$W^{2,1+\varepsilon}$ ESTIMATES FOR THE MONGE-AMPÈRE EQUATION

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ABSTRACT. We study strictly convex Alexandrov solutions u of the real Monge-Ampère equation $\det(\nabla^2 u) = f$, where f is measurable, positive, and bounded away from 0 and ∞. Under only these assumptions we prove interior $W^{2,1+\varepsilon}$-regularity of u.

CONTENTS

1. INTRODUCTION

In this paper we study solutions $u\colon \Omega \to \mathbb{R}$ of the classical real Monge-Ampère equation

$$\det(\nabla^2 u) = f \qquad \text{on } \Omega\,. \tag{1.1}$$

We will permanently assume that Ω is a convex bounded open set in $\mathbb{R}^n$, that $n \in \mathbb{N}$ is arbitrary, and that $f\colon \Omega \to (0, \infty)$ is Lebesgue measurable.

The first general existence theorems for solutions u of (1.1), coupled with the Dirichlet boundary condition

$$u = \psi \qquad \text{on } \partial\Omega\,, \tag{1.2}$$

are due to Alexandrov [2] and Bakelman [4]. We limit our discussion to the related class of generalized convex solutions, nowadays known as Alexandrov solutions, and recall that for strictly convex Ω, $f \in L^1(\Omega)$, and $\psi \in C^0(\partial\Omega)$ there exists a unique Alexandrov solution u of (1.1)-(1.2); see for instance [29, Theorem 4.1] or [24, Theorem 1.6.2].

With this existence statement at hand it is natural to investigate the regularity[1] of solutions, and we refer to the monographs [22, Chapter 17] and [24, Chapter 4] and the papers [27, 28, 13, 12, 32, 33] for classical (C^2 or better) regularity results. Here, we restate two more recent results of Caffarelli [8, 9], which require only very

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[1]Every convex function is locally Lipschitz continuous and has measures as distributional second derivatives. Here we are concerned with properties which go beyond this generic regularity.

mild assumptions on f. Indeed, for Alexandrov solutions u of (1.1)-(1.2) with zero boundary values $\psi \equiv 0$ he proved: if f is continuous, then there holds $u \in \mathrm{W}^{2,p}_{\mathrm{loc}}(\Omega)$ for all $p \in [1,\infty)$; if f solely satisfies

$$0 < \lambda \le f \le \Lambda < \infty \text{ on } \Omega\,, \tag{1.3}$$

for some constants λ and Λ, then one still has $u \in \mathrm{C}^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$.

These results leave open the question for the regularity of the second derivatives of u under the sole assumption (1.3) (compare [3] for a discussion). Very recently, De Philippis & Figalli [14] addressed this problem and established $u \in \mathrm{W}^{2,1}_{\mathrm{loc}}(\Omega)$ and $|\nabla^2 u| \log^k(2+|\nabla^2 u|) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ for all $k \in \mathbb{N}$ in this situation. In contrast, for every $p > 1$ Wang [34] provided counterexamples of Alexandrov solutions $u \notin \mathrm{W}^{2,p}_{\mathrm{loc}}(\Omega)$. However, in these examples the ratio $\frac{\Lambda}{\lambda}$ of the bounds in (1.3) blows up for $p \searrow 1$, and thus — as pointed out in [14] — one may still hope for regularity $u \in \mathrm{W}^{2,1+\varepsilon}_{\mathrm{loc}}(\Omega)$ with some positive ε *depending* on $\frac{\Lambda}{\lambda}$. Taking [14] as a starting point we will prove here that this is indeed the case.

Before stating our result we mention that the described regularity theory for solutions of (1.1)-(1.2) is valid not only for zero boundary values, but more generally for boundary data of class $\mathrm{C}^{1,\beta}$ with some $\beta \in (\frac{n-2}{n},1)$. Indeed, Caffarelli [7, Corollary 4] (compare also [33, Theorem 1.3] and [24, Theorem 5.4.7]) showed that under the $\mathrm{C}^{1,\beta}$-hypothesis[2] u is necessarily strictly convex, and once strict convexity is achieved interior regularity results can be reduced to the case of zero boundary values — by local renormalization on sections; see for instance [24, Chapter 3.2].

We will now impose strict convexity of u as an assumption (which is automatically valid in case of $\mathrm{C}^{1,\beta}$ boundary data). Then we can formulate our main result in the following localized fashion which avoids any reference to the boundary condition (1.2) at all:

Theorem 1.1. *Suppose that $u\colon \Omega \to \mathbb{R}$ is a strictly convex Alexandrov solution of* (1.1)*, where f satisfies* (1.3)*. Then we have $u \in \mathrm{W}^{2,1+\varepsilon}_{\mathrm{loc}}(\Omega)$ for some positive constant ε, which depends only on n, λ, and Λ.*

Moreover, for every section (see Definition 2.3) $S_u(x_0,r_0) \subset\subset \Omega$ and every $\gamma > 0$ there exists some $\varrho_0(n,\lambda,\Lambda,S_u(x_0,r_0),\gamma) > 0$ with the following property: whenever we have $\mathrm{dist}(S_u(x,2r),\mathbb{R}^n \setminus S_u(x_0,r_0)) \ge \gamma$ for another section $S_u(x,2r)$ with $0 < r < \varrho_0$, then there holds[3]

$$\fint_{S_u(x,\sigma r)} |\nabla^2 u|^{1+\varepsilon}\,\mathrm{d}y \le C\left(\fint_{S_u(x,2r)} \Delta u\,\mathrm{d}y\right)^{1+\varepsilon}. \tag{1.4}$$

with constants $\sigma(n,\lambda,\Lambda) \in (0,1)$ and $C(n,\lambda,\Lambda) \in [1,\infty)$,

The proof of Theorem 1.1 will be carried out in Section 5 of this paper.

The integrability improvement of the theorem carries over to a couple of related problems where the Monge-Ampère equation (1.1) naturally arises. Two such instances are $\mathrm{W}^{2,1+\varepsilon}$-regularity for the boundary of convex sets with prescribed Gauss curvature (the Minkowski problem) and $\mathrm{W}^{1,1+\varepsilon}$-regularity for optimal transport maps (with respect to the quadratic cost) to convex targets. We will not enter into

[2]Explicit examples show that this hypothesis is sharp; see the discussion in [24, Chapter 5.5].

[3]Here, we use $|\nabla^2 u| := \sqrt{\sum_{i,j=1}^n (\partial_i\partial_j u)^2}$ for the Hilbert-Schmidt norm of the Hessian and $\Delta u := \sum_{i=1}^n \partial_i\partial_i u$ for the Laplacian as usual. We remark that due to the convexity of u we have $|\nabla^2 u| \le \Delta u \le \sqrt{n}|\nabla^2 u|$.

the details here, but refer the reader to [7, 8, 14] and [10, 3, 14], respectively, for further discussion and references.

The basic idea in the proof of Theorem 1.1 is to combine the key estimate [14, Lemma 3.4] of De Philippis & Figalli with a variant of Gehring's higher integrability lemma [19]. More precisely, [14, Lemma 3.4] yields a certain (weak-)L^1-estimate for a maximal function, and we will show that this estimate is sufficient to apply an endpoint version of Gehring's lemma, occasionally called Fefferman's lemma; see [16, 5]. Roughly speaking the latter lemma states that either a reverse $\mathrm{L}\log\mathrm{L}$-inequality[4] $\|h\|_{\mathrm{L}\log\mathrm{L}} \le C\|h\|_{\mathrm{L}^1}$ or[5] an L^1-estimate $\|\mathrm{M}h\|_{\mathrm{L}^1} \le C\|h\|_{\mathrm{L}^1}$ for a maximal function $\mathrm{M}h$ (on all cubes or balls) already implies $\mathrm{L}^{1+\varepsilon}$-integrability of h.

