Geometric analysis
of fractional phase transition interfaces

Giovanni Franzina & Enrico Valdinoci

1 The fractional Laplacian operator

This note is devoted to report some recent advances concerning the fractional powers of the Laplace operator and some related problems arising in pde's and geometric measure theory. Namely, the s-Laplacian of a (sufficiently regular) function u can be defined as an integral in the principal value sense by the formula

$$\begin{aligned} -(-\Delta)^s u(x) &:= C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{u(x+y)-u(x)}{|y|^{n+2s}}\,dy \\ &:= C_{n,s}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon} \frac{u(x+y)-u(x)}{|y|^{n+2s}}\,dy. \end{aligned} \tag{1}$$

where $s\in(0,1)$, and

$$C_{n,s}=\pi^{-2s+n/2}\frac{\Gamma(n/2+s)}{\Gamma(-s)}$$

is a normalization constant (blowing up as $s\to 1^-$ and $s\to 0^+$, because of the singularities of the Euler Γ-function). Note that the integral here is singular in the case $s\geq 1/2$, but it converges if $s<1/2$ as it can be estimated via an elementary argument by splitting the domain of integration.

We point out that an equivalent definition may be given by integrating against a singular kernel, which suitably averages a second-order incremental quotient. Indeed, thanks to the symmetry of the kernel under the map $y\mapsto -y$, by performing a standard change of variables, one obtains

$$\int_{\mathbb{R}^n\setminus B_\varepsilon} \frac{u(x+y)-u(x)}{|y|^{n+2s}}\,dy=\frac{1}{2}\int_{\mathbb{R}^n\setminus B_\varepsilon} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy,$$

for all $\varepsilon > 0$; thus (1) becomes

$$-(-\Delta)^s u(x) = \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy. \tag{2}$$

It is often convenient to use the expression in (2), that deals with a convergent Lebesgue integral, rather than the one in (1), that needs a principal value to be well-posed.

The fractional Laplacian may be equivalently defined by means of the Fourier symbol $|\xi|^{2s}$ by simply setting, for every $s \in (0,1)$,

$$\mathcal{F}\Big((-\Delta)^s u\Big) = \big(|\xi|^{2s}(\mathcal{F}u)\big), \tag{3}$$

for all $u \in \mathcal{S}'(\mathbb{R}^n)$, and this occurs in analogy with the limit case $s = 1$, when (3) is consistent with the well-known behavior of the (distributional) Fourier transform $\mathcal{F}$ on Laplacians.

Note that this operator is invariant under the action of the orthogonal transformations in $\mathbb{R}^n$ and that the following scaling property holds:

$$(-\Delta)^s u_\lambda(x) = \lambda^{2s}((-\Delta)^s u)(\lambda x), \quad \text{for all } x \in \mathbb{R}^n, \tag{4}$$

were we denoted $u_\lambda(x) = u(\lambda x)$. For the basics of the fractional Laplace operators and related functional settings see, for instance, [33] and references therein.

After being studied for a long time in potential theory and harmonic analysis, fractional operators defined via singular integrals are riveting attention due to the pliant use that can be made of their nonlocal nature and their applications to models of concrete interest. In particular, equations involving the fractional Laplacian or similar nonlocal operators naturally surface in several applications. For instance, the s-Laplacian is an interesting specific example of infinitesimal generator for rotationally invariant $2s$-stable Lévy processes, taking the "hydrodynamic limit" of the discrete random walk with possibly long jumps: in this case the probability density is described by a fractional heat equation where the classical Laplace operator is replaced by the s-Laplacian, see e.g. [61] for details. An analogous fractional diffusion arises in the asymptotic analysis of the distribution associated with some collision operators in kinetic theory, see [47]. Also, lower dimensional obstacle problems and the fractional Laplacian were intensively studied, see e.g. [58]; for the regularity of the solutions and of that of the free boundary in the obstacle problem and in the thin obstacle problem, we refer to [21]. Nonlocal operators arise also in elasticity problems [57], and in several phenomena, such as water waves [25, 26], flame propagation [20], stability of matter [38], quasi-geostrophic flows [45], crystal dislocation [60], soft thin films [44], stratified materials [52] and others.

