Qualitative properties of evolution schemes in higher dimension domains

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Abstract

Results concerning geometric regularity of minimizers for average distance functional in the static case were first presented in [7], [8] and [9], while [10] does a survey. They mainly prove that such minimizers cannot contain crosses or loops, and must verify some regularity condition, in two dimension case; [14] extended those results to higher dimension case. In [11] and [12] these results were discussed for two types of discrete evolution schemes, in two dimensional domains. In this paper we analyze the case of higher dimensions, and prove that similar results, i.e. absence of loops and Ahlfors regularity, hold. Finally, we will show that when key geometric properties on the domain are not verified, Ahlfors regularity is not true anymore.

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1 Introduction

In [7], [8] and [9] the authors proved that in a sufficiently regular domain, under fairly general conditions, local minimizers for the average distance functional must verify certain geometric properties, namely the absence of loops and crosses. Moreover, they proved that (under some general conditions) these optimal sets are always finite union of Lipschitz curves, and verify Ahlfors regularity. A review was done in [10].

Let $\Omega \subseteq \mathbb{R}^N$ be a sufficiently regular domain, compact, connected, closure of an open set, we define

$$\mathcal{A}_l(\Omega) := \{S \subseteq \Omega : S \text{ compact, connected by path, } \dim_{\mathcal{H}} S = 1, \mathcal{H}^1(S) \leq l\},\ \mathcal{A}(\Omega) := \bigcup_{j\geq 0} \mathcal{A}_j(\Omega). \tag{1.1}$$

In the following we will omit the dependence on domain Ω if there is no risk of confusion.

Then given a non decreasing function $A : [0, \text{diam } \Omega] \longrightarrow [0, \infty)$, a measure $f \in L^1(\Omega, \mathcal{L}^N)$, where $\mathcal{L}^N$ denotes the Lebesgue measure in $\mathbb{R}^N$, we define the functional

$$F_f : \mathcal{A} \longrightarrow (0, \infty), \qquad F_f(S) := \int_\Omega A(\text{dist}(x, S))df(x), \tag{1.2}$$

where $\text{dist}(\cdot, \cdot)$ represents the path distance in Ω (i.e. for any couple of points $x_1, x_2 \in \Omega$, $\text{dist}(x_1, x_2)$ is given by $\min_{\beta \in \mathcal{P}(x_1,x_2)} \mathcal{H}^1(\beta([0,1]))$, with $\mathcal{P}(x_1, x_2)$ denoting the set of continuous path $\beta : [0,1] \longrightarrow \Omega$, $x_1, x_2 \in \beta([0,1])$). While $\text{dist}(\cdot, \cdot)$ a priori depends on the domain Ω, for sake of brevity we will omit this dependence when no risk of confusion arises (e.g. when the domain is clearly given).

This will be referred as *"average distance functional"*. The associated minimization problem, i.e. given $l > 0$ find an element

$$S_{opt} \in \text{argmin}_{\mathcal{A}_l} F_f$$

will be referred as *"average distance problem"*, and such S_{opt} as *"average distance minimizer"*.

This formulation arises from urban planning/network optimization problems. Indeed an easy interpretation is: given a region Ω, with population distribution f, transport cost A (i.e. $A(s)$ is the cost to cover a distance s), find the optimal optimal transportation network S_{opt} among all networks with length not exceeding l, minimizing the cost of reaching the network. For a wider overview see [8]

It has been proven in [14] that under quite general conditions any average distance minimizer could not contain loops; moreover, under additional summability of the measure f, those minimizers are Ahlfors regular.

1.1 Evolutions

The evolutionary variants of the average distance problem are the Euler schemes. Similarly, they arise from urban planning/network optimization problems, when the additional time variable and related constraints are considered. Given similar $\Omega \subseteq \mathbb{R}^N$, f, A as in the average distance problem, and an initial datum $S_0 \in \mathcal{A}$, a time step $\varepsilon > 0$, consider

$$\begin{cases} w(0) := S_0 \\ w(n) \in \text{argmin}_{\mathcal{A}_{\mathcal{H}^1(S_0)+n\varepsilon}} F_f \\ w(n) \supseteq w(n-1) \end{cases} . \tag{1.3}$$

In all the paper, when we will write *"$X \in$ argmin $\mathcal{G}$"*, where X is an element and $\mathcal{G}$ a functional, we will mean that X is an arbitrary element of argmin $\mathcal{G}$ ($\sharp$argmin $\mathcal{G} > 1$ is possible in general).

From the third condition every set in the evolution must contain all previous sets: this property will be called *"irreversibility"* in the following. This property can be used to model irreversible network expansion, e.g. rapid transit system expansion, where removing existing network is highly uneconomical. It has been proven in [12] that evolutions like (1.3) in the two dimension case verify the absence of loops, i.e. if S_0 does not contain loops, then $w(n)$ does not contain loops for any n, and the proof used an idea very similar to that used in [7]. In the same paper it has been proven that the absence of crosses is false, by showing an explicit counterexample.

In (1.3) the only constraint is the length of the evolving set at each step. Another type of evolution does not impose constraints on length, but requires a second function, which we will call *"dissipation"*:

Definition 1.1. *A function $D_\varepsilon : \mathcal{A} \times \mathcal{A} \longrightarrow [0, \infty]$ is a "dissipation" if it verifies:*

1. *for any $S \in \mathcal{A}$, $D_\varepsilon(S, S) = 0$,*
2. *for any $S_0, S_1, S_2 \in \mathcal{A}$, satisfying $S_0 \subseteq S_1$, $S_0 \subseteq S_2$ and $\mathcal{H}^1(S_2 \backslash S_0) \geq \mathcal{H}^1(S_1 \backslash S_0)$ inequality*

$$D_\varepsilon(S_2 \backslash S_0) \geq D_\varepsilon(S_1 \backslash S_0)$$

   holds.

The dissipation represents the "cost" to pass from one configuration to another. Given $\Omega \subseteq \mathbb{R}^N, f, A$ as in the average distance problem, an initial datum $S_0 \in \mathcal{A}$ parameter $\varepsilon > 0$, and a dissipation $D_\varepsilon : \mathcal{A} \times \mathcal{A} \longrightarrow [0, \infty]$ we consider

$$\begin{cases} w(0) := S_0 \\ w(n) \in \text{argmin}_{\mathcal{X} \supseteq w(n)} F_f(\mathcal{X}) + D_\varepsilon(\mathcal{X}, w(n)) \\ w(n) \supseteq w(n-1) \end{cases} . \tag{1.4}$$

It has been proven in [11] that, similarly to evolution (1.3), in the two dimension case the absence of loops is true (when initial datum does not contain loops), while the absence of crosses is false, by showing an explicit counterexample.

For both problems (1.3) and (1.4) a solution will be a sequence $\{w(k)\}_{k=0}^{\infty}$ of elements of $\mathcal{A}$, verifying the constraints and minimality properties imposed. In this paper we aim to extend some results about evolution cases to domain in higher dimensions. In particular we will prove that the absence of loops is valid even in higher dimensions, along with some weak analytic regularity (Ahlfors regularity).

It is worth mentioning that taking the limit $\varepsilon \to 0$ in (1.3) leads to the class of *quasi static* evolutions, while taking the limit $\varepsilon \to 0$ in (1.4) leads to the *dynamic evolutions* (with $D_\varepsilon(S_1, S_2) := \frac{d(S_1, S_2)^2}{2\varepsilon}$, where d is a suitable distance, being the most classic case). We refer to [3] for more discussion.

The paper will be structured as follows:

- Section 2 will recall estimates about the average distance functional in higher dimension domains,
- Section 3 will analyze the the absence of loops for solutions of problem (1.3) and (1.4),
- Section 4 will discuss Ahlfors regularity for solutions of problem (1.3) and (1.4),
- Section 5 will present some counterexamples to Ahlfors regularity of solutions, when key geometric regularity properties are not assumed.

1.2 Notations

Now we define the notion of *"loop"*:

Definition 1.2. *Let $\Omega \subseteq \mathbb{R}^N$ be a domain, $W \in \mathcal{A}$, we say that W is a "loop" if there exists an homeomorphism $\varphi : W \longrightarrow S^1 \subseteq \mathbb{R}^2$.*

We will use frequently the expression "adding a set I to the set Σ", with $I, \Sigma \in \mathcal{A}$: with this we will mean (and implicitly assume):

- $\Sigma \cup I \in \mathcal{A}$, $\mathcal{H}^1(I \cap \Sigma) = 0$,
- there exist points $x^* \in I \cap \Sigma$, $x' \in I \backslash \Sigma$ and a path $\gamma : [0,1] \longrightarrow I$ such that $\gamma(0) = x'$, $\gamma(1) = x^*$, $\gamma([0,1]) \cap W = \{x^*\}$.

Moreover, some symbols (e.g. $\varepsilon, \rho, \Omega, f$, etc.) will be used several times, in different statements: unless explicitly mentioned, if a symbol is used in two different Theorem/Proposition/Lemma/Definition, there is no connection between them. The only notable is $\mathcal{A}$, which (when there is no risk of confusion about the domain) will always refer to the set defined in (1.1).

We list some common used symbols:

- $\varepsilon, \eta, \xi, r, \rho$ to denote small positive numbers,
- n to denote integers, like mute counters or even the dimension,
- p to denote the summability class, and q the conjugate exponent of p,
- Ω to denote the domain,
- f to denote the measure,
- Λ to denote the Lipschitz constant.

1.3 Basic conditions

We have imposed in (1.2) that the average distance function (given a domain $\Omega \subseteq \mathbb{R}^N$, a function $A : [0, \text{diam}\,\Omega] \longrightarrow [0, \infty)$, a measure f) must have form

$$F_f(S) := \int_\Omega A(\text{dist}(x, S))df(x).$$

Under this generality very little can be said about its minimizers: indeed, if no additional condition is put, any set can be the optimal set for F_f. This because given an arbitrary set $X \in \mathcal{A}$, putting $\chi_X \cdot \mathcal{H}^1$ the Hausdorff measure restricted on X, i.e. $\chi_X \cdot \mathcal{H}^1(Y) := \mathcal{H}^1(Y \cap X)$ for any Lebesgue measurable $Y \subseteq \Omega$

$$F_{\chi_X \cdot \mathcal{H}^1}(X) = \int_\Omega A(\text{dist}(w, X))\chi_X(w)dw = 0.$$

The first condition we put is that we consider only measures f not charging ridges, i.e. given an arbitrary $W \in \mathcal{A}$, the set

$$\mathcal{R}_W := \{x \in \Omega : \text{ there exist distinct } y_1, y_2 \in W \text{ such that } \text{dist}(x, y_1) = \text{dist}(x, y_2) = \text{dist}(x, W)\}$$

is f-negligible. This is a quite weak condition, as from [13] these ridges are $(\mathcal{H}^1, 1)$-rectifiable. Thus any measure absolutely continuous with respect to the Lebesgue measure does not charge ridges.

Then some restrictions on the function A is required too: we will assume in all the paper, as done in [14]:

(α_1) $A : [0, \text{diam } \Omega] \longrightarrow \mathbb{R}$ is Lipschitz continuous with Lipschitz constant Λ, $A(0) = 0$, monotone increasing,

(α_2) for any $c > 0$ there exists $\lambda = \lambda(c) > 0$ such that $|A(x) - A(y)| \geq \lambda|x - y|$ whenever $|x - y| \in [c, \text{diam } \Omega]$.