For our purposes, such a result needs to be heavily adapted, and this will indeed be our main concern: on the one hand we need to customize it to estimates on Monge-Ampère sections (see Definition 2.3) instead of Euclidean cubes or balls. On the other hand, as usual in PDE problems, we need to deal with increasing supports in the starting reverse inequality. While these issues are well-understood for the more common versions of Gehring's lemma, both of them are new in case of the endpoint version, and in fact it seems that this version has not yet been applied to nonlinear PDEs at all.

Nonetheless, we will implement the required refinements, carrying out — in the spirit of Caffarelli & Gutiérrez [9, 11] — all our analysis on sections. In fact, we first follow the approach of Giaquinta & Modica [21], as described in [23, Chapter 6.4], to derive the Gehring type statement of Proposition 3.2 on sections. We stress that this first statement could alternatively be obtained by applying a result of Gianazza [20, Theorem 4.1] in the homogeneous space of [1], but we will provide a simpler self-contained proof in our situation, which also yields a more explicit control of the relevant constants. Once this proof is completed, we involve a modified idea of Fusco & Sbordone [18] and supply some additional arguments in order to reach the corresponding endpoint version. This last result, which is stated as Proposition 3.4, is the main technical contribution of the present paper.

Finally, we remark that our interest in endpoint versions of Gehring's lemma has originally emerged in connection with the $\mathrm{L}\log\mathrm{L}$-estimate of [6]; however, in that context, the applicability of Gehring type lemmas has remained an open problem.

The plan of the paper is as follows. In Section 2 we collect preliminaries on Alexandrov solutions, Monge-Ampère sections, and the Gehring improvement. In Section 3 we establish versions of Gehring's lemma for sections, and in Section 4 we deduce higher integrability estimates for smooth Alexandrov solutions from the maximal function estimate of [14]. Lastly, in Section 5 we conclude the proof of Theorem 1.1 by an approximation procedure.

After the completion of this paper the author was informed that De Philippis & Figalli & Savin [15] had just established the same regularity result for the Monge-Ampère equation. They used related, but somewhat different methods.

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[4] One may regard a reverse $\mathrm{L}\log\mathrm{L}$-inequality as a reverse Jensen inequality in the sense of [17, 18] for the function $A(s) := s\log(2+s)$. However, this choice of A does not satisfy the assumptions of the higher integrability results in these papers, and thus the endpoint version is not included there.

[5] By a classical result [30], the two alternative assumptions stated here are essentially equivalent.

2. Preliminaries

2.1. **General notation.** Our notation is mostly standard, and therefore we just highlight the following conventions at this point.

Dependencies of constants are often specified in brackets and are then understood as *full* dependencies; for instance $\varepsilon(n,\lambda,\Lambda)$ signifies that ε can be chosen depending *only* on n, λ, and Λ.

Angle brackets $\langle\,\cdot\,,\,\cdot\,\rangle$ are used for the Euclidean inner product on $\mathbb{R}^n$.

If E is a subset of $\mathbb{R}^n$, then we write χ_E for its characteristic function and $\overline{E}$ for its closure. Furthermore, $E \subset\subset \Omega$ means that $\overline{E}$ is compact and contained in Ω, and for $\widetilde{E} \subset \mathbb{R}^n$ and $y \in \mathbb{R}^n$ we set $\mathrm{dist}(\widetilde{E},E) := \inf\{\sqrt{\langle \widetilde{x}-x, \widetilde{x}-x\rangle} \,:\, x \in E,\, \widetilde{x} \in \widetilde{E}\}$ and $\mathrm{dist}(y,E) := \mathrm{dist}(\{y\},E)$.

For the n-dimensional Lebesgue measure on $\mathbb{R}^n$ we write $\mathcal{L}^n$, and if B is a measurable subset of $\mathbb{R}^n$ and f a measurable function on B, we abbreviate $|B| := \mathcal{L}^n(B)$, $\int_B f\,\mathrm{d}x := \int_B f\,\mathrm{d}\mathcal{L}^n$, and $⨍_B f\,\mathrm{d}x := \frac{1}{|B|}\int_B f\,\mathrm{d}x$.

Concerning derivatives our convention is the following one: we write $\partial u(x)$ for the (set-valued) subdifferential of u at x. Moreover, ∇u stands either for the classical gradient of u or for the distributional gradient of u *if* it is represented as a function. If the distributional gradient of u is represented by a measure, we denote this measure by $\mathrm{D}u$. Analogously, we use $\nabla^2 u$ and $\mathrm{D}^2 u$ for second derivatives as functions and measures, respectively.

We write $\mathrm{L}^p(\Omega)$, $\mathrm{W}^{k,p}(\Omega)$, $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$, and $\mathrm{W}^{k,p}_{\mathrm{loc}}(\Omega)$ for the common Lebesgue and Sobolev spaces and their localized variants, respectively. Moreover, $\mathrm{C}^k(\Omega)$ is the spaces of functions $u\colon \Omega \to \mathbb{R}$ with continuous derivatives up to order k (where as usual u and its derivatives need *not* be bounded on Ω). Finally, $u \in \mathrm{C}^k(\overline{\Omega})$ means that u can be extended to a function in $\mathrm{C}^k(\mathcal{N})$ for an open neighborhood $\mathcal{N}$ of $\overline{\Omega}$.

2.2. **Alexandrov solutions and Monge-Ampère sections.** Here we collect some preliminaries related to the Monge-Ampère equation. For a broader discussion of these topics we refer to [24].

Definition 2.1 (Monge-Ampère measures)**.** *Suppose that $u\colon \Omega \to \mathbb{R}$ is a convex function. The Monge-Ampère measure $\mathcal{M}u$ of u is defined by*[6]

$$\mathcal{M}u(B) := \Bigg|\bigcup_{x\in B} \partial u(x)\Bigg| \qquad \textit{for Borel subsets } B \textit{ of } \Omega\,.$$

Definition 2.2 (Alexandrov solutions)**.** *A convex function $u\colon \Omega \to \mathbb{R}$ is called an Alexandrov solution (or weak solution) of* (1.1) *if there holds*

$$\mathcal{M}u(B) = \int_B f\,\mathrm{d}x \qquad \textit{for all Borel subsets } B \textit{ of } \Omega\,.$$

Evidently, requiring $\mathcal{M}u = \mu$ one can define Alexandrov solutions of (1.1) with a Borel measure μ instead of f as the right-hand side of (1.1). Indeed, many of the previously mentioned results have been obtained in this more general framework with a doubling assumption on μ. We stress that $f\mathcal{L}^n$ is doubling under (1.3) and hence these results are available in our case, while there is no[7] hope for a corresponding extension of Theorem 1.1.

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[6] Notice that the set $\bigcup_{x\in B} \partial u(x)$ is Lebesgue measurable; see for instance [24, Theorem 1.1.13].

[7] As remarked in [14] there exist doubling singular measures μ, and for $n = 1$ one can solve $\mathrm{D}^2 u = \mu$ for any of them.

Definition 2.3 (sections)**.** *Suppose $u \in \mathrm{C}^1(\Omega)$. For $x \in \Omega$ and $r \in (0,\infty)$ the (cross-)section $S_u(x,r)$ is given by*

$$S_u(x,r) := \{y \in \Omega \,:\, u(y) < u(x) + \langle \nabla u(x), y-x\rangle + r\}\,.$$

Moreover, for $\tau \in (0,\infty)$ we use the convention

$$\tau S_u(x,r) := \{x + \tau(y-x) \,:\, y \in S_u(x,r)\}\,,$$

i. e. $\tau S_u(x,r)$ is the τ-dilation of $S_u(x,r)$ with respect to the center x.

