon\to0}\int_\Omega|\nabla u_\varepsilon|^2\varphi\,dx\geq c_0\int_\Omega|\nabla u|^2\varphi\,dx,$$

the second inequality following from $\nabla u_\varepsilon\rightharpoonup\nabla u_0=\nabla u$ in $L^2(\Omega;\mathbb{R}^n)$. Since this inequality holds true for every $\varphi\in C^\infty_c(\Omega)$, $\varphi\geq0$, we get that

$$\sigma_0\nabla u\cdot\nabla u\geq c_0|\nabla u|^2\quad\text{a.e. in }\Omega, \tag{3.16}$$

for every $u \in H^1_0(\Omega)$. Using the second condition in (2.2), we find

$$\int_\Omega (\sigma_0 \nabla u \cdot \nabla u)\varphi\, dx \geq c_1^{-1} \liminf_{\varepsilon \to 0} \int_\Omega |\sigma_\varepsilon \nabla u_\varepsilon|^2 \varphi\, dx \geq c_1^{-1} \int_\Omega |\sigma_0 \nabla u|^2 \varphi\, dx,$$

since $\sigma_\varepsilon \nabla u_\varepsilon \rightharpoonup \sigma_0 \nabla u_0 = \sigma_0 \nabla u$ in $L^2(\Omega;\mathbb{R}^n)$. From the previous inequality we deduce

$$\sigma_0 \nabla u \cdot \nabla u \geq c_1^{-1} |\sigma_0 \nabla u|^2 \quad \text{a.e. in } \Omega, \tag{3.17}$$

for every $u \in H^1_0(\Omega)$. Finally, (2.2) follows from (3.16) and (3.17) by taking u to be affine in an open set $\omega \subset\subset \Omega$.

We now prove that σ_ε^T H-converges to σ_0^T. Subtracting (3.9) from (3.8) gives $\nabla v_\varepsilon \rightharpoonup Ab_0 - B\nabla v_0$ weakly in $L^2(\Omega;\mathbb{R}^n)$; the latter combined with (3.1) imply that $\nabla v_0 = Ab - B\nabla v_0$. We deduce then

$$b_0 = \tilde{\sigma} \nabla v_0, \tag{3.18}$$

where

$$\tilde{\sigma} := A^{-1} + A^{-1}B. \tag{3.19}$$

Since $-\mathrm{div} b_\varepsilon = g$ by (2.16) and (2.17), we get $-\mathrm{div} b_0 = g$, so that (3.18) implies that v_0 is the solution to

$$\begin{cases} -\mathrm{div}(\tilde{\sigma} \nabla v_0) = g & \text{in } \Omega, \\ v_0 \in H^1_0(\Omega). \end{cases} \tag{3.20}$$

As in the previous part of the proof, this implies that σ_ε^T H-converges to $\tilde{\sigma}$. We want to prove that $\tilde{\sigma} = \sigma_0^T$.

To this end, we argue as in the previous step. Let $u, v \in H^1_0(\Omega)$. We choose $f := -\mathrm{div}(\sigma_0 \nabla u)$ and $g := -\mathrm{div}(\tilde{\sigma} \nabla v)$ and we consider the corresponding solutions u_ε and v_ε of (2.16). Since u coincides with the solution u_0 of (3.12) and v coincides with the solution v_0 of (3.20), the H-convergence of σ_ε entails

$$u_\varepsilon \rightharpoonup u_0 = u \quad \text{weakly in } H^1_0(\Omega) \quad \text{and} \quad \sigma_\varepsilon \nabla u_\varepsilon \rightharpoonup \sigma_0 \nabla u_0 = \sigma_0 \nabla u \quad \text{weakly in } L^2(\Omega;\mathbb{R}^n),$$

while the H-convergence of (σ_ε^T) yields

$$v_\varepsilon \rightharpoonup v_0 = v \quad \text{weakly in } H^1_0(\Omega) \quad \text{and} \quad \sigma_\varepsilon^T \nabla v_\varepsilon \rightharpoonup \tilde{\sigma} \nabla v_0 = \tilde{\sigma} \nabla v \quad \text{weakly in } L^2(\Omega;\mathbb{R}^n).$$

Let $\varphi \in C_c^\infty(\Omega)$; using φv_ε as test function in the equation for u_ε, we get

$$\int_\Omega f(\varphi v_\varepsilon)\, dx = \int_\Omega (\sigma_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon)\varphi\, dx + \int_\Omega \sigma_\varepsilon \nabla u_\varepsilon \cdot v_\varepsilon \nabla \varphi\, dx.$$

Therefore, appealing to the strong $L^2(\Omega)$ convergence of v_ε to v and using φv as a test function in (3.12), we obtain

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_\Omega (\sigma_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon)\varphi\, dx &= \int_\Omega f(\varphi v)\, dx - \int_\Omega \sigma_0 \nabla u \cdot v \nabla \varphi\, dx \\ = \int_\Omega \sigma_0 \nabla u \cdot \nabla(\varphi v)\, dx - \int_\Omega \sigma_0 \nabla u \cdot v\nabla\varphi\, dx &= \int_\Omega (\sigma_0 \nabla u \cdot \nabla v)\varphi\, dx. \end{aligned} \tag{3.21}$$

Moreover, arguing in a similar way, using now φu_ε as test function in the equation for v_ε, it is easy to show that

$$\lim_{\varepsilon \to 0} \int_\Omega (\sigma_\varepsilon^T \nabla v_\varepsilon \cdot \nabla u_\varepsilon)\varphi\, dx = \int_\Omega (\tilde{\sigma} \nabla v \cdot \nabla u)\varphi\, dx. \tag{3.22}$$