From above conditions (satisfied by several regular functions, like $A(x) := x^p$ for any $p \geq 1$) follows A injective on $[c', \text{diam}\Omega]$ for any $c' \in (0, \text{diam } \Omega)$.

Even if a priori the functional depends on A, for sake of brevity we will omit this dependence when no confusion arises (e.g. in a statement where A is given in the hypothesis).

Moreover, unless explicitly stated, we will always assume that the domain $\Omega \subseteq \mathbb{R}^N$ is *uniformly locally convex*, i.e.:

($*$) there exists positive constants ρ_0, m_-, m_+ such for any point $x \in \Omega$, $\rho \in (0, \rho_0)$, $\overline{B(x, \rho)} \cap \Omega$ is convex and there exists an homeomorphism $\varphi : \overline{B(x, \rho)} \cap \Omega \longrightarrow \overline{B(x, \rho)}$ such that

$$m_-\text{dist}(\varphi(x_1), \varphi(x_2)) \leq \text{dist}(x_1, x_2) \leq m_+\text{dist}(\varphi(x_1), \varphi(x_2)) \tag{1.5}$$

for any points $x_1, x_2 \in \overline{B(x, \rho)} \cap \Omega$.

2 Preliminaries

In this section we report some basic facts (for more details see [8], [9], [10] and [14]) about the average distance functional. Most of these will concern estimates on the average distance functional when small variations are done on a given set.

The above results were all proven with a similar technique: given a domain $\Omega \subseteq \mathbb{R}^N$, a measure f, suppose there exists $L > 0$ and an element

$$\Sigma_{opt} \in \text{argmin}_{\mathcal{A}_L} F_f$$

containing a loop $E \subseteq \Sigma_{opt}$. Then

1. Remove a suitable (small) set $J \subseteq E$, and estimate the difference $F_f(\Sigma_{opt}) - F_f(\Sigma_{opt} \setminus J)$,

2. Choose a suitable set $\Sigma' \in \mathcal{A}_L$ verifying $\Sigma' \supseteq \Sigma_{opt} \setminus J$, and estimate the difference $F_f(\Sigma_{opt} \setminus J) - F_f(\Sigma')$.

This procedure leads to $\Sigma' \in \mathcal{A}_L$ with $F_f(\Sigma') < F_f(\Sigma_{opt})$, contradicting $\Sigma_{opt} \in \text{argmin}_{\mathcal{A}_L} F_f$. In order to apply it, we need several estimates on F_f; most proofs can be found in [8], [9] and [14].

Lemma 2.1. *Given a domain $\Omega \subseteq \mathbb{R}^N$, a non negative measure f, a function $A : [0, diam\ \Omega] \longrightarrow \mathbb{R}$, for any elements $\Sigma_1, \Sigma_2 \in \mathcal{A}$ with $\Sigma_1 \subseteq \Sigma_2$ inequality*

$$F_f(\Sigma_2) \leq F_f(\Sigma_1)$$

holds. In other words, F_f is not decreasing with respect to the inclusion.

Moreover, suppose $\mathcal{H}^1(\Sigma_2 \backslash \Sigma_1) > 0$. Then inequality

$$F_f(\Sigma_2) < F_f(\Sigma_1)$$

holds.

Proof. The proof is very simple: $\Sigma_1 \subseteq \Sigma_2$ gives

$$\text{dist}(x, \Sigma_2) \leq \text{dist}(x, \Sigma_1)\ \forall x \in \Omega,$$

thus

$$A(\text{dist}(x, \Sigma_2)) \leq A(\text{dist}(x, \Sigma_1))\ \forall x \in \Omega,$$

and integrating on Ω

$$\int_\Omega A(\text{dist}(x, \Sigma_2)) df(x) \leq \int_\Omega A(\text{dist}(x, \Sigma_1)) df(x)\ \forall x \in \Omega.$$

For the second part, $\Sigma_1 \subsetneq \Sigma_2$ implies there exists an open set B such that for any $z \in B$ the inequality $\text{dist}(z, \Sigma_1) > \text{dist}(z, \Sigma_2)$ holds, so using the strict monotonicity of A, and integrating on Ω concludes the proof. □

This result has a first consequence: under these hypothesis on Ω, f, A, for any $l > 0$

$$\text{argmin}_{\mathcal{A}_l} F_f \subseteq \mathcal{A}_l \backslash \bigcup_{0 \leq j < l} \mathcal{A}_j.$$

The next three results are from [14], to which we refer for more details.

Lemma 2.2. *Let $\Omega \subseteq \mathbb{R}^N$ be the domain, f a measure, and $\Sigma \in \mathcal{A}$ containing a cross E. Then for any $x \in E$ there exist a sequence of open, connected sets $\{D_k\}_{k=0}^\infty$ such that:*

- *$x \in D_k$ for k sufficiently large,*
- *$E \backslash D_k$ connected,*
- *diam $D_k \to 0$ for $k \to \infty$.*

The next result estimates the difference for the average distance functional when some small set is removed from a loop.

Lemma 2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be the domain, f a given measure, $\Sigma \in \mathcal{A}$ containing a loop $E \subseteq \Sigma$. Then given $\beta \in (0,1]$, for $\mathcal{H}^1$-almost any point $x \in E$, for any $r > 0$ there exists $\rho \in (0,r)$ and $\Sigma' \in \mathcal{A}$ such that:*

- $\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) - \rho/2 + (16n^{3/2} + 2)\beta\rho$,
- $\Sigma \backslash \Sigma' \subseteq B(x,\rho)$, $\Sigma' \backslash \Sigma \subseteq B(x, 32n\rho)$,
- *dist*$(y,\Sigma') \leq$ *dist*(y,Σ) *for any* $y \notin B(x, 64n^{3/2}\rho)$,
- *dist*$(y,\Sigma') \leq$ *dist*$(y,\Sigma) + \rho$ *for any* $y \in B(x, 64n^{3/2}\rho)$.

This lemma mainly says that given a set $\Sigma \in \mathcal{A}$, it is possible to find a competitor for the loop with smaller Hausdorff measure, and the variation for the average distance functional is totally encompassed in $B(x, 64n^{3/2}\rho)$, while the "loss" in path is not greater than ρ.

Lemma 2.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a given domain, $l > 0$ a given value, f a given measure, Borel sets $H, K \subseteq \Omega$ such that $f(K) > 0$ and*

$$r := \inf\{\mathit{dist}(x,H) : x \in K\} > 0.$$

Then for any compact set $\Sigma \subseteq H$ with $\mathcal{H}^1(\Sigma) \leq l$ there exists for any ε sufficiently small a set $\Sigma' \supseteq \Sigma$ such that

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) + 2n\varepsilon,\ F_f(\Sigma') \leq F_f(\Sigma) - \frac{\lambda(r) f(K)}{32nl}\varepsilon^2.$$

This result affirms that adding some set with length ε sufficiently small to Σ, the gain for the average distance functional is comparable with ε^2 at least. In the two dimension case (see [7], [8] and [9] for instance) a stronger result holds: the gain for the average distance functional is comparable with $\varepsilon^{3/2}$ at least.

From these lemmata it has been proven in [14] an average distance minimizer Σ_{opt} could not contain loops:

1. Lemma 2.3 is used to estimate the variation for the average distance functional when some small set is removed,
2. then Lemma 2.4 gives a suitable competitor which contradicts the optimality of Σ_{opt}.

3 Absence of loops

In Section 2 we have very sketchily recalled the technique used in the proof of absence of loops for minimizers of the average distance problem in general N-dimension case, along with several preliminary lemmata. Here we will adapt the proof to the discrete Euler scheme evolutions, and prove that a similar result holds. The two dimension case has been already discussed in [12], for problem (1.3), in [11] for (1.4) and the absence of loops was effectively extended in both cases, while the absence of crosses proved false.

As we are considering evolutions like (1.3) or (1.4), it may be possible that at some step k the difference $w(k)\backslash w(k-1)$ is not connected: if this is the case, we can write that

$$w(k)\backslash w(k-1) = \bigcup_{i\in\mathcal{J}} \mathcal{C}_i$$

where $\mathcal{C}_i$ are its connected components and $\mathcal{J}$ is a suitable set of indexes. As $\mathcal{H}^1(w(k)\backslash w(k-1)) < \infty$, for at most countable h the component $\mathcal{C}_h$ verifies $\mathcal{H}^1(\mathcal{C}_h) > 0$, thus we can split the passage

$$w(k-1) \to w(k)$$

in

$$w(k-1) \to w(k-1)\cup\mathcal{C}_{i_1} \to w(k-1)\cup\mathcal{C}_{i_1}\cup\mathcal{C}_{i_2} \to w(k-1)\cup\mathcal{C}_{i_1}\cup\mathcal{C}_{i_2}\cup\mathcal{C}_{i_3} \to \cdots$$

where $\{i_s\}_{s=1}^\infty$ are indexes for which $\mathcal{H}^1(\mathcal{C}_{i_s}) > 0$, and analyze each single passage separately. This process leads in at most countable steps to $w(k)$, thus for any measure f the sequence $\{F_f(w(k-1)\cup\bigcup_{j=1}^h \mathcal{C}_{i_j})\}_{h=1}^\infty$ converges to $F_f(w(k))$, so in the this section (but not in Sections 4 and 5) we can assume $w(k)\backslash w(k-1)$ connected.

The absence of loops can be generalized to the N-dimension case, using a similar idea (but different estimates) from the two dimension case analyzed in [12]. The main result will be Theorem 3.3.

Lemma 3.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a given domain, f a given measure, A a given function, $S_0 \in \mathcal{A}$ with $F_f(S_0) < \infty$ and not containing loops, and $h > 0$ a given positive value. Then any element*

$$\Sigma_{opt} \in \{S \in \mathit{argmin}_{\mathcal{A}_{\mathcal{H}^1(S_0)+h}} F_f : S \supseteq S_0\}$$

is such that $\Sigma_{opt}\backslash S_0$ does not contain loops.

Proof. Suppose there exists an element

$$\Sigma_{opt} \in \{S \in \operatorname{argmin}_{\mathcal{A}_{\mathcal{H}^1(S_0)+h}} F_f : S \supseteq S_0\}$$

such that the difference $I := \Sigma_{opt}\backslash S_0$ contains a loop $E \subseteq I$. From Lemma 2.1 follows that such Σ_{opt} must verify $\mathcal{H}^1(\Sigma_{opt}) = \mathcal{H}^1(S_0) + h$. The goal will be creating a competitor $\Sigma' \in \mathcal{A}_{\mathcal{H}^1(S_0)+h}$ satisfying $F_f(\Sigma') < F_f(\Sigma_{opt})$.

The idea used here is similar to that used in [14] to prove the absence of loops in minimizers of the average distance problem (and in [12] for the two dimension case).

As $f(\Sigma_{opt}) = 0$ by hypothesis, there exists a not f-negligible compact set K such that $\Sigma_{opt}\cap K = \emptyset$, and put

$$R := \frac{1}{2}\min\{\mathrm{dist}(y, \Sigma_{opt}) : y \in K\} > 0.$$

We have supposed the existence of loop $E \subseteq \Sigma_{opt}$, thus $f(E) = 0$, and

$$\lim_{r\to 0^+} \frac{f(B(x,r))}{r} = 0$$

for $\mathcal{H}^1$-almost every $x \in E$ (see [2] for further details).