The fractional Laplacian presents several technical and conceptual difficulties. First of all, the operator is nonlocal, hence, during the computations, one needs to estimate also the contribution coming from far. Also, since integrating is usually harder than differentiating, constructing barriers and checking the existence of sub/supersolutions is often much harder than in the case of the Laplacian. Furthermore, a psychological unease may arise from the fact that an "integral" operator behaves in fact as a "differential" one. The counterpart of this difficulties lies in the nice averaging properties of the fractional Laplacian, that makes a function revert to its nearby mean, and this has somewhat to control the oscillations.

2 Back to the Laplacian case

2.1 The Allen-Cahn equation

In our opinion, an interesting topic of research involving the s-Laplacian, $s \in (0,1)$, concerns the analysis of the geometric properties of the solutions of the equation

$$(-\Delta)^s u = u - u^3. \tag{5}$$

The formal limit of equation (5) for $s \to 1^-$ is the well-studied Allen-Cahn (or scalar Ginzburg-Landau) equation

$$-\Delta u = u - u^3 \tag{6}$$

which describes (among other things) a two-phase model, where, roughly speaking, the pure phases correspond to the states $u \sim +1$ and $u \sim -1$ and the level set $\{u = 0\}$ is the interface which separates the pure phases.

2.2 A conjecture of De Giorgi

In 1978 De Giorgi [27] conjectured that if u is a smooth, bounded solution of equation (6) in the whole of $\mathbb{R}^n$ and it is monotone in the variable x_n (i.e. $\partial_{x_n} u(x) > 0$ for any $x \in \mathbb{R}^n$), then u depends in fact only on one Euclidean variable and its level sets are hyperplanes – at least if $n \leq 8$.

This conjecture seems to have a strong analogy with the celebrated Bernstein problem, which claims that all entire minimal graphs in $\mathbb{R}^n$ are hyperplanes: indeed, it is well-known that this property holds true for all $n \leq 8$ but it is false for $n \geq 9$.

Similarly, it is known that the conjecture of De Giorgi is true if $n \leq 3$: in the case $n = 2$ this was proved by N. Ghoussoub and C. Gui [43] (see also the paper of Berestycki, Caffarelli and Nirenberg [10]), whereas the result for $n = 3$ is due to

Ambrosio and Cabré [6] (see also the paper of Alberti, Ambrosio and Cabré [1]). In his PhD Thesis [51], Savin proved that the claim holds true also for $4 \leq n \leq 8$ under the additional assumption

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1, \qquad \text{for all } x' \in \mathbb{R}^{n-1}. \tag{7}$$

On the other hand the same conjecture turns out to be false in general if $n \geq 9$: the final answer was given by Del Pino, Kowalczyk and Wei, see [29, 30, 31] by manufacturing a solution to the Allen-Cahn equation, monotone in the direction of x_9, whose zero level set lies closely to the same graph exhibited as a counterexample to the Bernstein problem in $\mathbb{R}^9$ by Bombieri, De Giorgi and Giusti in [13].

Nevertheless, if one assumes the limit (7) to hold uniformly for $x' \in \mathbb{R}^{n-1}$, the conjecture (which follows under the name of Gibbons conjecture in this case and it has some importance in cosmology) is true for all dimension $n \in \mathbb{N}$, see [34, 9, 11]. We point out that the problem is still open in dimension $4 \leq n \leq 8$ if the extra assumptions are dropped. For more information, see [35].

3 Some research lines for $(-\Delta)^s$

3.1 The symmetry problem for $(-\Delta)^s$

One can ask a question similar to the one posed by De Giorgi for the fractional Laplacian: that is, given $s \in (0, 1)$ and u a smooth, bounded solution of

$$(-\Delta)^s u = u - u^3$$

in the whole of $\mathbb{R}^n$, with

$$\partial_{x_n} u > 0,$$

we may wonder whether or not u depends only on one Euclidean variable and its level sets are hyperplanes – at least in small dimension.