If u is an Alexandrov solution of the Monge-Ampère equation, then the $S_u(x,r)$ are called Monge-Ampère sections. It is clear that we generally have $x \in S_u(x,r) \subset S_u(x,R)$ for $0 < r < R$ and $\bigcup_{0<t<r} S_u(x,t) = S_u(x,r)$. If u is strictly convex, then it is also easy to see that $\overline{S_u(x,r)} \setminus S_u(x,r)$ has zero measure, and that $\overline{S_u(x,r)} = \bigcap_{t>r} S_u(x,t)$ holds whenever $S_u(x,r) \subset\subset \Omega$. In particular, combining these observations and the additivity of the Lebesgue measure we have:

Lemma 2.4 (continuous dependence)**.** *Suppose that $u \in \mathrm{C}^1(\Omega)$ is strictly convex and $S_u(x,R) \subset\subset \Omega$. Then $|S_u(x,r)|$ depends continuously on $r \in (0,R]$.*

Moreover, the sections have the convenient properties of the following proposition which is essentially taken from [14, Proposition 2.1]; see also [25] and [24, Chapter 3] for the original statements and the proofs on the whole $\mathbb{R}^n$.

Proposition 2.5 (properties of the sections)**.** *Suppose that $u \in \mathrm{C}^1(\Omega)$ is a strictly convex Alexandrov solution of* (1.1)*, where f satisfies* (1.3)*. Then there exists some $\theta(n,\lambda,\Lambda) > 1$ and for every $E \subset\subset \Omega$ there is a positive $\varrho(n,\lambda,\Lambda,\Omega,E)$ such that the following properties are valid:*

(A) $S_u(x,r) \subset\subset \Omega$ *for all $x \in E$ and $0 < r \le 2\varrho$,*

(B) $\tau S_u(x,r) \subset S_u(x,\tau r)$ *for all $\tau \in (0,1]$, $x \in E$, and $0 < r \le 2\varrho$,*

(C) *if $S_u(y,r) \cap S_u(x,r) \neq \emptyset$ holds for $x,y \in E$ and $0 < r \le 2\varrho/\theta$, then one has $S_u(y,r) \subset S_u(x,r\theta)$*

In the following we will refer to the properties of Proposition 2.5 only as (A), (B), and (C), and we will widely use them, where we consider the constant θ as fixed for the remainder of the paper.

As pointed out in [31, 11, 14] the preceding properties of sections suffice to use them like Euclidean balls in many regards. In particular, they allow to establish the following lemma; see [31, Chapter 1.3.1].

Lemma 2.6 (Lebesgue points)**.** *Under the assumption of Proposition 2.5 consider $G \in \mathrm{L}^1(\Omega)$. Then we have*

$$\lim_{r\searrow 0} \fint_{S_u(x,r)} G \,\mathrm{d}y = G(x) \qquad \text{for } \mathcal{L}^n\text{-a. e. } x \in \Omega\,.$$

Finally, we turn to [11, Lemma 1], which was originally formulated for an abstract system of sections. However, the assumptions of [11] on this abstract system are valid in our case by [24, Corollary 3.3.6, Theorem 3.3.8], and thus we may restate the lemma as follows.

Lemma 2.7 (Besicovitch type covering)**.** *Suppose that $u\colon \Omega \to \mathbb{R}$ is a strictly convex Alexandrov solution of* (1.1)*, where f satisfies* (1.3)*. Moreover, fix $E \subset\subset \Omega$, the corresponding ϱ from Proposition 2.5, and a subset A of E. Then, given for every $x \in A$ some $r_x \in (0,\varrho]$, one can find a finite or countable family $(x_i)_{i\in I}$ of points in A such that*

$$A \subset \bigcup_{i\in I} S_u(x_i, r_{x_i}) \qquad \textit{and} \qquad \sum_{i\in I} \chi_{S_u(x_i,\zeta r_{x_i})} \le \frac{1}{\zeta} \textit{ on } \Omega\,. \tag{2.1}$$

with a constant $\zeta(n,\lambda,\Lambda) \in (0,1]$.

2.3. **Gehring's improvement.** Here, we record two known lemmas. The first one is a simple version of [23, Lemma 6.3]. The second one is a particular case of a key lemma of Gehring [19] and corresponds to the choice $m{=}0$, $\omega{\equiv}0$ in [23, Proposition 6.1]. However, instead of using Riemann-Stieltjes integrals as in [23] we prefer to regard monotone functions φ on the real line as BV-functions, and thus consider the derivative $\mathrm{D}\varphi$ as a measure. Even besides that we have slightly modified the statements, and for completeness and convenience of the reader we decided to include the brief proofs.

Lemma 2.8. *Consider $q \in (0,\infty)$, a measurable function $G\colon S \to \mathbb{R}$ on a measurable subset S of $\mathbb{R}^n$ with $|S| < \infty$, and set*

$$\varphi(s) := |S \cap \{G > s\}| \tag{2.2}$$

for $s \ge 0$. Then for all $t \ge 0$ one has

$$-\int_{(t,\infty)} s^q \,\mathrm{d}\mathrm{D}\varphi(s) = \int_{S\cap\{G>t\}} G^q \,\mathrm{d}y\,,$$

where either both integrals are finite or both are infinite.

Proof. We note that φ is non-increasing and right-continuous with $\lim_{s\to\infty}\varphi(s) = 0$. Moreover, writing $\Phi(s) := s^q\varphi(s)$ we observe that Φ has locally bounded variation and that $\Phi(s) \le -\int_{(s,\infty)} r^q \,\mathrm{d}\mathrm{D}\varphi(r)$ holds. Now we deal with the case that the left-hand integral in the claim is finite for some — and consequently for all — $t \ge 0$. Then, from the preceding inequality for $\Phi(s)$ we infer $\lim_{s\to\infty}\Phi(s) = 0$ and thus

$$\Phi(t) + \int_{(t,\infty)} \mathrm{d}\mathrm{D}\Phi = 0 \qquad \text{for all } t \ge 0\,.$$

Moreover, using the Fubini-Tonelli theorem we have

$$\begin{aligned} \Phi(t) + \int_{(t,\infty)} \mathrm{d}\mathrm{D}\Phi &= t^q\varphi(t) + \int_t^\infty q s^{q-1}\varphi(s)\,\mathrm{d}s + \int_{(t,\infty)} s^q \,\mathrm{d}\mathrm{D}\varphi(s) \\ &= \int_{S\cap\{G>t\}} \left[t^q + \int_t^{G(y)} q s^{q-1}\,\mathrm{d}s\right]\mathrm{d}y + \int_{(t,\infty)} s^q\,\mathrm{d}\mathrm{D}\varphi(s) \\ &= \int_{S\cap\{G>t\}} G^q\,\mathrm{d}y + \int_{(t,\infty)} s^q\,\mathrm{d}\mathrm{D}\varphi(s)\,. \end{aligned} \tag{2.3}$$

Combining the previous equalities we arrive at the claim. To conclude the proof it remains to exclude the case that the right-hand integral in the claim is finite, but the left-hand integral is infinite. In this case (2.3) would still be valid and the common value of the expressions in (2.3) would be $-\infty$. In particular, we would get $\int_{(t,\infty)} \mathrm{d}\mathrm{D}\Phi = -\infty$, which contradicts $\Phi \ge 0$ on $(0,\infty)$. $\square$

Lemma 2.9. *Consider $K \geq 1$, $t_0 > 0$, and a non-increasing $\varphi\colon (t_0, \infty) \to [0, \infty)$ with $\lim_{t\to\infty} \varphi(t) = 0$. If one has*

$$-\int_{(t,\infty)} s \,\mathrm{dD}\varphi(s) \leq Kt\varphi(t) \qquad \text{for all } t \geq t_0\,, \tag{2.4}$$

then for some positive $\varepsilon(K)$ one also has

$$-\int_{(t_0,\infty)} s^{1+\varepsilon} \,\mathrm{dD}\varphi(s) \leq -2t_0^\varepsilon \int_{(t_0,\infty)} s \,\mathrm{dD}\varphi(s)\,. \tag{2.5}$$