Let be $\beta := \dfrac{1}{64n^{3/2}+8}$, and t a free parameter for now. Applying Lemma 2.3 yields to the existence of:

- $\rho \in (0,t)$ and $\Sigma' \in \mathcal{A}$ such that

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma_{opt}) - \rho/4.$$

Choose $x^* \in E$ such that $\lim_{r\to 0^+} \frac{f(B(x^*,r))}{r} = 0$, this leads to

$$\begin{aligned} F_f(\Sigma') &\leq F_f(\Sigma_{opt}) + \int_{B(x^*,64n^{3/2}\rho)} (A(\text{dist}(w,\Sigma_{opt})+\rho) - A(\text{dist}(w,\Sigma_{opt})))df(w) \\ &\leq F_f(\Sigma_{opt}) + \rho f(B(x^*,64n^{3/2}\rho))\Lambda \\ &= F_f(\Sigma_{opt}) + 64n^{3/2}\rho^2 \frac{f(B(x^*,64n^{3/2}\rho))}{64n^{3/2}\rho}\Lambda \end{aligned} . \quad (3.1)$$

Lemma 2.4 applied to Σ' gives the existence of a competitor Σ'' verifying

$$\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(\Sigma') + 2n\varepsilon \leq \mathcal{H}^1(\Sigma_{opt}) + 2n\varepsilon - \rho/4$$

and choosing $\varepsilon := \rho/8n$ this yields

$$\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(\Sigma_{opt}).$$

For the average distance functional

$$F_f(\Sigma'') \leq F_f(\Sigma') - \frac{\lambda(R)f(K)}{32n\mathcal{H}^1(\Sigma')}\frac{\rho^2}{64n^2} \quad (3.2)$$

holds. Combining (3.1) and (3.2), for ρ sufficiently small, Σ'' satisfies $\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(\Sigma_{opt})$ and $F_f(\Sigma'') < F_f(\Sigma_{opt})$. Finally, the competitor Σ'' contains S_0, thus is admissible. □

Lemma 3.2. *Let Ω be a given domain, f a given measure, A a given function, $S_0 \in \mathcal{A}$ with $F_f(S_0) < \infty$ and not containing loops, and $h > 0$ a given positive value. Consider an arbitrary element*

$$\Sigma_{opt} \in \{S \in \mathit{argmin}_{\mathcal{A}_{\mathcal{H}^1(S_0)+h}} F_f : S \supseteq S_0\}.$$

Suppose there exists a loop $E \in \Sigma_{opt}$, and let $\varphi : \mathbb{R}^2 \supset S^1 \longrightarrow E$ be an arbitrary homeomorphism. Then the set $V := \varphi^{-1}(E \cap (\Sigma_{opt}\backslash S_0))$ has non empty interior part.

Proof. From Lemma 3.1 follows that $E \nsubseteq \Sigma_{opt}\backslash S_0$. As by hypothesis $E \nsubseteq S_0$, then both $E \cap S_0$ and $E \cap \Sigma_{opt}\backslash S_0$ are non empty. So $V := \varphi^{-1}(E \cap (\Sigma_{opt}\backslash S_0)) \neq \emptyset$. Without loss of generality we can work with another homeomorphism ϕ satisfying:

1. $\phi : [0,1] \longrightarrow E$, $\phi(0) = \phi(1) = P \in E \cap S_0$,
2. $\phi_{|(0,1)} : (0,1) \longrightarrow E\backslash\{P\}$ is an homeomorphism.

This choice is due to technical reasons only, as it is easier to work with ϕ. Proving that V has non empty interior is equivalent to prove $W := \phi^{-1}(E \cap (\Sigma_{opt} \backslash S_0))$ has non empty interior. Suppose the opposite, i.e. W has empty interior (that is, as both ϕ and ϕ^{-1} are homeomorphism, $E \cap (\Sigma_{opt} \backslash S_0)$ has empty interior). From assumption (2) on ϕ this means $(E \cap (\Sigma_{opt} \backslash S_0)) \backslash \{P\}$ has empty interior in $E \backslash \{P\}$, or equivalently $(E \cap S_0) \backslash \{P\}$ dense in $E \backslash \{P\}$.

Since $E \backslash \{P\}$ dense in E, this leads to

$$\overline{(E \cap S_0) \backslash \{P\}} = \overline{E \backslash \{P\}} = E$$

which ultimately yields

$$\overline{E \cap S_0} = E$$

and considering E, S_0 are closed sets, $E \cap S_0 = E$ follows, contradicting the hypothesis. □

Notice that the parameter h almost plays no role in the proof: indeed this is the case, and will be used in the following result.

Theorem 3.3. *Let $\Omega \subseteq \mathbb{R}^N$ be a given domain, f a given measure, A a given function, $\varepsilon > 0$ a given time step $S_0 \in \mathcal{A}$ an initial datum with $F_f(S_0) < \infty$ and not containing loops, and consider*

$$\begin{cases} w(0) := S_0 \\ w(n+1) \in \textit{ argmin}_{\mathcal{A}_{\mathcal{H}^1(S_0)+(n+1)\varepsilon}} F_f \\ w(n+1) \supseteq w(n) \end{cases} . \tag{3.3}$$

Then for any $n \geq 0$ the set $w(n)$ does not contain loops.

Then let $D_\varepsilon : \mathcal{A} \times \mathcal{A} \longrightarrow [0, \infty]$ a dissipation, and consider

$$\begin{cases} w(0) := S_0 \\ w(n+1) \in \textit{ argmin}_{\mathcal{A}} F_f(\mathcal{X}) + D_\varepsilon(\mathcal{X}, w(n)) \\ w(n+1) \supseteq w(n) \end{cases} . \tag{3.4}$$

Then for any $n \geq 0$ the set $w(n)$ does not contain loops.

We have deliberately used the same notation in both cases: indeed the proof is somewhat similar, and unless explicitly specified, valid in both cases.

Proof. The proof is done by induction on n, and the first part is valid for both evolutions (3.3) and (3.4):

- by hypothesis $w(0) := S_0$ does not contain loops,
- suppose $w(n)$ does not contain loops.

The goal is to prove that $w(n+1)$ does not contain loops. Suppose the contrary, i.e. there exists a loop $S \subseteq w(n+1)$: this may lead to two possibilities:

1. $S \subseteq w(n+1) \backslash w(n)$,

or

2. $S \cap w(n+1) \backslash w(n)$ and $S \cap w(n)$ are non empty,

with the third possibility $S \subseteq w(n)$ excluded by inductive hypothesis.

Notice that by construction $w(n+1) \supseteq w(n)$, and

$$w(n+1) \in \{\mathcal{X} \in \text{ argmin}_{\mathcal{A}_{\mathcal{H}^1(w(n))+\varepsilon}} F_f : \mathcal{X} \supseteq w(n)\},$$

so hypothesis of Lemma 3.1 and 3.2 are applicable to both possibility (1) and (2). Applying Lemma 3.1 would lead immediately $S \nsubseteq w(n+1) \backslash w(n)$, thus possibility (1) is excluded.

Now possibility (2) remains. Here the proofs for (3.3) and (3.4) are slightly different.

1. Case (3.3).
   Let be $\phi : [0,1] \longrightarrow S$ an homeomorphism like that chosen in the proof of Lemma 3.2; applying the latter, $\phi^{-1}(E \cap (w(n+1) \backslash w(n)))$ is not empty, thus contains an open ball $(t^* - \rho, t^* + \rho) \subseteq \phi^{-1}(E \cap (w(n+1) \backslash w(n)))$, with $\rho > 0$. The image $\phi((t^* - \rho, t^* + \rho))$ is an open connected arc in $E \cap (w(n+1) \backslash w(n))$. Then it is possible to apply Lemma 2.3 and 2.4, similarly to what done in [14], and create a competitor $\Sigma' \in \{\mathcal{X} \in \mathcal{A}_{\mathcal{H}^1(w(n))+\varepsilon} : \mathcal{X} \supseteq w(n)\}$ with $F_f(\Sigma') < F_f(w(n+1))$, contradicting the optimality of $w(n+1)$.

2. Case (3.4).
   Using the same technique, i.e. construct a competitor Σ' satisfying $\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(w'(n+1))$ and $F_f(\Sigma') < F_f(w'(n+1))$, as done for case (3.3): this verifies

   (a) $w(n) \subseteq \Sigma'$,

   (b) $\mathcal{H}^1(\Sigma' \backslash w(n)) = \mathcal{H}^1(w(n+1) \backslash w(n))$,

   thus

   $$F_f(w(n+1)) + D_\varepsilon(w(n+1), w(n)) > F_f(\Sigma') + D_\varepsilon(\Sigma', w(n))$$

   which contradicts the optimality of $w(n+1)$.

Thus a loop is never present. □

In this proof it was possible to apply results from the average distance minimizers case almost without modifications, and the irreversibility condition in the evolution problems could be easily solved, as Lemma 2.3 and Lemma 2.4 state the existence of suitable competitors for $\mathcal{H}^1$-almost every $x \in E$, and these competitors differ from the original set in a small ball.

4 Ahlfors regularity

In [14] it has been proven that minimizers of the average distance functional exhibit Ahlfors regularity, when the measure considered verifies some summability properties. In this section we aim to extend these results to solutions of (1.3) and (1.4), by adapting the proof. We present now some preliminary results about Ahlfors regularity.

Definition 4.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a given domain, a set $\Sigma \in \mathcal{A}$ is "Ahlfors regular" is there exists $x_1, c_2 > 0$ such that*

$$c_1 \leq \frac{\mathcal{H}^1(\Sigma \cap B(x,r))}{r} \leq c_2 \tag{4.1}$$

for any $x \in \Sigma$, $r \in (0, diam\ \Sigma)$.

While a quite weak regularity property, it causes uniform rectifiability on Σ.

4.1 Discrete evolutions

In [14] it has been shown that the lower bound estimate in (4.1) is trivial: for any $r \leq \text{diam}\ \Sigma/2$ the border $\partial \overline{B(x,r)}$ must intersect Σ, thus $\mathcal{H}^1(\Sigma \cap B(x,r)) \geq r$, or equivalently

$$\frac{\mathcal{H}^1(\Sigma \cap B(x,r))}{r} \geq 1.$$

This argument applies to all elements of $\mathcal{A}$, independently from minimality properties.

Thus the "hard" part of the proof is the upper bound estimate, which uses several properties and results. As see in the following, condition $f \in L^1$ may be not sufficient, and extra summability is required.

We will first present some lemmata from [14], to which we refer for the proof.

Lemma 4.2. *Given natural numbers n and k, $x \in \mathbb{R}^n$, $\rho >$), points $\{z_i\}_{i=1}^k \subseteq \overline{B(x,\rho)}$, there exists $\Sigma \in \mathcal{A}(\overline{B(x,\rho)})$ such that*

- $z_i \in \Sigma$ *for* $i = 1, \cdots, k$,
- $\mathcal{H}^1(\Sigma) \leq C^* k^{\frac{n-1}{n}} \rho$, *where C^* depends only on n.*

We present a sketch of the proof, and refer to [14] for more details.