In this framework, the first positive answer was given in the pioneering work of Cabré and Solà-Morales [17] when $n = 2$ and $s = 1/2$.

The answer to this question is also positive when $n = 2$ for any $s \in (0, 1)$, see [59, 16], and when $n = 3$ and $s \in [1/2, 1)$, see [14, 15].

Moreover, as it happens in the classical case when $s = 1$, the answer is positive for any $n \in \mathbb{N}$ and any $s \in (0, 1)$ if one assumes the limit condition in (7) to hold uniformly for $x' \in \mathbb{R}^{n-1}$ (this is a byproduct of the results in [36, 16]).

The problem is open for $n \geq 4$ and any $s \in (0, 1)$, and even for $n = 3$ and $s \in (0, 1/2)$. No counterexample is known, in any dimension.

3.2 Γ-convergence for $(-\Delta)^s$

In this section we discuss the asymptotics of a variational problem related to the fractional Laplacian, and specifically to the fractional Allen-Cahn equation (5). Namely, we will consider the free energy defined by

$$\mathfrak{I}_s^\varepsilon(u,\Omega) = \varepsilon^{2s}\mathscr{K}_s(u,\Omega) + \int_\Omega W(u)\,dx, \tag{8}$$

where, at the right-hand side, the dislocation energy of a suitable double-well potential W vanishing at ± 1 (e.g., $W(u) = (1-u^2)^2/4$) is penalized by a small contribution given by the nonlocal functional

$$\mathscr{K}_s(u,\Omega) = \frac{1}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy + \int_\Omega\int_{\mathcal{C}\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy, \tag{9}$$

where $s \in (0,1)$. The presence of a nonlocal contribution here, due to the term $\mathscr{K}_s$, constitutes the main difference of $\mathfrak{I}_s^\varepsilon$ from the perturbed energies arising in the standard theory of phase transitions, such as the functional

$$\varepsilon^2\int_\Omega |\nabla u|^2\,dx + \int_\Omega W(u)\,dx,$$

where the potential energy, which is minimized by the system at the equilibrium, is compensated by a term proportional (up to a small factor representing the surface tension coefficient) to the perimeter of the interface between the two phases. In the nonlocal model in (8), this gradient term is replaced by the fractional Sobolev semi-norm of u, which is responsible for the effects of the long-range particle interactions and affects the interface between the two phases, which can have a *fractional dimension* (see [63, 62] for instance). Such models – or some variation of them obtained after replacing the singular kernel in (8) by a suitable anisotropic Kac potential – under suitable boundary conditions arise in the study of surface tension effects, and they have been investigated, jointly with the limit properties of functionals in the sense of Γ-convergence, by many authors: we mention [2, 3, 4, 4, 39, 40, 41, 42] among the others.

Moreover, as a robust Γ-convergence theory is available, it is customary to discuss the asymptotic behavior of the latter model problems as $\varepsilon \to 0^+$: for instance, in the classical case of $s = 1$, i.e. of the Allen-Cahn equation (6), the Γ-limit is strictly related to the perimeter of a set in the sense of De Giorgi (see [48, 49, 50]). On the other hand, when dealing with the nonlocal functional in (8), the Γ-convergence to the classical perimeter functional holds true provided that

$s \in [1/2, 1)$, while for all $s \in (0, 1/2)$ the Γ-limit is related to a suitable nonlocal perimeter that will be introduced and discussed in the incoming section 3.4. From a physical point of view, these results locate at $s = 1/2$ a critical threshold[1] for the size of the range of all possible interactions among particles contributing to affect the limit interface. We recall that in the nonlocal case, in analogy with the classical case $s = 1$, by the scaling property (4), the solution obtained passing to the microscopic variables $v(x) = u(x/\varepsilon)$ satisfies

$$\varepsilon^{2s}(-\Delta)^s v = v - v^3 \qquad \text{in } \mathbb{R}^n,$$

and the latter can be regarded to as the Euler-Lagrange equation associated with the variational problem of minimizing the scaled energy $\mathfrak{I}_s^\varepsilon(\cdot, \mathbb{R}^n)$ where the potential is $W(u) = (1 - u^2)^2/4$.