In particular, the integral on the left-hand side of (2.5) *is finite.*

Proof. Setting $\psi(t) := -\int_{(t,\infty)} s \,\mathrm{dD}\varphi(s) \geq 0$ we get $\mathrm{D}\psi = w \,\mathrm{dD}\varphi$, where $w(s) := s$. Moreover, we have $\psi(t) \leq Kt\varphi(t)$ by assumption. Now we first assume that $\varphi(t)$ vanishes for sufficiently large t. In this case we rely on the preceding observations and integrate by parts twice to obtain (at first for arbitrary $\varepsilon > 0$)

$$\begin{aligned}
-\int_{(t_0,\infty)} s^{1+\varepsilon} \,\mathrm{dD}\varphi(s) &= -\int_{(t_0,\infty)} s^\varepsilon \,\mathrm{dD}\psi(s) \\
&= t_0^\varepsilon \psi(t_0) + \varepsilon \int_{(t_0,\infty)} s^{\varepsilon-1} \psi(s) \,\mathrm{d}s \\
&\leq t_0^\varepsilon \psi(t_0) + \varepsilon K \int_{(t_0,\infty)} s^\varepsilon \varphi(s) \,\mathrm{d}s \\
&= t_0^\varepsilon \psi(t_0) - \frac{\varepsilon K}{\varepsilon+1} t_0^{1+\varepsilon} \varphi(t_0) - \frac{\varepsilon K}{\varepsilon+1} \int_{(t_0,\infty)} s^{1+\varepsilon} \,\mathrm{dD}\varphi(s)\,.
\end{aligned}$$

Now we choose ε such that $\frac{\varepsilon K}{\varepsilon+1} = \frac{1}{2}$, that is $\varepsilon := (2K-1)^{-1}$. Then we can absorb the last term and arrive at

$$-\int_{(t_0,\infty)} s^{1+\varepsilon} \,\mathrm{dD}\varphi(s) \leq 2t_0^\varepsilon \psi(t_0)\,,$$

which is just the claim. If $\varphi(t)$ does not vanish for large t, then we approximate φ by the functions $\varphi_T := \chi_{(t_0,T)}\varphi$ with $T \in (t_0, \infty)$. We observe that $\lim_{t\to\infty} \varphi(t) = 0$ implies $\mathrm{D}\varphi_T(\{T\}) \leq -\int_{[T,\infty)} s \,\mathrm{dD}\varphi(s)$, and deduce that (2.4) holds also with φ_T in place of φ (for $t \geq T$ this is trivial, for $t < T$ it comes from the preceding observation). Recalling what we already proved we infer that (2.5) is valid with φ_T in place of φ. Letting $T \to \infty$ we easily arrive at the final claim. □

3. Gehring type lemmas on Monge-Ampère sections

In the following we fix a strictly convex Alexandrov solution $u \in \mathrm{C}^1(\Omega)$ of (1.1), where f satisfies (1.3), and we briefly write $S(x, r)$ instead of $S_u(x, r)$ for the sections of u. Moreover, we also fix $E \subset\subset \Omega$ and the corresponding ϱ from Proposition 2.5.

We start with a preparatory lemma which already contains an integrability improvement.

Lemma 3.1. *For a measurable subset S of E and $0 \leq G \in \mathrm{L}^1(S)$ suppose that there are a constant $L \geq 1$, a level $t_0 > 0$, and for each $x \in S$ some $R_x \in (0, \varrho]$*

such that $S(x,R_x)\subset S$ *and*

$$\fint_{S(x,R_x)} G\,\mathrm{d}y \le t_0\,, \tag{3.1}$$

$$\fint_{S(x,r)} G\,\mathrm{d}y \le L \inf_{S(x,r)} G \quad \textit{for all } r\in(0,R_x]\,. \tag{3.2}$$

Then we have $G\in \mathrm{L}^{1+\varepsilon}(S)$ *and*

$$\int_S G^{1+\varepsilon}\,\mathrm{d}y \le 2t_0^\varepsilon \int_S G\,\mathrm{d}y$$

for some positive constant $\varepsilon(n,\lambda,\Lambda,L)$*.*

Proof. We introduce the abbreviation

$$m_x(r) := \fint_{S(x,r)} G\,\mathrm{d}y$$

and record $m_x(R_x)\le t_0$ for all $x\in S$ by assumption. For the moment we fix a level $t\ge t_0$, and we introduce

$$A := \Big\{x\in S\,:\,\limsup_{r\searrow 0} m_x(r) > Lt\Big\}.$$

Since $m_x(r)$ depends continuously on r by Lemma 2.4, for every $x\in A$ we can find some $r_x\in(0,R_x]$ with $m_x(r_x)=Lt$, that is

$$\fint_{S(x,r_x)} G\,\mathrm{d}y = Lt\,.$$

Now by Lemma 2.7 there exists a finite or countable family $(x_i)_{i\in I}$ of points in A with the covering properties (2.1). Moreover, by Lemma 2.6 and the definition of A we see that $G\le Lt$ holds $\mathcal{L}^n$-a. e. on $S\setminus A$. Altogether we therefore get

$$\begin{aligned}\int_{S\cap\{G>Lt\}} G\,\mathrm{d}y &\le \int_A G\,\mathrm{d}y\\ &\le \sum_{i\in I}\int_{S(x_i,r_{x_i})} G\,\mathrm{d}y\\ &= Lt\sum_{i\in I}|S(x_i,r_{x_i})| \le \frac{Lt}{\zeta^n}\sum_{i\in I}|S(x_i,\zeta r_{x_i})|\,,\end{aligned} \tag{3.3}$$

where we also employed (B) in the last step. By our assumption (3.2) we have

$$\inf_{S(x_i,r_{x_i})} G \ge \frac{1}{L}\fint_{S(x_i,r_{x_i})} G\,\mathrm{d}y = t$$

for all $k\in\mathbb{N}$ and in particular $G\ge t$ on all sections $S(x_i,\zeta r_{x_i})$. Moreover, we know $S(x_i,r_{x_i})\subset S(x_i,R_{x_i})\subset S$. Combining this with the bounded-overlap property from (2.1) we can further estimate the right-hand side of (3.3) ending up with

$$\int_{S\cap\{G>Lt\}} G\,\mathrm{d}y \le \frac{Lt}{\zeta^{n+1}}|S\cap\{G\ge t\}|\,.$$

In addition, we trivially have

$$\int_{S\cap\{Lt\ge G>t\}} G\,\mathrm{d}y \le Lt|S\cap\{G\ge t\}|\,,$$

and combining the last two estimates we find

$$\int_{S\cap\{G>t\}} G\,\mathrm{d}y \le Kt|S\cap\{G\ge t\}|\,, \tag{3.4}$$

where $K := \frac{L}{\zeta^{n+1}} + L$ depends only on n, λ, Λ, and L. Since $t \ge t_0$ is arbitrary, we can replace the right-hand side of (3.4) with $Kt|S\cap\{G>t\}|$ by a continuity argument. Then by Lemma 2.8 we write (3.4) as

$$-\int_{(t,\infty)} s\,\mathrm{dD}\varphi(s) \le Kt\varphi(t) \qquad \text{for } t\ge t_0$$

with φ from (2.2). Noting $\lim_{t\to\infty}\varphi(t) = 0$ we can apply Lemma 2.9, and we deduce

$$-\int_{(t_0,\infty)} s^{1+\varepsilon}\,\mathrm{dD}\varphi(s) \le -2t_0^{\varepsilon}\int_{(t_0,\infty)} s\,\mathrm{dD}\varphi(s)$$

for some positive $\varepsilon(n,\lambda,\Lambda,L)$. Using Lemma 2.8 once more this can be rewritten as

$$\int_{S\cap\{G>t_0\}} G^{1+\varepsilon}\,\mathrm{d}y \le 2t_0^{\varepsilon}\int_{S\cap\{G>t_0\}} G\,\mathrm{d}y\,.$$

Combining the previous inequality with the trivial estimate

$$\int_{S\cap\{G\le t_0\}} G^{1+\varepsilon}\,\mathrm{d}y \le t_0^{\varepsilon}\int_{S\cap\{G\le t_0\}} G\,\mathrm{d}y$$

we arrive at the claim. □

Next we will remove the assumption (3.1) in the previous lemma coming out with the following Gehring type statement. A similar result with an inf-inequality on standard cubes is implicitly contained in the work of Muckenhoupt [26] and is stated as [18, Theorem 1.4]. Moreover, as already mentioned in the introduction the following statement could essentially be obtained from [20, Theorem 4.1] and [1], but the following self-contained proof is — in our situation — more manageable than the one in [20].