Proof. (Sketch) Upon translation and rescaling we can suppose $x = (0, \cdots, 0)$, $\rho = 1/2$ and $\{z_i\}_{i=1}^k \subseteq [0,1]^n$. Let Γ_j be a uniform one dimensional grid with step j ($\{(x_1, \cdots, x_n) : jx_i \in \mathbb{N}$ for at least $n - 1$ indexes$\}$): we have

$$H^1(\Gamma_j) \leq n(j+1)^{n-1}, \qquad \max_{y \in [0,1]^n} \text{dist}(y, \Gamma_j) \leq \frac{\sqrt{n}}{2j}. \tag{4.2}$$

Let $z_{i,j}$ one arbitrary projection of z_i on Γ_j for $j = 1, \cdots, k$, and put

$$\Gamma_j^* := \Gamma_j \cup \bigcup_{i=1}^k \{sz_i + (1-s)z_{i,j} : s \in [0,1]\}. \tag{4.3}$$

It is obvious that $z_i \in \Gamma_j^*$ for any i, j; from (4.2) inequality

$$\mathcal{H}^1(\Gamma_j^*) \leq n(j+1)^{n-1} + \frac{k\sqrt{n}}{2j}$$

follows, and choice $j := [k^{1/n}]$ gives

$$\mathcal{H}^1(\Gamma^*_{[k^{1/n}]}) \leq n([k^{1/n}]+1)^{n-1} + \frac{k\sqrt{n}}{2[k^{1/n}]}$$

which concludes the proof. □

Remark 4.3. *Let $M \subseteq \mathbb{R}^n$ be convex set, and assume there exists a homeomorphism $\varphi : M \longrightarrow \overline{B(x,\rho)}$ verifying:*

- *there exists $m_1, m_2 > 0$ such that*

$$m_1 dist(\varphi(z_1), \varphi(z_2)) \leq dist(z_1, z_2) \leq m_2 dist(\varphi(z_1), \varphi(z_2)) \tag{4.4}$$

for any $z_1, z_2 \in M$.

Then the conclusion of Lemma 4.2 can be applied for points $\{y_i\}_{i=1}^K \subseteq M$: indeed given k points $\{z_i\}_{i=1}^k$ of M, upon translation and rescaling, we can apply Lemma 4.2 to points $\{\varphi(z_i)\}_{i=1}^k$ in the domain $[0,1]^n$. Using the same construction, let be Γ_j the same set defined in the proof of Lemma 4.2, and put

$$\Gamma'_j := \Gamma_j \cup \bigcup_{i=1}^k \{s\varphi(z_i) + (1-s)z_{\varphi,i,j} : s \in [0,1]\}$$

where $z_{\varphi,i,j}$ denote an arbitrary projection of $\varphi(z_i)$ on Γ_j.

Now it is clear that $\varphi^{-1}(\Gamma_j) \subseteq M$, as well $\varphi^{-1}(\{s\varphi(z_i) + (1-s)z_{\varphi,i,j} : s \in [0,1]\}) \subseteq M$ for any $i = 1, \cdots, k$. From (4.4) there exists m'_1, m'_2 such that

$$m'_1 \mathcal{H}^1(\Gamma_j) \leq \mathcal{H}^1(\varphi^{-1}(\Gamma_j)) \leq m'_2 \mathcal{H}^1(\Gamma_j)$$

and

$$m'_1(dist(\varphi(z_i), z_{\varphi,i,j})) \leq dist(z_i, \varphi^{-1}(z_{\varphi,i,j})) \leq m'_2(dist(\varphi(z_i), z_{\varphi,i,j}))$$

thus the same conclusion of Lemma 4.2 holds for points $\{z_i\}_{i=1}^k \subseteq M$.

Lemma 4.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a given domain, $\Sigma \in \mathcal{A}$, then for any $x \in \Sigma$ there exists $\Sigma' \in \mathcal{A}$ such that for any $\rho > 0$*

- $\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B(x,\rho)) + C(\left(\frac{\mathcal{H}^1(\Sigma \cap B(x,2\rho))}{2\rho}\right)^{\frac{n-1}{n}} + 1)\rho$*,*
- $\Sigma \backslash \Sigma' \subseteq B(x, 2\rho)$*,* $\Sigma' \backslash \Sigma \subseteq B(x, 8\sqrt{n}\rho)$*,*
- *$dist(z, \Sigma') < dist(z, \Sigma)$ for any $z \notin B(x, 4n\rho)$,*
- *$dist(z, \Sigma') \leq dist(z, \Sigma) + \rho$ for any $z \in B(x, 4n\rho)$.*

where C is a positive constant depending only on n.

Lemma 4.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a given domain, $\Sigma \in \mathcal{A}$ and suppose there exists $r > 0$ such that for any $x \in \Sigma$, $0 < \rho < r$ the inequality*

$$\frac{\mathcal{H}^1(\Sigma \cap B(x,\rho))}{\rho} \leq a \frac{\mathcal{H}^1(\Sigma \cap B(x,2\rho))}{2\rho}^{\alpha} + b$$

holds for some fixed $a > 0, b \geq 0, \alpha \in (0,1)$. Then there exists a constant $K = K(a,b,\alpha,r,\mathcal{H}^1(\Sigma))$ such that

$$\frac{\mathcal{H}^1(\Sigma \cap B(x,\rho))}{\rho} \leq K.$$

The proof for the evolution cases must deal with the irreversibility conditions. We recall a brief sketch for the proof in the averager distance case (and refer to [14] for more details):

Theorem 4.6. *Let be $\Omega \subseteq \mathbb{R}^n$ a given domain, $f \in L^p$, $p \geq \dfrac{n}{n-1}$ a given measure, $A : [0, \text{diam } \Omega] \longrightarrow [0,\infty)$ a given function, and $\Sigma_{opt} \in \text{argmin}_{\mathcal{A}_L} F_f$ for some $L \geq 0$. Then Σ_{opt} is Ahlfors regular.*

Proof. First suppose $L > 0$, otherwise Σ_{opt} is a single point.

1. $f(\Sigma_{opt}) = 0$, thus there exists a compact set K with $f(K) > 0$ and $K \cap \Sigma_{opt} = \emptyset$. This can be chosen as $K := \Omega \backslash (\Sigma_{opt})_{2c}$, with $c \in (0, \text{diam } \Sigma_{opt})$ and $(\Sigma_{opt})_{2c} := \{y \in \Omega : \text{dist}(y, \Sigma_{opt}) < 2c\}$; choose a small $\rho > 0$;

2. let be Σ' the competitor given in Lemma 4.4, and using Hölder inequality follows

$$F_f(\Sigma') \leq F_f(\Sigma_{opt}) + 2\Lambda\rho f(B(x,4n\rho)) \leq F_f(\Sigma_{opt}) + 2\Lambda\rho ||f||_{L^p}^{1/p} \mathcal{L}^n(B(x,4n\rho))^{1/q}$$

   where $\mathcal{L}^n(B(x,4n\rho))$ clearly has order $O(\rho^n)$,

3. inequality

$$\mathcal{H}^1(\Sigma') \geq \mathcal{H}^1(\Sigma_{opt} \cap B(x,\rho)) - \rho H(\frac{\mathcal{H}^1(\Sigma_{opt} \cap B(x,2\rho))}{2\rho} + 1) \tag{4.5}$$

   holds, and two cases arise:

   (a) if

$$\mathcal{H}^1(\Sigma_{opt} \cap B(x,\rho)) - \rho H(\frac{\Sigma_{opt} \cap B(x,2\rho)}{2\rho} + 1) \leq 0$$

   Lemma 4.5 concludes $\dfrac{\mathcal{H}^1(\Sigma_{opt} \cap B(x,\rho))}{\rho} \leq K'$ for some $K' > 0$.

   (b) if $\mathcal{H}^1(\Sigma_{opt} \cap B(x,\rho)) - \rho H(\dfrac{\Sigma_{opt} \cap B(x,2\rho)}{2\rho} + 1) > 0$ then for ρ sufficiently small inclusion $\Sigma' \subseteq \{z \in \Omega : \text{dist}(x, \Sigma_{opt}) < c\}$ holds. Applying Lemma 2.4 yields to the existence of a set $\Sigma'' \in \mathcal{A}$ such that

$$F_f(\Sigma'') \leq F_f(\Sigma') - H'(\mathcal{H}^1(\Sigma' \cap B(x,\rho)) - \rho H''(\left(\frac{\mathcal{H}^1(\Sigma' \cap B(x,2\rho))}{2\rho}\right)^{\frac{n-1}{n}} + 1))^2 \tag{4.6}$$

where H', H'' are positive constants not dependent on ρ and x. Combining

$$F_f(\Sigma') \leq F_f(\Sigma_{opt}) + O(\rho^{\frac{n}{q}+1}) \tag{4.7}$$

with (4.6) and the optimality of Σ_{opt} (i.e. $F_f(\Sigma_{opt}) \leq F_f(\Sigma''))$ yields

$$\mathcal{H}^1(\Sigma' \cap B(x,\rho)) - \rho H''(\left(\frac{\mathcal{H}^1(\Sigma' \cap B(x,2\rho))}{2\rho}\right)^{\frac{n-1}{n}} + 1) \leq H^* \rho^{\frac{n}{2q}-\frac{1}{2}} \leq H^*(\text{diam } \Sigma_{opt})^{\frac{n}{2q}-\frac{1}{2}}$$

with H^* independent from x and ρ, and applying Lemma 4.5 concludes the proof.

□

Lemma 4.4 cannot be used when irreversibility condition is added. A weaker variant is required.

Lemma 4.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a given domain, $\Sigma_* \in \mathcal{A}$ Ahlfors regular, $\Sigma \supseteq \Sigma_*$, then for any $x \in \Sigma$ there exists $\Sigma' \in \mathcal{A}$, $\Sigma' \supseteq \Sigma_*$, such that for any $\rho > 0$*

- $\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B(x,\rho)) + C(\left(\frac{\mathcal{H}^1(\Sigma \cap B(x,2\rho))}{2\rho}\right)^{\frac{n-1}{n}} + 1)\rho,$
- $\Sigma \backslash \Sigma' \subseteq B(x, 2\rho)$, $\Sigma' \backslash \Sigma \subseteq B(x, 8\sqrt{n}\rho)$,
- *$dist(z, \Sigma') < dist(z, \Sigma)$ for any $z \notin B(x, 4n\rho)$,*
- *$dist(z, \Sigma') \leq dist(z, \Sigma) + \rho$ for any $z \in B(x, 4n\rho)$.*

where C is a positive constant depending only on n and Σ_*.

Proof. The proof uses an idea similar to that found for 4.4 (see [14] for instance), with corrections due to irreversibility condition.