To make all these statements precise, first of all one has to specify the metric space where the functionals involved in the Γ-limit are defined: we will denote by X the metric space given by the set

$$\{u \in L^\infty(\mathbb{R}^n) : \|u\|_\infty \leq 1\},$$

equipped with the convergence in $L^1_{loc}(\mathbb{R}^n)$. For all $\varepsilon > 0$ we define a functional $\mathscr{F}_s^\varepsilon : X \to \mathbb{R} \cup \{+\infty\}$ by setting

$$\mathscr{F}_s^\varepsilon(u, \Omega) = \begin{cases} \varepsilon^{-2s}\mathfrak{I}_s^\varepsilon(u, \Omega), & \text{if } s \in (0, 1/2), \\ |\varepsilon \log \varepsilon|^{-1}\mathfrak{I}_s^\varepsilon(u, \Omega), & \text{if } s = 1/2, \\ \varepsilon^{-1}\mathfrak{I}_s^\varepsilon(u, \Omega), & \text{if } s \in (1/2, 1). \end{cases}$$

The proof of the following Theorems 1 and 2 can then be found in [53].

Theorem 1. *Let Ω be a bounded open set in $\mathbb{R}^n$, $\varepsilon \in (0, 1]$ and $s \in [1/2, 1)$. Then, there exists a constant $c_* > 0$, possibly depending on W and s, such that the functional $\mathscr{F}_s^\varepsilon$ of X to $\mathbb{R} \cup \{+\infty\}$ defined by* (8) *Γ-converges in X to the functional $\mathscr{F}_L$ of X to $\mathbb{R} \cup \{+\infty\}$ defined by*

$$\mathscr{F}_L(u, \Omega) = \begin{cases} c_* \operatorname{Per}(E, \Omega), & \text{if } u_{|\Omega} = \chi_E - \chi_{\mathcal{C}E}, \text{for some } E \subset \Omega, \\ +\infty, & \text{otherwise}, \end{cases} \tag{10}$$

for[2] *all $u \in X$.*

[1]The reader has noticed that this is the same threshold for the known results discussed in Section 3.1 when $n = 3$. On the other hand, while the threshold $s = 1/2$ is optimal here, the optimality for the results of Section 3.1 is completely open. Further regularity results for the limit interface when $s \to (1/2)^-$ will be discussed in Section 3.5.

[2]As usual, χ_E denotes the characteristic function of the set E.

Theorem 2. *Let Ω be a bounded open set in $\mathbb{R}^n$, $\varepsilon \in (0,1]$ and $s \in (0,1/2)$. Then, the functional $\mathscr{F}_s^\varepsilon$ of X to $\mathbb{R} \cup \{+\infty\}$ defined by* (8) Γ*-converges in X to the functional $\mathscr{F}_L$ of X to $\mathbb{R} \cup \{+\infty\}$ defined by*

$$\mathscr{F}_{NL}^\varepsilon(u,\Omega) = \begin{cases} \mathscr{K}_s(u,\Omega), & \text{if } u_{|\Omega} = \chi_E - \chi_{\complement E}, \text{for some } E \subset \Omega, \\ +\infty, & \text{otherwise }, \end{cases} \tag{11}$$

for all $u \in X$.

The proof of the Γ-convergence stated in Theorem 2 for the case $s \in (0,1/2)$ turns out to be quite direct, while a much finer analysis is needed in order to deal with the case $s \in [1/2,1)$. The claim of Theorem 1, besides giving an explicit Γ-limit, says that a localization property occurs as ε goes to zero when $s \in [1/2,1)$.