Proposition 3.2. *Consider a section $S(x_0,r_0)\subset E$ with $0<r_0<2\varrho/\theta^2$ and $0\le g\in \mathrm{L}^1(S(x_0,r_0))$. Suppose that there is a constant $M\ge 1$ such that for each enclosed section $S(x,r)\subset S(x_0,r_0)$ there holds*

$$\fint_{S(x,r)} g\,\mathrm{d}y \le M\inf_{S(x,r)} g\,.$$

Then we have $g\in \mathrm{L}^{1+\varepsilon}(S(x_0,r_0/\theta))$ and

$$\fint_{S(x_0,r_0/\theta)} g^{1+\varepsilon}\,\mathrm{d}y \le 2\theta^{4n+5n\varepsilon}\left(\fint_{S(x_0,r_0)} g\,\mathrm{d}y\right)^{1+\varepsilon}$$

for some positive constant $\varepsilon = \varepsilon(n,\lambda,\Lambda,M)$.

Proof. We abbreviate

$$S := S(x_0,r_0)\,,$$

for $x\in S$ we set

$$\delta(x) := \sup\{r\in(0,\infty) : S(x,r)\subset S\}\,, \tag{3.5}$$

$$v(x) := |S(x,\delta(x)/\theta)|\,,$$

and we notice $\delta(x) \le r_0\theta < 2\varrho/\theta$, since $S = S(x_0, r_0) \subset S(x, r_0\theta)$ by (C). Moreover, for all $x \in S$ we claim

$$\delta(x)/\theta \le \delta(y) \le \delta(x)\theta \qquad \text{for } y \in S(x, \delta(x)/\theta)\,, \tag{3.6}$$

$$v(x)/\theta^{2n} \le v(y) \le v(x)\theta^{2n} \qquad \text{for } y \in S(x, \delta(x)/\theta)\,. \tag{3.7}$$

We first prove (3.6). By (C) and (3.5) we have $S(y, \delta(x)/\theta) \subset S(x, \delta(x)) \subset S$. Using (3.5) once more we infer the left-hand bound in (3.6). In order to derive the right-hand bound we take an arbitrary $0 < r < 2\varrho/\theta - r_0\theta$. Then we have $S(x, \delta(x) + r) \not\subset S$ by (3.5), $S(x, \delta(x){+}r) \subset S(y, (\delta(x){+}r)\theta)$ by (C), and consequently $S(y, (\delta(x){+}r)\theta) \not\subset S$. Since r can be made arbitrarily small, this and (3.5) yield the right-hand bound in (3.6). Next, we establish (3.7). Indeed, using in turn (B), (3.6), (C), and again (B) we have $\theta^{-1}S(y, \delta(y)/\theta) \subset S(y, \delta(y)/\theta^2) \subset S(y, \delta(x)/\theta) \subset S(x, \delta(x)) \subset \theta S(x, \delta(x)/\theta)$, which yields $\theta^{-n}v(y) \le \theta^n v(x)$ and gives the right-hand bound in (3.7). Similarly, by (C), (3.6), and (B) we get $S(x, \delta(x)/\theta) \subset S(y, \delta(x)) \subset S(y, \delta(y)\theta) \subset \theta^2 S(y, \delta(y)/\theta)$ and the left-hand bound in (3.7).

Now we exploit the preceding observations in the context of the proposition: for all $x \in S$ we have $S(x, \delta(x)/\theta) \subset S$ by (3.5), and in view of (3.6) and (3.7) our assumptions imply

$$\fint_{S(x,r)} gv \,\mathrm{d}y \le M\theta^{4n} \inf_{S(x,r)} gv \qquad \text{for all } r \in (0, \delta(x)/\theta]$$

and

$$\fint_{S(x,\delta(x)/\theta)} gv \,\mathrm{d}y = \int_{S(x,\delta(x)/\theta)} \frac{v(y)}{v(x)} g(y) \,\mathrm{d}y \le \theta^{2n} \int_S g \,\mathrm{d}y\,,$$

still for all $x \in S$. In conclusion, we can apply Lemma 3.1 with the choices $G := gv$, $R_x := \delta(x)/\theta$, $L := M\theta^{4n}$, and $t_0 := \theta^{2n} \int_S g \,\mathrm{d}y$. We deduce $gv \in \mathrm{L}^{1+\varepsilon}(S)$ and

$$\int_S (gv)^{1+\varepsilon} \,\mathrm{d}y \le 2\left(\theta^{2n} \int_S g \,\mathrm{d}y\right)^{\varepsilon} \int_S gv \,\mathrm{d}y \tag{3.8}$$

for some positive $\varepsilon = \varepsilon(n, \lambda, \Lambda, M)$. Recalling $S = S(x_0, r_0)$ we notice $\delta(x_0) = r_0$ and $v(x_0) = |S(x_0, r_0/\theta)|$. Applying (3.7) with $x = x_0$ we get the lower bound $v(y) \ge \theta^{-2n}|S(x_0, r_0/\theta)|$ for all $y \in S(x_0, r_0/\theta)$. Moreover, by definition we have the upper bound $v(y) \le |S|$ for all $y \in S$, and plugging both bounds into (3.8) we find $g \in \mathrm{L}^{1+\varepsilon}(S(x_0, r_0/\theta))$ and

$$|S(x_0, r_0/\theta)|^{2+\varepsilon} \fint_{S(x_0,r_0/\theta)} g^{1+\varepsilon} \,\mathrm{d}y \le 2\theta^{2n(1+2\varepsilon)} |S|^{2+\varepsilon} \left(\fint_S g \,\mathrm{d}y\right)^{1+\varepsilon}.$$

In view of $|S| = |S(x_0, r_0)| \le \theta^n |S(x_0, r_0/\theta)|$ we arrive at the claim. □

Now we specify our terminology for maximal functions on sections.

Definition 3.3 (maximal function)**.** *Consider $S(x, R) \subset E$ and $0 < R < 2\varrho/\theta$. For every $h \in \mathrm{L}^1(S(x, R\theta))$ we define a maximal function $\mathrm{M}_R h \colon S(x, R) \to \mathbb{R}$ by*[8]

$$\mathrm{M}_R h(y) := \sup_{r \in (0,R]} \fint_{S(y,r)} |h| \,\mathrm{d}y \qquad \textit{for } y \in S(x, R)\,.$$

[8]Notice that in the case of the definition we have $S(y, r) \subset S(y, R) \subset S(x, R\theta)$ by (C) and thus $|h|$ is indeed defined on $S(y, r)$.

Next we establish the aimed endpoint version of Gehring's lemma on sections, formulated with the maximal functions of Definition 3.3. The statement allows for increasing supports in the starting inequality and this feature seems to be new even for Euclidean balls, that is in the special case $u(x) = \langle x, x\rangle$. The proof is inspired by an idea of [18] and aims at applying Proposition 3.2 to a suitable maximal function.

Proposition 3.4. *For a section $S(x_0, 2r_0) \subset E$ with $0 < r_0 < \varrho/\theta^2$ and some function $0 \le h \in \mathrm{L}^1(S(x_0, 2r_0))$ suppose that there is a constant $N \ge 1$ such that for each section $S(x, r) \subset S(x_0, r_0/\theta^3)$ there holds*

$$\fint_{S(x,r)} \mathrm{M}_r h \,\mathrm{d}y \le N \fint_{S(x,2r\theta)} h \,\mathrm{d}y\,. \tag{3.9}$$

Then we have $h \in \mathrm{L}^{1+\varepsilon}(S(x_0, r_0/\theta^4))$ and

$$\fint_{S(x_0,r_0/\theta^4)} h^{1+\varepsilon} \,\mathrm{d}y \le C\bigg(\fint_{S(x_0,2r_0)} h \,\mathrm{d}y\bigg)^{1+\varepsilon} \tag{3.10}$$

with positive constants $\varepsilon(n, \lambda, \Lambda, N)$ and $C(n, \lambda, \Lambda, N)$.