Given a point $x \in \Sigma$, $\rho \in (0, \delta)$ (δ given by condition $(*)$), put $k(x,\rho) := \sharp\{\Sigma \cap \partial B(x,\rho)\}$; from coarea formula

$$\mathcal{H}^1(\Sigma \cap B(x,2\rho)) \geq \int_0^{2\rho} k(x,t)dx \geq \int_\rho^{2\rho} k(x,t)dt$$

which implies there exists $t \in [\rho, 2\rho]$ such that

$$k(x,t) \leq 2\frac{\mathcal{H}^1(\Sigma \cap B(x,2\rho))}{\rho}.$$

Lemma 4.2, with condition $(*)$ and Remark 4.3 guarantee the existence of $\Sigma_0(t) \in \mathcal{A}$ such that $\{\Sigma \cap B(x,t)\} \subseteq \Sigma_0(t)$, and $\mathcal{H}^1(\Sigma_0(t)) \leq C^*(n)k(x,t)^{\frac{n-1}{n}}t$.

Let be $\Sigma_1(t) := x + \cup_{j=1}^n \{se_j : s \in [-t,t]\}$, where e_j denotes the j-th unit vector ($e_j = (0, \cdots, 0, 1, 0, \cdots, 0)$, with the only "1" occupying the j-th place).

Some discussion about $\Sigma_1(t)$ is required, as we have only $\Sigma_1(t) \subseteq \overline{B(x,t)}$ but not $\Sigma_1(t) \subseteq \Omega$, thus we should prove $\Sigma_1(t) \cap \Omega$ connected first. Thus given an arbitrary point $z_0 \in (\Sigma_1(t)\backslash\{x\}) \cap \Omega$, there exists $t(z_0) \in [-t, t]$ and $j(z_0) \in \{1, \cdots, n\}$ such that $z_0 = x + t(z_0)e_{j(z_0)}$, and since $\overline{B(x,t)} \cap \Omega$ convex by condition $(*)$, $\{x + ue_{j(z_0)} : u \in [0, t(z_0)]\} \subseteq \Omega$ follows. This guarantees that every point $z \in \Sigma_1(t) \cap \Omega$ is connected by a path (as $\{x + ue_{j(z_0)} : u \in [0, t(z_0)]\} \subseteq \Omega$) to $x \in \Omega$, thus $\Sigma_1(t) \cap \Omega$ is connected. In the following we will write $\Sigma_1(t)$ instead of $\Sigma_1(t) \cap \Omega$.

Upon a rotation $\Sigma_0(t) \cap \Sigma_1(t) \neq \emptyset$. Put

$$\Sigma' := \Sigma \backslash B(x,t) \cup (\Sigma_* \cap B(x,t)) \cup \Sigma_0(t) \cup \Sigma_1(t),$$

and inequality

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B(x,t)) + \mathcal{H}^1(\Sigma_* \cap B(x,t)) + \mathcal{H}^1(\Sigma_0(t)) + \mathcal{H}^1(\Sigma_1(t))$$

follows.

By construction $\mathcal{H}^1(\Sigma_1(t)) \leq 4n^{3/2}t$; combining

$$\mathcal{H}^1(\Sigma_0(t)) \leq C^*(n)k(x,t)^{\frac{n-1}{n}}$$

given by Lemma 4.2 and

$$k(x,t) \leq 2\frac{\mathcal{H}^1(\Sigma \cap B(x,2\rho))}{\rho},$$

inequality

$$\mathcal{H}^1(\Sigma_0(t)) \leq 2C^*(n)\left(\frac{\mathcal{H}^1(\Sigma \cap B(x,2\rho))}{2\rho}\right)^{\frac{n-1}{n}} t$$

follows, yielding

$$\begin{aligned}
\mathcal{H}^1(\Sigma') &\leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B(x,t)) + \mathcal{H}^1(\Sigma_* \cap B(x,t)) + 2C^*(n)\left(\frac{\mathcal{H}^1(\Sigma \cap B(x,2\rho))}{2\rho}\right)^{\frac{n-1}{n}} t + 4n^{3/2}t \\
&\leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B(x,t)) + \mathcal{H}^1(\Sigma_* \cap B(x,t)) + 4C^*(n)\left(\frac{\mathcal{H}^1(\Sigma \cap B(x,2\rho))}{2\rho}\right)^{\frac{n-1}{n}} \rho + 8n^{3/2}\rho \\
&= \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B(x,\rho)) + \mathcal{H}^1(\Sigma_* \cap B(x,t)) + \left(4C^*(n)\left(\frac{\mathcal{H}^1(\Sigma \cap B(x,2\rho))}{2\rho}\right)^{\frac{n-1}{n}} + 8n^{3/2}\right)\rho
\end{aligned}.$$

As Σ_* is Ahlfors regular by hypothesis, there exists $K > 0$ such that

$$\frac{\mathcal{H}^1(\Sigma_* \cap B(x,t))}{t} \leq K,$$

thus

$$\begin{aligned}
\mathcal{H}^1(\Sigma') &\leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B(x,\rho)) + Kt\left(4C^*(n)\left(\frac{\mathcal{H}^1(\Sigma \cap B(x,2\rho))}{2\rho}\right)^{\frac{n-1}{n}} + 8n^{3/2}\right)\rho \\
&\leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \cap B(x,\rho)) + \left(2K + 4C^*(n)\left(\frac{\mathcal{H}^1(\Sigma \cap B(x,2\rho))}{2\rho}\right)^{\frac{n-1}{n}} + 8n^{3/2}\right)\rho
\end{aligned}$$

and putting $C := 2K + 4C^*(n)\left(\frac{\mathcal{H}^1(\Sigma \cap B(x,2\rho))}{2\rho}\right)^{\frac{n-1}{n}} + 8n^{3/2}$ concludes the proof. □

Notice that assumption $(*)$ is crucial, as guarantees $\Sigma_0(t), \Sigma_1(t) \subseteq \Omega$ for all ρ sufficiently small.
Now we can present the result about evolution cases:

Theorem 4.8. *Let be $\Omega \subseteq \mathbb{R}^N$ a given domain, $f \in L^p$ with $p > \dfrac{N}{N-1}$ a given measure, $A : [0, diam] \longrightarrow [0, \infty)$ a given function, $S_0 \in \mathcal{A}$ an Ahlfors regular initial datum, $\varepsilon > 0$ a given time step, and consider*

$$\begin{cases} w(0) := S_0 \\ w(n+1) \in \mathit{argmin}_{\mathcal{A}_{\mathcal{H}^1(S_0)+(n+1)\varepsilon}} F_f \\ w(n+1) \supseteq w(n) \end{cases} . \tag{4.8}$$

Then for any n the set $w(n)$ is Ahlfors regular.

Let $D_\varepsilon : \mathcal{A} \times \mathcal{A} \longrightarrow [0, \infty]$ a dissipation, and consider

$$\begin{cases} w(0) := S_0 \\ w(n+1) \in \mathit{argmin} F(S) + D_\varepsilon(S, w(n)) \\ w(n+1) \supseteq w(n) \end{cases} . \tag{4.9}$$

Then for any n the set $w(n)$ is Ahlfors regular.

Notice that we deliberately omitted using different notations for the two cases: indeed the proof is similar, and unless specified, will be intended valid in both cases.

Proof. Similarly to Theorem 3.3 on the absence of loops, the proof is done by induction. By hypothesis $w(0) := S_0$ is Ahlfors regular. Suppose that $w(n)$ is Ahlfors regular, and the goal is to prove $w(n+1)$ is Ahlfors regular too.

First notice that $f(w(n+1)) = 0$ forces the existence of a compact set $K \subseteq \Omega$ with $f(K) > 0$ (similarly to what done in the proof of Theorem 4.6, available in [14], the choice $K := \Omega \backslash \{\omega \in \Omega : \mathrm{dist}(\omega, w(n+1)) < 2c\}$ is acceptable for some $c \in (0, \mathrm{diam}\, w(n+1))$.

Consider a point $y \in w(n+1)$. Applying Lemma 4.7 (with $\Sigma_* = w(n)$) yields the existence of $\Sigma' \in \mathcal{A}$ verifying

- $\Sigma' \supseteq w(n)$,
- inequality

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(w(n+1)) - \mathcal{H}^1(w(n+1) \cap B(y, \rho)) + C\left(\left(\frac{\mathcal{H}^1(w(n+1) \cap B(y, 2\rho))}{2\rho}\right)^{\frac{N-1}{N}} + 1\right)\rho$$

  for some $C > 0$ depending on N and $w(n)$.

Moreover

$$\begin{aligned} F_f(\Sigma') \leq F_f(w(n+1)) + 2\Lambda\rho f(B(y, 4N\rho)) &\leq F_f(w(n+1)) + 2\Lambda ||f||^{1/p}_{L^p(B(y,4N\rho))} |B(y, 4N\rho)|^{1/q} = F_f(w(n+1) \\ &= F_f(w(n+1)) + C'\rho^{\frac{N}{q}+1} \end{aligned} \tag{4.10}$$

for ρ sufficiently small (so that condition $(*)$ applies), with Λ denoting the Lipschitz constant of A and C' a constant not dependent on y and ρ.

Then the argument found in Theorem 4.6 follows:

$$\mathcal{H}^1(w(n+1)) - \mathcal{H}^1(\Sigma') \geq \mathcal{H}^1(w(n+1) \cap B(y,\rho)) - \rho C((\frac{\mathcal{H}^1(w(n+1) \cap B(y,2\rho))}{2\rho})^{\frac{N-1}{N}} + 1)$$

and if

$$\mathcal{H}^1(w(n+1) \cap B(y,\rho)) - \rho C((\frac{\mathcal{H}^1(w(n+1) \cap B(y,2\rho))}{2\rho})^{\frac{N-1}{N}} + 1) \leq 0$$

Lemma 4.5 (applied with $a = C$, $\alpha = \frac{N-1}{N}$, $b = C$, $r = \text{diam}\, w(n+1), \Sigma = w(n+1)$) concludes the proof. If

$$\mathcal{H}^1(w(n+1) \cap B(y,\rho)) - \rho C((\frac{\mathcal{H}^1(w(n+1) \cap B(y,2\rho))}{2\rho})^{\frac{N-1}{N}} + 1) > 0$$

then put

$$\xi := (\mathcal{H}^1(w(n+1) \cap B(y,\rho)) - \rho C((\frac{\mathcal{H}^1(w(n+1) \cap B(y,2\rho))}{2\rho})^{\frac{N-1}{N}} + 1))/2N;$$

using Lemma 2.4 there exists $\Sigma'' \in \mathcal{A}$, $\Sigma'' \supseteq \Sigma'$, such that

$$F_f(\Sigma'') \leq F_f(\Sigma') - C_1\xi^2, \qquad \mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(\Sigma') + 2N\xi \tag{4.11}$$

with $C_1 > 0$ not dependent on y and ρ, thus combined with $\mathcal{H}^1(w(n+1)) - \mathcal{H}^1(\Sigma') \geq 2N\xi$ gives

$$\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(w(n+1)). \tag{4.12}$$

Now a slightly different argument has to be made for the two cases:

- if considering evolution (4.8): combining $\Sigma'' \supseteq \Sigma' \supseteq w(n)$, (4.12) and $w(n+1) \in \text{argmin}_{S \supset w(n), \mathcal{H}^1(S) \leq \mathcal{H}^1(w(n))+\varepsilon} F_f$, we get $F_f(\Sigma'') \geq F_f(w(n+1))$,
- if considering evolution (4.9): combining $\Sigma'' \supseteq \Sigma' \supseteq w(n)$ and (4.12) (along with the definition of dissipation) leads to
$$D_\varepsilon(\Sigma'', w(n)) \leq D_\varepsilon(w(n+1), w(n)),$$
thus $F_f(\Sigma'') \geq F_f(w(n+1))$ is required to satisfy
$$F_f(\Sigma'') + D_\varepsilon(\Sigma'', w(n)) \geq F_f(w(n+1)) + D_\varepsilon(w(n+1), w(n)).$$