Moreover, the following pre-compactness criterion was proved in [53]:

Theorem 3. *Let Ω be a bounded open set in $\mathbb{R}^n$, and $s \in (0,1)$. For all sequences $\{u_\varepsilon\}_{\varepsilon>0}$ such that*

$$\sup_{\varepsilon \in (0,1]} \mathscr{F}_s^\varepsilon(u_\varepsilon,\Omega) < \infty,$$

by possibly passing to a subsequence we have that

$$u_\varepsilon \to u_* = \chi_E - \chi_{\complement E}, \qquad \text{in } L^1(\Omega), \tag{12}$$

for some $E \subset \Omega$.

Actually, when dealing with minimizers, the convergence in (12) can be enhanced, thanks to the optimal uniform density estimates of [54]. Indeed, it turns out that the level sets of the ε-minimizers u_ε of the scaled functional $\mathscr{F}_s^\varepsilon$ converge to ∂E locally uniformly. The content of such density estimates will be briefly described in the next section.

3.3 Density estimates for the level sets

One of the interesting feature of the solutions to the fractional Allen-Cahn equation is that it is possible to gain informations on the measure occupied by their level sets in a given ball, and this may be interpreted as the probability of finding a phase in a fixed region. In the classical case $s = 1$, corresponding to the Allen-Cahn equation in (6), these density estimates were proved by Caffarelli and Cordoba [18]. A nonlocal counterpart indeed holds, as the following result states, in the case of the fractional Laplacian. The proof, for which we refer to the paper of the second author and Savin [54], makes use of a fine analysis on a weighted double integral and the measure theoretical properties of of the minimizers; an alternative proof based on a fractional Sobolev inequality is contained in [55].

Theorem 4. *Let $R > 0$ and u be a minimizer of the functional $\mathfrak{I}_s^1(\cdot, B_R)$. Then, there exist a function $\bar{R} : (-1,1) \times (-1,1) \to (0, +\infty)$ and a positive constant $\bar{c} > 0$, depending on n, s, W, such that if*

$$u(0) > \theta_1,$$

then

$$|\{u > \theta_2\} \cap B_R| \geq \bar{c} R^n, \tag{13}$$

provided $R \geq \bar{R}(\theta_1, \theta_2)$.

By a scaling argument it follows that if u_ε minimizes the functional $\mathscr{F}_s^\varepsilon(\cdot, B_r)$ and $u_\varepsilon(0) > \theta_1$, then

$$|\{u_\epsilon > \theta_2\} \cap B_r| \; \geq \; \bar{c}\, r^n,$$

provided that $\bar{R}\varepsilon \leq r$.

3.4 Few words on the nonlocal perimeter

Let $s \in (0,\, 1/2)$ and E be a measurable subset of $\mathbb{R}^n$. The s-perimeter of a set E in Ω is defined by

$$\begin{aligned} \mathrm{Per}_s(E, \Omega) \;\; := & \;\; \int_{E \cap \Omega} \int_{(\mathcal{C}E) \cap \Omega} \frac{1}{|x-y|^{n+2s}}\, dy\, dx \\ & + \int_{E \cap \Omega} \int_{(\mathcal{C}E) \cap (\mathcal{C}\Omega)} \frac{1}{|x-y|^{n+2s}}\, dy\, dx \\ & + \int_{E \cap (\mathcal{C}\Omega)} \int_{(\mathcal{C}E) \cap \Omega} \frac{1}{|x-y|^{n+2s}}\, dy\, dx, \end{aligned} \tag{14}$$

for all bounded and connected open set Ω in $\mathbb{R}^n$.

We point out that, in the notation of Section 3.2, we have

$$\mathrm{Per}_s(E, \Omega) = \mathscr{K}_s(\chi_E, \Omega).$$

The properties of the above fractional perimeter were studied in [62, 63] where a generalized co-area formula was established and some nonlocal functionals defined similarly as in (14) were used to define a suitable concept of *fractal dimension*.