Before coming to the proof of the proposition we record some preliminaries on the sections in the statement. We first notice that by assumption and (C) we have $S(x, r) \subset S(x_0, r_0/\theta^3) \subset S(x, r_0/\theta^2)$, which implies $r \le r_0/\theta^2$. Relying on (C) once more we get $S(x, 2r\theta) \subset S(x, 2r_0/\theta) \subset S(x_0, 2r_0)$. Consequently, h is well-defined on $S(x, 2r\theta)$ and $\mathrm{M}_r h$ is well-defined on $S(x, r)$; so, the integrals in the assumption (3.9) are meaningful.

Proof of Proposition 3.4. We define $H \in \mathrm{L}^1(\Omega)$ by

$$H := \chi_{S(x_0,2r_0)} h\,.$$

Since we assume $r_0 < \varrho/\theta^2$, we have $S(x_0, 2r_0\theta^2) \subset \Omega$ by (A). In particular, H is defined on $S(x_0, 2r_0\theta^2)$ and $\mathrm{M}_{2r_0\theta} H$ is defined on $S(x_0, 2r_0\theta)$ and consequently on all sections in the following argument. Now we fix an enclosed section $S(x, r) \subset S(x_0, r_0/\theta^3)$ as in the statement. By Definition 3.3 we find

$$\mathrm{M}_{2r_0\theta} H(y) \le \mathrm{M}_r h(y) + \sup\bigg\{\fint_{S(y,\widetilde r)} H \,\mathrm{d}z \,:\, r < \widetilde r \le 2r_0\theta\bigg\} \qquad \text{for } y \in S(x, r)\,,$$

where we also used that $S(y, r) \subset S(y, r_0/\theta^2) \subset S(x_0, r_0/\theta)$ by (C) and thus $H = h$ on such sections. We integrate this inequality over $S(x, r)$ and use the assumption (3.9) to obtain

$$\fint_{S(x,r)} \mathrm{M}_{2r_0\theta} H \,\mathrm{d}y \le (N{+}1) \sup\bigg\{\fint_{S(y,\widetilde r)} H \,\mathrm{d}z \,:\, y{\in}S(x, r),\ r \le \widetilde r \le 2r_0\theta\bigg\}\,, \tag{3.11}$$

where we exploited $H = h$ on $S(x, 2r\theta) \subset S(x_0, 2r_0)$. Now we deal with the integrals on the right-hand side of (3.11), first in the case $r \le \widetilde r \le 2r_0/\theta$. For arbitrary $y, \widetilde y \in S(x, r)$ we then have

$$S(y, \widetilde r) \subset S(x, \widetilde r\theta) \subset S(\widetilde y, \widetilde r\theta^2) \tag{3.12}$$

by (C). As a consequence (C) and (B) also yield $S(\widetilde y, \widetilde r\theta^2) \subset S(y, \widetilde r\theta^3) \subset \theta^3 S(y, \widetilde r)$ and

$$|S(\widetilde y, \widetilde r\theta^2)| \le \theta^{3n} |S(y, \widetilde r)|\,. \tag{3.13}$$

Combining (3.12) and (3.13) we have

$$\fint_{S(y,\widetilde{r})} H\,\mathrm{d}z \leq \theta^{3n} \fint_{S(\widetilde{y},\widetilde{r}\theta^2)} H\,\mathrm{d}z\,. \tag{3.14}$$

Since $\widetilde{y}$ was arbitrary in $S(x,r)$ and $\widetilde{r}\theta^2 \leq 2r_0\theta$, we have proved in particular

$$\fint_{S(y,\widetilde{r})} H\,\mathrm{d}z \leq \theta^{3n} \inf_{S(x,r)} \mathrm{M}_{2r_0\theta} H \tag{3.15}$$

for all $y \in S(x,r)$ and $r \leq \widetilde{r} \leq 2r_0/\theta$. Next we come back to the remaining case on the right-hand side of (3.11), that is $y \in S(x,r)$ and $2r_0/\theta < \widetilde{r} \leq 2r_0\theta$. We again take an arbitrary $\widetilde{y} \in S(x,r)$ and get

$$\fint_{S(y,\widetilde{r})} H\,\mathrm{d}z \leq \frac{1}{|S(y,2r_0/\theta)|} \int_{S(\widetilde{y},2r_0\theta)} H\,\mathrm{d}z\,,$$

since we have $S(x_0,2r_0) \subset S(\widetilde{y},2r_0\theta)$ by (C) and thus $H \equiv 0$ outside $S(\widetilde{y},2r_0\theta)$. In view of $y \in S(\widetilde{y},2r_0\theta)$ we also infer $S(\widetilde{y},2r_0\theta) \subset S(y,2r_0\theta^2) \subset \theta^3 S(y,2r_0/\theta)$ from (C) and (B). Using the corresponding bound for the volumes and arguing as for (3.15) we find

$$\fint_{S(y,\widetilde{r})} H\,\mathrm{d}z \leq \theta^{3n} \fint_{S(\widetilde{y},2r_0\theta)} H\,\mathrm{d}z\,. \tag{3.16}$$

Consequently, (3.15) is also valid in the present case. Thus we can control the right-hand side of (3.11) by (3.15) in all cases, and we get

$$\fint_{S(x,r)} \mathrm{M}_{2r_0\theta} H\,\mathrm{d}y \leq (N{+}1)\theta^{3n} \inf_{S(x,r)} \mathrm{M}_{2r_0\theta} H\,.$$

Now we recall that $S(x,r)$ was arbitrary in $S(x_0,r_0/\theta^3)$. In particular, we can take $S(x,r) = S(x_0,r_0/\theta^3)$ to infer $g := \mathrm{M}_{2r_0\theta}H \in \mathrm{L}^1(S(x_0,r_0/\theta^3))$. In conclusion, we are in the position to apply Proposition 3.2 to g on $S(x_0,r_0/\theta^3)$ with $M := (N{+}1)\theta^{3n}$. We end up with $\mathrm{M}_{2r_0\theta}H \in \mathrm{L}^{1+\varepsilon}(S(x_0,r_0/\theta^4))$ and

$$\fint_{S(x_0,r_0/\theta^4)} (\mathrm{M}_{2r_0\theta}H)^{1+\varepsilon}\,\mathrm{d}y \leq 2\theta^{4n+5n\varepsilon} \left(\fint_{S(x_0,r_0/\theta^3)} \mathrm{M}_{2r_0\theta} H\,\mathrm{d}y \right)^{1+\varepsilon}, \tag{3.17}$$

where $\varepsilon(n,\lambda,\Lambda,N) > 0$. Next we establish an estimate for the right-hand side of (3.17). To this end we revisit (3.11), (3.14), and (3.16) with $x = x_0$, $r = r_0/\theta^3$, and $\widetilde{y} = x_0$ coming out with

$$\fint_{S(x_0,r_0/\theta^3)} \mathrm{M}_{2r_0\theta} H\,\mathrm{d}y \leq (N{+}1)\theta^{3n} \sup \left\{ \fint_{S(x_0,\widetilde{r}\theta^2)} H\,\mathrm{d}y \,:\, r_0/\theta^3 \leq \widetilde{r} \leq 2r_0/\theta \right\}.$$

Recalling $H = \chi_{S(x_0,2r_0)}h$ and using the inclusion $(2\theta)^{-1}S(x_0,2r_0) \subset S(x_0,r_0/\theta)$ from (B) this gives

$$\fint_{S(x_0,r_0/\theta^3)} \mathrm{M}_{2r_0\theta} H\,\mathrm{d}y \leq 2^n(N{+}1)\theta^{4n} \fint_{S(x_0,2r_0)} h\,\mathrm{d}y\,.$$