Thus in both cases the competitor Σ'' verifies $F_f(\Sigma'') \geq F_f(w(n+1))$, and from now the proof returns to be valid in both cases. Combining (4.10) and $F_f(\Sigma'') \geq F_f(w(n+1))$ leads to

$$\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(\Sigma') + 2N\xi \qquad F_f(w(n+1)) + C'\rho^{\frac{N}{q}+1} - C_1\xi^2 \geq F_f(\Sigma'') \geq F_f(w(n+1)).$$

From direct computation

$$(\mathcal{H}^1(w(n+1) \cap B(y,\rho)) - \rho C((\frac{\mathcal{H}^1(w(n+1) \cap B(y,2\rho))}{2\rho})^{\frac{N-1}{N}} + 1))^2 \leq C'\rho^{\frac{N}{q}+1},$$

thus

$$\mathcal{H}^1(w(n+1)\cap B(y,\rho)) - \rho C\left(\left(\frac{\mathcal{H}^1(w(n+1)\cap B(y,2\rho))}{2\rho}\right)^{\frac{N-1}{N}}+1\right) \leq C'\rho^{\frac{N}{2q}-\frac{1}{2}}$$

and by hypothesis $\frac{N}{2q} \geq \frac{1}{2}$, thus forcing

$$\rho^{\frac{N}{2q}-\frac{1}{2}} \leq (\text{diam } w(n+1))^{\frac{N}{2q}-\frac{1}{2}}$$

and Lemma 4.5 concludes the proof. □

Notice that in the proof the value of time step $\varepsilon > 0$ plays almost no role: indeed it holds for any Euler scheme (having form (4.8) or (4.9)).

In [8] it has been proven that Ahlfors regularity for average distance minimizers in $\mathbb{R}^2$ requires weaker conditions on the measure compared to Theorem 4.6: indeed L^p summability with $p > 4/3$ is enough, instead of L^p, $p > 2$ given in Theorem 4.6. This is due to a result similar to Lemma 2.4, which proves stronger in the two dimension case and weaker in higher dimensions:

Lemma 4.9. *Let $\Omega \subseteq \mathbb{R}^N$ a given domain, $f << \mathcal{L}^N$ a measure, $A : [0, diam\ \Omega] \longrightarrow [0,\infty)$ a function, and $\{\Sigma_k\}_{k=0}^{\infty}$ a sequence of closed sets with $\bigcap_{k=0}^{\infty}\Sigma_k \neq \emptyset$. Let be T the set of points $y \in \Omega$ such that*

$$0 < dist(y, \bigcap_{k=0}^{\infty}\Sigma_k) < dist(y, \Sigma_k \backslash \bigcap_{k=0}^{\infty}\Sigma_k)\ \forall k \geq 0$$

and suppose $f(T) > 0$. Then there exists ξ_0 for every $\xi \in (0,\xi_0)$ there exists a segment $I_\xi \in A_\xi \backslash \bigcup_{0\leq j<\xi} A_j$ such that

$$F_f(\Sigma_k \cup I_\xi) \leq F_f(\Sigma_k) - C\xi^{\frac{N+1}{2}}$$

for any k, where $C > 0$ is a constant not dependent on ξ_0, ξ, k.

The proof can be found in [8].

A similar sharper estimate holds for solutions of (4.8) and (4.9):

Theorem 4.10. *Let be $\Omega \subseteq \mathbb{R}^2$ a given domain, $f \in L^p$, $p > \dfrac{4}{3}$ a given measure, $A : [0, diam\ \Omega] \longrightarrow [0,\infty)$ a function, $S_0 \in \mathcal{A}$ an Ahlfors regular initial datum, $\varepsilon > 0$ a given time step, $D_\varepsilon : \mathcal{A}\times\mathcal{A} \longrightarrow [0,\infty]$ a dissipation, and consider evolution*

$$\begin{cases} w(0) := S_0 \\ w(n+1) \in \mathit{argmin}_{\mathcal{A}_{\mathcal{H}^1(S_0)+(n+1)\varepsilon}} F_f \\ w(n+1) \supseteq w(n) \end{cases} . \tag{4.13}$$

Then for any n the set $w(n)$ is Ahlfors regular.

Now consider evolution

$$\begin{cases} w(0) := S_0 \\ w(n+1) \in \mathit{argmin} F(S) + D_\varepsilon(S, w(n)) \\ w(n+1) \supseteq w(n) \end{cases} . \tag{4.14}$$

Then for any n the set $w(n)$ is Ahlfors regular.

As in Theorem 4.8, we deliberately use the same notations for both cases. Here we will give two different proofs. The first one is closely based on the idea used in Theorem 4.8, while the second strongly relies on the fact $\Omega \subseteq \mathbb{R}^2$.

Proof. A first proof can be obtained by closely following the proof of Theorem 4.8: by hypothesis X_0 is Ahlfors regular, and suppose $w(n)$ is Ahlfors regular. The goal is to prove $w(n+1)$ Ahlfors regular.

All passages before (4.11) follow without modifications (as they are true even for the two dimension case). The difference is that in the two dimension case, estimate (4.11) can be replaced by the stronger variant given by Lemma 4.9: putting $N = 2$ (we deliberately use N to put in evidence the sharper estimate given by Lemma 4.9),

$$F_f(\Sigma'') \leq F_f(\Sigma') - C_1(\mathcal{H}^1(w(n+1) \cap B(y,\rho)) - \rho C((\frac{\mathcal{H}^1(w(n+1) \cap B(y,2\rho))}{2\rho})^{\frac{N-1}{N}} + 1))^{3/2} \quad (4.15)$$

where $C_1, C > 0$ are constants not dependent on y, ρ, and this leads to (combined with $F_f(\Sigma'') \geq F_f(w(n+1))$, see the proof of Theorem 4.8 for more details)

$$(\mathcal{H}^1(w(n+1) \cap B(y,\rho)) - \rho C((\frac{\mathcal{H}^1(w(n+1) \cap B(y,2\rho))}{2\rho})^{\frac{N-1}{N}} + 1))^{3/2} \leq C' \rho^{\frac{N}{q}+1}$$

with $C' > 0$ a constant not depending on y and ρ, thus

$$\mathcal{H}^1(w(n+1) \cap B(y,\rho)) - \rho C((\frac{\mathcal{H}^1(w(n+1) \cap B(y,2\rho))}{2\rho})^{\frac{N-1}{N}} + 1) \leq C' \rho^{\frac{2N}{3q} - \frac{1}{3}}$$

and by hypothesis $\frac{2N}{3q} \geq \frac{1}{3}$ holds, finally yielding

$$\rho^{\frac{2}{3q} - \frac{1}{3}} \leq (\operatorname{diam} w(n+1))^{\frac{2N}{3q} - \frac{1}{3}}$$

and Lemma 4.5 concludes the proof. □

The second proof strongly relies on $\Omega \subseteq \mathbb{R}^2$:

Proof. Similarly to the previous proof, it is done by induction on n: by hypothesis S_0 is Ahlfors regular, and suppose $w(n)$ Ahlfors regular. The goal is to prove $w(n+1)$ Ahlfors regular.

Consider an arbitrary $\rho < \dfrac{\varepsilon}{2\pi + 3}$. Put $U := w(n+1) \backslash w(n)$, clearly

$$\mathcal{H}^1(w(n+1) \cap B(y,\rho)) = \mathcal{H}^1(w(n) \cap B(\rho)) + \mathcal{H}^1(U \cap B(y,\rho)).$$

If $\mathcal{H}^1(U \cap B(y,\rho)) \geq M\rho$ with $M > 2\pi + 3$, then consider the competitor

$$\Sigma' := w(n+1) \backslash (U \cap B(y,\rho)) \cup (\partial \overline{B(y,\rho)} \cap \Omega) \cup Seg \cup Seg'$$

with Seg a suitable chord of $\overline{B(y,\rho)}$, and Seg' a suitable radius.

Condition $(*)$ gives that it is possible to choose Seg, Seg' such that $Seg, Seg'\Omega$, and $\overline{B(y,\rho)} \cap \Omega$ connected; then with basic topological considerations compactness and connection follow.

Considering $M > 2\pi + 3$, this competitor verifies

$$\mathcal{H}^1(\Sigma') \leq \mathcal{H}^1(w(n+1)) - \mathcal{H}^1(U \cap B(y,\rho)) + (2\pi+3)\rho \leq \mathcal{H}^1(w(n+1));$$

points $Q \in \Omega$ projecting to $w(n+1)\backslash(U \cap B(y,\rho))$ (i.e $Q \in \{x \in \Omega : \exists P \in w(n+1)\backslash(U \cap B(y,\rho))$ such that $\text{dist}(x, w(n+1)) = \text{dist}(x,P)\}$) verify $\text{dist}(Q, w(n+1)) \geq \text{dist}(Q, \Sigma')$.

Thus $|F_f(w(n+1)) - F_f(\Sigma')| \leq \rho f(B(y,\rho)) \leq C\rho^{1+\frac{2}{q}}$, with C positive constant not dependent on y, ρ. Applying Lemma 4.9, there exists $\Sigma'' \supseteq \Sigma'$ such that $F_f(\Sigma'') \leq F_f(\Sigma') - C^*\rho^{3/2}$ where C^* is a positive constant not dependent on y and ρ. As by hypothesis we have $p > 4/3$, thus $q < 4$, there exists $\rho > 0$ (independent from ρ and y) such that for any $\rho < \rho_0$ inequality $F_f(w(n+1)) > F_f(\Sigma'')$ follows, contradicting $w(n+1) \in \text{argmin}_{\mathcal{A}_{\mathcal{H}^1(S_0)+(n+1)\varepsilon}} F_f$ and concluding the proof for (4.13).

For case (4.14), as we have $\mathcal{H}^1(\Sigma'') \leq \mathcal{H}^1(w(n+1))$, inequality

$$D_\varepsilon(\Sigma'', w(n)) \leq D_\varepsilon(w(n+1), w(n))$$

holds, which combined with $F_f(w(n+1)) > F_f(\Sigma'')$ yields

$$F_f(w(n+1)) + D_\varepsilon(w(n+1), w(n)) > F_f(\Sigma'') + D_\varepsilon(\Sigma'', w(n))$$

contradicting $w(n+1) \in \text{argmin} F(S) + D_\varepsilon(S, w(n))$.

Thus for any $\rho < \rho_0$, $M > 2\pi + 3$, we have

$$\mathcal{H}^1(U \cap B(y,\rho)) \leq M\rho;$$

as by inductive hypothesis $w(n)$ is Ahlfors regular, there exists $C_n > 0$ such that

$$\mathcal{H}^1(w(n) \cap B(y,\rho)) \leq C_n\rho$$

finally yielding

$$\mathcal{H}^1(w(n+1) \cap B(y,\rho)) \leq (C_n + 2\pi + 3)\rho$$

and the proof is complete. □

Notice that the proof relies on two fundamental properties of two dimension domains:

1. Lemma 4.9 gives a sharper estimate than Lemma 2.4,
2. a Hausdorff one-dimensional Jordan curve can divide the rest of the domain in two connected components.