As customary in calculus of variations, it worths considering the problem of finding local minimizers of the functional $\mathrm{Per}_s(\cdot, \Omega)$, *i.e.* measurable sets E in $\mathbb{R}^n$ such that

$$\mathrm{Per}_s(E, \Omega) \leq \mathrm{Per}_s(F, \Omega), \quad \text{for all } F \subset \mathbb{R}^n \text{ such that } E \cap \mathcal{C}\Omega = F \cap \mathcal{C}\Omega. \tag{15}$$

If E satisfies (15) (with $\mathrm{Per}_s(E,\Omega) < +\infty$), we say that E is s-minimal in Ω. Since the functional $\mathrm{Per}_s(\cdot,\Omega)$ is lower semi-continuous thanks to Fatou's Lemma, the existence of a minimizer E such as in (15) follows by the compact fractional Sobolev embedding. These nonlocal minimal surfaces were introduced by Caffarelli, Roquejoffre and Savin in [19], were the regularity issues were studied; among the several results therein obtained, we may mention the following

Theorem 5. *Let $\Omega = B_1$ be the ball of radius 1 centered at the origin in $\mathbb{R}^n$. If E is a measurable subset of $\mathbb{R}^n$ such that* (15) *holds, then $\partial E \cap B_{1/2}$ is locally a $\mathscr{C}^{1,\alpha}$ hypersurface out of a closed a set $N \subset \partial E$ having finite* $(n-2)$*-dimensional Hausdorff measure.*

In order to prove Theorem 5, the strategy in [19] is to obtain some density estimates and a geometric *clean ball condition* for all the s-minimizers $E \subset \mathbb{R}^n$. Namely, there is a universal constant $c \in (0,1)$ such that for all points $\xi \in \partial E$ and all $r \in (0,c)$ one can find balls $B_{cr}(a) \subset E \cap B_r(\xi)$, and $B_{cr}(b) \subset (\mathcal{C}E) \cap B_r(\xi)$.

Several improvements of this regularity results have been obtained in the recent years, see [22, 7, 23, 24, 56]. For instance, Caffarelli and the second author have shown in [22] that the constant c of the clean ball condition is uniform as $s \to (1/2)^-$ and they deduced several limit properties of nonlocal minimal surfaces that can be summarized in the following

Theorem 6. *Let $R' > R > 0$, $\{s_k\}_{k\in\mathbb{N}} \subset (0,1/2)$ be a sequence converging to $1/2$ and let us denote by E_k an s_k-minimal set in $B_{R'}$, for all $k \in \mathbb{N}$. Then the following hold:*

- *by possibly passing to a subsequence, the set $\{E_k\}_{k\in\mathbb{N}}$ converge to some limit set E uniformly in B_R;*
- *E is a set of minimal classical perimeter in B_R.*

The convergence of minimizers stated in Theorem 6, was obtained under mild assumptions via dual convergence techniques in the paper by Ambrosio, de Philippis and Martinazzi [7], where the authors proved the equi-coercivity and the Γ-convergence of the fractional perimeter (up to the scaling factor $\omega_{n-1}^{-1}(1/2-s)$) to the classical perimeter in the sense of De Giorgi, whence they also deduced a local convergence result for minimizers.

In the case of a general (smooth) subset $E \subset \mathbb{R}^n$ – possibly being not s-minimal – the s-perimeter can however be related to the classical perimeter in a ball, as the following pointwise convergence result of [22] states:

Theorem 7. *Let $R > 0$ and $E \subset \mathbb{R}^n$ be such that $\partial E \cap B_R$ is $\mathscr{C}^{1,\alpha}$ for some $\alpha \in (0,1)$. Then, there exists a countable subset $N \subset (0,R)$ such that*

$$\lim_{s \to (1/2)^-} (1/2 - s)\mathrm{Per}_s(E, B_r) = \omega_{n-1}\mathrm{Per}(E, \Omega), \tag{16}$$

for all $r \in (0,1) \setminus N$.