Then (3.21) and (3.22) yield

$$\int_\Omega (\sigma_0 \nabla u \cdot \nabla v)\varphi\, dx = \int_\Omega (\tilde{\sigma} \nabla u \cdot \nabla v)\varphi\, dx \quad \text{for every } \varphi \in C_c^\infty(\Omega).$$

Arguing as in the previous proof of (2.2) we deduce from this equality that

$$\sigma_0\xi\cdot\eta=\tilde\sigma\eta\cdot\xi\quad\text{a.e. in }\Omega,$$

for every $\xi,\eta\in\mathbb{R}^n$. This implies that $\tilde\sigma=\sigma_0^T$ a.e. in Ω which concludes the proof of the theorem. □

Given $\sigma_0\in\mathcal{M}(c_0,c_1,\Omega)$, the matrix Σ_0 and the functionals Q_0, F_0, and $F_0^{\lambda,\mu}$ are defined as in (2.6), (2.10), (2.12), and (2.13) with $\sigma=\sigma_0$.

Theorem 3.2. *Let (σ_ε) be a sequence in $\mathcal{M}(c_0,c_1,\Omega)$ and let $\sigma_0\in\mathcal{M}(c_0,c_1,\Omega)$. The following conditions are equivalent:*

(a) *σ_ε H-converges to σ_0;*
(b) *σ_ε^T H-converges to σ_0^T;*
(c) *F_ε $\Gamma(d)$-converges to F_0;*
(d) *$F_\varepsilon^{\lambda,\mu}$ $\Gamma(w)$ to $F_0^{\lambda,\mu}$ for every $\lambda,\mu\in H^{-1}(\Omega)$.*

Proof. The equivalence between (a) and (b) follows immediately from Theorem 3.1. The implication (c) $\Rightarrow$ (d) is given by Theorem 2.2. The implication (d) $\Rightarrow$ (a) is obtained in the proof of Theorem 3.1. It remains to prove that (a) and (b) imply (c). By Theorem 2.1 we may assume that F_ε $\Gamma(d)$-converges to $F_{\mathbf{M}}$ where $\mathbf{M}\in L^\infty(\Omega;\mathbb{R}^{2n\times 2n})$ is a positive definite, symmetric matrix satisfying the coercivity condition (2.4).

To prove that $\mathbf{M}\in SO(n,n)$ we consider the block decomposition (2.9). In Theorem 3.1 we proved that $\sigma_0=A^{-1}-A^{-1}B$ and $\sigma_0^T=\tilde\sigma=A^{-1}+A^{-1}B$; hence, we immediately deduce that

$$AB^T+BA=0\quad\text{a.e. in }\Omega.\tag{3.23}$$

It remains to prove the second condition in (2.8). Let us fix $f,g\in H^{-1}(\Omega)$ and let $u_\varepsilon,v_\varepsilon,a_\varepsilon,b_\varepsilon$ be as in (2.16) and (2.17). By (2.11), (2.15), (3.4), and (3.5) using only the second component we get

$$B_\varepsilon^T(a_\varepsilon+b_\varepsilon)+C_\varepsilon(u_\varepsilon-v_\varepsilon)=\sigma_\varepsilon\nabla u_\varepsilon-\sigma_\varepsilon^T\nabla v_\varepsilon\ \rightharpoonup\ B^T(a_0+b_0)+C(\nabla u_0-\nabla v_0)\tag{3.24}$$

$$B_\varepsilon^T(a_\varepsilon-b_\varepsilon)+C_\varepsilon(u_\varepsilon+v_\varepsilon)=\sigma_\varepsilon\nabla u_\varepsilon+\sigma_\varepsilon^T\nabla v_\varepsilon\ \rightharpoonup\ B^T(a_0-b_0)+C(\nabla u_0-\nabla v_0)\tag{3.25}$$

weakly in $L^2(\Omega;\mathbb{R}^n)$. Then, adding up (3.24) and (3.25) we get

$$\sigma_\varepsilon\nabla u_\varepsilon\rightharpoonup B^Ta_0+C\nabla u_0$$

weakly in $L^2(\Omega;\mathbb{R}^n)$; on the other hand, since $\sigma_\varepsilon\nabla u_\varepsilon=a_\varepsilon\rightharpoonup a_0$ weakly in $L^2(\Omega;\mathbb{R}^n)$, we obtain

$$a_0=B^Ta_0+C\nabla u_0.$$

Since in the proof of Theorem 3.1 we already showed that $a_0=(A^{-1}-A^{-1}B)\nabla u_0$, we finally obtain

$$(I-B^T)(A^{-1}-A^{-1}B)\,\nabla u_0=C\,\nabla u_0\quad\text{a.e. in }\Omega.$$

Therefore, suitably choosing f as in (3.13) and arguing as in the proof of Theorem 3.1 we can easily deduce that

$$(I-B^T)(A^{-1}-A^{-1}B)\,\xi=C\,\xi\quad\text{a.e. in }\Omega,\text{ for every }\xi\in\mathbb{R}^n,$$

thus, by the arbitrariness of $\xi\in\mathbb{R}^n$, we get

$$(I-B^T)(A^{-1}-A^{-1}B)=C\quad\text{a.e. in }\Omega.$$

The latter combined with (3.23) leads to

$$AC+B^2=I\quad\text{a.e. in }\Omega.\tag{3.26}$$

Eventually, by (3.23) and (3.26) we can apply [2, Proposition 3.1] and we deduce that $\mathbf{M}\in SO(n,n)$ a.e. in Ω and that $\mathbf{M}$ is equal to the matrix $\mathbf{\Sigma}$ defined in (2.3) with $\sigma=A^{-1}-A^{-1}B$. Since we have also $\sigma_0=A^{-1}-A^{-1}B$, we conclude that $\mathbf{M}=\mathbf{\Sigma}_0$. □

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