5 Counterexamples

In Section 4 we have proven that solutions for evolution (1.3) and (1.4), when the initial datum is Ahlfors regular, are always Ahlfors regular too. An important condition about domain was $(*)$, as it allowed us to apply Lemmas 4.2. In this section we prove that condition $(*)$ is utterly essential, without which Theorems 4.8 and 4.10 are false.

In the following condition $(*)$ will not be assumed.

5.1 Discrete evolutions

Given $\alpha \in (1,2)$ (and this α will be fixed in all the section), $k \in \mathbb{N}$, impose a cartesian coordinate system in $\mathbb{R}^2$ (see Figure 1), and define sets

$$C_k := \left\{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} \in \left[\frac{1}{k^\alpha}, \frac{1}{k^\alpha} + 4^{-k}\right]\right\}$$

and L_k the rectangle of $\mathbb{R}^2$ with vertexes:

- $\left(\frac{1}{k^\alpha}, 0\right)$, $\left(\frac{1}{k^\alpha}, \frac{1}{4^{k+1}}\right)$, $\left(\frac{1}{(k+1)^\alpha}, 0\right)$, $\left(\frac{1}{(k+1)^\alpha}, \frac{1}{4^{k+1}}\right)$ if k even,
- $\left(-\frac{1}{k^\alpha}, 0\right)$, $\left(-\frac{1}{k^\alpha}, -\frac{1}{4^{k+1}}\right)$, $\left(-\frac{1}{(k+1)^\alpha}, 0\right)$, $\left(-\frac{1}{(k+1)^\alpha}, -\frac{1}{4^{k+1}}\right)$ if k odd.

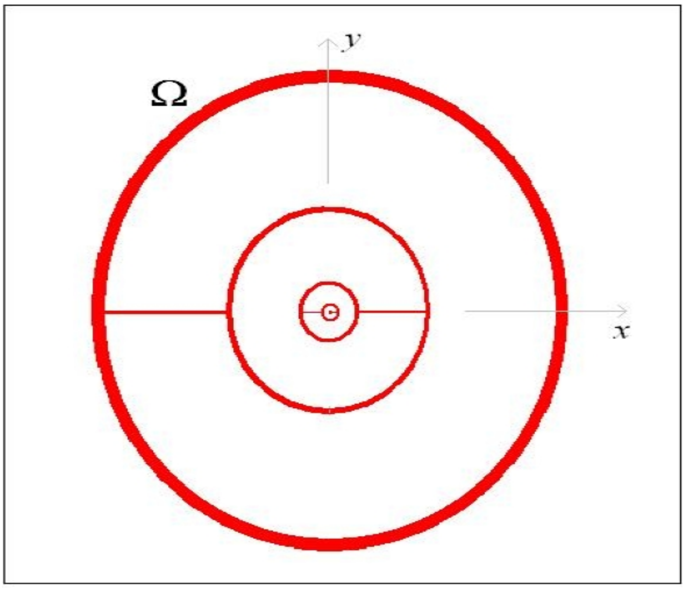

Fig. 1: This is a schematic representation of Ω.

Let

$$\Omega := \bigcup_{k=1}^{\infty} C_k \cup \{(0,0)\} \cup \bigcup_{k=1}^{\infty} L_k$$

be our domain, endowed with the geodesic distance (i.e. the distance between two points $x_1, x_2 \in \Omega$ is given by the length of the shortest path $\beta : [0,1] \longrightarrow \Omega$, $\beta(0) = x_1$, $\beta(1) = x_2$).

Lemma 5.1. *The set Ω is sequentially compact.*

Proof. The proof is straightforward, using basic topological considerations. Let $\{x_j\}_{j=0}^{\infty} \subseteq \Omega$ be an arbitrary sequence. If $I := \{i : x_i = (0,0)\}$ verifies $\sharp I = \infty$ then $\{x_i\}_{i \in I}$ is a converging sequence.

IF $\sharp I < \infty$, we can consider the sequence $\{x_j\}_{j=0}^{\infty}\backslash\{x_i\}_{i\in I}$ since removing finitely many elements from a sequence has no influence on Cauchy condition. Thus without loss of generality we will assume $I = \emptyset$ in the following. The following dichotomy is possible:

1. if there exists $M > 0$ such that $\{x_j\}_{j=0}^{\infty} \subseteq \bigcup_{i=1}^{M}(C_i \cup L_i)$, then $\{x_j\}_{j=0}^{\infty}$ admits a convergent subsequence, since $\bigcup_{i=1}^{M}(C_i \cup L_i)$ is a finite union of compact sets,
2. if a similar M does not exist, then for any $K > 0$ we have
$$\{x_j\}_{j=0}^{\infty}\backslash \bigcup_{s=0}^{K}(C_s \cup L_s) \neq \emptyset,$$
or equivalently
$$\{x_j\}_{j=0}^{\infty} \cap \bigcup_{s=K+1}^{\infty}(C_s \cup L_s) \neq \emptyset,$$
and as $\bigcup_{s=K+1}^{\infty}(C_s \cup L_s) \subseteq B((0,0), K^{-\alpha})$ for any $K > 0$,
$$\{x_j\}_{j=0}^{\infty} \cap B((0,0), K^{-\alpha}) \neq \emptyset.$$
Thus there exists a subsequence $\{x_{j_g}\}_{g=0}^{\infty} \subseteq \{x_j\}_{j=0}^{\infty}$ converging to $(0,0)$.

Thus Ω is sequentially compact. □

Now we provide some estimate on the distance between two points in Ω.

Lemma 5.2. *Given arbitrary $a, b \in \mathbb{N}$, $a < b$, for any couple of points $x_1 \in C_a$, $x_2 \in C_b$ inequality*

$$\frac{2}{3}\pi \sum_{j=a+1}^{b-2} \frac{1}{j^{\alpha}} \leq \mathit{dist}(x_1, x_2) \leq \frac{4}{3}\pi \sum_{j=a-1}^{b} \frac{1}{j^{\alpha}} \tag{5.1}$$

holds.

Proof. The proof is split on several passages:

- We first estimate $\operatorname{dist}(C_k, C_{k+1})$ for a given $k \in \mathbb{N}$.

  By construction for any $k \in \mathbb{N}$ we have
$$\operatorname{dist}(C_k, C_{k+1}) \geq \frac{1}{k^{\alpha}} - \frac{1}{(k+1)^{\alpha}} - \frac{1}{4^{k+1}}$$
.

  On the other hand:

- if k even, there exists $\gamma : [0,1] \longrightarrow \Omega$, $\gamma(0) = (k^{-\alpha}, 0) \in C_k \cap L_k$, $\gamma(1) = ((k+1)^{-\alpha}, 0) \in C_{k+1} \cap L_k$, as $\gamma(s) := (1-s)(k^{-\alpha}, 0) + s((k+1)^{-\alpha}, 0)$ is admissible due to convexity of L_k
- if k odd there exists $\gamma' : [0,1] \longrightarrow \Omega$, $\gamma'(0) = (-k^{-\alpha}, 0) \in C_k$, $\gamma'(1) = (-(k+1)^{-\alpha}, 0) \in C_{k+1}$, as $\gamma'(s) := (1-s)(-k^{-\alpha}, 0) + s(-(k+1)^{-\alpha}, 0)$ is admissible due to convexity of L_k.

thus in both cases $\text{dist}(C_k, C_{k+1}) \leq k^{-\alpha} - (k+1)^{-\alpha}$, and

$$\frac{1}{k^\alpha} - \frac{1}{(k+1)^\alpha} - \frac{1}{4^{k+1}} \leq \text{dist}(C_k, C_{k+1}) \leq \frac{1}{k^\alpha} - \frac{1}{(k+1)^{-\alpha}} \tag{5.2}$$

holds.

- Now we have to estimate $\text{dist}(L_k, L_{k+1})$ for a given $k \in \mathbb{N}$.

By construction the only way to connect arbitrary points $p_0 \in L_k$ and $p_1 \in L_{k+1}$ is through a path $\beta : [0,1] \longrightarrow \Omega$ verifying $\beta([0,1]) \cap L_k \supseteq \{p_0\}$, $\beta([0,1]) \cap L_{k+1} \supseteq \{p_1\}$, and this path must "pass through" C_{k+1}.

As p_0 and p_1 are almost antipodal (i.e. $\text{dist}(p_0, -p_1) \leq 2 \cdot 4^{-(k+1)}$, where $-p_1$ denotes the point symmetric to p_1 with respect to $(0,0)$), any such path β' must verify

$$\frac{2\pi}{3(k+1)^\alpha} \leq \frac{\pi}{(k+1)^{-\alpha}} - \frac{2}{4^{k+1}} \leq \mathcal{H}^1(\beta([0,1])).$$

On the other hand, as both $p_0, p_1 \in C_{k+1}$, the path β can be chosen verifying

$$\mathcal{H}^1(\beta([0,1])) \leq \frac{\pi}{(k+1)^\alpha} + \frac{2}{4^{k+1}} \leq \frac{4\pi}{3(k+1)^\alpha},$$

thus

$$\frac{2\pi}{3(k+1)^\alpha} \leq \text{dist}(L_k, L_{k+1}) \leq \frac{4\pi}{3(k+1)^\alpha} \tag{5.3}$$

Similarly, given $x_1 \in C_a$, $x_2 \in C_b$, we have $\text{dist}(x_1, L_a) \leq \dfrac{4\pi}{3a^\alpha}$ and $\text{dist}(x_2, L_{b-1}) \leq \dfrac{4\pi}{3b^\alpha}$. Combining with (5.2) and (5.3), with simple algebraic passages, leads to

$$\frac{2}{3}\left(\frac{1}{a^\alpha} - \frac{1}{b^\alpha} + \pi \sum_{j=a+1}^{b-1} \frac{1}{j^\alpha}\right) \leq \text{dist}(x_1, x_2) \leq \frac{4}{3}\left(\frac{1}{a^\alpha} - \frac{1}{b^\alpha} + \pi \sum_{j=a}^{b} \frac{1}{j^\alpha}\right)$$

and the thesis follows with simple estimates. □

If we let $b \to \infty$, point x_2 converges to $(0,0)$, and (5.1) reads

$$\frac{2}{3}\pi \sum_{j=a+1}^{\infty} \frac{1}{j^\alpha} \leq \text{dist}(x_1, (0,0)) \leq \frac{4}{3}\pi \sum_{j=a-1}^{\infty} \frac{1}{j^\alpha}. \tag{5.4}$$

Notice that although we had better estimates for (5.3), the less accurate one is sufficient for our goals.

Before proceeding with the main result, another important lemma is required.