The presence of the normalizing factor $(1/2 - s)$, vanishing as $s \to (1/2)^-$, in front of the fractional perimeter here is consistent with the fact that the first integral in (14) diverges when $s = 1/2$ unless either $E \cap \Omega$ or $\mathcal{C}E \cap \Omega$ is empty (see [12], and the remark after Theorem 1 in [7]). In order to prove Theorem 7 one has to estimate the integral contribution to the s-perimeter coming from the smooth transition surface, whereas the boundary contributions are responsible for the possible presence of the negligible set N of radii where (16) may fail. Namely, it turns out that the $(n - s)$-dimensional fractional perimeter of the set E inside a ball B_r, *i.e.*

$$\mathrm{Per}_s^L(E, B_r) = \int\limits_{E \cap B_r} \int\limits_{(\mathcal{C}E) \cap B_r} \frac{1}{|x - y|^{n+2s}}\, dy\, dx,$$

up to constants, does indeed approach the classical perimeter, whereas the integral contributions

$$\mathrm{Per}_s^{NL}(E, B_r) = \int\limits_{E \cap B_r} \int\limits_{(\mathcal{C}E) \cap (\mathcal{C}B_r)} \frac{1}{|x - y|^{n+2s}}\, dy\, dx + \int\limits_{E \cap (\mathcal{C}B_r)} \int\limits_{(\mathcal{C}E) \cap B_r} \frac{1}{|x - y|^{n+2s}}\, dy\, dx$$

can be estimated by the $(n - 1)$-dimensional Hausdorff measure of $\partial E \cap \partial B_r$, which vanishes for all radii r out of a countable subset of $\mathbb{R}$.

Some words are also in order about the asymptotic behaviour of the s-perimeter as $s \to 0^+$. The results proved in the paper [46] imply that

$$\lim_{s \to 0^+} s\mathrm{Per}_s(E, \mathbb{R}^n) = n\omega_n |E|, \tag{17}$$

where $|\cdot|$ here stands for the n-dimensional Lebesgue measure. If Ω is any bounded open set in $\mathbb{R}^n$, it is easily seen that the subadditivity property of $\mathrm{Per}_s(\cdot, \Omega)$, is preserved by taking the limit and thus

$$\mu(E) = \lim_{s \to 0^+} s\mathrm{Per}_s(E, \Omega) \tag{18}$$

defines a subadditive set function on the family $\mathscr{E}$ of sets E such that the limit (18) exists. For istance, all the bounded measurable sets E such that $\mathrm{Per}_s(E, \Omega) < \infty$ for some $s \in (0, 1/2)$ turn out to be included in $\mathscr{E}$. Unfortunately, μ is not a measure, as it was shown in the very recent work [28]. However, in the same paper it is proved that μ is finitely additive on the bounded and separeted subsets or $\mathbb{R}^n$ belonging to $\mathscr{E}$ and, in turn, it coincides with a rescaled Lebesgue measure of the intersection $E \cap \Omega$, namely

$$\mu(E) = n\omega_n |E \cap \Omega|,$$

provided $E \in \mathscr{E}$ is bounded. Furthermore, it can be proved that whenever $E \in \mathscr{E}$ and the integrals

$$\int_{E\cap(\mathcal{C}B_1)} \frac{1}{|y|^{n+2s}}\, dy$$

do converge to some limit $\alpha(E)$ as s tends to zero, then

$$\mu(E) = \Big(n\omega_n - \alpha(E)\Big)|E \cap \Omega| + \alpha(E)|\Omega \cap \mathcal{C}E|.$$

We end this section with the following result.

Theorem 8. *Let $s_o \in (0, 1/2)$, $s \in [s_o, 1/2)$ and E be s-minimal. There exists a universal $\varepsilon_* > 0$, possibly depending on s_o but independent of s and E, such that if*

$$\partial E \cap B_1 \subset \Big\{x : |x_n| \le \varepsilon_*\Big\} \tag{19}$$

then $\partial E \cap B_{1/2}$ is a $\mathscr{C}^\infty$-graph in the n-th Euclidean direction.