Combining the last estimate with (3.17) we find

$$\fint_{S(x_0,r_0/\theta^4)} (\mathrm{M}_{2r_0\theta}H)^{1+\varepsilon}\,\mathrm{d}y \leq 2^{n+1+n\varepsilon}(N{+}1)^{1+\varepsilon}\theta^{8n+9n\varepsilon} \left(\fint_{S(x_0,2r_0)} h\,\mathrm{d}y \right)^{1+\varepsilon}.$$

By the definition of H and Lemma 2.6 we additionally know that $0 \le h \le H \le \mathrm{M}_{2r_0\theta}H$ holds $\mathcal{L}^n$-a. e. on $S(x_0, r_0/\theta^4)$. Thus, we have $h \in \mathrm{L}^{1+\varepsilon}(S(x_0, r_0/\theta^4))$ and we finally arrive at (3.10) with $C = 2^{n+1+n\varepsilon}(N+1)^{1+\varepsilon}\theta^{8n+9n\varepsilon}$. □

4. $\mathrm{W}^{2,1+\varepsilon}$ ESTIMATES FOR C^2-SOLUTIONS

In this section we permanently fix a strictly convex Alexandrov solution $u \in \mathrm{C}^2(\Omega)$ of (1.1), we assume (1.3), we abbreviate $S(x,r) := S_u(x,r)$, and we also fix $E \subset\subset \Omega$ and the corresponding ϱ from Proposition 2.5.

The next result is the starting point for our application of the Gehring type results. It is a slightly reformulated version of the key lemma [14, Lemma 3.4] of De Philippis & Figalli.

Theorem 4.1 ($\mathrm{W}^{2,1}$ estimate, [14])**.** *For every section $S(x,r) \subset E$ with $r < \varrho/\theta$ one has*

$$|S(x,r) \cap \{\mathrm{M}_r \nabla^2 u \ge t\}| \le \Gamma |S(x, 2r\theta) \cap \{|\nabla^2 u| \ge \Gamma^{-1} t\}| \qquad \text{for all } t \ge 0$$

with a constant $\Gamma(n, \lambda, \Lambda) \ge 1$.

Theorem 4.1 can be proved along the lines of [14] without any essential modification of the respective arguments. We do not provide a detailed proof here, but we only comment on the slight differences between the above statement and the one in [14]. Indeed, [14, Lemma 3.4] was formulated with an r-independent maximal function (M_ϱ in the present terminology), for solutions with zero boundary values on a normalized convex domain $\mathcal{U}$, and for the dilations $\frac{1}{2}\mathcal{U}$ and $\frac{3}{4}\mathcal{U}$ in place of $S(x,r)$ and $S(x, 2r\theta)$, respectively. However, for $y \in S(x,r)$ we have $S(y, 2r) \subset S(x, 2r\theta)$ by (C); that means for all sections $S(y,r)$ in the definition of $\mathrm{M}_r \nabla^2 u(y)$ the enlarged section $S(y, 2r)$ is still contained in $S(x, 2r\theta)$, and this inclusion suffices for the relevant arguments of [14]. Let us stress in particular that the section $S(x,r)$ in Theorem 4.1 need not be normalized and Γ is independent of $S(x,r)$.

Combining Theorem 4.1 with our Gehring type result in Proposition 3.4 we readily derive an higher integrability estimate for $\nabla^2 u$.

Theorem 4.2 ($\mathrm{W}^{2,1+\varepsilon}$ estimate)**.** *For every section $S(x_0, 2r_0) \subset E$ with $r_0 < \varrho/\theta^2$ we have*

$$\fint_{S(x_0, r_0/\theta^4)} |\nabla^2 u|^{1+\varepsilon} \,\mathrm{d}y \le C \left(\fint_{S(x_0, 2r_0)} |\nabla^2 u| \,\mathrm{d}y \right)^{1+\varepsilon}$$

with positive constants $\varepsilon(n, \lambda, \Lambda)$ and $C(n, \lambda, \Lambda)$.

Proof. We consider an arbitrary section $S(x,r) \subset S(x_0, r_0)$. By (C) we then have $S(x,r) \subset S(x_0, r_0) \subset S(x, r_0\theta)$ and hence $r \le r_0\theta < \varrho/\theta$. In particular, we can apply Theorem 4.1 on $S(x,r)$, and using also the Fubini-Tonelli theorem we have

$$\begin{aligned}
\int_{S(x,r)} \mathrm{M}_r \nabla^2 u \,\mathrm{d}y &= \int_0^\infty |S(x,r) \cap \{\mathrm{M}_r \nabla^2 u \ge t\}| \,\mathrm{d}t \\
&\le \Gamma \int_0^\infty |S(x, 2r\theta) \cap \{|\nabla^2 u| \ge \Gamma^{-1} t\}| \,\mathrm{d}t \\
&= \Gamma^2 \int_{S(x, 2r\theta)} |\nabla^2 u| \,\mathrm{d}y \,.
\end{aligned}$$

Taking into account $S(x,2r\theta)\subset 2\theta S(x,r)$ by (B) and $|S(x,2r\theta)|\leq 2^n\theta^n|S(x,r)|$ we also get

$$\fint_{S(x,r)}\mathrm{M}_r\nabla^2 u\,\mathrm{d}y\leq 2^n\theta^n\Gamma^2\fint_{S(x,2r\theta)}|\nabla^2 u|\,\mathrm{d}y$$

for all $S(x,r)\subset S(x_0,r_0)$ and in particular for $S(x,r)\subset S(x_0,r_0/\theta^3)$. Consequently, we can apply Proposition 3.4 with $h:=|\nabla^2 u|$ and $N:=2^n\theta^n\Gamma^2$. It follows that $\nabla^2 u$ is in $\mathrm{L}^{1+\varepsilon}(S(x_0,r_0/\theta^4))$ with the claimed estimate. □

5. Proof of the main result

In order to establish Theorem 1.1 it remains to remove in Theorem 4.2 the C^2-assumption of the previous section. Indeed, it seems that this can be achieved, if one first uses the main result of [14] and then partially revisits the arguments of [14] once more to obtain Theorem 4.1 and Theorem 4.2 without the smoothness assumption. However, such a reasoning is somewhat awkward, and thus we prefer to follow a different line of proof via an approximation procedure. Notice that the approximation requires some care, since each approximating function brings its own sections. For this reason we now return to the full notation of Definition 2.3, indicating the function in question with an index, and we record the following simple lemma which will be useful in order to relate the sections of different functions.

Lemma 5.1. *Suppose that $v\in\mathrm{C}^1(\Omega)$ is strictly convex. If for $t\in\mathbb{R}$ and $p\in\mathbb{R}^n$ we have*

$$A_v(p,t):=\{y\in\Omega\,:\,v(y)<\langle p,y\rangle+t\}\subset\subset\Omega$$

and $A_v(p,t)\neq\emptyset$, then there exist $x\in A_v(p,t)$ and $r>0$ such that

$$A_v(p,t)=S_v(x,r)\,,\qquad \nabla v(x)=p\,,\qquad \textit{and}\qquad v(x)+r=\langle p,x\rangle+t\,.$$

Proof. By the strict convexity of v there is a unique $r>0$ such that the graph of $y\mapsto\langle p,y\rangle+t-r$ is a tangent hyperplane at the graph of v, and the two graphs touch each other in a single point $(x,v(x))=(x,\langle p,x\rangle+t-r)$. □

Proof of Theorem 1.1. In order to work with the sections of u, as introduced in Definition 2.3, we will make use of the fact that $u\in\mathrm{C}^1(\Omega)$ by [9, Theorem 2]. Fixing a section $S_0:=S_u(x_0,r_0)\subset\subset\Omega$ and $\gamma>0$ we set

$$E:=\{x\in\Omega\,:\,\mathrm{dist}(x,\mathbb{R}^n\backslash S_0)>\gamma\}\subset\subset S_0\,. \tag{5.1}$$

We denote by ϱ the positive number from Proposition 2.5, corresponding to the strictly convex domain S_0 (in place of Ω) and its subset E, and we record that in the present setup ϱ depends only on n, λ, Λ, S_0, and γ. Our aim is to establish the estimate of Theorem 4.2 for u on another fixed section

$$S_u(x,2r)\subset E$$

with $r<\varrho/\theta^2$, even though u need not be C^2.