Lemma 5.3. *Any element of* $\mathcal{A}(\Omega)\backslash\mathcal{A}_0(\Omega)$ *containing* $(0,0)$ *is not Ahlfors regular.*

Proof. Let $W \in \mathcal{A}(\Omega)\backslash\mathcal{A}_0(\Omega)$ be an arbitrary element, and H the smallest index for which $W \cap C_H \neq \emptyset$ (if such H does not exist, i.e. $W \cap C_g = \emptyset$ for any $g \in \mathbb{N}$, would lead $W = \{(0,0)\}$ contradicting $\mathcal{H}^1(W) > 0$), and choose $X \in W \cap C_H$: as $W \supseteq (0,0)$, there exists a path $\varphi : [0,1] \longrightarrow W$ with $\varphi(0) = X$, $\varphi(1) = (0,0)$. From (5.4) we have

$$\frac{2}{3}\pi \sum_{j=H+1}^{\infty} \frac{1}{j^\alpha} \leq \text{dist}(X,(0,0)) \leq \frac{4}{3}\pi \sum_{j=H-1}^{\infty} \frac{1}{j^\alpha},$$

and using

$$\sum_{i=n}^{\infty} \frac{1}{i^\alpha} \geq \frac{1}{\alpha-1}\frac{1}{(n+1)^{\alpha-1}}, \tag{5.5}$$

we get

$$\frac{2}{3}\pi\frac{1}{\alpha-1}\frac{1}{(H+2)^{\alpha-1}} \leq \text{dist}(X,(0,0)).$$

From the construction of Ω, for any $n \geq 0$ there exists $X_n \in \varphi([0,1]) \cap C_{H+n}$. From (5.4) and (5.5) we have that

$$\frac{2}{3}\pi\frac{1}{\alpha-1}\frac{1}{(H+n+2)^{\alpha-1}} \leq \text{dist}(X_n,(0,0)),$$

holds for an $n \geq 1$.

Let $r_s := \frac{1}{s^\alpha}$: for any $k \geq 1$

$$\begin{aligned}
\frac{\mathcal{H}^1(W \cap B((0,0),r_{H+k}))}{r_{H+k}} &\geq \frac{\mathcal{H}^1(\varphi([0,1]) \cap B((0,0),r_{H+k}))}{r_{H+k}} \\
&\geq \frac{\text{dist}(X_k,(0,0))}{r_{H+k}} \\
&\geq \frac{1}{r_{H+k}}\frac{2}{3}\pi\frac{1}{\alpha-1}\frac{1}{(H+k+2)^{\alpha-1}} \\
&= \frac{2}{3}\pi\frac{1}{\alpha-1}\frac{(H+k)^\alpha}{(H+k+2)^{\alpha-1}}
\end{aligned}$$

leading do

$$\lim_{k\to\infty} \frac{\mathcal{H}^1(W \cap B((0,0),r_{H+k}))}{r_{H+k}} \geq \lim_{k\to\infty} \frac{2}{3}\pi\frac{1}{\alpha-1}\frac{(H+k)^\alpha}{(H+k+2)^{\alpha-1}} = \infty,$$

thus W cannot be Ahlfors regular. □

Now we can present the main result of this section. Given a parameter $\varepsilon > 0$ consider the evolution

$$\begin{cases} w(0) := \{(0,0)\} \\ w(n+1) \in \text{argmin}_{\mathcal{A}(\Omega)_{(n+1)\varepsilon}} F \\ w(n+1) \supseteq w(n) \end{cases} \tag{5.6}$$

where

$$F : \mathcal{A}(\Omega) \longrightarrow (0, \infty), \qquad F(S) := \int_\Omega \text{dist}(x, S)dx.$$

Proposition 5.4. *For any parameter $\varepsilon > 0$, any solution $\{w(j)\}_{j=0}^{\infty}$ of* (5.6) *is such that $w(1)$ is not Ahlfors regular.*

Proof. From

$$\text{dist}(x, (0,0)) \leq \text{dist}(x, w(1)) + \max_{y \in w(1)} \text{dist}(y, (0,0))$$

integrating on Ω leads to

$$\begin{aligned} F(\{(0,0)\}) - F(w(1)) &= \int_\Omega \text{dist}(x, (0,0))dx - \int_\Omega \text{dist}(x, w(1))dx \\ &\leq \int_\Omega \text{dist}(x, w(1)) + \max_{y \in w(1)} \text{dist}(y, (0,0))dx - \int_\Omega \text{dist}(x, w(1))dx \ \cdot \\ &\leq \max_{y \in w(1)} \text{dist}(y, (0,0))\mathcal{L}^2(\Omega) \end{aligned}$$

As $\mathcal{H}^1(w(1)) = \varepsilon$ is admissible, we can choose $S \in \mathcal{A}_\varepsilon$ containing $(0,0)$ such that $\max_{y \in S} \text{dist}(S, (0,0)) = \varepsilon$, with $\{z \in \Omega : \text{dist}(z, (0,0)) = \varepsilon\}$ is not empty as $t \mapsto \text{dist}(t, (0,0))$ is continuous on Ω. Let H be the smallest index for which $S \cap C_H \neq \emptyset$: this forces $S \cap L_H \neq \emptyset$; moreover, as S intersects both C_H and C_{H+1}, $\text{diam}(S \cap L_H) \geq \frac{1}{2}\left(\frac{1}{H^\alpha} - \frac{1}{(H+1)^\alpha}\right)$, thus the set

$$\{w \in \Omega : \text{dist}(w, (0,0)) < \text{dist}(w, S))\}$$

contains at least $\{w \in L_H : \text{dist}(w, S) \leq \frac{1}{2}\text{dist}(C_{H+2}, (0,0))\}$, which has positive measure. Thus $F(S) < F(\{(0,0)\})$, and $w(1) \neq \{(0,0)\}$ as by definition $w(1) \in \text{argmin}_{S' \supseteq (0,0),\ \mathcal{H}^1(S') \leq \varepsilon} F(S')$. Lemma 5.3 concludes the proof. □

5.2 Minimizing movements

In (5.6) we can let the time step ε go to 0: this yields the "continuous" variant of Euler schemes, i.e. minimizing movements.

Definition 5.5. *Given a metric space (X, τ), a functional $\mathcal{F}$, an initial datum $u_0 \in X$, a time $T > 0$, a function $\Sigma : [0,T] \longrightarrow X$ is a minimizing movement with initial datum u_0 if there exists a sequence $\{\varepsilon_k\}_{k=0}^{\infty} \downarrow 0$ and Euler schemes*

$$\begin{cases} w(0) := u_0 \\ w(n+1) \in \textit{argmin}_{Const}\mathcal{F} \end{cases},$$

where $Const$ are constraints (potentially dependent on many quantities), with associated functions

$$\Sigma_{\varepsilon_k} : [0,T] \longrightarrow X, \qquad \Sigma_{\varepsilon_k}(t) := w([t/\varepsilon_k])$$

satisfying

$$\Sigma(s) = \lim_{k\to\infty} \Sigma_{\varepsilon_k}(s)$$

for any $s \in [0, T]$.

In our case $(X, \tau) = (\mathcal{A}(\Omega), d_{\mathcal{H}})$, $\mathcal{F} = F$, $u_0 = \{(0,0)\}$, and Euler schemes considered have form

$$\begin{cases} w(0) := \{(0,0)\} \\ w(n+1) \in \text{argmin}_{S\supseteq w(n), \mathcal{H}^1(S)\leq (n+1)\varepsilon} F(S) \end{cases}, \tag{5.7}$$

and associated functions

$$\Sigma_\varepsilon : [0,T] \longrightarrow \mathcal{A}(\Omega), \qquad \Sigma_\varepsilon(t) := w([t/\varepsilon]) \tag{5.8}$$

Existence of minimizing movements is discussed in [5], to which we refer for more details: here we limit to remark that

1. the domain $\mathcal{A}(\Omega)$ is sequentially compact,
2. for any sequences $\{X_k\}_{k=0}^\infty \subseteq \mathcal{A}(\Omega)$ converging to $X \in \mathcal{A}(\Omega)$, $\{Y_k\}_{k=0}^\infty \subseteq \mathcal{A}(\Omega)$ converging to $Y \in \mathcal{A}(\Omega)$, verifying $X_k \subseteq Y_k$ for any k, inclusion $X \subseteq Y$ holds,
3. any non decreasing function $\psi : \mathbb{R} \longrightarrow \mathcal{A}(\Omega)$ admits at most countable discontinuity points: indeed let $\{x_i\}_{i\in I}$ the set of discontinuity points of ψ, then putting

   $$\phi : \mathbb{R} \longrightarrow \mathbb{R}, \qquad \phi(x) := \mathcal{H}^1(\psi(x))$$

   leads to $\{x_i\}_{i\in I}$ discontinuity points for ϕ, possible only if I is at most countable.

These conditions are sufficient to guarantee the existence of minimizing movements (see [5] for more details): given an initial datum X_0, for every sequence $\{\varepsilon_k\}_{k=0}^\infty \downarrow 0$ there exists a subsequence $\{\varepsilon_{k_h}\}_{h=0}^\infty$, Euler schemes

$$\begin{cases} w(0) := X_0 \\ w(n+1) \in \text{argmin}_{S\supseteq w(n), \mathcal{H}^1(S)\leq \mathcal{H}^1(X_0)+(n+1)\varepsilon_{k_h}} F(S) \end{cases},$$

with associated functions

$$\Sigma_{\varepsilon_{k_h}} : [0,T] \longrightarrow \mathcal{A}(\Omega), \qquad \Sigma_{\varepsilon_{k_h}}(t) := w([t/\varepsilon_{k_h}]),$$

and a function $\Sigma : [0,T] \longrightarrow \mathcal{A}(\Omega)$ such that

$$\Sigma(s) = \lim_{h\to\infty} \Sigma_{\varepsilon_{k_h}}(s)\ \forall s \in [0,T].$$

Proposition 5.6. *Given a time* $T \geq 0$*, a minimizing movement* $\Sigma : [0,T] \longrightarrow \mathcal{A}(\Omega)$ *with* $\Sigma(0) = \{(0,0)\}$*, for any* $s \in [0,T]$ *the set* $\Sigma(s)$ *is not Ahlfors regular unless* $\Sigma(s) = \{(0,0)\}$.

Proof. As $\Sigma : [0,T] \longrightarrow \mathcal{A}(\Omega)$ is a minimizing movement, there exists a sequence $\{\varepsilon_k\}_{n=0}^{\infty} \downarrow 0$, Euler schemes

$$\begin{cases} w(0) := \{(0,0)\} \\ w(n+1) \in \text{argmin}_{S \supseteq w(n), \mathcal{H}^1(S) \leq (n+1)\varepsilon_k} F(S) \end{cases},$$

with associated functions

$$\Sigma_{\varepsilon_k} : [0,T] \longrightarrow \mathcal{A}(\Omega), \qquad \Sigma_{\varepsilon_k}(t) := w([t/\varepsilon_k])$$

such that

$$\Sigma(s) = \lim_{k\to\infty} \Sigma_{\varepsilon_k}(s)$$

for any $s \in [0,T]$. Thus $\Sigma(s) \supseteq \{(0,0)\}$ for any $s \in [0,T]$.

Choose an arbitrary $t \in [0,T]$: if $\Sigma(t) \neq \{(0,0)\}$, then $\mathcal{H}^1(\Sigma(t)) > 0$, and $\Sigma(t) \supseteq \{(0,0)\}$. Applying Lemma 5.3 yields $\Sigma(t)$ not Ahlfors regular. □

Notice that hypothesis $\Sigma : [0,T] \longrightarrow \mathcal{A}(\Omega)$ is minimizing movement is used only to guarantee $(0,0) \in \Sigma(s)$ for any $s \in [0,T]$, allowing us to apply Lemma 5.3.

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