Such a uniform flatness property was shown in [23], with a proof of the $\mathscr{C}^{1,\alpha}$ regularity, and the $\mathscr{C}^\infty$ smoothness for all s-minimal $\mathscr{C}^{1,\alpha}$-graphs was proved in the recent paper [8] by the second author in collaboration with Barrios Barrera and Figalli.

This result improves some previous work in [19] where a similar condition was also obtained with $\varepsilon_* > 0$ possibly depending on s. The independence of s will be crucial for the subsequent Theorems 10, 11 and 12.

3.5 Singularities of nonlocal minimal surfaces

As it was discussed in the previous section 3.4, the s-perimeter Γ-converges, up to scaling factors, to the classical classical perimeter and the s-minimal sets converge to the classical minimal surfaces. Thus, the regularity of the classical minimal surfaces in suitably low dimension naturally drops hints that the regularity results available for nonlocal minimal surfaces might not be sharp. Namely, by Theorem 5, the boundary of any s-minimal set E in Ω is a $\mathscr{C}^{1,\alpha}$ manifold out of a set N, which is somehow negligible. The question is whether or not

$$N = \varnothing \tag{20}$$

As far as we know, the only occurrence in which (20) can be proved for any $s \in (0, 1/2)$ is in dimension 2:

Theorem 9. *Let $s \in (0, 1/2)$. If $E \subset \mathbb{R}^2$ is a s-minimal cone in $\mathbb{R}^2$ then E is a half-plane.*

We refer to the recent paper by the second author and O.Savin [56] for the details of the proof, in which a central role is played by some estimates related to a compactly supported domain-perturbation of the cone which is almost linear about the origin. Thus, through a classical dimension reduction argument due to Federer [37] – that was adapted to the fractional case in [19, Theorem 10.3] – it is possible to give an improved estimate for the size of the singular set by replacing $n-2$ with $n-3$ in Theorem 5.

The question of the possible existence of s-minimal cones different from half-space is open in higher dimension, for any $s \in (0, 1/2)$. On the other hand, when s is sufficiently close to $1/2$ the s-minimal sets inherit some of the regularity properties of the classical minimal surfaces. The full regularity results summarized in the following Theorems follow by combinining the rigidity results of [23, Theorem 3,4,5], where the $\mathscr{C}^{1,\alpha}$ regularity is proved, and the Schauder-type estimates recently provided in the paper [8].

Theorem 10. *Let $n \leq 7$. There exists $\varepsilon > 0$ such that, if $2s \in (1-\varepsilon, 1)$, then any s-minimal surface in $\mathbb{R}^n$ is locally a $\mathscr{C}^\infty$-hypersurface for some $\alpha \in (0, 1)$.*

Theorem 11. *There exists $\varepsilon > 0$ such that, if $2s \in (1-\varepsilon, 1)$, then any s-minimal surface in $\mathbb{R}^8$ is locally a $\mathscr{C}^\infty$-hypersurface out of a countable set.*

Theorem 12. *Let $n \geq 9$. There exists $\varepsilon > 0$ such that, if $2s \in (1-\varepsilon, 1)$, then any s-minimal surface in $\mathbb{R}^n$ is locally a $\mathscr{C}^\infty$-hypersurface out of a set whose Hausdorff measure is at most $n-8$.*

Of course , we think that it would be desirable to determine sharply the above ε, to better analyze the regularity theory of nonlocal minimal surfaces for other ranges of $s \in (0, 1/2)$ and $n \in \mathbb{N}$, and to understand the possible relation between this regularity theory and the symmetry property of minimal or monotone solutions of the fractional Allen-Cahn equation.

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Giovanni Franzina
Dipartimento di Matematica
Università degli Studi di Trento
Via Sommarive 14
38050 Povo, Trento (Italy)
giovanni.franzina@unitn.it

Enrico Valdinoci
Dipartimento di Matematica
Università degli Studi di Milano
Via Cesare Saldini 50
20133 Milano (Italy)
enrico.valdinoci@unimi.it