To this end we first choose a sequence of smooth functions $f_k\in\mathrm{C}^\infty(\overline{S_0})$ such that $\lambda\leq f_k\leq\Lambda$ holds on S_0 with the constants from (1.3) and such that f_k converges to f in $\mathrm{L}^1(S_0)$; for instance the f_k can be taken as mollifications of f. By the existence and interior regularity result [13, Theorem 3] we then find Alexandrov solutions $u_k\in\mathrm{C}^2(S_0)$ of

$$\det(\nabla^2 u_k)=f_k\qquad\text{on }S_0$$

with $u_k = u$ on ∂S_0 (here we do not need to worry about the regularity of the boundary datum, since u coincides with an affine function on the boundary of the section S_0 and we can thus renormalize to zero boundary values). We record that the u_k — as C^2 solutions — are obviously strictly convex. By [29, Theorem 3.7] and the uniqueness of Alexandrov solutions (see for instance [24, Corollary 1.4.7]), u_k converges to u uniformly on S_0. Passing to a subsequence we assume

$$\sup_{S_0} |u_k - u| \le 1/k\,.$$

Following [14] we now combine [24, Lemma 3.2.1] and [24, Theorem 1.4.2] to see that the gradients ∇u_k are uniformly bounded on every compact subset of S_0. By the divergence theorem this implies the bound $\sup_{k\in\mathbb{N}} \int_{S_u(x,2r)} \Delta u_k \,\mathrm{d}y < \infty$ for the Laplacians (notice that $\Delta u_k \ge 0$ because u_k is convex). Passing to a subsequence we can therefore assume that $(\Delta u_k)\mathcal{L}^n$ weak-$*$-converges to a non-negative measure μ on $\overline{S_u(x,2r)}$, which represents on $S_u(x,2r)$ the distributional Laplacian of u.

Next we assume that k is sufficiently large ($k \ge 8r$ suffices). We observe

$$S_u(x, 2r{-}2/k) \subset \{y \in S_0 \,:\, u_k(y) < u(x) + \langle \nabla u(x), y{-}x\rangle + 2r - 1/k\} \subset S_u(x, 2r)\,.$$

Here, by Lemma 5.1 the intermediate set in the last formula can be written as a section $S_{u_k}(x_k, 2r_k)$ of u_k, so that we get

$$S_u(x, 2r{-}2/k) \subset S_{u_k}(x_k, 2r_k) \subset S_u(x, 2r)\,, \tag{5.2}$$

$\nabla u_k(x_k) = \nabla u(x)$, and $u_k(x_k) + 2r_k = u(x) + \langle \nabla u(x), x_k{-}x\rangle + 2r - 1/k$. By the convexity of u the last equality implies in particular

$$2r_k \le u(x_k) + 2r - 1/k - u_k(x_k) \le 2r\,.$$

Similarly, the rearranged equality $u_k(x_k) + \langle \nabla u_k(x_k), x{-}x_k\rangle + 2r_k = u(x) + 2r - 1/k$ and the convexity of u_k yield

$$2r_k \ge u(x) + 2r - 1/k - u_k(x) \ge 2r - 2/k\,,$$

and in conclusion r_k converges to r. Furthermore, Theorem 4.2, applied to the approximations u_k yields

$$\begin{aligned} \fint_{S_{u_k}(x_k, r_k/\theta^4)} |\nabla^2 u_k|^{1+\varepsilon}\,\mathrm{d}y &\le C\bigg(\fint_{S_{u_k}(x_k,2r_k)} |\nabla^2 u_k|\,\mathrm{d}y\bigg)^{1+\varepsilon} \\ &\le C\bigg(\fint_{S_{u_k}(x_k,2r_k)} \Delta u_k\,\mathrm{d}y\bigg)^{1+\varepsilon}, \end{aligned} \tag{5.3}$$

where we also exploited that $|\nabla^2 u_k| \le \Delta u_k$ due to convexity. In order to convert this into an estimate on the sections of u we now record

$$\begin{aligned} &S_u(x, 2r - 2/k - r_k(2{-}\theta^{-4})) \\ &\qquad \subset \{y \in S_0 \,:\, u_k(y) < u(x) + \langle \nabla u(x), y{-}x\rangle + 2r - 1/k - r_k(2{-}\theta^{-4})\} \\ &\qquad = \{y \in S_0 \,:\, u_k(y) < u_k(x_k) + \langle \nabla u_k(x_k), y{-}x_k\rangle + r_k/\theta^4\} \\ &\qquad = S_{u_k}(x_k, r_k/\theta^4)\,. \end{aligned}$$

By a quite similar computation we get

$$S_{u_k}(x_k, r_k/\theta^4) \subset S_u(x, 2r - r_k(2{-}\theta^{-4}))\,.$$

Now we use both the inclusions from (5.2) on the right-hand side of (5.3), and the two last computations on the left-hand side. Hence we arrive at

$$\frac{1}{|S_u(x,2r-r_k(2-\theta^{-4}))|}\int_{S_u(x,2r-2/k-r_k(2-\theta^{-4}))}|\nabla^2 u_k|^{1+\varepsilon}\,\mathrm{d}y \leq C\bigg(\frac{1}{|S_u(x,2r-2/k)|}\int_{S_u(x,2r)}\Delta u_k\,\mathrm{d}y\bigg)^{1+\varepsilon}. \tag{5.4}$$

Taking into account Lemma 2.4 we observe that the right-hand side of (5.4) is majorized by $C\Big(\frac{\mu(\overline{S_u(x,2r)})}{|S_u(x,2r)|}\Big)^{1+\varepsilon}$ in the limit $k\to\infty$. Since r_k converges to r, (5.4) consequently gives $\sup_{k\in\mathbb{N}}\int_P|\nabla^2 u_k|^{1+\varepsilon}\,\mathrm{d}y<\infty$ for every $P\subset\subset S_u(x,r/\theta^4)$, and passing once more to a subsequence $\nabla^2 u_k$ converges weakly in $\mathrm{L}^{1+\varepsilon}_{\mathrm{loc}}(S_u(x,r/\theta^4),\mathbb{R}^{n\times n})$. Thus, we have $u\in\mathrm{W}^{2,1+\varepsilon}_{\mathrm{loc}}(S_u(x,r/\theta^4))$, the limit in the preceding convergence is identified as $\nabla^2 u$, and sending $k\to\infty$ in (5.4) we finally get

$$⨍_{S_u(x,r/\theta^4)}|\nabla^2 u|^{1+\varepsilon}\,\mathrm{d}y\leq C\bigg(\frac{\mu(\overline{S_u(x,2r)})}{|S_u(x,2r)|}\bigg)^{1+\varepsilon}. \tag{5.5}$$

Recalling that $S_0=S_u(x_0,r_0)\subset\subset\Omega$, $\gamma>0$, and $S_u(x,2r)\subset\subset E$ (with $r<\varrho/\theta^2$ and E defined in (5.1)) are all arbitrary, we find $u\in\mathrm{W}^{2,1+\varepsilon}_{\mathrm{loc}}(\Omega)$. Going back in our above arguments we thus have $\mu=(\Delta u)\mathcal{L}^n$, and (5.5) reduces to the claim (1.4) with $\sigma:=1/\theta^4$. All in all, we have verified Theorem 1.1 with the constant C from Theorem 4.2 and $\varrho_0:=\varrho/\theta^2$. □

Remark 5.2. *If u is not in $\mathrm{C}^1(\Omega)$, one can still define sections allowing for an arbitrary $p\in\partial u(x)$ in place of $\nabla u(x)$; see* [24, Definition 3.1.1]. *We believe that working with this more general definition one can avoid the usage of the C^1-regularity result* [9, Theorem 2] *in the above proof.*

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